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ON A CLOSE-TO-CONVEX ANALOGUE OF CERTAIN STARLIKE FUNCTIONS

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Abstract

For f analytic in the unit disk $\mathbb{D}$, we consider the close-to-convex analogue of a class of starlike functions introduced by R. Singh ['On a class of star-like functions', *Compos. Math.* **19**(1) (1968), 78–82]. This class of functions is defined by $|zf'(z)/g(z) - 1| < 1$ for $z \in \mathbb{D}$, where g is starlike in $\mathbb{D}$. Coefficient and other results are obtained for this class of functions.

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1. Preliminaries

Let $\mathcal{H}$ denote the class of functions f analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ the subclass of $\mathcal{H}$ consisting of functions normalised by $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S} \subset \mathcal{A}$ be the class of univalent (that is, one-to-one) functions in $\mathbb{D}$. Any function $f \in \mathcal{A}$ has the series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ of starlike functions. It is well known that $f \in \mathcal{S}^*$ if and only if $\mathfrak{Re}(zf'(z)/f(z)) > 0,\ z \in \mathbb{D}$. Denote by C the subclass of $\mathcal{S}^*$ of convex functions. It is well known that $f \in \mathcal{S}^*$ if and only if $f(z) = zg'(z)$, for some $g \in C$. By $\mathcal{P}$ we denote the class of Carathéodory functions p which are analytic in $\mathbb{D}$ and satisfy the condition $\mathfrak{Re}(p(z)) > 0$ for $z \in \mathbb{D}$, with

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{1.2}$$

Suppose now that f is analytic in $\mathbb{D}$. Then f is close to convex if and only if there exist $\alpha \in (-\pi/2, \pi/2)$ and a function $g \in \mathcal{S}^*$ such that $\mathfrak{Re}(e^{i\alpha}\, zf'(z)/g(z)) > 0,\ z \in \mathbb{D}$.

When $\alpha = 0$, we denote this class of close-to-convex functions by $\mathcal{K}$, and note that $\mathcal{S}^* \subset \mathcal{K} \subset \mathcal{S}$.

Suppose next that $f \in \mathcal{A}$ and is given by (1.1) and satisfies $|zf'(z)/f(z) - 1| < 1$ for $z \in \mathbb{D}$. This class of functions $\mathcal{S}_u^*$ was introduced in 1968 by Singh [9]. It is clear that $\mathcal{S}_u^* \subset \mathcal{S}^*$. Singh [9] showed that if $f \in \mathcal{S}_u^*$, then $|a_n| \leq 1/(n-1)$ for $n \geq 2$, and that this inequality is sharp. Other properties of functions in $\mathcal{S}_u^*$ were also given in [9].

We now define the close-to-convex analogue of the class $\mathcal{S}_u^*$ as follows.

Definition 1.1. We say that $f \in \mathcal{K}_u$ if $f \in \mathcal{A}$ and there exists $g \in \mathcal{S}^*$ such that

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < 1, \quad z \in \mathbb{D}.$$

Again it is clear that $\mathcal{S}_u^* \subset \mathcal{K}_u \subset \mathcal{K} \subset \mathcal{S}$. Although $\mathcal{K}_u$ represents the natural close-to-convex analogue of $\mathcal{S}_u^*$, we shall see that obtaining sharp estimates for the coefficients represents a much more difficult problem. We note that this phenomenon is often reflected in extending results from $\mathcal{S}^*$ to $\mathcal{K}$ and will see in this paper that the class $\mathcal{K}_u$ gives rise to some significant and interesting problems.

2. Lemmas

A function ω is called a Schwarz function if $\omega \in \mathcal{H}$, $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \mathbb{D}$. We denote the class of Schwarz functions by Ω.

Note that for $p \in \mathcal{P}$ given by (1.2), we can write $p(z) = (1 + \omega(z))/(1 - \omega(z))$, for some $\omega \in \Omega$. So writing

$$\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n \tag{2.1}$$

and equating coefficients gives

$$p_1 = 2\omega_1, \quad p_2 = 2\omega_2 + 2\omega_1^2. \tag{2.2}$$

We will need the following lemmas.

Lemma 2.1 [2, page 78]. *Let $\omega \in \Omega$ be given by (2.1). Then*

$$|\omega_{2n-1}| \leq 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - \cdots - |\omega_n|^2 \quad \textit{for } n = 2, 3, \ldots,$$
$$|\omega_{2n}| \leq 1 - |\omega_1|^2 - |\omega_2|^2 - |\omega_3|^2 - \cdots - |\omega_{n-1}|^2 - |\omega_n|^2 \quad \textit{for } n = 1, 2, 3, \ldots.$$

Lemma 2.2 [3]. *Let $\omega \in \Omega$ be given by (2.1). If $\mu \in \mathbb{C}$, then*

$$|\omega_2 - \mu\,\omega_1^2| \leq \max\{1, |\mu|\}. \tag{2.3}$$

Using (2.2) and (2.3) immediately gives the following result.

Lemma 2.3 [5]. *Let $p \in \mathcal{P}$ be given by (1.2). Then for $\mu \in \mathbb{C}$,*

$$|p_2 - \mu p_1^2| \leq 2\max\{1, |2\mu - 1|\}.$$

The inequality is sharp for each complex μ.

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LEMMA 2.4 [7, 10]. *Let $p \in \mathcal{P}$ be given by (1.2). Then for $n \geq 1$, $|p_n| \leq 2$ and*

$$\left|p_2 - \tfrac{1}{2}p_1^2\right| \leq 2 - \tfrac{1}{2}\left|p_1^2\right|.$$

The following Fekete–Szegö type inequalities for $g \in \mathcal{S}^*$ due to Keogh and Merkes [3] will be used extensively in Section 5.

LEMMA 2.5 [3]. *Let $g \in \mathcal{S}^*$ be given by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2.4}$$

Then for any $\mu \in \mathbb{C}$,

$$|b_3 - \mu b_2^2| \leq \max\{1, |4\mu - 3|\},$$
$$|b_3 - \mu b_2^2| \leq 1 + (|4\mu - 3| - 1)\frac{|b_2|^2}{4}. \tag{2.5}$$

Both inequalities are sharp.

We will also use the following lemma concerning functions in $\mathcal{P}$, the proof of which follows easily from Lemma 2.5.

LEMMA 2.6. *Let $p \in \mathcal{P}$. Then for any $t \in \mathbb{C}$,*

$$|p_2 - tp_1^2| \leq 2 + (|2t - 1| - 1)\frac{|p_1|^2}{2}. \tag{2.6}$$

The inequality is sharp.

PROOF. For $p \in \mathcal{P}$, there exists a function $g \in \mathcal{S}^*$ given by (2.4) such that

$$p(z) = \frac{zg'(z)}{g(z)}, \quad z \in \mathbb{D}.$$

Thus $b_2 = p_1$ and $b_3 = \frac{1}{2}(p_2 + p_1^2)$. Substituting in (2.5) gives

$$|p_2 - (2\mu - 1)p_1^2| \leq 2 + (|4\mu - 3| - 1)\frac{|p_1|^2}{2},$$

for all complex μ. Writing $\mu = (t + 1)/2$ gives (2.6) for all complex t. The function $p(z) = (1 + z)/(1 - z)$ shows that the result is sharp for $|2t - 1| \geq 1$ and the function $p(z) = (1 + z^2)/(1 - z^2)$ shows the sharpness for $|2t - 1| \leq 1$. □

LEMMA 2.7 [1], [8, page 67]. *Suppose that $f \in \mathcal{S}$ and $z = re^{i\theta} \in \mathbb{D}$. If*

$$m'(r) \leq |f'(z)| \leq M'(r),$$

where $m'(r)$ and $M'(r)$ are real-valued functions of r in $[0, 1)$, then

$$\int_0^r m'(t)\, dt \leq |f(z)| \leq \int_0^r M'(r)\, dt.$$

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LEMMA 2.8 (Baernstein's theorem, [2, page 198]). *Let $\Phi(x)$ be a convex nondecreasing function for $-\infty < x < \infty$. If $f \in \mathcal{S}$, then*

$$\int_0^{2\pi} \Phi(\ln|f(re^{i\theta})|)\,d\theta \le \int_0^{2\pi} \Phi(\ln|k(re^{i\theta})|)\,d\theta,$$

where $k(z) = z/(1-z)^2$. If equality holds for some r in $(0,1)$ and strictly convex Φ, then $f(z) = \eta k(\overline{\eta} z)$ for some $|\eta| = 1$.

LEMMA 2.9 [2, page 200]. *If $f \in C$, then*

$$\int_0^{2\pi} |f'(z)|^\lambda\,d\theta \le \int_0^{2\pi} |F'(z)|^\lambda\,d\theta$$

for all $\lambda \ge 0$, $0 < r < 1$, where $F(z) = z/(1-z)$ and $z = re^{i\theta}$.

LEMMA 2.10 [6, page 70]. *Suppose that $h \in \mathcal{H}$ is convex and univalent and $P \in \mathcal{H}$ satisfies $\Re\mathfrak{e}(P(z)) > 0$ for $z \in \mathbb{D}$. If $p \in \mathcal{H}$, then*

$$p(z) + P(z)\cdot zp'(z) \prec h(z) \Rightarrow p(z) \prec h(z).$$

3. Distortion theorems and integral means

THEOREM 3.1. *If $f \in \mathcal{K}_u$ and $z = re^{i\theta}$, $0 \le r < 1$, then*

$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2}, \tag{3.1}$$

$$\frac{2r}{1+r} - \log(1+r) \le |f(z)| \le \frac{2r}{1-r} + \log(1-r). \tag{3.2}$$

The inequalities are sharp.

PROOF. Write

$$f'(z) = \frac{g(z)}{z}(1+\omega(z)), \tag{3.3}$$

for some $g \in \mathcal{S}^*$ and some $\omega \in \Omega$. It is well known that for $g \in \mathcal{S}^*$, with $z = re^{i\theta}$, $0 \le r < 1$,

$$\frac{1}{(1+r)^2} \le \left|\frac{g(z)}{z}\right| \le \frac{1}{(1-r)^2}. \tag{3.4}$$

Thus, using the Schwarz lemma,

$$1 - r \le |1+\omega(z)| \le 1 + r, \tag{3.5}$$

and so from (3.3), using (3.4) and (3.5), we immediately obtain (3.1).

The inequalities in (3.1) are sharp when $f_1 \in \mathcal{K}_u$ is given by

$$f_1'(z) = \frac{g_0(z)}{z}(1+z) \quad \text{and} \quad g_0(z) = \frac{z}{(1-z)^2},$$

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in which case

$$f_1'(-r) = \frac{1-r}{(1+r)^2} \quad \text{and} \quad f_1'(r) = \frac{1+r}{(1-r)^2}.$$

Clearly (3.2) follows from Lemma 2.7, since $\mathcal{K}_u \subset \mathcal{S}$. The upper bound in (3.2) is sharp for $f_1 \in \mathcal{K}_u$ and the lower bound for $f_2 \in \mathcal{K}_u$, where

$$f_1(z) = \frac{2z}{1-z} + \log(1-z) \quad \text{and} \quad f_2(z) = \frac{2z}{1+z} - \log(1+z). \qquad \square$$

In the following two integral mean inequalities, the function f_1 shows that the orders of growth as $r \to 1$ are best possible (see [4, page 96]). However, the inequalities are not sharp.

Theorem 3.2. *Let $f \in \mathcal{K}_u$ be given by (1.1). Then with $z = re^{i\theta} \in \mathbb{D}$,*

$$\int_0^{2\pi} |f(z)|\, d\theta \le 2\pi \log \frac{1}{1-r}, \tag{3.6}$$

$$\int_0^{2\pi} |f'(z)|\, d\theta \le \frac{2\pi}{1-r}. \tag{3.7}$$

Proof. Write

$$f'(z) = h'(z)(1+\omega(z)), \tag{3.8}$$

for some $h \in C$ and some $\omega \in \Omega$. Integrating (3.8) and using the Schwarz lemma,

$$\begin{aligned}\int_0^{2\pi} |f(z)|\, d\theta &= \int_0^{2\pi} \left| \int_0^r h'(\rho e^{i\theta})[1+\omega(\rho e^{i\theta})]\, d\rho \right| d\theta \\ &\le \int_0^r (1+\rho) \int_0^{2\pi} |h'(\rho e^{i\theta})|\, d\theta\, d\rho.\end{aligned}$$

Applying Lemma 2.9 with $n = 2$ and using Parseval's theorem,

$$\int_0^{2\pi} |f(z)|\, d\theta \le \int_0^r \int_0^{2\pi} \frac{1+\rho}{|1-\rho e^{i\theta}|^2}\, d\theta\, d\rho = \int_0^r \frac{2\pi}{1-\rho}\, d\rho = 2\pi \log \frac{1}{1-r}.$$

This gives (3.6). In order to prove (3.7), we write

$$f'(z) = \frac{g(z)}{z}(1+\omega(z)), \tag{3.9}$$

for some $g \in \mathcal{S}^*$ and some $\omega \in \Omega$. Integrating (3.9) and using the Schwarz lemma,

$$\int_0^{2\pi} |f'(z)|\, d\theta = \int_0^{2\pi} \left| \frac{g(re^{i\theta})}{re^{i\theta}} [1+\omega(re^{i\theta})] \right| d\theta \le \frac{1+r}{r} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta.$$

Applying Lemma 2.8,

$$\int_0^{2\pi} |f'(z)|\, d\theta \le \frac{1+r}{r} \int_0^{2\pi} |g(re^{i\theta})|\, d\theta \le \frac{1+r}{r} \frac{2\pi r}{1-r^2} = \frac{2\pi}{1-r},$$

which is (3.7). $\square$

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Let $C(r, f)$ denote the image of the circle $|z| = r < 1$ under the mapping f and $L(r, f)$ the length of $C(r, f)$. We immediately deduce the following corollary.

Corollary 3.3. *Let $f \in \mathcal{K}_u$. Then with $z = re^{i\theta} \in \mathbb{D}$,*

$$L(r, f) = \int_0^{2\pi} |zf'(z)|\, d\theta \le \frac{2\pi r}{1 - r}.$$

Theorem 3.4. *Let $f \in \mathcal{K}_u$. Then with $z = re^{i\theta} \in \mathbb{D}$,*

$$\int_0^{2\pi} |f'(z) - h'(z)|\, d\theta \le \frac{2\pi r}{1 - r^2}, \tag{3.10}$$

where $h \in C$ and $h'(z) = f'(z)/(1 + \omega(z))$ for some $\omega \in \Omega$. Inequality (3.10) is sharp.

Proof. Write (3.8) as $f'(z) - h'(z) = h'(z)\omega(z)$, with $\omega \in \Omega$. By the Schwarz lemma,

$$|f'(z) - h'(z)| = |h'(z)\omega(z)| \le |zh'(z)| = |g(z)|, \tag{3.11}$$

for some $g \in \mathcal{S}^*$. Integrating (3.11) and applying Lemma 2.8,

$$\int_0^{2\pi} |f'(z) - h'(z)|\, d\theta \le \int_0^{2\pi} |g(z)|\, d\theta = \frac{2\pi r}{1 - r^2}.$$

Choosing $h(z) = z/(1 - z)$ and $\omega(z) = z$, we have $f(z) = 2z/(1 - z) + \log(1 - z)$, which gives equality in (3.10). □

4. Coefficients

Singh [9] was able to use the method of Clunie to obtain sharp coefficient estimates for functions in $\mathcal{S}_u^*$. Since this is not possible in $\mathcal{K}_u$, the problem of extending the coefficient inequalities in [9] to the class $\mathcal{K}_u$ appears not to be straightforward, with exact bounds not easy to find. We give the following results.

Theorem 4.1. *Let $f \in \mathcal{K}_u$ be given by (1.1). Then*

$$|a_2| \le \frac{3}{2}, \quad |a_3| \le \frac{5}{3}, \quad |a_4| \le \frac{7.3731\ldots}{4} = 1.8443\ldots, \quad |a_5| \le \frac{8}{5} + \frac{3}{5\sqrt[3]{4}} = 1.97\ldots.$$

The inequalities for $|a_2|$ and $|a_3|$ are sharp.

Proof. Write

$$zf'(z) = g(z)[1 + \omega(z)], \tag{4.1}$$

for some $g \in \mathcal{S}^*$ and some $\omega \in \Omega$. Equating coefficients in (4.1) and using (2.1) and (2.4) gives

$$2a_2 = b_2 + w_1, \tag{4.2}$$
$$3a_3 = b_3 + b_2w_1 + w_2, \tag{4.3}$$
$$4a_4 = b_4 + b_3w_1 + b_2w_2 + w_3, \tag{4.4}$$

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where $|b_n| \le n$ and $|w_n| \le 1$ for $n \ge 1$. Therefore (4.2) gives

$$2|a_2| \le |b_2| + |w_1| \Rightarrow 2|a_2| \le 3.$$

Now write $x_1 = |w_1|$, $x_2 = |w_2|$ and $x_3 = |w_3|$. From (4.3),

$$3|a_3| \le |b_3| + |b_2||w_1| + |w_2|,$$

so that Lemma 2.1 implies

$$3|a_3| \le 3 + 2|w_1| + (1 - |w_1|^2) \le 5,$$

since $0 \le 4 + 2x_1 - x_1^2 \le 5$ for $x_1 \in [0, 1]$.

The bound for $|a_4|$ is more complicated. Again from (4.4) and Lemma 2.1,

$$4|a_4| \le |b_4| + |b_3|w_1| + |b_2||w_2| + |w_3| \le 4 + 3x_1 + 2x_2 + x_3,$$

and so

$$0 \le x_1 \le 1, \quad x_2 \le 1 - x_1^2, \quad x_3 \le 1 - x_1^2 - x_2^2.$$

We therefore need to find

$$\max_H g(x_1, x_2, x_3),$$

where $g(x_1, x_2, x_3) = 4 + 3x_1 + 2x_2 + x_3$ and

$$H = \{(x_1, x_2, x_3) : \ x_1 \le 1, \ x_2 \le 1 - x_1^2, \ x_3 \le 1 - x_1^2 - x_2^2\}.$$

It is clear that the maximum over H occurs on the boundary ∂H which we now consider. If $x_3 = 1 - x_1^2 - x_2^2$ and $x_2 = 1 - x_1^2$, then

$$g(x_1, x_2, x_3) = 6 + 3x_1 - x_1^2 - x_1^4, \quad 0 \le x_1 \le 1.$$

Solving this equation (using Wolfram Alpha),

$$\max\{6 + 3x_1 - x_1^2 - x_1^4 : \ 0 \le x_1 \le 1\} = 7.3731\ldots \quad \text{at } x_1 = 0.72808\ldots,$$

where

$$7.3731\ldots = \frac{1}{24}\left\{148 - \frac{968}{\sqrt[3]{54181 + 2259\sqrt{753}}} + \sqrt[3]{54181 + 2259\sqrt{753}}\right\},$$

$$0.72808\ldots = \frac{\sqrt[3]{27 + \sqrt{753}}}{2\sqrt{39}} - \frac{1}{\sqrt[3]{3(27 + \sqrt{753})}}.$$

Hence $|a_4| \le \frac{1}{4} \cdot 7.3731\ldots = 1.8443\ldots$. Applying the same method for a_5 gives

$$5|a_5| \le 8 + \frac{3}{\sqrt[3]{4}} = 9.889\ldots, \quad \text{that is, } |a_5| \le 1.97\ldots.$$

The inequalities for a_2 and a_3 are sharp when

$$f(z) = \frac{2z}{1 - z} + \log(1 - z) = z + \sum_{n=2}^{\infty} \frac{2n - 1}{n} z^n. \qquad \square$$

Nonsharp bounds for $|a_n|$ when $n \geq 5$ can be obtained by the techniques used in the proof of Theorem 4.1. However, the analysis becomes more involved as n increases, and requires computer-aided numerical methods.

Inequalities for the coefficients of close-to-convex functions can exhibit unpredictable behaviour (compare the solution to the Fekete–Szegö problem [3]). On the basis of the extremal function for the coefficients a_2 and a_3 above, the obvious conjecture is the following, but this may prove not to be correct.

CONJECTURE 4.2. Let $f \in \mathcal{K}_u$ be given by (1.1). Then for $n \geq 2$,

$$|a_n| \leq \frac{2n-1}{n}.$$

A simple consequence of Corollary 3.3 shows that the coefficients a_n of functions f in $\mathcal{K}_u$ are bounded. To see this, let $f \in \mathcal{K}_u$ be given by (1.1). From Corollary 3.3, with $z = re^{i\theta} \in \mathbb{D}$,

$$n|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)|\, d\theta \leq \frac{1}{r^{n-1}(1-r)}.$$

Choosing $r = 1 - 1/n$ gives $n|a_n| \leq n(1-1/n)^{-n}$ and, since $(1-1/n)^{-n}$ decreases for $n \geq 2$, we obtain $|a_n| \leq 4$.

5. Fekete–Szegö theorems

THEOREM 5.1. *Let $f \in \mathcal{K}_u$ be given by (1.1) and let $\mu \in \mathbb{R}$.*

(1) *If $\mu \leq 0$, then*

$$|a_3 - \mu a_2^2| \leq \tfrac{5}{3} - \tfrac{9}{4}\mu. \tag{5.1}$$

(2) *If $0 \leq \mu \leq 2/3$, then*

$$|a_3 - \mu a_2^2| \leq \frac{2(10 - 18\mu + 9\mu^2)}{3(4 - 3\mu)}.$$

(3) *If $2/3 \leq \mu \leq 1$, then*

$$|a_3 - \mu a_2^2| \leq \tfrac{2}{3}. \tag{5.2}$$

(4) *If $1 \leq \mu \leq 10/9$, then*

$$|a_3 - \mu a_2^2| \leq \frac{3\mu - 5}{3(3\mu - 4)}. \tag{5.3}$$

(5) *If $\mu \geq 10/9$, then*

$$|a_3 - \mu a_2^2| \leq \tfrac{9}{4}\mu - \tfrac{5}{3}. \tag{5.4}$$

Inequalities (5.1), (5.2) and (5.4) are sharp.

PROOF. Since $f \in \mathcal{K}_u$, we can write

$$zf'(z) = g(z)\left[\frac{2p(z)}{1 + p(z)}\right], \tag{5.5}$$

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where $p \in \mathcal{P}$ and $g \in \mathcal{S}^*$. Equating coefficients in (5.5), using (1.2) and (2.4) gives

$$a_3 - \mu a_2^2 = \frac{1}{3}\Big(b_3 - \frac{3b_2^2\mu}{4}\Big) + \frac{b_2 p_1}{12}(2-3\mu) + \frac{1}{6}\Big(p_2 - \frac{p_1^2}{2}\Big) - \frac{1}{16}p_1^2\mu, \tag{5.6}$$

$$= \frac{1}{3}\Big(b_3 - \frac{3b_2^2\mu}{4}\Big) + \frac{b_2 p_1}{12}(2-3\mu) + \frac{1}{6}\Big(p_2 - \frac{4+3\mu}{8}p_1^2\Big). \tag{5.7}$$

We now treat the five cases in the theorem.

Case 1: $\mu \le 0$. We use (5.6) with $|p_1| = x$. From Lemmas 2.4 and 2.5, since $|b_2| \le 2$,

$$\begin{aligned} |a_3 - \mu a_2^2| &= \Big|\frac{1}{3}\Big(b_3 - \frac{3b_2^2\mu}{4}\Big) + \frac{b_2 p_1}{12}(2-3\mu) + \frac{1}{6}\Big(p_2 - \frac{p_1^2}{2}\Big) - \frac{1}{16}p_1^2\mu\Big| \\ &\le \frac{1}{3}|3\mu - 3| + \frac{1}{6}|2-3\mu|x + \frac{1}{6}\Big|2 - \frac{x^2}{2}\Big| - \frac{1}{16}x^2\mu \\ &= \frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\Big(2 - \frac{x^2}{2}\Big) - \frac{1}{16}x^2\mu, \end{aligned} \tag{5.8}$$

where $x \in [0,2]$. Since the right-hand side of (5.8) increases with respect to $x \in [0,2]$,

$$|a_3 - \mu a_2^2| \le \Big[\frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\Big(2 - \frac{x^2}{2}\Big) - \frac{1}{16}x^2\mu\Big]_{x=2} = \frac{5}{3} - \frac{9\mu}{4}.$$

The result is sharp for $b_3 = 3, b_2 = p_1 = p_2 = 2$ in (5.6), that is, $g(z) = z/(1-z)^2$, $p(z) = (1+z)/(1-z)$.

Case 2: $0 \le \mu \le 2/3$. We again use (5.6) with $x = |p_1|$, which gives

$$|a_3 - \mu a_2^2| \le \frac{1}{3}(3-3\mu) + \frac{1}{6}(2-3\mu)x + \frac{1}{6}\Big(2 - \frac{x^2}{2}\Big) + \frac{1}{16}x^2\mu.$$

This expression has a maximum value at $x = 4(3\mu-2)/(3\mu-4)$ in $[0,2]$, so the bound for $0 \le \mu \le 2/3$ follows.

Case 3: $2/3 \le \mu \le 1$. Applying (2.5) and (2.6) in (5.7),

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{1}{3}\Big(1 + (|3\mu - 3| - 1)\frac{|b_2|^2}{4}\Big) + \frac{|b_2 p_1|}{12}|2-3\mu| \\ &\quad + \frac{1}{6}\Big(2 + \Big(\Big|\frac{4+3\mu}{4} - 1\Big| - 1\Big)\frac{|p_1|^2}{2}\Big) \\ &\le \frac{1}{3}\Big(1 - \frac{3\mu-2}{4}|b_2|^2\Big) + \frac{3\mu-2}{12}|p_1||b_2| + \frac{1}{6}\Big(2 - \frac{4-3\mu}{4}\frac{|p_1|^2}{2}\Big) \\ &= -\frac{3\mu-2}{12}|b_2|^2 + \frac{3\mu-2}{12}|p_1||b_2| - \frac{4-3\mu}{48}|p_1|^2 + \frac{2}{3} \\ &= \frac{3\mu-2}{12}\Big(-y^2 + xy - \frac{4-3\mu}{4(3\mu-2)}x^2\Big) + \frac{2}{3}, \end{aligned} \tag{5.9}$$

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where $y = |b_2| \in [0, 2]$, $x = |p_1| \in [0, 2]$. If $\mu = 2/3$, (5.2) follows at once from (5.9).

If $\mu \neq 2/3$, we divide by $3\mu - 2$, so that it suffices to show that

$$F(x, y) = -y^2 + xy - \frac{4 - 3\mu}{4(3\mu - 2)} x^2 \leq 0 \quad \text{for } 2/3 < \mu \leq 1, y \in [0, 2], x \in [0, 2].$$

Since $F(x, y)$ has no critical points in $(0, 2) \times (0, 2)$, we only need to check that $F(x, y) \leq 0$ when $x = 0$ or $y = 0$, which is trivial, and when $x = 2$ or $y = 2$. If $x = 2$,

$$F(2, y) = -y^2 + 2y - \frac{4 - 3\mu}{3\mu - 2} = -(y - 1)^2 - \frac{6(1 - \mu)}{3\mu - 2} \leq 0 \quad \text{when } 2/3 < \mu \leq 1,$$

and if $y = 2$,

$$F(x, 2) = -2(2 - x) - \frac{4 - 3\mu}{4(3\mu - 2)} x^2 \leq 0 \quad \text{when } 2/3 < \mu \leq 1,$$

which establishes (5.2). To show the result is sharp we choose $b_2 = 0$, $b_3 = 1$, $p_1 = 0$ and $p_2 = 2$ in (5.7), that is, $g(z) = z/(1 - z^2)$, $p(z) = (1 + z^2)/(1 - z^2)$.

Case 4: $1 \leq \mu \leq 10/9$. Applying (2.5) and (2.6) in (5.7) gives, for all $\mu \geq 1$,

$$\begin{aligned}
|a_3 - \mu a_2^2| &\leq \frac{1}{3}\Big(1 + (|3\mu - 3| - 1)\frac{|b_2|^2}{4}\Big) + \frac{|b_2 p_1|}{12}|2 - 3\mu| \\
&\quad + \frac{1}{6}\Big(2 + \Big(\Big|\frac{4 + 3\mu}{4} - 1\Big| - 1\Big)\frac{|p_1|^2}{2}\Big) \\
&\leq \frac{1}{3}\Big(1 - \frac{4 - 3\mu}{4}|b_2|^2\Big) + \frac{3\mu - 2}{12}|p_1||b_2| + \frac{1}{6}\Big(2 - \frac{4 - 3}{4}\frac{|p_1|^2}{2}\Big) \\
&= -\frac{4 - 3\mu}{12}|b_2|^2 + \frac{3\mu - 2}{12}|p_1||b_2| - \frac{4 - 3\mu}{48}|p_1|^2 + \frac{2}{3} \\
&= \frac{4 - 3\mu}{48}\Big(-4y^2 + \frac{4(3\mu - 2)}{4 - 3\mu}xy - x^2\Big) + \frac{2}{3} := F(x, y),
\end{aligned}$$

where $y = |b_2| \in [0, 2]$, $x = |p_1| \in [0, 2]$. Thus to prove (5.3) it suffices to establish that

$$F(x, y) = \frac{2}{3} + \frac{4 - 3\mu}{48}\Big(- 4y^2 + \frac{4(3\mu - 2)}{4 - 3\mu}xy - x^2\Big) \leq \frac{2}{3} + \frac{\mu - 1}{4 - 3\mu} \tag{5.10}$$

for $1 \leq \mu \leq 10/9$, $y \in [0, 2]$ and $x \in [0, 2]$.

Again $F(x, y)$ has no critical points in $(0, 2) \times (0, 2)$, so we only need to check that $F(x, y) \leq 0$ when $x = 0$ or $y = 0$, and when $x = 2$ or $y = 2$. It is clear from (5.10) that in these four cases $F(x, y)$ attains the greatest value when $x = 2$, and

$$\begin{aligned}
\max_{0 \leq y \leq 2} F(2, y) &= \frac{2}{3} + \Big[\frac{4 - 3\mu}{48}\Big(- 4y^2 + \frac{8(3\mu - 2)}{4 - 3\mu}y - 4\Big)\Big]_{y=(3\mu-2)/(4-3\mu)} \\
&= \frac{2}{3} + \frac{\mu - 1}{4 - 3\mu} = \frac{3\mu - 5}{3(3\mu - 4)}.
\end{aligned}$$

This gives (5.3).

Case 5: $\mu \geq 10/9$. From (5.7) with $x = |p_1|$ and $y = |b_2|$,

$$|a_3 - \mu a_2^2| \leq \frac{1}{3}\Big(1 + (3\mu - 4)\frac{y^2}{4}\Big) + \frac{xy}{12}(3\mu - 2) + \frac{1}{6}\Big(2 - \frac{x^2}{2}\Big) + \frac{\mu x^2}{16} := H(x, y).$$

Since the only critical point of $H(x, y)$ is at $x = y = 0$ and $H(0, 0) = 2/3$, we only need to check the boundary values of $H(x, y)$ on $[0, 2] \times [0, 2]$. Observe that:

(i) $H(0, y) = \frac{1}{3} + \frac{1}{3}(1 + \frac{1}{4}(3\mu - 4)y^2) \leq \frac{1}{3}(3\mu - 2) \leq \frac{9}{4}\mu - \frac{5}{3}$ when $\mu \geq \frac{10}{9}$ and $0 \leq y \leq 2$;

(ii) $H(2, y) = \frac{1}{4}\mu + \frac{1}{6}(3\mu - 2)y + \frac{1}{3}(1 + \frac{1}{4}(3\mu - 4)y^2)$, which increases on $y \in [0, 2]$ and so again $H(2, y) \leq \frac{9}{4}\mu - \frac{5}{3}$;

(iii) $H(x, 0) = \frac{2}{3} + \frac{1}{48}(3m - 4)x^2$ and $H'(x, 0) = 0$ when either $x = 0$ or $\mu = \frac{4}{3}$, but $H(\frac{4}{3}, 0) = \frac{2}{3} \leq \frac{1}{4}\mu - \frac{5}{3}$ so we only need to consider $x = 0$ and $x = 2$, where $H(0, 0) = \frac{2}{3}$ and $H(2, 0) = \frac{1}{3} + \frac{1}{4}\mu \leq \frac{9}{4}\mu - \frac{5}{3}$, giving the result in this case;

(iv) $H(x, 2) = \frac{1}{3}(3\mu - 3) + \frac{1}{6}(3\mu - 2)x + \frac{1}{16}\mu x^2 + \frac{1}{6}(2 - \frac{1}{2}x^2)$ is increasing for $x \in [0, 2]$ when $\mu \geq \frac{10}{9}$ and $H(2, 2) = \frac{9}{4}\mu - \frac{5}{3}$, which completes the proof.

The result is sharp on choosing $b_3 = 3, b_2 = p_1 = p_2 = 2$ in (5.8), that is, $g(z) = z/(1 - z)^2$, $p(z) = (1 + z)/(1 - z)$. □

The following Fekete–Szegö theorem for complex μ is probably not sharp.

THEOREM 5.2. *Let $f \in \mathcal{K}_u$ be given by (1.1). For $\mu \in \mathbb{C}$,*

$$|a_3 - \mu a_2^2| \leq \tfrac{1}{3}[\max\{1, |4\mu_1 - 3|\} + \max\{1, |2\mu_2 - 1|\} + |2 - 3\mu|], \tag{5.11}$$

where $\mu_1 = 3\mu/4$ and $\mu_2 = (4 + 3\mu)/8$.

PROOF. From (5.7),

$$\begin{aligned}|a_3 - \mu a_2^2| &\leq \frac{1}{3}\Big|b_3 - \frac{3\mu}{4}b_2^2\Big| + \frac{1}{12}|b_2 p_1||2 - 3\mu| + \frac{1}{6}\Big|p_2 - \frac{4 + 3\mu}{8}p_1^2\Big| \\ &\leq \frac{1}{3}\Big|b_3 - \frac{3\mu}{4}b_2^2\Big| + \frac{1}{3}|2 - 3\mu| + \frac{1}{6}\Big|p_2 - \frac{4 + 3\mu}{8}p_1^2\Big|.\end{aligned}$$

Applying Lemmas 2.3 and 2.5 gives (5.11). □

6. The radius of convexity and starlikeness

A number $r_0 \in [0, 1]$ is called the radius of convexity of a particular subclass of $\mathcal{A}$ if r_0 is the largest number such that $\Re\mathrm{e}(1 + zf''(z)/f'(z)) > 0$ for all f in the subclass and $|z| < r_0$. It was shown in [9] that the radius of convexity for functions in $\mathcal{S}_u^*$ is $(\sqrt{13} - 3)/2$. We now show that when $f \in \mathcal{K}_u$, the radius of convexity is $1/3$.

THEOREM 6.1. *The radius of convexity of $\mathcal{K}_u$ is* $1/3$.

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PROOF. For $f \in \mathcal{K}_u$, we write $zf'(z) = g(z)[1 + \omega(z)]$, for some $g \in \mathcal{S}^*$ and some $\omega \in \Omega$. Thus

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{z\omega'(z)}{1 + \omega(z)}. \tag{6.1}$$

It is well known (see [10]), that for $g \in \mathcal{S}^*$, with $z = re^{i\theta}$ and $0 \le r < 1$,

$$\Re e\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1 - r}{1 + r}.$$

Also from the Schwarz lemma, $|\omega(z)| \le |z| = r$ and from [2, page 77],

$$|\omega'(z)| \le \frac{1 - |\omega(z)|^2}{1 - |z|^2} = \frac{1 - |\omega(z)|^2}{1 - r^2}. \tag{6.2}$$

Thus from (6.1), for $z = re^{i\theta}$ and $0 \le r < 1$,

$$\begin{aligned}
\Re e\left\{1 + \frac{zf''(z)}{f'(z)}\right\} &\ge \Re e\left\{\frac{zg'(z)}{g(z)}\right\} - \left|\frac{z\omega'(z)}{1 + \omega(z)}\right| \\
&\ge \frac{1 - r}{1 + r} - \frac{r}{1 - |\omega(z)|}|\omega'(z)| \\
&\ge \frac{1 - r}{1 + r} - \frac{r}{1 - |\omega(z)|}\,\frac{1 - |\omega(z)|^2}{1 - r^2} \\
&= \frac{1 - r}{1 + r} - \frac{r(1 + |\omega(z)|)}{1 - r^2} \\
&\ge \frac{1 - r}{1 + r} - \frac{r(1 + r)}{1 - r^2} = \frac{1 - 3r}{1 - r^2} > 0,
\end{aligned}$$

when $r \in [0, 1/3)$. Thus the radius of convexity for the class $\mathcal{K}_u$ is at least $1/3$.

To see that this is the largest such radius, consider $f_0 \in \mathcal{K}_u$ at the point $z = -r$ where f_0 is defined by $f_0'(z) = (1 + z)/((1 - z)^2$. □

A number $r_0 \in [0, 1]$ is called the radius of starlikeness of a particular subclass of functions in $\mathcal{A}$ if r_0 is the largest number such that $\Re e(zf'(z)/f(z)) > 0$ for all f in that subclass and $|z| < r_0$.

THEOREM 6.2. *The radius of starlikeness of* $\mathcal{K}_u$ *is not smaller than* $\sqrt{2} - 1$.

PROOF. For $f \in \mathcal{K}_u$,

$$\frac{f'(z)}{g'(z)} \prec 1 + z =: h(z), \tag{6.3}$$

for some $g \in C$ and where h is convex and univalent. Write $p(z) = f(z)/g(z)$ and $P(z) = g(z)/zg'(z)$. Since $g \in C$, (6.3) becomes $p(z) + P(z) \cdot zp'(z) \prec h(z)$, where $\Re e(P(z)) > 0$ for $z \in \mathbb{D}$. Thus from Lemma 2.10,

$$\frac{f(z)}{g(z)} = 1 + \omega(z),$$

for some $g \in C$ and some $\omega \in \Omega$. This gives

$$\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{z\omega'(z)}{1+\omega}.$$

It is well known that if $g \in C$, with $z = re^{i\theta}$, $0 \le r < 1$, then

$$\Re e\left\{\frac{zg'(z)}{g(z)}\right\} \ge \frac{1}{1+r}.$$

Again using the Schwarz lemma and (6.2), we obtain

$$\begin{aligned}\Re e\left\{\frac{zf'(z)}{f(z)}\right\} &\ge \Re e\left\{\frac{zg'(z)}{g(z)}\right\} - \left|\frac{z\omega'(z)}{1+\omega(z)}\right| \ge \frac{1}{1+r} - \frac{r}{1-|\omega(z)|}|\omega'(z)| \\ &\ge \frac{1}{1+r} - \frac{r}{1-|\omega(z)|}\frac{1-|\omega(z)|^2}{1-r^2} = \frac{1}{1+r} - \frac{r(1+|\omega(z)|)}{1-r^2} \\ &\ge \frac{1}{1+r} - \frac{r(1+r)}{1-r^2} = \frac{1-2r-r^2}{1-r^2} > 0\end{aligned}$$

when $r \in [0, \sqrt{2}-1)$. Thus the radius of starlikeness of $\mathcal{K}_u$ is at least $\sqrt{2}-1$. □

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