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Level-Dependent Annuities: Defaults of Multiple Degrees

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Abstract

Motivated by the effect on valuation of stopped or reduced debt coupon payments from a company in financial distress, we value a level-dependent annuity contract where the annuity rate depends on the value of an underlying asset process. The range of possible values of this asset is divided into a finite number of regions, with constant annuity rates within each region. We present closed-form formulas for the market value of level-dependent annuities contracts when the market value of the underlying asset is assumed to follow a geometric Brownian motion. Such annuities occur naturally in models of debt with credit risk in financial economics. Our results are applied for valuing both corporate debt with suspended interest payments under the U.S. Chapter 11 provisions and loans with contractual level-dependent interest rates.

I. Introduction

A number of financial contracts have payments contingent upon the market value of a financial asset. Whereas a standard *annuity* contract is characterized by repeated constant payments, we analyze annuities where the payment rate is constant for a given "financial health" of the obligor. Financial health is in our model proxied by the total market value of financial assets, that is, we assume a prespecified number of financial health categories where the annuity payment rate is fixed within each category, but may vary between categories. We denote this contract a *level-dependent annuity* (see Figure 1 for an illustration).

This paper is motivated by corporate debt, where we may observe stopped or reduced coupon payments for an issuer in financial distress. Financial distress occurs before, but does not necessarily lead to, liquidation of the company. Irrespective of the specific causes of reduced payments, they represent a challenge for the valuation of corporate debt. Chapter 11 of the U.S. bankruptcy code is an important example of regulations that allow a company to default without

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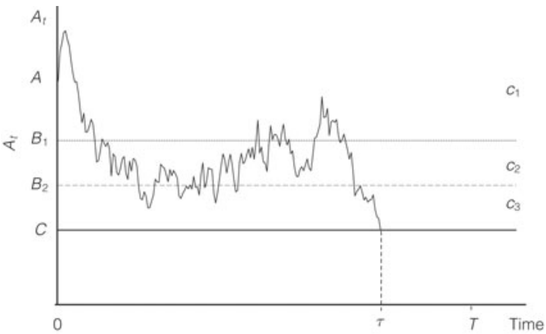
FIGURE 1

A Level-Dependent Annuity with 2 Barriers and Liquidation

Figure 1 shows an illustration of a level-dependent annuity, where $n = 2$. The graph contains an example of a path of A_t and indicates in which regions the annuity rates are c_1, c_2, and c_3, respectively. Also, A, B_1, B_2, C, T, and τ are depicted.



necessarily being liquidated. Strategic debt service (see, e.g., Anderson and Sundaresan (1996), Mella-Barral and Perraudin (1997)) is an example of a different situation with rationally reduced coupon payments.

We value level-dependent annuity contracts. Both the level-dependent continuous payment rates and the financial asset levels are assumed to be exogenous.[1] We derive closed-form solutions for the market values of level-dependent annuity contracts both in the cases of finite and infinite horizons. In order to interpret the financial asset levels as various degrees of financial distress, the natural assumption is that the initial asset value is above these levels. Our approach is general, and our formulas can be extended to other assumptions regarding the initial asset level as well.

Mathematically, we solve a boundary value problem (see, e.g., Øksendal (2005), chap. 9), where the underlying asset is assumed to be a geometric Brownian motion. First, we find the market value of the level-dependent annuity contract in the case of an infinite horizon using the standard assumption of *smooth pasting* (see, e.g., Dixit and Pindyck (1994)). The level-dependent annuity contract can be considered as a portfolio of simple annuities. The market value of the level-dependent annuity is calculated as the sum of the market values of these annuities. In the case of a finite horizon, we apply the standard argument that a finite-horizon annuity may be considered as an immediate-starting infinite-horizon (ISIH) annuity from which another infinite-horizon annuity starting at a future fixed time T, a forward-starting infinite-horizon (FSIH) annuity, is subtracted.

We apply our results to a model of corporate debt and for pricing 2 different loans. Our corporate debt model includes optimal default and liquidation levels. This application is motivated by the recent paper by Broadie, Chernov, and Sundaresan (BCS) (2007). Our model has closed-form solutions, and we find endogenous default and liquidation levels by maximizing the value of equity. For analytical tractability, we model the cash flows in default differently from BCS.

[1]We apply our results in a corporate debt model with endogenous asset levels in Section VIII.

Their model closely captures actual default procedures in the U.S. Our model, while not including all the specifics of U.S. legislation, still produces similar results in closed form.

Our second application is a valuation model of loans with contractual level-dependent interest rates (i.e., increasing interest rates for decreasing credit quality). Lenders' 2 primary concerns are whether borrowers are able to service and repay a loan and the recovery of the loan in case of default. Both of these concerns may be reflected in the contractual interest rates of the debt. Corporate debt with performance pricing (see, e.g., Asquith, Beatty, and Weber (2005)) is an example where the interest rates depend on the company cash flow, a measure of payment ability. Residential mortgage is another example where interest rates depend on the value of the collateral, a measure of recovery if the loan is defaulted. Our results show that lenders of residential mortgages may have incentives to prefer the more volatile segments of the real estate market. To the best of our knowledge, our valuation approach is novel, although these loans are common.

In the applications we assume that the exogenous stochastic process represents the market price process of a financial asset rather than a common financial market factor (e.g., an interest rate). This choice implies that contract-specific risk, and not general market risk as such, is viewed as more important for default and/or liquidation.

To illustrate some aspects of our results we include a simple, initial example. A standard infinite-horizon continuous annuity rate of 5 (without liquidation risk) has time 0 value of 100, given a continuous risk-free interest rate of 0.05. Consider an issuing company with current market value of 100, which is liquidated if its asset value decreases to 30. Our example illustrated in Table 1 divides the value of this annuity into components related to the loss connected to liquidation, the value above and below an additional barrier of 60, and a possible finite maturity.

TABLE 1

Decomposition of an Annuity

Table 1 presents the annuity value decomposition based on the parameter values $\mu = 0.02$, $\sigma = 0.20$, $r = 0.05$, $A = 100$, $B = 60$, $C = 30$, $c = 5$, and $T = 10$.

Annuity	Finite Horizon $[0, T]$	Forward Starting $[T, \infty)$	Infinite Horizon $[0, \infty)$
(1) Above annuity with liquidation	35.25	39.97	75.22
(2) Below annuity with liquidation	3.59	6.29	9.88
(3) Above + below annuity	38.84	46.26	85.10
(4) Value of liquidation loss	0.51	14.39	14.90
(5) Value of risk-free annuity	39.35	60.65	100.00

The values of ISIH above and below annuities with liquidation risk (rows (1) and (2) of Table 1) are calculated using equations (12) and (13) in this paper, respectively. The values of FSIH above and below annuities are calculated using equations (15) and (16), respectively. The values of finite-horizon annuities maturing at time T are found by deducting the value of an FSIH annuity starting at time T from an ISIH annuity, or calculated directly from expressions (17) and (18).

The values in row (3) of Table 1 are found by adding the values in rows (1) and (2). The value of the infinite-horizon annuity in row (3) is calculated as

$$\left(\frac{c}{r}\right)\left(1-\left(\frac{A}{C}\right)^{-\beta}\right),$$

where β is given in expression (5), using results from Black and Cox (1976). The value of the forward-starting annuity in row (3) is calculated as

$$\left(\frac{c}{r}\right)\left(e^{-rT}Q_g-\left(\frac{A}{C}\right)^{-\beta}Q_g^{\beta}\right),$$

where Q_g and Q_g^{β} are given in expressions (B-2) and (B-10) in Appendix B. The 1st term is interpreted as the time 0 value of a time T FSIH annuity. The 2nd term is the time 0 market value of an FSIH annuity, starting at the time of liquidation given that liquidation occurs after time T. The time 0 value of the finite-horizon annuity in row (3) is then calculated as the difference between the infinite-horizon annuity and the forward-starting annuity using these formulas.

The values of the loss due to liquidation (row (4) of Table 1) are calculated as the differences between the values of the respective annuities with (row (3)) and without (row (5)) risk of liquidation. In row (5), the infinite-horizon case, the value of an immediate-starting risk-free annuity is simply (c/r), while the value of a forward-starting risk-free annuity is the same value discounted (i.e., $e^{-rT}(c/r)$). The finite-horizon annuity in row (5) is then calculated as the usual difference, $(c/r)(1-e^{-rT})$.

This paper is organized as follows. Section II contains the setup and main result. Section III treats the case of ISIH annuities. Section IV develops the results for the case of FSIH annuities. In Section V the results from Sections III and IV are combined into finite-horizon annuities. Whereas Sections III and IV treat simpler annuities, Section VI extends these results to infinite-horizon level-dependent annuities. Finite versions of level-dependent annuities are developed in Section VII. Section VIII contains 2 applications. Conclusions and areas for further research are found in Section IX. Some technical results are collected in Appendices A–C.

II. Setup and Main Result

A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ is given. In particular, Q represents a fixed *equivalent martingale measure*. We furthermore impose the standard frictionless, continuous time market assumptions of financial economics (see, e.g., Duffie (2001)).

We assume that the underlying asset process is given by a geometric Brownian motion

$$dA_t = \mu A_t dt + \sigma A_t dW_t, \tag{1}$$

where the initial value $A_0 = A$ is a constant. Here the drift parameter μ and volatility parameter σ are constants, and W_t represents a standard Brownian motion.

Let T be the finite time horizon, let the constant C be an absorbing barrier, and define the stopping time τ as

$$\tau = \inf\{t \geq 0, A_t = C\}. \tag{2}$$

We interpret C as the liquidation barrier, and τ as the time of liquidation.

There are n additional constant levels or nonabsorbing barriers $B_1, \ldots, B_n$ so that $B_1 > \cdots > B_n > C$. For notational convenience we let $B_0 = \infty$ and, in the case with liquidation risk, $B_{n+1} = C$, or, in the case without liquidation risk, $B_{n+1} = 0$, respectively. The constant annuity rate is c_1 when $A_t > B_1$, c_{i+1} when $B_i > A_t > B_{i+1}$, $i = 1, \ldots, n-1$, and c_{n+1} when $B_n > A_t > B_{n+1}$. All c_is are constants. The initial value of the asset process is by assumption above the highest barrier (i.e., $A > B_1$).

Let r be the constant risk-free interest rate. Note that we restrict $\mu \leq r$.

The most general contract we consider is a finite-horizon, level-dependent annuity with liquidation risk. The time 0 market value of this contract is

$$M^T(A) = \mathrm{E}\left[\int_0^{\tau \wedge T} \sum_{i=0}^{n} c_{i+1} e^{-rs} 1\{B_i > A_s > B_{i+1}\} ds\right],$$

where $1\{\cdot\}$ denotes the standard indicator function, and $\mathrm{E}[\cdot]$ denotes the expectation under the equivalent martingale measure. Furthermore, $B_{n+1} = C$.

Our main valuation result is that

$$M^T(A) = \left(\frac{c_1}{r}\right) - \left(\frac{c_{n+1}}{r}\right)\left[\left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta} + e^{-rT} Q_g\right] + \sum_{i=1}^{n}\left(\frac{c_i - c_{i+1}}{r}\right)\left(\psi_i - Q_{gg}(B_i) e^{-rT}\right), \tag{3}$$

where

$$\psi_i = \frac{\alpha\left(\frac{A}{B_i}\right)^{-\beta}\left(Q_{gg}^{\beta}(B_i) - 1\right) - \beta\left(\frac{A}{B_i}\right)^{\alpha} Q_{lg}^{\alpha}(B_i) - \beta\left(\frac{A}{C}\right)^{-\beta}\left(\frac{C}{B_i}\right)^{\alpha} Q_l^{\beta}}{\alpha + \beta},$$

$$\alpha = \frac{\frac{1}{2}\sigma^2 - \mu + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \quad (> 1) \quad \text{and} \tag{4}$$

$$\beta = \frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{\left(\frac{1}{2}\sigma^2 - \mu\right)^2 + 2\sigma^2 r}}{\sigma^2} \quad \left(> \left(\frac{2\mu}{\sigma^2}\right) > 0\right). \tag{5}$$

The probability $Q_l^{\beta} = Q^{\beta}(\tau \leq T) = 1 - Q_g^{\beta}$, where $Q_g^{\beta} = Q^{\beta}(\tau > T)$ is given in expression (B-10) in Appendix B. Furthermore, the probabilities $Q_g = Q(\tau > T)$,

$Q^{\beta}_{gg}(B_i) = Q^{\beta}(A_T > B_i, \tau > T)$, $Q^{\alpha}_{lg}(B_i) = Q^{\alpha}(A_T \leq B_i, \tau > T)$, and $Q_{gg}(B_i) = Q(A_T > B_i, \tau > T)$ are given in expressions (B-2), (B-11), (B-8), and (B-3) in Appendix B, respectively. Here Q^{α} and Q^{β} represent probability measures equivalent to Q; see Appendix A for details.

The 1st term of expression (3) represents the time 0 value of an infinite-horizon annuity rate c_1 without liquidation risk. The negative of the 2nd term represents the time 0 value of an FSIH annuity rate c_{n+1} without liquidation risk, starting either at the time of liquidation τ or time T, whichever comes first. Roughly interpreted, the remaining terms represent correction terms of the total time 0 value due to the multiple annuity rate levels between c_1 and c_{n+1}.

III. ISIH Claims

In this section we consider ISIH claims, assuming that $T = \infty$. Let f be the time 0 market value of an arbitrary infinite-horizon claim on A_t, and denote the 1st- and 2nd-order partial derivatives by $f_A = \partial f / \partial A$ and $f_{AA} = \partial^2 f / \partial A^2$, respectively. Then the partial differential equation (see, e.g., Merton (1974)),

$$\text{(6)} \qquad \frac{1}{2}\sigma^2 A^2 f_{AA} + \mu A f_A - rf + c(A) \quad = \quad 0$$

holds, subject to appropriate boundary conditions. Here $c(A)$ represents the annuity payment rate (to be interpreted as dividends or coupons, depending on the nature of the claim) to the owner of the claim f. The general solution to the homogeneous part, obtained by letting $c(A) = 0$, of equation (6) is

$$\text{(7)} \qquad f^*(A) \quad = \quad K_1 A^{\alpha} + K_2 A^{-\beta},$$

where α and β are given in expressions (4) and (5), respectively, and the constants K_1 and K_2 are determined by boundary conditions. The general solution to equation (6) is $f(A) = f^*(A) + f^s(A)$, where $f^s(A)$ is any *special solution* of equation (6).

We denote initial market values by capital letters, possibly with subscripts, for example, U, or $U(A, B)$ to emphasize the dependence on the initial value of the process and on the barrier B.

A. The Value of 1 at the Initial Hit of a Barrier

Let U denote the time 0 market price of a claim that pays 1 when $A_t = B$ for the first time. Here

$$\text{(8)} \qquad U(A,B) \quad = \quad \begin{cases} U^a = \left(\dfrac{A}{B}\right)^{-\beta}, & \text{when } A \geq B, \\ U^b = \left(\dfrac{A}{B}\right)^{\alpha}, & \text{when } A \leq B. \end{cases}$$

The superscripts a and b signify that A_t hits the barrier from *above* or *below*, respectively. These results are standard, but we include a proof for the completeness of the exposition.

Proof. U does not pay any coupon, so $c(A)=0$ in expression (6). U^a is calculated from equation (7) using the boundary conditions $\lim_{A\to\infty} U^a = 0 \Rightarrow K_1 = 0$ and $U^a(B)=1$. U^b is calculated from the boundary conditions $\lim_{A\to 0} U^b=0 \Rightarrow K_2=0$ and $U^b(B) = 1$. □

We remark that $U(\cdot, B)$ is continuous at B, but does not satisfy the *smooth pasting* condition at B.

B. Infinite-Horizon Annuities Without Liquidation Risk

1. The Value of an Infinite-Horizon Above Annuity Without Liquidation Risk

Let V_A denote the time 0 market price of an annuity that pays the rate c when $A_t > B$ (*above annuity*):

$$(9) \quad V_A(A,B) = \begin{cases} V_A^a = \left(\frac{c}{r}\right)\left(1 - \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}\right), & \text{when } A \geq B, \\ V_A^b = \left(\frac{c}{r}\right)\left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha}, & \text{when } A \leq B. \end{cases}$$

Observe that $V_A^b = 0$ when $B = \infty$.

Proof. V_A pays c only when $A_t > B$, so in expression (6), $c(A) = c$ when $A_t > B$, and $c = 0$ otherwise. Observe that $f^s(A) = (c/r)$ solves equation (6) when $A > B$. The relevant boundary conditions are $\lim_{A\to\infty} V_A^a = (c/r) \Rightarrow K_1 = 0$ and $\lim_{A\to 0} V_A^b = 0 \Rightarrow K_2 = 0$. To determine K_2 for V_A^a and K_1 for V_A^b, we require continuity and smooth pasting at B (i.e., $V_A^a(B) = V_A^b(B)$ and $(\partial/\partial A)V_A^a(B) = (\partial/\partial A)V_A^b(B)$). □

2. The Value of an Infinite-Horizon Below Annuity Without Liquidation Risk

Let V_B denote the time 0 market price of an annuity that pays c when $A_t < B$ (*below annuity*):

$$(10) \quad V_B(A,B) = \begin{cases} V_B^a = \left(\frac{c}{r}\right)\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}, & \text{when } A \geq B, \\ V_B^b = \left(\frac{c}{r}\right)\left(1 - \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha}\right), & \text{when } A \leq B. \end{cases}$$

Observe that $V_B^b = (c/r)$ when $B = \infty$. Also observe that $V_B^b = (c/r) - V_A^b$, if $A < B$, an infinite-horizon annuity with payments below B equals an infinite-horizon annuity from which an infinite-horizon annuity with payments only above B is subtracted. Also observe that $V_A^a = (c/r) - V_B^a$, if $A > B$, an infinite-horizon annuity with payments above B equals an infinite-horizon annuity from which an infinite-horizon annuity with payments only below B is subtracted.

Proof. V_B pays c only when $A_t < B$, so in expression (6), $c(A) = c$ when $A_t < B$, and $c = 0$ otherwise. Observe that $f^s(A) = (c/r)$ solves equation (6) when $A < B$. The relevant boundary conditions are $\lim_{A\to\infty} V_B^a=0 \Rightarrow K_1=0$ and $\lim_{A\to 0} V_B^b = 0 \Rightarrow K_2 = 0$. To determine K_2 for V_B^a and K_1 for V_B^b we also here require continuity and smooth pasting at B (i.e., $V_B^a(B)=V_B^b(B)$ and $(\partial/\partial A)V_B^a(B)=(\partial/\partial A)V_B^b(B)$). □

C. Infinite-Horizon Annuities With Liquidation Risk

Let D_j denote the value of a claim V_j where $j \in \{A, B\}$, including liquidation risk. Using economic arguments we show in the proof that

$$D_j(A,B) \quad = \quad V_j(A,B) - V_j^b(C,B)U^a(A,C). \tag{11}$$

Proof. Upon liquidation (i.e., at time τ), the value of the claim V_j is $V_j^b(C, B_i)$. Because $C < B_i$ for all $i \leq n$, $V_j = V_j^b$. Therefore $V_j^b(C, B_i)$ represents the reduction in value of the claim V_j due to liquidation at the time of liquidation. The initial value of this claim is found by discounting by $U = U^a$ because $A > C$. □

1. The Value of an Infinite-Horizon Above Annuity in the Case With Liquidation Risk

$$D_A(A,B) \tag{12}$$
$$= \begin{cases} D_A^a = \left(\frac{c}{r}\right)\left[1 - \left(\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{-\beta} + \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{\alpha}\right)\left(\frac{A}{C}\right)^{-\beta}\right], \\ \quad \text{when } A \geq B, \\ D_A^b = \left(\frac{c}{r}\right)\left(\frac{\beta}{\alpha+\beta}\right)\left[\left(\frac{A}{B}\right)^{\alpha} - \left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta}\right], \\ \quad \text{when } C \leq A \leq B. \end{cases}$$

The first 2 terms in the case where $A \geq B$, and the 1st term in the case where $C \leq A \leq B$, are identical to the corresponding annuities without liquidation risk. The final terms in both cases are identical and equal (the negative of) the value of an above annuity below the barrier when $A = C$ multiplied by $U^a(A, C)$, the value of 1 upon liquidation. In the case where $B = C$, the results collapse to the standard Black and Cox (1976) result for infinite-horizon debt with liquidation risk.

2. The Value of an Infinite-Horizon Below Annuity in the Case With Liquidation Risk

$$D_B(A,B) \tag{13}$$
$$= \begin{cases} D_B^a = \left(\frac{c}{r}\right)\left[\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{-\beta} + \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{\alpha} - 1\right]\left(\frac{A}{C}\right)^{-\beta}, \\ \quad \text{when } A \geq B, \\ D_B^b = \left(\frac{c}{r}\right)\left[1 - \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha} - \left(1 - \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{\alpha}\right)\left(\frac{A}{C}\right)^{-\beta}\right], \\ \quad \text{when } C \leq A \leq B. \end{cases}$$

As for the previous annuity, the last term in both these expressions can be interpreted as the value of an infinite horizon below annuity below the barrier when $A = C$ multiplied by the value of 1 upon liquidation. Both of these results can alternatively be derived by solving equation (6) with appropriate boundary conditions.

IV. FSIH Annuities

In this section we calculate the time 0 market values of infinite-horizon annuities that start at a future time $T > 0$. For a general forward-starting claim $\nu_j(A_T, B)$, the time 0 value $\xi_j(A, B)$ is calculated as

$$\xi_j(A,B) = \mathrm{E}[e^{-rT}\nu_j(A_T,B)].$$

A. FSIH Annuities Without Liquidation Risk

1. FSIH Above Annuity Without Liquidation Risk

Denote the time 0 market value of an FSIH above annuity by W_A. Then

$$(14)\quad W_A(A,B) = \left(\frac{c}{r}\right)\left(e^{-rT}Q_g(B) - \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta} Q_g^{\beta}(B) + \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha} Q_l^{\alpha}(B)\right),$$

where $Q_g(B) = Q(A_T > B) = 1 - Q_l(B)$, $Q_g^{\beta}(B) = Q^{\beta}(A_T > B) = 1 - Q_l^{\beta}(B)$. Here $Q_l(B)$, $Q_l^{\beta}(B)$, and $Q_l^{\alpha}(B) = Q^{\alpha}(A_T \leq B)$ are defined in expressions (B-1), (B-5), and (B-9) in Appendix B, respectively.

Proof.

$$\begin{aligned}
W_A &= \mathrm{E}[e^{-rT}V_A(A_T,B)] \\
&= \mathrm{E}\left[e^{-rT}\left(V_A^a(A_T,B)1\{A_T > B\} + V_A^b(A_T,B)1\{A_T < B\}\right)\right], \\
&= \mathrm{E}\left[e^{-rT}\left(\frac{c}{r}\right)\left(\left(1-\left(\frac{\alpha}{\alpha+\beta}\right)U^a(A_T,B)\right)1\{A_T > B\}\right)\right] \\
&\quad + \mathrm{E}\left[e^{-rT}\left(\frac{c}{r}\right)\left(\left(\frac{\beta}{\alpha+\beta}\right)U^b(A_T,B)1\{A_T < B\}\right)\right], \\
&= \left(\frac{c}{r}\right)\left(e^{-rT}Q(A_T > B) - \left(\frac{\alpha}{\alpha+\beta}\right)P_1(A,B) + \left(\frac{\beta}{\alpha+\beta}\right)P_2(A,B)\right),
\end{aligned}$$

where $P_1(A,B)$ and $P_2(A,B)$ are defined in Appendix A, and the event Z is specialized to $\{A_T > B\}$ for $P_1(A,B)$ and $\{A_T \leq B\}$ for $P_2(A,B)$. The result follows from expressions (A-1) and (A-2) in Appendix A. □

2. FSIH Below Annuity Without Liquidation Risk

Denote the time 0 market value of an FSIH below annuity by W_B. Then

$$W_B(A,B) = \left(\frac{c}{r}\right)\left(e^{-rT}Q_l(B) + \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta} Q_g^{\beta}(B) - \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha} Q_l^{\alpha}(B)\right).$$

Proof.

$$\begin{aligned} W_B &= \mathrm{E}[e^{-rT}V_B(A_T,B)] \\ &= \mathrm{E}\left[e^{-rT}\left(V_B^a(A_T,B)1\{A_T>B\}+V_B^b(A_T,B)1\{A_T<B\}\right)\right], \\ &= \mathrm{E}\left[e^{-rT}\left(\frac{c}{r}\right)\left(\left(\frac{\alpha}{\alpha+\beta}\right)U^a(A_T,B)1\{A_T>B\}\right)\right] \\ &\quad +\mathrm{E}\left[e^{-rT}\left(\frac{c}{r}\right)\left(\left(1-\left(\frac{\beta}{\alpha+\beta}\right)U^b(A_T,B)\right)1\{A_T<B\}\right)\right], \end{aligned}$$

using similar definitions of Z as in the previous proof. The result follows from expressions (A-1) and (A-2) in Appendix A. □

B. FSIH Annuities With Liquidation Risk

Denote by $\xi(A,B)$ the time 0 market value of a general FSIH annuity $\nu(A_T,B)$ delivered at time T upon no prior liquidation. Then

$$\xi(A,B) = \mathrm{E}[e^{-rT}\nu(A_T,B)1\{\tau>T\}].$$

1. FSIH Above Annuity With Liquidation Risk

Denote the time 0 market value of an FSIH above annuity with liquidation risk by Y_A. Then

$$\begin{aligned} (15)\quad & Y_A(A,B) \\ &= \left(\frac{c}{r}\right)\left(e^{-rT}Q_{gg}(B)-\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}Q_{gg}^{\beta}(B)\right) \\ &\quad +\left(\frac{c}{r}\right)\left(\left(\frac{\beta}{\alpha+\beta}\right)\left[\left(\frac{A}{B}\right)^{\alpha}Q_{lg}^{\alpha}(B)-\left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta}Q_{g}^{\beta}\right]\right), \end{aligned}$$

where $Q_{gg}(B)=Q(A_T>B,\tau>T)$, $Q_{gg}^{\beta}(B)=Q^{\beta}(A_T>B,\tau>T)$, $Q_{lg}^{\alpha}(B)=Q^{\alpha}(A_T\leq B,\tau>T)$, and $Q_g^{\beta}=Q^{\beta}(\tau>T)$ are given in expressions (B-3), (B-11), (B-8), and (B-10) in Appendix B, respectively.

Proof.

$$\begin{aligned} Y_A &= \mathrm{E}[e^{-rT}D_A(A_T)1\{\tau>T\}] \\ &= \mathrm{E}\left[e^{-rT}\left(D_A^a(A_T,B)1\{A_T>B\}+D_A^b(A_T,B)1\{A_T<B\}\right)1\{\tau>T\}\right]. \quad \square \end{aligned}$$

2. FSIH Below Annuity With Liquidation Risk

Denote the time 0 market value of an FSIH below annuity with liquidation risk by Y_B. Then

$$\begin{aligned} (16)\quad & Y_B(A,B) \\ &= \left(\frac{c}{r}\right)\left(e^{-rT}Q_{lg}(B)+\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}Q_{gg}^{\beta}(B)\right) \end{aligned}$$

$$+\left(\frac{c}{r}\right)\left(\left(\frac{\beta}{\alpha+\beta}\right)\left[\left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta}Q_g^{\beta}-\left(\frac{A}{B}\right)^{\alpha}Q_{lg}^{\alpha}(B)\right]\right.$$
$$\left.-\,Q_g^{\beta}\left(\frac{A}{C}\right)^{-\beta}\right),$$

where $Q_{lg}(B) = Q(A_T \leq B, \tau > T)$ is given in expression (B-4) in Appendix B.

Proof.

$$\begin{aligned} Y_B &= \mathrm{E}\left[e^{-rT} D_B(A_T) 1\{\tau > T\}\right] \\ &= \mathrm{E}\left[e^{-rT}\left(D_B^a(A_T,B)1\{A_T > B\} + D_B^b(A_T,B)1\{A_T < B\}\right)1\{\tau > T\}\right]. \quad \square \end{aligned}$$

As in the example in the Introduction, we calculate the time 0 market value of an FSIH annuity with liquidation risk as

$$Y_A(A,B) + Y_B(A,B) \quad = \quad \left(\frac{c}{r}\right)\left(e^{-rT}Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_g^{\beta}\right).$$

The 1st term is interpreted as the time 0 value of a time T FSIH annuity. The 2nd term is the time 0 market value of an FSIH annuity, starting at the time of liquidation given that liquidation occurs after time T. From Appendix A we know that $P_3(A,C) = (A/C)^{-\beta}Q_g^{\beta}$ can be interpreted as the value of 1 unit account payable at liquidation, only if liquidation occurs after time T.

V. Finite-Horizon Above and Below Annuities

In this section we show how the previous infinite-horizon annuities can be combined into annuities with a finite horizon. We assume in this section that $A > B$.

A. Finite-Horizon Annuities Without Liquidation Risk

1. Finite-Horizon Above Annuity Without Liquidation Risk

The time 0 market price of a finite-horizon above annuity without liquidation risk is calculated as

$$\begin{aligned} V_A^T(A,B) &= V_A(A,B) - W_A(A,B) \\ &= \left(\frac{c}{r}\right) - \left(\frac{c}{r}\right)\left(e^{-rT}Q_g(B) + \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta} Q_l^{\beta}(B)\right. \\ &\qquad \left. + \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha} Q_l^{\alpha}(B)\right). \end{aligned}$$

2. Finite-Horizon Below Annuity Without Liquidation Risk

The time 0 market price of a finite-horizon below annuity without liquidation risk is calculated as

$$V_B^T(A,B) \quad = \quad V_B(A,B) - W_B(A,B)$$

$$= \left(\frac{c}{r}\right)\left(-e^{-rT}Q_l(B) + \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta} Q_l^{\beta}(B) + \left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha} Q_l^{\alpha}(B)\right).$$

Observe that the time 0 value of a finite-horizon annuity that pays the rate c both above and below B is $V_A^T(A,B) + V_B^T(A,B) = (c/r)(1-e^{-rT})$, a familiar result.

B. Finite-Horizon Annuities With Liquidation Risk

1. Finite-Horizon Above Annuity With Liquidation Risk

The time 0 market price of a finite-horizon above annuity with liquidation risk is calculated as

$$D_A^T(A,B) = D_A(A,B) - Y_A(A,B) = \left(\frac{c}{r}\right)\gamma, \tag{17}$$

where

$$\gamma = 1 - \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}(1-Q_{gg}^{\beta}(B)) - \left(\frac{\beta}{\alpha+\beta}\right)\left(\left(\frac{A}{B}\right)^{\alpha} Q_{lg}^{\alpha}(B) + \left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta}\right) - e^{-rT}Q_{gg}(B).$$

2. Finite-Horizon Below Annuity With Liquidation Risk

The time 0 market price of a finite-horizon below annuity with liquidation risk is calculated as

$$D_B^T(A,B) = D_B(A,B) - Y_B(A,B) = \left(\frac{c}{r}\right)\eta, \tag{18}$$

where

$$\eta = \left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}(1-Q_{gg}^{\beta}(B)) + \left(\frac{\beta}{\alpha+\beta}\right)\left(\left(\frac{A}{B}\right)^{\alpha} Q_{lg}^{\alpha}(B) + \left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta}\right) - e^{-rT}Q_{lg}(B) - \left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta}.$$

As indicated in the Introduction, we calculate the time 0 market value of an ISFH annuity with liquidation risk as

$$D_A^T(A,B) + D_B^T(A,B) = \left(\frac{c}{r}\right)\left(1 - e^{-rT}Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta}\right).$$

The 1st term represents the time 0 market value of an ISIH annuity without liquidation risk. The 2nd term represents (the negative of) the time 0 market value of a time T FSIH annuity without liquidation risk. The final term represents (the negative of) the time 0 market value of an FSIH annuity, starting at the time of liquidation, but only if liquidation occurs before time T.

VI. Infinite-Horizon Level-Dependent Annuities

In this section we consider annuities with multiple barriers and possibly different coupon rates in each of the regions defined by these barriers, as explained in Section II. We use the results from Sections III and IV as our building blocks. To incorporate level-dependent annuities, we formally assume that the parameter c in the previous formulas equals 1 and multiply by the region-specific coupon rate c_i. This approach is without any loss of generality. For simplicity we only treat the case where $A > B_1$.

A. ISIH Level-Dependent Annuities

1. ISIH Level-Dependent Annuity Without Liquidation Risk

The time 0 market value $\hat{M}^{\infty}(A)$ of an infinite-horizon level-dependent annuity in the case of no liquidation risk is

$$(19) \qquad \hat{M}^{\infty}(A) = \left(\frac{c_1}{r}\right) - \left(\frac{\alpha}{\alpha+\beta}\right)\sum_{i=1}^{n}\left(\frac{c_i - c_{i+1}}{r}\right)\left(\frac{A}{B_i}\right)^{-\beta}.$$

Proof. The time 0 market value of an annuity c_{i+1} that is paid only when $B_i \leq A_t \leq B_{i+1}$, for $i = 0, \ldots, n$, is $(V_A(A, B_{i+1}) - V_A(A, B_i))c_{i+1}$. The time 0 market value of a level-dependent annuity is found by simply adding such annuities, that is,

$$\hat{M}^{\infty}(A) = \sum_{i=0}^{n}\left(V_A(A, B_{i+1}) - V_A(A, B_i)\right) c_{i+1}.$$

Observe that $V_A(A, B_0) = 0$ and that $V_A(A, B_{n+1}) = (c_{n+1}/r)$. The formula follows by direct calculations using expression (9) with $c = 1$. □

2. ISIH Level-Dependent Annuity With Liquidation Risk

Denote the time 0 value of the infinite-horizon version of the level-dependent annuity in the case of liquidation risk by $M^{\infty}(A)$.

The time 0 market value of an infinite-horizon level-dependent annuity in the case of liquidation risk is

$$(20) \qquad M^{\infty}(A) = \left(\frac{c_1}{r}\right) - \left(\frac{c_{n+1}}{r}\right)\left(\frac{A}{C}\right)^{-\beta} - \sum_{i=1}^{n}\left(\frac{c_i - c_{i+1}}{r(\alpha+\beta)}\right)\left(\alpha\left(\frac{C}{B_i}\right)^{-\beta} + \beta\left(\frac{C}{B_i}\right)^{\alpha}\right)\left(\frac{A}{C}\right)^{-\beta}.$$

Observe that expression (20) is reduced to expression (19) for $C = 0$.

Proof. As in the previous proof, we may write

$$M^{\infty}(A) \quad = \quad \sum_{i=0}^{n} \left(D_A(A,B_{i+1}) - D_A(A,B_i)\right) c_{i+1},$$

where $D_A(A,B_i)$ is given in expression (12) and $B_{n+1} = C$. Observe that $D_A(A,B_0) = 0$ and that $D_A(A,B_{n+1}) = D_A(A,C) = (c_{n+1}/r)(1-(A/C)^{-\beta})$. The formula follows by direct calculations with $c = 1$. □

B. FSIH Level-Dependent Annuities

1. FSIH Level-Dependent Annuity Without Liquidation Risk

The time 0 market value $\hat{M}_T^{\infty}(A)$ of an infinite, time T FSIH level-dependent annuity in the case of no liquidation risk is

$$\text{(21)} \quad \hat{M}_T^{\infty}(A) \quad = \quad \left(\frac{c_{n+1}}{r}\right) e^{-rT} - \sum_{i=1}^{n} \left(\frac{c_i - c_{i+1}}{r}\right) \left(\lambda_i - Q_g(B_i)e^{-rT}\right),$$

where

$$\lambda_i \quad = \quad \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} Q_g^{\beta}(B_i) - \beta \left(\frac{A}{B_i}\right)^{\alpha} Q_l^{\alpha}(B_i)}{\alpha + \beta}.$$

Proof. Similarly to the previous proofs,

$$\hat{M}_T^{\infty}(A) \quad = \quad \sum_{i=0}^{n} \left(W_A(A,B_{i+1}) - W_A(A,B_i)\right) c_{i+1},$$

where $W_A(A,B_i)$ is given in expression (14) and $B_{n+1} = 0$. Observe that $W_A(A,B_0) = 0$ and that $W_A(A,B_{n+1}) = W_A(A,0) = (c_{n+1}/r)e^{-rT}$. The formula follows by direct calculations with $c = 1$. □

2. FSIH Level-Dependent Annuity With Liquidation Risk

The time 0 market value $M_T^{\infty}(A)$ of a time T FSIH level-dependent annuity in the case of liquidation risk is

$$\text{(22)} \quad M_T^{\infty}(A) \quad = \quad \left(\frac{c_{n+1}}{r}\right) \left[e^{-rT} Q_g - \left(\frac{A}{C}\right)^{-\beta} Q_g^{\beta}\right] - \sum_{i=1}^{n} \left(\frac{c_i - c_{i+1}}{r}\right) \left(\kappa_i - Q_{gg}(B_i)e^{-rT}\right),$$

where

$$\kappa_i \quad = \quad \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} Q_{gg}^{\beta}(B_i) - \beta \left(\frac{A}{B_i}\right)^{\alpha} Q_{lg}^{\alpha}(B_i) + \beta \left(\frac{A}{C}\right)^{-\beta} Q_g^{\beta} \left(\frac{C}{B_i}\right)^{\alpha}}{\alpha + \beta}.$$

Proof. Similarly to the previous proofs,

$$D_{0,T}^{\infty}(A) = \sum_{i=0}^{n} \left(Y_A(A, B_{i+1}) - Y_A(A, B_i)\right) c_{i+1},$$

where $Y_A(A, B_i)$ is given in expression (15) and $B_{n+1}=C$. Observe that $Y_A(A, B_0)=0$ and that $Y_A(A, B_{n+1}) = Y_A(A, C) = (c_{n+1}/r)[e^{-rT}Q(\tau > T) - (A/C)^{-\beta}Q^{\beta}(\tau > T)]$. The formula follows by direct calculations with $c = 1$. □

Also here observe that expression (22) is reduced to expression (21) for $C=0$.

VII. Finite-Horizon Level-Dependent Annuities

A. Finite-Horizon Level-Dependent Annuity Without Liquidation Risk

The time 0 market value $\hat{M}^T(A)$ of a finite-horizon level-dependent annuity in the case of no liquidation risk is

$$\begin{aligned} \hat{M}^T(A) &= \hat{M}^{\infty}(A) - \hat{M}_T^{\infty}(A), \\ &= \left(\frac{c_1}{r}\right) - \left(\frac{c_{n+1}}{r}\right) e^{-rT} + \sum_{i=1}^{n} \left(\frac{c_i - c_{i+1}}{r}\right) \left(\phi_i - Q_g(B_i) e^{-rT}\right), \end{aligned}$$

where

$$\phi_i = -\frac{\left(\alpha \left(\frac{A}{B_i}\right)^{-\beta} Q_l^{\beta}(B_i) + \beta \left(\frac{A}{B_i}\right)^{\alpha} Q_l^{\alpha}(B_i)\right)}{\alpha + \beta}.$$

B. Finite-Horizon Level-Dependent Annuity With Liquidation Risk

The time 0 market value $M^T(A)$ of a finite-horizon level-dependent annuity in the case of liquidation risk is

$$\begin{aligned} M^T(A) &= M^{\infty}(A) - M_T^{\infty}(A) \\ &= \left(\frac{c_1}{r}\right) - \left(\frac{c_{n+1}}{r}\right) \left[\left(\frac{A}{C}\right)^{-\beta} Q_l^{\beta} + e^{-rT} Q_g\right] \\ &\quad + \sum_{i=1}^{n} \left(\frac{c_i - c_{i+1}}{r}\right) \left(\psi_i - Q_{gg}(B_i) e^{-rT}\right), \end{aligned}$$

where

$$\psi_i = \frac{\alpha \left(\frac{A}{B_i}\right)^{-\beta} \left(Q_{gg}^{\beta}(B_i) - 1\right) - \beta \left(\frac{A}{B_i}\right)^{\alpha} Q_{lg}^{\alpha}(B_i) - \beta \left(\frac{A}{C}\right)^{-\beta} \left(\frac{C}{B_i}\right)^{\alpha} Q_l^{\beta}}{\alpha + \beta}.$$

This result is already presented in expression (3) in Section II but is included here for completeness.

VIII. Selected Applications

A. Optimal Debt and Equity Valuation in a Model With Chapters 7 and 11

1. Introduction

U.S. bankruptcy legislation distinguishes between a company's default on its obligations; that is, bankruptcy, and its final liquidation, referred to as *Chapter 11* and *Chapter 7*, respectively. In theoretical models, default and liquidation often correspond to specific levels (barriers) of a company's asset value.

The results from our model are qualitatively similar to the results of the recent BCS (2007) model, which includes separate default and liquidation barriers, and both the BCS model and our model are based on the seminal Leland (1994) paper. Compared to the Leland model, we add a default region defined by the default and the liquidation barriers. When the asset value of the company is within the default region, the company is said to be in Chapter 11. In Chapter 11 we assume that the continuous coupon payment rate is reduced from c to θc, where $0 \leq \theta \leq 1$, and a distress cost ω, $0 \leq \omega \leq 1$, proportional to the value of the company. The optimal asset default level is endogenized by maximizing the equity value of the company. The reduced coupon payment is traded off against the distress cost. This maximization also determines the liquidation level, as in Leland.

In comparison, in the BCS (2007) model a company in Chapter 11 accumulates unpaid coupons and dividends as *arrears*. If the asset value of the company subsequently increases above the default level, the arrears are distributed subject to a prespecified level of forgiveness from debtholders to equityholders. Liquidation occurs either when the asset value of the company hits the liquidation barrier or after some maximum time spent in default.

The reduced coupon payment rate in our model may be seen as a stylized result of debt renegotiations. Observe that the coupon reduction θ is a parameter of the model, hence interpreting the model literally implies that the debt renegotiating is prewired at time 0. By comparison, the BCS (2007) model assumes a fixed level of forgiveness after the default period. Both models therefore exclude the possibility of dynamic debt renegotiations. The BCS model includes key features of the well-regulated default procedures of the U.S. market including, for example, maximum time spent in default. Bankruptcy procedures vary between jurisdictions and our model, excluding some of the U.S.-specific features, may better reflect practice outside the U.S. As mentioned, the results of our model are, perhaps surprisingly, similar to the results of the BCS model, suggesting that the differences between the models have limited economic importance.

In our model both the default and the liquidation levels are found by maximizing the value of equity. This approach acknowledges the equityholders' control rights in a solvent company. Finally, we solve our model in closed form, whereas the BCS (2007) model is solved numerically, using the binomial approach of Broadie and Kaya (2007).

2. The Model

Recall that in models with a lognormal state variable, the time t EBIT (earnings before interest and taxes) can be calculated from the "unlevered value" of the company as $A_t(r-\mu)$.

The time 0 market value of equity, assuming $A > B$, is

$$\begin{aligned} E(A) &= \mathrm{E}\left[\int_0^{\tau} e^{-rs}\left(A_s(r-\mu)-c\right)1_{\{A_s \geq B\}} ds\right] \\ &\quad + \mathrm{E}\left[\int_0^{\tau} e^{-rs}\left((1-\omega)(r-\mu)A_s-\theta c\right)1_{\{B>A_s>C\}} ds\right], \end{aligned}$$

where τ represents the liquidation time, formally defined in expression (2). Here $A_s(r-\mu)-c$ is the cash flow to equity at time s, payable as long as the company is not in default. Also, $(1-\omega)(r-\mu)A_s-\theta c$ represents the cash flow to equity in default. The parameter ω, expressed as a fraction of the EBIT cash flow, can be interpreted as a proportional distress cost carried in full by equity when the company is in default. The parameter θ represents the fraction of the contractual coupon payment debtholders receive in default. The remaining parameters are given in Table 2. Note that to facilitate comparison we disregard taxes.

TABLE 2

Parameter Values Used in Example

Table 2 presents the parameter values used in the comparison of our model with the BCS (2007) model. All numerical parameter values are chosen to be economically equivalent to the corresponding BCS parameter values.

Parameters	Values	Explanations
μ	0.01	Drift of asset process
σ	0.20	Volatility of asset process
r	0.05	Risk-free interest rate
A	100.00	Total asset value at time 0
c	3.00	Coupon payment for $A_t > B$
k	0.50	Liquidation cost parameter
θ	0.50	Coupon fraction paid in default
ω	0.50	Distress cost parameter

In our model the time 0 market value of equity is calculated as

$$E(A) = A\pi_1 - \left(\frac{c}{r}\right)\pi_2 + (1-\omega)A\pi_3 - \left(\frac{\theta c}{r}\right)\pi_4, \tag{23}$$

where

$$\begin{aligned} \pi_1 &= \left(\frac{r-\mu}{A}\right)\mathrm{E}\left[\int_0^{\tau} A_s e^{-rs} 1_{\{A_s \geq B\}} ds\right] \\ &= 1-\left(\frac{\alpha-1}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta-1} - \left(\frac{\beta+1}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{\alpha-1}\left(\frac{A}{C}\right)^{-\beta-1}, \end{aligned}$$

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and where α and β are defined in expressions (4) and (5), respectively. We recognize $A\pi_1$ as the time 0 market value of the company EBIT when the company is not in default. This is a version of an above annuity with liquidation risk. The result is proved in Appendix C.

$$\begin{aligned}\pi_2 &= \left(\frac{r}{c}\right)\mathrm{E}\left[\int_0^{\mathcal{T}} ce^{-rs}1_{\{A_s\geq B\}}ds\right]\\ &= 1-\left(\frac{\alpha}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta}-\left(\frac{\beta}{\alpha+\beta}\right)\left(\frac{C}{B}\right)^{\alpha}\left(\frac{A}{C}\right)^{-\beta}.\end{aligned}$$

Here $(c/r)\pi_2$ is the time 0 market value of the coupon payments when the company is not in default and is given in expression (12). This is an above annuity with liquidation risk.

$$\begin{aligned}\pi_3 &= \left(\frac{r-\mu}{(1-\omega)A}\right)\mathrm{E}\left[\int_0^{\mathcal{T}} A_s e^{-rs}1_{\{B>A_s>C\}}ds\right]\\ &= 1-\pi_1-\left(\frac{A}{C}\right)^{-\beta-1}.\end{aligned}$$

The expression $(1-\omega)A\pi_3$ represents the time 0 market value of the *distressed* EBIT cash flow when the company is in default. This is a version of a below annuity with liquidation risk. The result is also proved in Appendix C.

$$\pi_4 = \left(\frac{r}{\theta c}\right)\mathrm{E}\left[\int_0^{\mathcal{T}} ce^{-rs}1_{\{B>A_s>C\}}ds\right] = 1-\pi_2-\left(\frac{A}{C}\right)^{-\beta}.$$

Finally, we recognize $(\theta c/r)\pi_4$ as the time 0 market value of the reduced coupon payments when the company is in default, given in expression (13). This is a below annuity with liquidation risk.

The time 0 market value of debt, assuming $A>B$, is

$$D(A) = \mathrm{E}\left[\int_0^{\mathcal{T}} e^{-rs}(c1_{\{A_s\geq B\}}+\theta c1_{\{B>A_s>C\}})ds\right]+(1-k)\mathrm{E}[e^{-\mathcal{T}r}C].$$

The 1st term represents the time 0 market value of the coupons received when the company is not in default plus the fraction of the coupons received when the company is in default. The last term represents the time 0 market value of the payoff to debtholders upon liquidation, where k represents a proportional deadweight liquidation loss. We also assume absolute priority upon liquidation (i.e., no payoff to equityholders).

The previous expectation can be calculated as

$$D(A) = \left(\frac{c}{r}\right)\pi_2+\left(\frac{\theta c}{r}\right)\pi_4+(1-k)C\left(\frac{A}{C}\right)^{-\beta}.$$

The total time 0 market value of the company is thus

$$v(A) = E(A)+D(A) = A[\pi_1+(1-\omega)\pi_3]+(1-k)C\left(\frac{A}{C}\right)^{-\beta}.$$

The parameters ω and k, reflecting losses in default and liquidation, explain why $v(A)$ is less than A. By letting $\omega=k=0$, $v(A)=A$.

3. Numerical Example Including Endogenous Chapters 7 and 11 Asset Levels

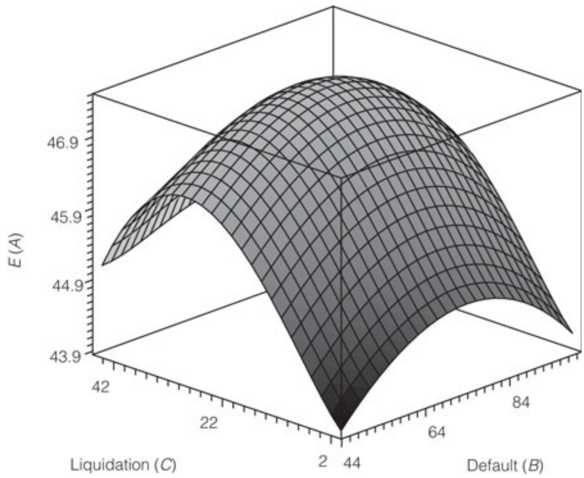
In this subsection we use the parameter values given in Table 2. To facilitate comparison with the results of the BCS (2007) model, the parameter values are either identical to or a logical equivalent to the base case parameters used in BCS.

In Figure 2 we calculate the time 0 market value of equity as a function of default and liquidation asset levels. We see that there exists a unique, global maximum point (i.e., a unique combination of default and liquidation levels that maximizes the time 0 market value of equity).

FIGURE 2

The Time 0 Market Value of Equity as a Function of Default and Liquidation Asset Levels

Figure 2 shows a plot of expression (23). Parameter values are given in Table 2.



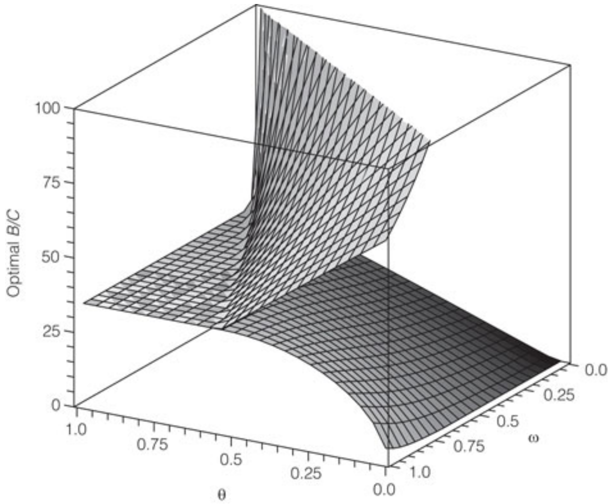
A natural question is how sensitive the optimal default and liquidation levels in Figure 2 are to the choice of the Chapter 11 parameters θ, fraction of coupon paid, and ω, distress cost. This question is addressed in Figure 3, where the upper surface is the optimal default level (B) and the lower surface is the optimal liquidation level (C). The graph shows that for a large θ (i.e., a small reduction in coupon rate) and a large value of ω (i.e., a large distress cost), the optimal values of B and C are identical. In these cases there is no default region, and our model gives the same results as the Leland (1994) model. For a large reduction in coupon rate and a small distress cost, it is optimal to enter Chapter 11 immediately. Our model gives the same result in these situations as the Leland model with a reduced coupon θc. However, it is clear from the figure that there exist optimal values of B, strictly larger than the optimal values of C and, thus, not resulting in immediate default, for a number of combinations of θ and ω.

In Table 3 we compare the results from our model with the BCS (2007) model, presented relative to the similar quantity from the Leland (1994) model. Both models produce similar values for equity and total capital. The default level and the coupon spread are somewhat higher in our model. Although both models

FIGURE 3

Optimal Default (B) and Liquidation (C) Asset Levels

Figure 3 displays the optimal values of default B and liquidation C levels maximizing the time 0 value of equity in expression (23) as a function of the Chapter 11 parameters θ, fraction of coupon paid in default, and ω, distress cost in default. The upper surface represents the optimal B, the lower surface represents the optimal C. Parameter values are given in Table 2.



optimize the value of equity, our equity value is marginally higher, at the expense of a lower total company value, compared to the BCS model. The higher optimal default level in our model naturally corresponds to a higher coupon spread.

TABLE 3

A Comparison with the BCS Model

Table 3 presents a comparison of some of the results from the BCS (2007) model with our model. The numbers in the BCS column are picked as the right-most values of the graphs in BCS (Fig. 4, p. 1359) . This figure is based on equity maximization, where the right-most values of the graphs correspond to the longest maximum period in Chapter 11, a property that is not built into our model. Table 3 shows the results expressed relative to the corresponding results from the Leland (1994) model, using the liquidation level as the reference for B. Parameter values are given in Table 2.

	Our Model	BCS
B	2.18	1.75
$E(A)$	1.03	1.02
$V(A)$	0.96	0.98
Spread	1.64	1.35

B. Debt With Contractual Level-Dependent Interest Rates

We present 2 examples of loans where the contractual interest rates vary by the borrower's credit quality. Lenders' 2 primary concerns are whether borrowers are able to service and repay a loan and the amount of recovery of the loan in case of liquidation. The contractual interest rates of the debt reflect both these concerns. Our example of corporate debt with performance pricing has interest rates depending on the company cash flow (earnings before interest, taxes, depreciations, and amortizations (EBITDA)), a measure of payment ability. Our example of residential mortgage has interest rates depending on the value of the collateral, a measure of recovery if the loan is liquidated.

In our valuation application we assume that the loans have fixed principals and no installments before maturity. The total time 0 value of a loan consists of the time 0 market value of the level-dependent interest payments, the time 0 market value of the repayment of the principal at time T (in case of no liquidation), and the time 0 market value of the realized recovery (in case of liquidation):

$$L(A) \quad = \quad M^T(A) + De^{-rT}Q_g + (1-k)DV_0, \tag{24}$$

where $M^T(A)$ (with n equal to the number of barriers) is given in expression (3), Q_g is given in equation (B-2), and

$$V_0 \quad = \quad \mathrm{E}^Q[e^{-r\tau}1\{\tau \leq T\}] \quad = \quad e^{b(z-w)}N\left(\frac{b-wT}{\sqrt{T}}\right) + e^{b(z+w)}N\left(\frac{b+wT}{\sqrt{T}}\right),$$

where $z=(\mu-(1/2)\sigma^2)/\sigma$, $w=\sqrt{z^2+2r}$, and $b=\ln(C/A)/\sigma$ (see Lando (2004), App. B). Here $N(\cdot)$ denotes the cumulative standard normal distribution function, k can be interpreted as the proportional deadweight loss in case of liquidation, and D is the principal of the loan.

In this application the barrier C is interpreted as the liquidation criteria of the loan but is not necessarily connected to the overall liquidation of the borrower (as in the previous application), and we do not distinguish between default and liquidation; that is, we analyze *contractually* level-dependent market interest rates as opposed to an assumed effect of financial distress (Chapter 11) exemplified in the previous application. In principle our 2 applications could be combined into 1 application, incorporating level dependency due to both contractual interest rates and financial distress.

1. Performance Pricing in Bank Debt Contracts

Performance pricing is a relatively new provision in bank debt contracts (see, e.g., Asquith et al. (2005)). Performance pricing establishes ex ante how changes in indicators of credit quality impact the interest rates of a loan. In this example we value a loan with interest-increasing performance pricing, that is, interest rates increase as the company's credit quality, measured by the debt/EBITDA ratio, deteriorates. This contract is described in Table 4.

TABLE 4

Example of Performance Pricing Contract

Table 4 presents the performance pricing grid from Appendix B in Asquith et al. (2005). The numbers in the LIBOR Plus column are denoted as basis points.

Level	Debt/EBITDA Ratio	EBITDA/Debt Ratio	LIBOR Plus
(1)	< 1.00	> 1.00	75
(2)	$[1.00, 1.25)$	$(0.80, 1.00]$	100
(3)	$[1.25, 1.75)$	$(0.57, 0.80]$	125
(4)	$[1.75, 2.20)$	$(0.45, 0.57]$	150

In this application we assume that the EBITDA/debt ratio is the underlying process in expression (1). A covenant in this contract requires that the

EBITDA/debt ratio[2] should be maintained above 0.45. This corresponds to a liquidation level of $C = 0.45$ in our model. We apply the terms given in Table 4 and assume the same parameter values for μ, σ, r, and k as in Table 2. Furthermore, we assume that $T = 5$, $D = 100$, and the initial EBITDA-to-loan ratio is $A = 1.25$ (implying a loan-to-EBITDA ratio of 0.80). For simplicity we assume that the coupon rates in each region equal the sum of the risk-free rate and the "London Interbank Offered Rate (LIBOR) plus"-interest margin.

From expression (24), using $n = 3$, we calculate the time 0 value of the loan as $L(1.25) = 106.36$. This result implies a *gross* time 0 value of the loan of 6.36, while to estimate the *net* time 0 value of the loan to the lender one also needs to deduct items like administrative costs and capital charges.

In order to quantify the value of default risk and of increased interest rates for decreasing EBITDA-to-loan ratios, a relevant benchmark is the value of a 5-year fixed default-free loan with constant coupon rate of 5.75. Using the previous parameters, the present value of this loan is 103.32, lower than the 106.36 value of the loan including both default risk and increasing coupons. Maintaining the coupon rate of 5.75, but introducing default risk as above, the value of the loan is 102.12. Thus, in this example the added value of increasing, level-dependent interest rates is 4.24.

2. Residential Mortgages With Level-Dependent Interest Rates

The loan-to-collateral value of a residential mortgage impacts its interest rates. A lower loan-to-collateral value implies a lower interest rate, reflecting the lender's reduced loss in case of default and subsequent recovery of the collateral. Potential borrowers are typically offered a menu with interest rates increasing in the loan-to-collateral ratio;[3] an example is given in Table 5.

TABLE 5

Example of Residential Mortgage Contract

Table 5 presents the collateral dependent interest rates for private client residential mortgages at www.postbanken.no (July 2008).

i	Loan-to-Collateral Ratio	Collateral-to-Loan Ratio	Interest Rate c_i
(1)	$< 60\%$	> 1.67 (B_1)	6.95%
(2)	$60\% - 80\%$	$1.25 - 1.67$	7.43%
(3)	$> 80\%$	< 1.25 (B_2)	8.24%

In this application we assume that the collateral-to-loan ratio is the underlying process in expression (1). We assume that $C = 1$ (i.e., the bank terminates the contract the first time the value of the collateral hits the value of the mortgage).

[2]For simplicity we assume that the company's debt consists of only this loan.

[3]Our model assumes a constant risk-free interest rate. In most markets observed floating mortgage rates and (short) risk-free interest rates tend to move in a correlated manner. For consistency, the rates included in Table 5 are valid for adjustable interest rate (floating) mortgages. In our opinion, treating both the current short-term risk-free and the current floating mortgage rates as constant at time 0 for the duration of the mortgage is preferable compared to, for example, using the market terms for mortgages with fixed rates for a fixed number of years.

We assume[4] that $\mu = 0.04$ and $\sigma = 0.08$. Furthermore, we apply the terms given in Table 5 and assume that $r = 0.05$, $k = 0$, $T = 10$, $D = 100$, and the initial collateral-to-loan ratio is $A = 1.75$ (implying a loan-to-collateral ratio of 57%). From expression (24), using $n = 2$, we calculate the total time 0 value of the mortgage as $L(1.75) = 115.73$.

In order to quantify the value of default risk and of increased interest rates for decreasing collateral-to-loan ratios, a relevant benchmark is the value of a 10-year fixed default-free mortgage with coupon rate of 6.95. Using the previous parameters, the present value of this mortgage is 115.35, marginally lower than the 115.73 value of the mortgage including both default risk and increasing coupons.

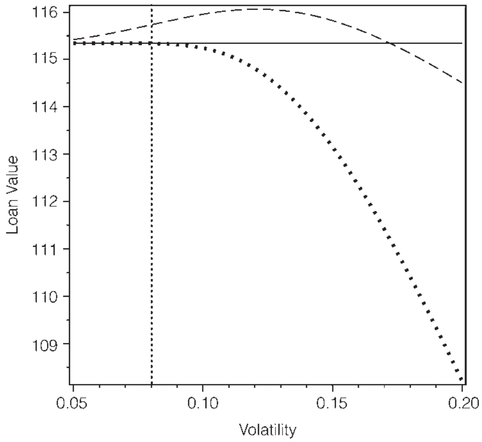
In Figure 4 we plot the values of

i) a fixed interest default-free mortgage,
ii) a fixed interest mortgage including default risk, and
iii) a level-dependent interest mortgage including default risk

FIGURE 4

Mortgage Values as Functions of Volatility

The black horizontal line represents the value of a fixed coupon risk-free mortgage. The dotted line represents the value of a fixed coupon mortgage with liquidation risk, calculated from the extended Black and Cox (1976) formulas presented in Section I. The dashed line represents the value of a level-dependent interest mortgage for increasing volatility using expression (24) with $n = 2$. All values are presented as functions of the volatility. The other parameter values are $A = 1.75$, $\mu = 0.04$, $C = 1$, $r = 0.05$, $k = 0$, $T = 10$, and $D = 100$. Furthermore, B_1, B_2, c_1, c_2, and c_3 are from Table 5. The vertical dotted line represents the estimated volatility at 0.08 of the Norwegian residential market.



for increasing volatility. For volatilities larger than approximately 0.10, the value of the fixed interest mortgage including default risk ii) is naturally well below the value of the risk-free benchmark i). However, the value of the multiple level interest mortgage iii) is higher than the risk-free benchmark i) for all volatilities

[4]These values are estimated from data on Norwegian residential real estate prices from the period 1850–2005 from Norges Bank, most kindly provided to us by Anders Øksendal. Some additional, casual testing using more recent sets of data indicates that these parameter values seem to be in a reasonable range.

up to approximately 0.17. This illustrates, first, that the increasing coupons for lower collateral-to-loan ratios more than compensate the lender for the default risk, and second, that the level-dependent interest mortgage has a higher value than the risk-free mortgage for all realistic values of volatility (< 0.17).

As a final point, observe that the value of the level-dependent interest mortgage is maximized for a volatility of approximately 0.125. This observation provides the lender with incentives to target the riskier segment of the mortgage market, given that the Norwegian housing market volatility is approximately 0.08. A moderately higher volatility increases the probability of receiving the higher coupon rates, whereas for a larger increase in volatility (> 0.17) the increased default risk dominates the increased interest.

IX. Conclusions and Areas of Further Research

We present closed-form solutions in a continuous time no-arbitrage model for the market value of level-dependent annuities motivated by debt contracts with separate default and liquidation risk.

We believe that our results have relevance for valuing a number of different financial contracts. This paper includes applications to standard corporate debt and to loans with contractual level-dependent interest rates. Our model of corporate debt includes endogenous default and liquidation levels found by maximizing the value of equity. We apply our results for valuing both performance-priced corporate loans and residential mortgages with interest rates depending on the value of the collateral.

Preferred stock, issued by banks and insurance companies as part of their risk capital, is another example of possible applications. The issuer of such claims is entitled to drop coupon payments in financial distress, potentially conditional upon no dividend payments on common stock (see, e.g., Emanuel (1983), Mjøs and Persson (2010)). Novel structured financial products and special purpose vehicles are other likely areas of application.

An area of further research may be optimal capital structure, including taxes and level-dependent claims. Another area is the valuation of equity-type claims with some form of level dependency (e.g., performance-based compensation contracts).

Our main contributions are general valuation results for level-dependent annuities and some preliminary results for areas where our results may be applied.

Appendix A. Some Standard Valuation Results

In Appendix A, we apply the change of measure technique introduced in finance by Jamshidian (1989) and Geman, El Karoui, and Rochet (1995).

Let Z be any $\mathcal{F}_T$-measurable event. Denote its associated indicator function by 1_Z.

First, the time 0 market value of a claim with time T market value $U^a(A_T, B)$, given in expression (8), receivable at time T only if the event Z occurs, is

$$\text{(A-1)} \qquad \begin{aligned} P_1(A,B) &= \mathrm{E}[e^{-rT}U^a(A_T,B)1_Z] \\ &= U^a(A,B)\mathrm{E}\left[1_Z e^{-(1/2)\sigma^2\beta^2 T-\sigma\beta W_T}\right] \end{aligned}$$

$$= \quad U^a(A,B)Q^\beta(Z)$$
$$= \quad \left(\frac{A}{B}\right)^{-\beta} Q^\beta(Z),$$

where the probability measure Q^β is defined by $\partial Q^\beta/\partial Q = \exp(-½\sigma^2\beta^2 T - \sigma\beta T)$, and the dynamics of A_t under Q^β are $dA_t = (\mu - \sigma^2\beta)A_t dt + \sigma A_t dW_t$ (abusing notation by letting W_t also denote a standard Brownian motion under Q^β).

Similarly, the time 0 market value of a claim with time T market value $U^b(A_T, B)$, given in expression (8), receivable at time T only if the event Z occurs, is

(A-2)
$$\begin{aligned} P_2(A,B) &= \mathrm{E}[e^{-rT}U^b(A_T,B)1_Z] \\ &= U^b(A,B)\mathrm{E}\left[1_Z e^{-(1/2)\sigma^2\alpha^2 T + \sigma\alpha W_T}\right] \\ &= U^b(A,B)Q^\alpha(Z) \\ &= \left(\frac{A}{B}\right)^{\alpha} Q^\alpha(Z), \end{aligned}$$

where the probability measure Q^α is defined by $\partial Q^\alpha/\partial Q = \exp(-½\sigma^2\alpha^2 T + \sigma\alpha T)$ and the dynamics of A_t under Q^α is $dA_t = (\mu + \sigma^2\alpha)A_t dt + \sigma A_t dW_t$ (repeatedly abusing notation by letting W_t also denote a standard Brownian motion under Q^α).

The time 0 market value of a claim that pays 1 upon liquidation (when A_t hits C) if liquidation occurs after time T is

$$\begin{aligned} P_3(A,C) &= \mathrm{E}[e^{-rT}U^a(A_T,C)1\{\tau > T\}] \\ &= U^a(A,C)\mathrm{E}\left[e^{-(1/2)\sigma^2\beta^2 T - \sigma\beta W_T}1\{\tau > T\}\right] \\ &= U^a(A,C)Q^\beta(\tau > T) \\ &= \left(\frac{A}{C}\right)^{-\beta} Q^\beta(\tau > T). \end{aligned}$$

Finally, the time 0 market value of a claim that pays 1 upon liquidation (when A_t hits C) if liquidation occurs before time T is

$$P_4(A,C) \quad = \quad U^a(A,C)Q^\beta(\tau \le T) \quad = \quad \left(\frac{A}{C}\right)^{-\beta} Q^\beta(\tau \le T).$$

Appendix B. Some Standard Probability Results

In Appendix B, we consider the following process under different probability measures. Consider

$$X_t \quad = \quad \ln(A_t) \quad = \quad \ln(A) + \hat{\mu}t + \sigma W_t,$$

where W_t is defined under a fixed probability measure P, and $\ln(A)$, $\hat{\mu}$, and σ are constants. The process X_t represents the logarithmic version of the process A_t used in this paper. Define the stopping time

$$\tau \quad = \quad \inf\{t : A_t = C)\}.$$

The following results are standard:

$$P_g \quad = \quad P(\tau > T) = N\left(\frac{\ln\left(\frac{A}{C}\right) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - \left(\frac{A}{C}\right)^{\left(-\frac{2\hat{\mu}}{\sigma^2}\right)} N\left(-\frac{\ln\left(\frac{A}{C}\right) - \hat{\mu}T}{\sigma\sqrt{T}}\right),$$

$$
\begin{aligned}
P_{gg}(B) &= P(A_T > B, \tau > T) \\
&= N\left(\frac{\ln\left(\frac{A}{B}\right) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - \left(\frac{A}{C}\right)^{\left(-\frac{2\hat{\mu}}{\sigma^2}\right)} N\left(-\frac{\ln\left(\frac{A}{C}\right) + \ln\left(\frac{B}{C}\right) - \hat{\mu}T}{\sigma\sqrt{T}}\right).
\end{aligned}
$$

Observe that $\lim_{B\downarrow C} P(X_t > \ln(B), \tau > T) = P(\tau > T)$. Trivially,

$$
\begin{aligned}
P_{lg}(B) &= P(A_T < B, \tau > T) = N\left(\frac{\ln\left(\frac{A}{C}\right) + \hat{\mu}T}{\sigma\sqrt{T}}\right) - N\left(\frac{\ln\left(\frac{A}{B}\right) + \hat{\mu}T}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{A}{C}\right)^{\left(-\frac{2\hat{\mu}}{\sigma^2}\right)} \left(N\left(-\frac{\ln\left(\frac{A}{C}\right) + \ln\left(\frac{B}{C}\right) - \hat{\mu}T}{\sigma\sqrt{T}}\right) \right. \\
&\quad \left. - N\left(-\frac{\ln\left(\frac{A}{C}\right) - \hat{\mu}T}{\sigma\sqrt{T}}\right)\right).
\end{aligned}
$$

Here $N(\cdot)$ denotes the cumulative standard normal distribution function. The notation P_g is used for the univariate distribution of the stopping time τ, the g signifies that τ is *greater* than T. The notation $P_{gg}(B)$ is used for the joint distribution between A_T and τ, and footscript gg indicates that A_T is *greater* than the value in the parenthesis B and that τ is *greater* than T. Similarly, an occurrence of l in the footscript signifies that the relevant variable is *lower* than some value. For example the notation $P_l(B)$ is used for the univariate distribution of A_T, and the l signifies that the probability of the event A_T is *lower* than B. A similar notation is used throughout.

1. Probability Measure *Q*

Under the probability measure Q,

$$\hat{\mu} = \mu - \frac{1}{2}\sigma^2.$$

(B-1) $$Q_l(B) = Q(A_T \leq B) = N(-d_3),$$

where

$$d_3 = \frac{\ln\left(\frac{A}{B}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}},$$

(B-2) $$Q_g = Q(\tau > T) = N(d_1) - \left(\frac{A}{C}\right)^{\alpha-\beta} N(-d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{A}{C}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln\left(\frac{A}{C}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Also,

$$\text{(B-3)} \quad Q_{gg}(B) = Q(A_T > B, \tau > T) = N(d_3) - \left(\frac{A}{C}\right)^{\alpha-\beta} N(-d_4),$$

$$d_4 = \frac{\ln\left(\frac{A}{C}\right) + \ln\left(\frac{B}{C}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \quad \text{and}$$

$$\text{(B-4)} \quad Q_{lg}(B) = Q(A_T < B, \tau > T) = N(d_1) - N(d_3) + \left(\frac{A}{C}\right)^{\alpha-\beta}(N(-d_4) - N(-d_2)).$$

2. Probability Measure Q^α

Under the probability measure Q^α,

$$\hat{\mu} = \mu + \sigma^2\alpha - \frac{1}{2}\sigma^2.$$

$$\text{(B-5)} \quad Q_l^\alpha(B) = Q^\alpha(A_T \leq B) = N(-d_3^\alpha),$$

where

$$d_3^\alpha = \frac{\ln\left(\frac{A}{B}\right) + \left(\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

$$\text{(B-6)} \quad Q_g^\alpha = Q^\alpha(\tau > T) = N(d_1^\alpha) - \left(\frac{A}{C}\right)^{-(\alpha+\beta)} N(-d_2^\alpha),$$

where

$$d_1^\alpha = \frac{\ln\left(\frac{A}{C}\right) + \left(\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and}$$

$$d_2^\alpha = \frac{\ln\left(\frac{A}{C}\right) - \left(\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Also,

$$\text{(B-7)} \quad Q_{gg}^\alpha(B) = Q^\alpha(A_T > B, \tau > T) = N(d_3^\alpha) - \left(\frac{A}{C}\right)^{-(\alpha+\beta)} N(-d_4^\alpha),$$

where

$$d_4^\alpha = \frac{\ln\left(\frac{A}{C}\right) + \ln\left(\frac{B}{C}\right) - \left(\mu + \sigma^2\alpha - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and}$$

$$\text{(B-8)} \quad Q_{lg}^\alpha(B) = Q^\alpha(A_T < B, \tau > T) = N(d_1^\alpha) - N(d_3^\alpha) + \left(\frac{A}{C}\right)^{-(\alpha+\beta)}(N(-d_4^\alpha) - N(-d_2^\alpha)).$$

3. Probability Measure Q^β

Under the probability measure Q^β,

$$\hat{\mu} = \mu - \sigma^2\beta - \frac{1}{2}\sigma^2.$$

$$\text{(B-9)} \quad Q_l^\beta(B) = Q^\beta(A_T \leq B) = N\left(-d_3^\beta\right),$$

where

$$d_3^\beta = \frac{\ln\left(\frac{A}{B}\right) + \left(\mu - \sigma^2\beta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

$$\text{(B-10)} \quad Q_g^\beta = Q^\beta(\tau > T) = N\left(d_1^\beta\right) - \left(\frac{A}{C}\right)^{(\alpha+\beta)} N\left(-d_2^\beta\right),$$

where

$$d_1^\beta = \frac{\ln\left(\frac{A}{C}\right) + \left(\mu - \sigma^2\beta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and}$$

$$d_2^\beta = \frac{\ln\left(\frac{A}{C}\right) - \left(\mu - \sigma^2\beta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}.$$

Also,

$$\text{(B-11)} \quad Q_{gg}^\beta(B) = Q^\beta(A_T > B, \tau > T) = N\left(d_3^\beta\right) - \left(\frac{A}{C}\right)^{(\alpha+\beta)} N\left(-d_4^\beta\right),$$

where

$$d_4^\beta = \frac{\ln\left(\frac{A}{C}\right) + \ln\left(\frac{B}{C}\right) - \left(\mu - \sigma^2\beta - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \quad \text{and}$$

$$\text{(B-12)} \quad Q_{lg}^\beta(B) = Q^\beta(A_T < B, \tau > T) = N\left(d_1^\beta\right) - N\left(d_3^\beta\right) + \left(\frac{A}{C}\right)^{(\alpha+\beta)} \left(N\left(-d_4^\beta\right) - N\left(-d_2^\beta\right)\right).$$

Appendix C. Valuation of Claims With Stochastic Dividend Rates

We calculate the value of above and below claims with stochastic dividend rates. In the main body of this paper we have treated deterministic dividend rates, interpretable as coupon rates. As opposed to these coupon rates, the dividend rates in this case are functions of A_t in expression (1) and, thus, random. With $c(A) = A$ we solve the differential equation (6) and obtain the time 0 value of ISIH above and below claims with stochastic dividend rates as

$$\text{(C-1)} \quad \hat{V}_A(A,B) = \begin{cases} \hat{V}_A^a = \left(\frac{A}{r-\mu}\right)\left(1 - \left(\frac{\alpha-1}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{-\beta-1}\right), & \text{when } A \geq B, \\ \hat{V}_A^b = \left(\frac{A}{r-\mu}\right)\left(\frac{\beta+1}{\alpha+\beta}\right)\left(\frac{A}{B}\right)^{\alpha-1}, & \text{when } A \leq B, \end{cases}$$

and

$$(\text{C-2})\quad \hat{V}_B(A,B) = \begin{cases} \hat{V}_B^a = \left(\dfrac{A}{r-\mu}\right)\left(\dfrac{\alpha-1}{\alpha+\beta}\right)\left(\dfrac{A}{B}\right)^{-\beta-1}, & \text{when } A \geq B, \\ \hat{V}_B^b = \left(\dfrac{A}{r-\mu}\right)\left(1-\left(\dfrac{\beta+1}{\alpha+\beta}\right)\left(\dfrac{A}{B}\right)^{\alpha-1}\right), & \text{when } A \leq B. \end{cases}$$

Expressions (C-1) and (C-2) are reduced to expressions (9) and (10) by letting $\mu = 0$, replacing A by c, replacing $(\alpha-1)$ by α, and replacing $(\beta+1)$ by β. The first 2 replacements are logical; μ represents the drift of the dividend, which is 0 for a fixed coupon, and A is the initial value of the dividend rates in this case (comparable with c in the case with a deterministic coupon). The 2 latter replacements do not have any immediate logical interpretations but are due to the more general dividend rate in this case.

The expressions for π_1 and π_3 in Section VIII.A.3 are easily calculated from equation (11), using the $\hat{V}$s from expressions (C-1) and (C-2).

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