Laser and Particle Beams (2017), *35*, 137–144.

doi:10.1017/S0263034616000902

Growth of spike in relativistic Gaussian laser beam in a plasma and its effect on third-harmonic generation

N. AHMAD,[1] S.T. MAHMOUD,[1] AND G. PUROHIT[2]

[1]Department of Physics, College of Science, UAE University, PO Box 15551 Al-Ain, United Arab Emirates
[2]Department of Physics, Laser-Plasma Computational Laboratory, DAV (PG) College, Dehradun, Uttarakhand, India

(Received 24 August 2016; Accepted 15 December 2016)

Abstract

A paraxial ray formalism is developed to study the evolution of an on axis intensity spike on a Gaussian laser beam in a plasma dominated by relativistic and ponderomotive non-linearities. Ion motion is taken to be frozen. A single beam width parameter characterizes the evolution of the spike. The spike introduces two competing influences: diffraction divergence and self-convergence. The former grows with the reduction in spot size of the spike, while the latter depends on the gradient in non-linear permittivity. Parameter $\delta = (\omega_p r_{00}/c)\, a_{00}/(3.5\, r_{00}/r_{01})$ characterizes the relative importance of the two, where r_{01} and r_{00} are the spike and main beam radii, ω_p is the plasma frequency, and a_{00} is the normalized laser amplitude. For $\delta > 1$, the intensity ripple causes faster self-focusing of the beam; higher the ripple amplitude stronger the focusing. In the opposite limit, diffraction divergence increases more rapidly, slowing down the self-focusing of the beam. As the beam intensity rises due to self-focusing, it causes stronger generation of the third harmonic.

Keywords: Gaussian laser beam; Self-focusing; Non-linear plasma physics; Paraxial theory; Harmonic generation

1. INTRODUCTION

The propagation of laser beams of finite size is affected by diffraction divergence. In a non-linear medium, this effect is countered by non-linear refraction and the beam may be self-focused. In plasmas, non-linearity arises due to variety of sources, for example, Ohmic heating, ponderomotive force, relativistic mass modification, and tunnel ionization; hence, self-focusing and filamentation (Sodha *et al.*, 1976; Leemans *et al.*, 1992; Esarey *et al.*, 1996; Borghesi *et al.*, 1998; Hafizi *et al.*, 2000; Liu & Tripathi, 2001; Lushnikov & Rose, 2006) offer a fascinating study. These phenomena manifest in modifying a host of non-linear phenomena, such as parametric instabilities, harmonic generation (Sprangle & Esarey, 1991; Zhou *et al.*, 1996; Ganeev *et al.*, 2009; Singhal *et al.*, 2009), super continuum generation, self-generated magnetic fields, THz generation, and charged particle acceleration (Fuchs *et al.*, 2006; Wang *et al.*, 2013; Arefiev *et al.*, 2014). While most studies on self-focusing deal with Gaussian beams, some have explored non-Gaussian beams (Misra & Mishra, 2009; Gill *et al.*, 2010; Patil *et al.*, 2012). Sodha *et al.* (2009*b*) have studied self-focusing/defocusing of dark hollow Gaussian beams in plasma considering collisional, ponderomotive, and relativistic non-linearities.

The growth of filamentation instability in plasmas has received considerable interest in the field of laser-driven fusion. Kaw *et al.* (1973) studied the growth of a small amplitude ripple on a plane uniform wave front. The ripple modulates the refractive index of the plasma and it acquires a maximum value where amplitude is the maximum. As the beam propagates the amplitude maxima propagate with smaller phase velocity than the neighboring rays. As a consequence the wave front acquires a curvature and the beam gets focused around intensity maxima by pulling energy from its neighborhood. Thus the perturbation grows with time.

Experimental evidences (Loy & Shen, 1969; Chiligarian, 1968; Abbi & Mahr, 1971) suggest that the filamentation in non-linear media is caused by the presence of intensity spikes on a smooth looking irradiance distribution in the plane, transverse to the direction of propagation. Thus, the growth of a Gaussian ripple on a plane uniform beam has been extensively studied (Pandey *et al.*, 1990; Sharma *et al.*, 2004; Sodha *et al.*, 2006; Sodha & Faisal, 2008; Singh *et al.*, 2009). The growth of the ring ripple on a Gaussian beam has also been investigated (Sodha *et al.*, 1992, 1981, 2004; Asthana *et al.*, 1999) in a paraxial

Address correspondence and reprint requests to: N. Ahmad, Department of Physics, College of Science, UAE University, PO Box 15551 Al-Ain, UAE. E-mail: nafis.jmi@gmail.com

approximation. The approach is based on the formulation of Akhmanov *et al*. (1968) and its development by Sodha *et al*. (1974, 1976). Recently, Purohit *et al*. (2015) have used higher-order paraxial theory to investigate the growth of the ring ripple and its effect on the propagation of a ring ripple in a plasma when both relativistic and ponderomotive non-linearities are simultaneously operative. However, two different beam width parameters f_0 and f_1 are required for the main beam and the ripple, respectively. We have employed a much transparent paraxial theory that employs only one beam width parameter f, which characterizes both the main laser beam and the ripple. It brings out with clarity effects of diffraction and non-linear refraction how they are affected by the ripple. Sun *et al*. (1987) have brought out an interesting feature of self-focusing where short pulse laser creates complete electron evacuation on the axis due to ponderomotive force-driven expulsion of electrons.

Generation of harmonics of electromagnetic waves in plasmas is an important issue in high-power laser–plasma interaction. Since most of the electromagnetic beams have Gaussian distribution of irradiance along the wave front, so there is need to take into account this non-uniformity in the theory of harmonic generation. The magnitude of the generated harmonics is higher in case of non-uniform irradiance. Sodha *et al.* (2009*a*, *b*) studied the third-harmonic generation caused by the self-focusing of a Gaussian beam in a collisional plasma. Kaur *et al*. (2010) have analyzed the effect of self-focusing on resonant third-harmonic generation of laser in a rippled density plasma in which the self-focusing of the laser enhances the third-harmonic power. Liu and Tripathi (2008) have studied resonant third-harmonic generation of a Gaussian laser pulse in a rippled density plasma created by machining beam.

In this paper, we examine the growth of a Gaussian perturbation of small radius on a high-power Gaussian laser beam in a plasma properly accounting for the transfer of energy from the main beam to the spike. The motivation for this study arose from a recent experiment on proton–boron nuclear fusion by Picciotto *et al*. (2014). They employed hydrogen enriched silicon targets with a layer of boron dopant of thickness 100 nm at a depth of 190 nm. A linearly polarized long pulse laser of temporal width 0.3 ns, wavelength 1.315 μm, spot size 80 μm, and moderate intensity $3 \times 10^{16}\ \mathrm{W/cm^2}$ was obliquely impinged on the target at 30° angle of incidence. The experiment reported high-yield production of alpha particles, 10^9 particles per steradian via proton–boron fusion. The fusion requires protons of energy 0.6 MeV. Such proton energies, according to mechanism of hot electron sheath acceleration discussed by Gitomer *et al*. (1986) would require hot electron temperature $T_\mathrm{h} \sim 100$ KeV. The given intensity is just about the border line for this temperature. Thus it appears enhancement in laser amplitude is taking-place in the large expanding plasma. One probable scenario is filamentation.

We invoke twin non-linearities, arising due to relativistic increase in electron mass and ponderomotive force-induced electron density cavitation, to cause spatial growth of a Gaussian spike on a Gaussian laser beam. In Section 2, we discuss the plasma equilibrium in the presence of a laser beam and deduce the non-linear plasma permittivity treating ions to be immobile. In Section 3, the wave equation in Wentzel-Kramers-Brillouin (WKB) and paraxial ray approximations has been solved. A function f is introduced that characterizes the spot size of the perturbation and also the amplitude of the laser on the axis of Gaussian perturbation. The equation governing f is solved numerically. In Section 4, the third-harmonic generation of a relativistic Gaussian laser beam in a plasma have been studied in the limit of normalized laser amplitude $a < 1$. We obtained the third-harmonic non-linear current density and solved the wave equation for the third-harmonic amplitude under paraxial ray approximation. The results are discussed in Section 5.

2. NON-LINEAR DIEECTRIC CONSTANT

Consider a singly ionized plasma of electron density n_0^0. A linearly polarized Gaussian laser beam with a small coaxial ripple superimposed on the main beam propagates along the z-axis through it with electric field

$$\vec{E} = \hat{x} A\,(r, z)\, e^{-i(\omega t - kz)}, \tag{1}$$

$$A|^2_{z=0} = A^2_{00}(1 + \alpha\, e^{-r^2/r^2_{01}})\, e^{-r^2/r^2_{00}},$$

where r_{00} and r_{01} are the initial beam radius (spot size) of the main Gaussian laser beam and the ripple and α is the fractional intensity of the spike. For $z > 0$, following Akhmanov *et al*. (1968) and Sodha *et al*. (1974) one may write in the paraxial ray approximation

$$A_0^2 = (A^2_{00}/f^2)(1 + \alpha e^{-r^2/r^2_{01}f^2})\, e^{-r^2/r^2_{00}f^2}, \tag{2}$$

where $f(z)$ is the beam width parameter to be determined later.

The laser imparts oscillatory velocity to electrons $\vec{v} = e\vec{E}/mi\omega\gamma$ and exerts a ponderomotive force on them (primarily in the transverse direction), following Liu and Tripathi (1995)

$$\vec{F}_\mathrm{p} = e\nabla\varphi_\mathrm{p},$$

$$\varphi_\mathrm{p} = -\frac{mc^2}{e}(\gamma - 1), \tag{3}$$

where $\gamma = (1 + a^2/2)^{1/2}$, $a = e|A|/m\omega c$, c is the speed of light in vacuum, and $-e$ and m are the charge and rest mass of an electron. As the electrons are displaced, a space charge field $\vec{E}_\mathrm{s} = e\nabla\varphi_\mathrm{s}$ is created in the ripple region. In the quasi-steady state, the space charge force on electrons balances the ponderomotive force, that is, $\varphi_\mathrm{s} = -\varphi_\mathrm{p}$. Using

https://doi.org/10.1017/S0263034616000902 Published online by Cambridge University Press

this in the Poisson's equation, $\nabla^2\varphi_s = (e/\varepsilon_0)(n_e - n_0^0)$, the modified electron density can be written as

$$n_e = n_0^0\left(1 + \frac{m\varepsilon_0 c^2}{n_0^0 e^2}\nabla^2\gamma\right), \tag{4}$$

where ε_0 is the permittivity of free space.

The effective relative permittivity of the plasma can be written as

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2\gamma}\frac{n_e}{n_0^0} = \varepsilon_{00} + \Phi(E.E^*),$$

$$\varepsilon_{00} = 1 - \frac{\omega_p^2}{\omega^2},$$

$$\Phi = \frac{\omega_p^2}{\omega^2}\left(1 - \frac{n_e}{n_0^0\gamma}\right). \tag{5}$$

The relativistic factor γ can be expanded in powers of r as

$$\gamma = \gamma_{00} - \gamma_{02}\frac{r^2}{f^2} + \gamma_{04}\frac{r^4}{f^4}, \tag{6}$$

where

$$\gamma_{00} = \left[1 + \frac{a_{00}^2}{2f^2}(1+\alpha)\right]^{1/2},$$

$$\gamma_{02} = \frac{a_{00}^2}{4f^2\gamma_{00}}\left(\frac{\alpha}{r_{01}^2} + \frac{1+\alpha}{r_{00}^2}\right),$$

$$\begin{aligned}\gamma_{04} = {}& \frac{a_{00}^2}{4f^2\gamma_{00}}\left(\frac{\alpha}{2r_{01}^4} + \frac{1+\alpha}{2r_{00}^4} + \frac{\alpha}{r_{01}^2 r_{00}^2}\right)\\ & - \frac{a_{00}^4}{32f^4\gamma_{00}^3}\left(\frac{\alpha}{r_{01}^2} + \frac{1+\alpha}{r_{00}^2}\right)^2.\end{aligned} \tag{7}$$

Using Eq. (6), $\nabla^2\gamma$ can be written as

$$\nabla^2\gamma = -4\frac{\gamma_{02}}{f^2} + 16\gamma_{04}\frac{r^2}{f^4}, \tag{8}$$

From Eq. (4) one may recall that since n_e cannot be negative, one must have for the validity of the treatment $(c^2/\omega_p^2)\,\nabla^2\gamma + 1 > 0$, that is, $4c^2\gamma_{02}/\omega_p^2 f^2 < 1$ or

$$\left(\frac{a_{00}^2}{f^4\gamma_{00}}\right)\left(1 + \alpha + \alpha\frac{r_{00}^2}{r_{01}^2}\right) < \frac{r_{00}^2\omega_p^2}{c^2}. \tag{9}$$

This inequality guarantees $n_e \geq 0$ at $r = 0$. Of course when $n_e > 0$ at $r = 0$, it is so all over for all values of r.

The non-linear permittivity of the plasma can be written as

$$\varepsilon = \varepsilon_0 - \varepsilon_2\frac{r^2}{f^2}, \tag{10}$$

where

$$\varepsilon_0 = 1 - \frac{\omega_p^2}{\omega^2\gamma_{00}}\left(1 - 4\frac{\gamma_{02}c^2}{\omega_p^2 f^2}\right),$$

$$\varepsilon_2 = \frac{\omega_p^2}{\omega^2\gamma_{00}}\left[\frac{\gamma_{02}}{\gamma_{00}}\left(1 - 4\frac{\gamma_{02}c^2}{\omega_p^2 f^2}\right) + 16\frac{\gamma_{04}c^2}{\omega_p^2 f^2}\right].$$

3. EVOLUTION OF THE SPIKE

The wave equation governing the propagation of the laser beam in a low-density plasma is

$$\nabla^2\vec{E} + \frac{\omega^2}{c^2}\varepsilon\vec{E} = 0. \tag{11}$$

Substituting for $\vec{E}$ from Eq. (1) and using WKB approximation, Eq. (11) takes the form

$$2ik\frac{\partial A}{\partial z} + \nabla_\perp^2 A + \frac{\omega^2}{c^2}\Phi A = 0, \tag{12}$$

where $k = (\omega/c)(1 - \omega_p^2/\omega^2)^{1/2}$, $\omega_p = (n_0^0 e^2/\varepsilon_0 m)^{1/2}$ and we have ignored the $\nabla(\nabla.\vec{E})$ term.

We write $A = A_0 \exp(ikS)$, where A_0 and S are real functions of r and z, and separate out the real and imaginary parts of Eq. (12),

$$\frac{\partial A_0^2}{\partial z} + \nabla_\perp^2 S A_0^2 + \frac{\partial S}{\partial r}\frac{\partial A_0^2}{\partial r} = 0, \tag{13}$$

$$2\frac{\partial S}{\partial z} + \left(\frac{\partial S}{\partial r}\right)^2 - \frac{\Phi}{\varepsilon_0} = \frac{1}{k^2 A_0}\nabla_\perp^2 A_0. \tag{14}$$

In the paraxial ray approximation, we expand the eikonal S as

$$S = S_0(z) + S_2(z)\frac{r^2}{2}, \tag{15}$$

and introduce a function $f(z)$ such that

$$S_2 = \frac{1}{f}\frac{df}{dz}, \tag{16}$$

where f is the beam width parameter for slowly converging/diverging fields. Using these in Eq. (13), one obtains the solution for A_0^2 given by Eq. (2) exactly. Using Eqs (2), (15), and (16) in Eq. (14) and collecting the coefficients of r^2 on both sides, we get the equation governing f

$$\frac{d^2f}{d\xi^2} = \frac{1}{f^3}\left[1 + \frac{\alpha}{(1+\alpha)}\left\{\frac{2r_{00}^2}{r_{01}^2} + \frac{r_{00}^4}{r_{01}^4}\frac{(3+\alpha)}{(1+\alpha)}\right\}\right] - \frac{\omega^2}{c^2}\frac{\varepsilon_2}{f}r_{00}^4, \tag{17}$$

where $\xi = z/R_d$, $R_d = kr_{00}^2$. Equation (17) is a non-linear ordinary differential equation governing the behavior of dimensionless beam width parameter f as a function of normalized distance of propagation. The first term on the right-hand-side represents diffraction of the ripple imposed on the laser beam, while the second term represents non-linear self-focusing, arising due to relativistic and ponderomotive non-linearities. The focusing/defocusing of the beam is determined by the relative magnitudes of these terms. For an initially plane wave front $f = 1$ and $df/dz = 0$ at $\xi = 0$.

For $\alpha \le 0.2$, $r_{01}/r_{00} \le 0.3$, Eq. (17) takes the form

$$\frac{d^2f}{d\xi^2} = \frac{1}{f^3}\left(1 + 3\alpha\frac{r_{00}^4}{r_{01}^4}\right) - \frac{\Omega_p^2 a_{00}^2}{4\gamma_{00}^3 f^3}\left(1 + \alpha\frac{r_{00}^2}{r_{01}^2} + \psi\right), \tag{18}$$

$$\psi = \frac{1}{2\Omega_p^2}\left[\frac{1 + a_{00}^2/4f^2}{\gamma_{00}} + \alpha\gamma_{00}\frac{r_{00}^4}{r_{01}^4} - \frac{2a_{00}^2}{f^4\gamma_{00}}\left(1 + \alpha\frac{r_{00}^4}{r_{01}^4}\right)^2\right],$$

where $\Omega_p = r_{00}\omega_p/c$. Further, for beams of large spot size $\Omega_p^2 \gg 1$, $\Psi \ll 1$. At low beam intensities $a_{00}^2/2f^2 < 1$, Eq. (18) simplifies to

$$\frac{d^2f}{d\xi^2} = \frac{1}{f^3}\left(1 + 3\alpha\frac{r_{00}^4}{r_{01}^4}\right) - \frac{\Omega_p^2 a_{00}^2}{4f^3}\left(1 + \alpha\frac{r_{00}^2}{r_{01}^2}\right), \tag{19}$$

giving

$$f^2 = 1 - \beta\xi^2, \tag{20}$$

where

$$\beta = \frac{\Omega_p^2 a_{00}^2}{4} - 1 + \frac{\alpha\, r_{00}^2}{4r_{01}^2}\left(\Omega_p^2 a_{00}^2 - 12\frac{r_{00}^2}{r_{01}^2}\right).$$

For self-focusing β must be positive. Ripple aids self-focusing when $\Omega_p^2 a_{00}^2 > 12r_{00}^2/r_{01}^2$. In the opposite limit, ripple suppresses self-focusing due to enhanced diffraction divergence of the axial portion of the laser beam.

We have solved Eq. (20) numerically for typical parameters. Figure 1 shows the variation of beam width parameter, f, as a function of normalized distance of propagation, ξ, for $\alpha = 0,\ 0.05,\ 0.1$, $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$, $(r_{00}\omega_p/c) = 100$. One observes that self-focusing starts earlier and is stronger when the spike intensity is increased from zero to 0.1, that is, the spike aids self-focusing of the laser beam; higher the value of the spike amplitude, stronger the self-focusing. For 1 μm wavelength laser of spot size 100 μm, the laser travels a distance of 1256 μm as a self-focused beam.

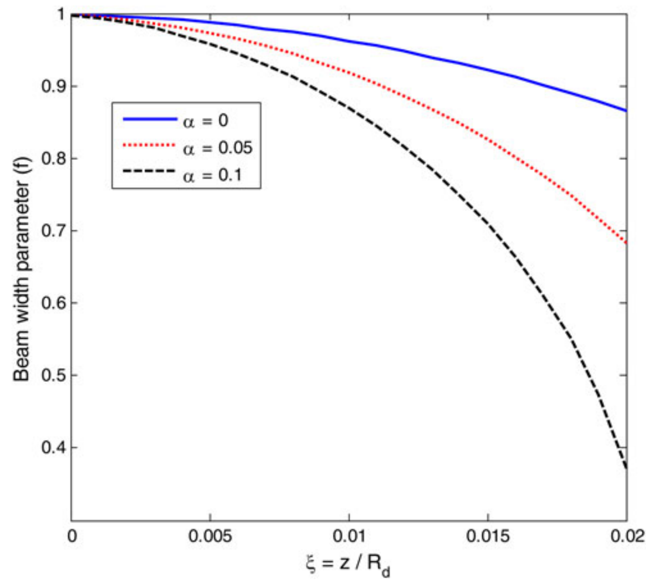

Fig. 1. Beam width parameter f plotted against the normalized distance of propagation ξ in relativistic plasma for different spike intensities $\alpha = 0, 0.05, 0.1$ and for the following parameters $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$, $r_{00}\omega_p/c = 100$.

Figure 2 shows the variation of f with ξ for $\alpha = 0.1$, $r_{01}/r_{00} = 0.1$, $(r_{00}\omega_p/c) = 100$, and different values of normalized main beam amplitude, $a_{00} = 0.4,\ 0.5,\ 0.6$. The self-focusing starts earlier when the value of the normalized amplitude of the main laser is increased due to the

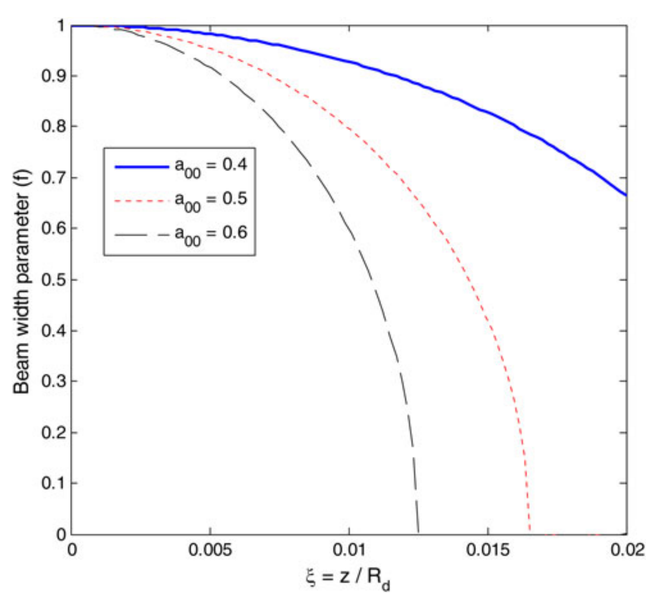

Fig. 2. Beam width parameter f plotted against the normalized distance of propagation ξ in relativistic plasma for main laser intensities $a_{00} = 0.4, 0.5, 0.6$ and for the following parameters $\alpha = 0.1$, $r_{01}/r_{00} = 0.1$, $r_{00}\omega_p/c = 100$.

https://doi.org/10.1017/S0263034616000902 Published online by Cambridge University Press

predominance of the non-linear term over the diffraction term. One may also note that the distance of self-focusing decreases with the increase in the intensity of the main beam. This is similar to the pinching effect of self-generated quasi-stationary magnetic field at relativistic intensity, which adds to self-focusing (Pukhov & Meyer-ter-Vehn, 1996).

Figure 3 shows the variation of f with ξ for different values of normalized plasma density$(r_{00}\omega_p/c) = 50,\ 75,\ 100$ when the other parameters are: $\alpha = 0.1$, $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$. The self-focusing is stronger at higher plasma density due to the enhancement in the non-linear term. Figure 4 shows the variation of f with ξ for different values of the ratio of the size of the spike to main beam radius $r_{01}/r_{00} = 0.1$, 0.15, 0.2, when $\alpha = 0.1$, $a_{00} = 0.5$, $(r_{00}\omega_p/c) = 100$. The self-focusing starts earlier when the value of r_{01}/r_{00} is decreased.

We have also plotted the normalized radial laser intensity profile as a function of r/r_{00} using Eq. (2) in Figure 5a and 5b, for $\alpha = 0.1$, $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$, $(r_{00}\omega_p/c) = 100$, at $(z = 0,\ 0.02R_d)$. The axial intensity increases while the radial width shrinks. The ripple becomes even narrower.

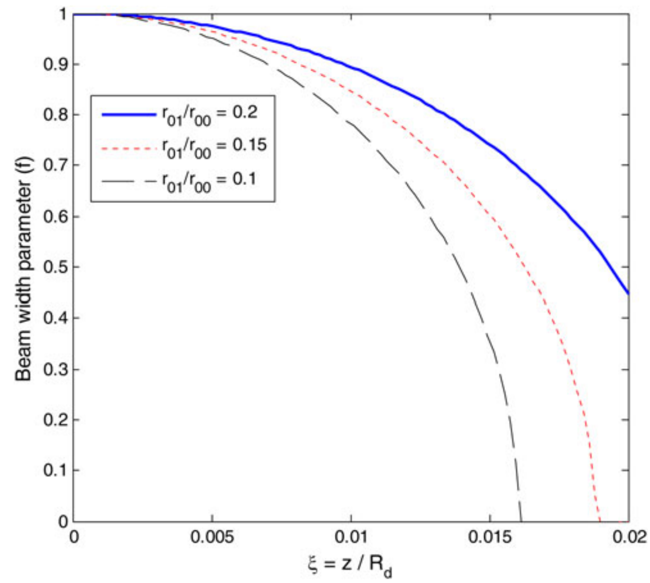

Fig. 4. Beam width parameter f plotted against the normalized distance of propagation ξ in relativistic plasma for the following parameters $r_{01}/r_{00} = 0.1, 0.15, 0.2$, $\alpha = 0.1$, $a_{00} = 0.5$, $r_{00}\omega_p/c = 100$.

4. THIRD-HARMONIC GENERATION

So far we have considered only the time average ponderomotive force on electrons. The laser also exerts a second-harmonic ponderomotive force and gives rise to third-harmonic generation. Here we study this phenomenon in the limit of $a < 1$.

The laser field, given by Eq. (1), imparts oscillatory velocity to electrons, $\vec{v}_\omega = e\vec{E}_\omega/mi\omega\gamma$, and exerts a

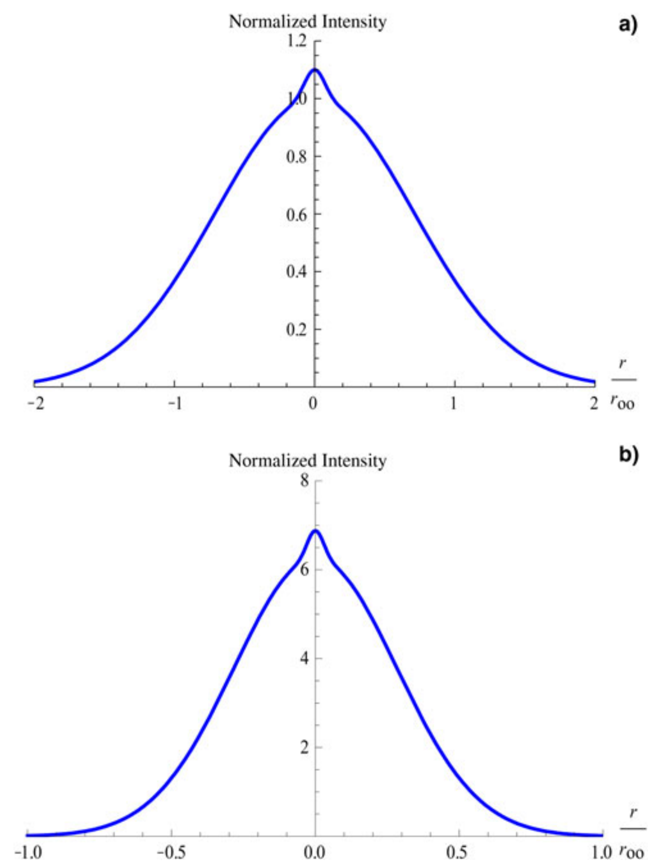

Fig. 5. (a). Plot of the normalized radial laser intensity profiles as a function of r/r_{00}, for $\alpha = 0.1$, $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$, $(r_{00}\omega_p/c) = 100$, at $z = 0$. (b) Plot of the normalized radial laser intensity profiles as a function of r/r_{00}, for $\alpha = 0.1$, $a_{00} = 0.5$, $r_{01}/r_{00} = 0.2$, $(r_{00}\omega_p/c) = 100$, at $z = 0.02R_d$.

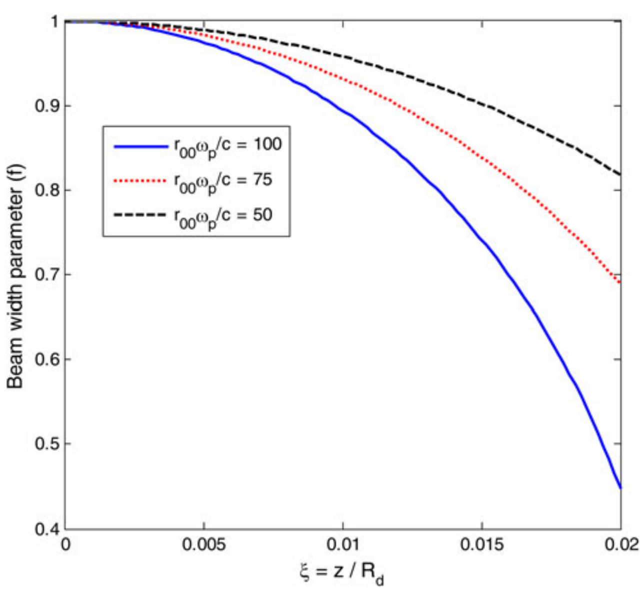

Fig. 3. Beam width parameter f plotted against the normalized distance of propagation ξ in relativistic plasma for the following parameters $r_{00}\omega_p/c = 50, 75, 100$, $\alpha = 0.1$, $r_{01}/r_{00} = 0.2$, $a_{00} = 0.5$.

https://doi.org/10.1017/S0263034616000902 Published online by Cambridge University Press

ponderomotive force on them at the second harmonic

$$\vec{F}_{2\omega} = -\frac{e}{2c}\vec{v}_\omega \times \vec{B}_\omega = -\hat{z}\frac{e^2E_x^2}{2mi\omega c}\eta, \tag{21}$$

where we have employed $\vec{B}_\omega = c\vec{k}\times\vec{E}/\omega$ for the magnetic field of the wave and $\eta = kc/\omega$ is the refractive index. The relativistic factor γ can be written as

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} = \gamma_0 + \gamma_{2\omega}e^{-2i(\omega t - kz)}, \tag{22}$$

where $\gamma_0 = (1 + |v_\omega|^2/4c^2) \approx 1$, and $\gamma_{2\omega} = v_\omega^2/4c^2\gamma_0$.

Using $\vec{F}_{2\omega}$ in the equation of motion one obtains the second-harmonic oscillatory velocity

$$\vec{v}_{2\omega}^{NL} = -\frac{\vec{F}_{2\omega}}{2im\omega} = -\frac{a^2c}{4\gamma_0}\eta\, e^{-2i(\omega t-kz)}. \tag{23}$$

This velocity produces second-harmonic density perturbation through the equation of continuity

$$n_{2\omega}^{NL} = n_0^0\frac{\vec{k}.\vec{v}_{2\omega}^{NL}}{\omega} = -\frac{n_0^0a^2}{4}\eta\, e^{-2i(\omega t-kz)}. \tag{24}$$

The density perturbation gives rise to second-harmonic space charge field $\vec{E}_{2\omega} = -\nabla\Phi_2$. The field in turn produces linear perturbations in electron velocity and density

$$\vec{v}_{2\omega}^{L} = \frac{e\vec{E}_{2\omega}}{mi\omega},$$

$$n_{2\omega}^{L} = \frac{4k^2\chi_2\varepsilon_0}{e}\Phi_2, \tag{25}$$

where $\chi_2 = -\omega_p^2/4\omega^2\gamma_0$. Using density perturbations in the Poisson's equation $\nabla^2\Phi_2 = e(n_{2\omega}^L + n_{2\omega}^{NL})/\varepsilon_0$ and replacing ∇^2 by $-4k^2$, we obtain

$$n_{2\omega} = n_{2\omega}^L + n_{2\omega}^{NL} = -\frac{n_0^0a^2}{4\varepsilon_2}.$$

where $\varepsilon_2 = 1 + \chi_2$.

The response of electrons at the third harmonic is governed by the equation of motion,

$$m\frac{\partial}{\partial t}\left(\gamma_0\,\vec{v}_{3\omega} + \frac{1}{2}\gamma_{2\omega}\,\vec{v}_\omega\right) = -e\vec{E}_{3\omega} - \frac{e}{2}\,\vec{v}_{2\omega}\times\vec{B}_\omega, \tag{26}$$

Assuming $\gamma_0 \approx 1$, we obtain

$$\vec{v}_{3\omega} = \frac{a^3c}{8i}\left(1 + \frac{k^2c^2}{3\omega^2}\right)e^{-3i(\omega t-kz)} \cong \frac{a^3c}{6i}e^{-3i(\omega t-kz)}, \tag{27}$$

where $\eta \approx 1$ in a low-density plasma. The third-harmonic velocity gives rise to third-harmonic current density. Besides this, the density perturbation at 2ω frequency beats with $\vec{v}_\omega$ to produce non-linear current density at the third harmonic,

$$\vec{J}_{3\omega}^{NL} = -\frac{1}{2}n_{2\omega}e\;\vec{v}_\omega - \frac{1}{2}n_0^0e\;\vec{v}_{3\omega} = -\hat{x}\,i\frac{n_0^0e\,a^3c}{48}e^{-3i(\omega t-kz)}. \tag{28}$$

The linear current density due to the self-consistent third-harmonic field $\vec{E}_{3\omega}$ is

$$\vec{J}_{3\omega}^{L} = -\frac{n_0^0e^2\,\vec{E}_{3\omega}}{3m\,i\,\omega}. \tag{29}$$

The wave equation governing the generation of third-harmonic field is

$$\nabla^2\vec{E}_{3\omega} + k_3^2\,\vec{E}_{3\omega} = -\frac{3i\omega}{\varepsilon_0c^2}\vec{J}_{3\omega}^{NL}. \tag{30}$$

We may write $\vec{E}_{3\omega} = \hat{x}A_{3\omega}(r,\ z)\,e^{-3i(\omega t-kz)}$, where $k_3 = (3\omega/c)(1 - \omega_p^2/9\omega^2)^{1/2}$.

Using Eq. (28) for $\vec{J}_{3\omega}^{NL}$, Eq. (30) gives

$$\frac{A_3}{A} = \frac{i\beta_1e^{-i\beta_1z}}{128}\int_0^z\frac{a_{00}^2}{f^3}(1+\alpha)e^{i\beta_1z}dz, \tag{31}$$

where

$$\beta_1 = -\frac{4}{3}\frac{\omega_p^2}{\omega c(1-\omega_p^2/\omega^2)^{1/2}},\; f = (1-\beta\xi^2)^{1/2}.$$

The third-harmonic power conversion efficiency is related to the amplitude ratio as

$$\eta_p = A_3^2/3A^2.$$

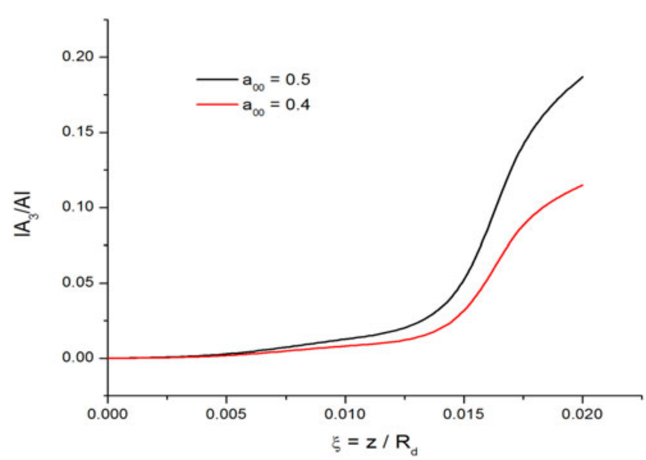

Fig. 6. Ratio of third-harmonic amplitude to main laser amplitude plotted against the normalized distance of propagation ξ for the following parameters $a_{00} = 0.4, 0.5$, $r_{01}/r_{00} = 0.2$, $\alpha = 0.1$, $r_{00}\omega_p/c = 100$.

https://doi.org/10.1017/S0263034616000902 Published online by Cambridge University Press

The factor of $1/3$ arises due to the reduced spot size of the third harmonic.

We have solved Eq. (31) numerically, in conjunction with Eq. (20) governing the beam width parameter, for the following parameters: $a_{00} = 0.4, 0.5$ $\alpha = 0.1$, $(r_{00}\omega_p/c) = 100$, $r_{01}/r_{00} = 0.2$. Figure 6 shows the variation of normalized third-harmonic amplitude with the normalized distance of propagation when self-focusing effect is taken into account. The harmonic amplitude rises with distance rather rapidly due to the self-focusing of the laser beam. At higher laser amplitude, third-harmonic generation is stronger.

5. CONCLUSIONS

The presence of an axial spike in a Gaussian laser beam has significant effect on the self-focusing of the beam due to relativistic and ponderomotive non-linearities. When the spike amplitude is increased, there is stronger self-focusing depending on the ratio of the size of the spike to main beam radius r_{01}/r_{00}, normalized laser amplitude a_{00}, and normalized plasma frequency $\omega_p r_{00}/c$. It is observed that self-focusing starts earlier and is stronger when the spike intensity is increased from zero to 0.1. It is also found that the self-focusing of the main beam starts earlier when the value of the normalized amplitude of the main laser beam a_{00}, as well as normalized plasma frequency $\omega_p r_{00}/c$ are increased. The focusing length $z = \xi R_d$ decreases with increasing a_{00}. For $a_{00} > 1$, the electrons are completely evacuated from the channel and present paraxial theory does not hold. It is also noted that self-focusing is stronger when the spike radius to main beam radius ratio r_{01}/r_{00} is decreased. The problem is relevant in the context of proton–boron fusion experiment by Picciotto *et al*. (2014), as one gets intensity enhancement of the ripple by $1/f^2 \approx 4$. Thus, the effective intensity of the laser is enhanced to 1.2×10^{17} W/cm^2 from 3×10^{16} W/cm^2. At this intensity electron temperature is also enhanced by a similar factor; hence, 600 KeV are likely to produced that can cause proton–boron fusion.

The enhanced self-focusing of the laser beam manifests in the rise of third-harmonic generation efficiency. Third-harmonic generation could be a valuable diagnostic for filamentation. The results of the present analysis are useful in understanding the physics of intense laser plasma interaction and find the application in the high-power laser-driven fusion and particle acceleration process.

For long pulses the present treatment may be extended to study thermal self-focusing where ion motion becomes important. Thermal nonlinearity arises due to Ohmic heating and can be ignored for pulses shorter than collisional heating time.

ACKNOWLEDGMENT

The authors are grateful to Professor V. K. Tripathi, IIT Delhi, for the helpful discussions. The authors are also grateful to United Arab Emirates University for financial support under grant UPAR (2014)-31S164.

REFERENCES

Abbi, S.C. & Mahr, H. (1971). Correlation of filaments in nitrobenzene with laser spikes. *Phys. Rev. Lett*. **26**, 604.

Akhmanov, S.A., Sukhorukov, A.P. & Khokhlov, R.V. (1968). Self focusing and diffraction of light in a nonlinear medium. *Sov. Phys. – Usp*. **10**, 609–636.

Arefiev, A.V., Khudik, V.N. & Schollmeier, M. (2014). Enhancement of laser-driven electron acceleration in an ion channel. *Phys. Plasmas* **21**, 033104.

Asthana, M.V., Giulietti, A., Varshney, D. & Sodha, M.S. (1999). Relativistic self-focusing of a rippled laser beam in a plasma. *J. Plasma Phys*. **62**, 389–396.

Borghesi, M., Mackinnon, A.J., Gaillard, R., Willi, O., Pukhov, A. & Mayer-ter-Vehn, J. (1998). Large quasistatic magnetic fields generated by a relativistically intense laser pulse propagating in a preionized plasma. *Phys. Rev. Lett*. **80**, 5137.

Chiligarian, Y.S. (1968). Self-focusing of inhomogeneous laser beams and its effect on stimulated scattering. *Zh. Eksp. Teor. Fiz*. **55**, 1589.

Esarey, E., Sprangle, P., Krall, J. & Ting, A. (1996). Overview of plasma-based accelerator concepts. *IEEE Trans. Plasma Sci*. **24**, 252.

Fuchs, J., Antici, P., d'Humieres, E., Lefebvre, E., Borghesi, M., Brambrink, E., Cecchetti, C.C., Kaluza, M., Malka, V., Manclossi, M., Meyroneinc, S., Mora, P., Schreiber, J., Toncian, T., Pepin, H. & Audebert, P. (2006). Laser-driven proton scaling laws and new paths towards energy increase. *Nat. Phys*. **2**, 48–54.

Ganeev, R.A., Bom, L.B.E., Abdul-Hadi, J., Wong, M.C.H., Brichta, J.P., Bhardwaj, V.R. & Ozaki, T. (2009). Higher-order harmonic generation from fullerene by means of the plasma harmonic method. *Phys. Rev. Lett*. **102**, 013903.

Gill, T.S., Mahajan, R. & Kaur, R. (2010). Relativistic and ponderomotive effects on evolution of dark hollow Gaussian beams in a plasma. *Laser Part. Beams* **28**, 521–529.

Gitomer, S.J., Jones, R.D., Begay, F., Ehler, A.W., Kephart, J.F. & Kristal, R. (1986). Fast ions and hot electrons in the laser–plasma interaction. *Phys. Fluids* **29**, 2679–2688.

Hafizi, B., Ting, A., Sprangle, P. & Hubbard, R.F. (2000). Relativistic focusing and ponderomotive channeling of intense laser beams. *Phys. Rev. E* **62**, 4120.

Kaur, S., Yadav, S. & Sharma, A.K. (2010). Effect of self-focusing on resonant third harmonic generation of laser in a rippled density plasma. *Phys. Plasmas* **17**, 053101.

Kaw, P., Schmidt, G. & Wilcox, T. (1973). Filamentation and trapping of electromagnetic radiation in plasmas. *Phys. Fluids* **16**, 1522.

Leemans, W.P., Clayton, C.E., Mori, W.B., Marsh, K.A., Kaw, P.K., Dyson, A., Joshi, C. & Wallace, J.M. (1992). Experiments and simulations of tunnel-ionized plasmas. *Phys. Rev. A* **46**, 1091.

Liu, C.S. & Tripathi, V.K. (2001). Self-focusing and frequency broadening of an intense short-pulse laser in plasmas. *J. Opt. Soc. Am. A* **7**, 1714.

Liu, C.S. & Tripathi, V.K. (1995). Interaction of electromagnetic waves with electron beams and plasmas. Singapore: World Scientific.

Liu, C.S. & Tripathi, V.K. (2008). Third harmonic generation of a short pulse laser in a plasma density ripple created by a machining beam. *Phys. Plasmas* **15**, 023106.

Loy, M.M.T. & Shen, Y.R. (1969). Small-scale filaments in liquids and tracks of moving foci. *Phys. Rev. Lett*. **22**, 994.

Lushnikov, P.M. & Rose, H.A. (2006). How much laser power can propagate through fusion plasma? *Plasma Phys. Control. Fusion* **48**, 1501.

Misra, S. & Mishra, S.K. (2009). Focusing of dark hollow Gaussian electromagnetic beams in a plasma with relativistic-ponderomotive regime. *Progr. Electromagn. Res. B* **16**, 291–309.

Pandey, H.D., Tripathi, V.K. & Sodha, M.S. (1990). Growth of a spike on a laser beam in a plasma. *Phys. Fluids B* **2**, 1221.

Patil, S.D., Takale, M.V., Navare, S.T., Fulari, V.J. & Dongare, M.B. (2012). Relativistic self-focusing of cosh-Gaussian laser beams in a plasma. *Opt. Laser Technol*. **44**, 314–317.

Picciotto, A., Margarone, D., Velyhan, A., Bellutti, P., Krasa, J., Szydlowsky, A., Bertuccio, G., Shi, Y., Mangione, A., Prokupek, J., Malinowska, A., Krousky, E., Ullschmied, J., Laska, L., Kucharik, M. & Korn, G. (2014). Boron-proton nuclear-fusion enhancement induced in Boron-Doped silicon targets by low-contrast pulsed laser. *Phys. Rev. X* **4**, 031030.

Pukhov, A. & Meyer-ter-Vehn, J. (1996). Relativistic magnetic self-channeling of light in near-critical plasma: Three dimensional particle-in-cell simulation. *Phys. Rev. Lett*. **76**, 3975.

Purohit, G., Rawat, P., Chauhan, P. & Mahmoud, S.T. (2015). Higher-order paraxial theory of the propagation of ring rippled laser beam in plasma: Relativistic ponderomotive regime. *Phys. Plasmas* **22**, 052116.

Sharma, A., Verma, M.P., Prakash, G. & Sodha, M.S. (2004). Three regimes of growth of a Gaussian ripple on a uniform plane electromagnetic wave front in a plasma. *J. Appl. Phys*. **95**, 2963.

Singh, A., Aggarwal, M. & Gill, T.S. (2009). Dynamics of Gaussian spikes on Gaussian laser beam in relativistic plasma. *Laser Part. Beams* **27**, 587–593.

Singhal, H., Arora, V., Rao, B.S., Naik, P.A., Chakravarty, U., Khan, R.A. & Gupta, P.D. (2009). Dependence of high-order harmonic intensity on the length of preformed plasma plumes. *Phys. Rev. A* **79**, 023807.

Sodha, M.S. & Faisal, M. (2008). Propagation of high power electromagnetic beams in overdense plasmas: Higher order paraxial theory. *Phys. Plasmas* **15**, 033102.

Sodha, M.S., Faisal, M. & Verma, M.P. (2009). Effect of self-focusing on third harmonic generation by a Gaussian beam in a collisional plasma. *Phys. Plasmas* **16**, 082304.

Sodha, M.S., Mishra, S.K. & Misra, S. (2009*b*). Focusing of dark hollow Gaussian electromagnetic beams in a plasma. *Laser Part. Beams* **27**, 57–68.

Sodha, M.S., Ghatak, A.K. & Tripathi, V.K. (1974). Self Focusing of Laser Beams in Dielectrics, Semiconductors and Plasmas. Delhi: Tata-McGraw-Hill.

Sodha, M.S., Ghatak, A.K. & Tripathi, V.K. (1976). Self focusing of laser beams in plasmas and semiconductors. *Progr. Opt*. **13**, 169.

Sodha, M.S., Konar, S. & Maheshwari, K.P. (1992). Steady-state self-focusing of rippled laser beams in plasmas: Arbitrary nonlinearity. *J. Plasma Phys*. **48**, 107–118.

Sodha, M.S., Sharma, A. & Agarwal, S.K. (2006). Focusing of electromagnetic beams in collisional plasmas, with finite thermal conduction. *Phys. Plasmas* **13**, 083105.

Sodha, M.S., Sharma, A., Prakash, G. & Verma, M.P. (2004). Growth of a ring ripple on a Gaussian beam in a plasma. *Phys. Plasmas* **11**, 3023.

Sodha, M.S., Singh, T., Singh, D.P. & Sharma, R.P. (1981). Growth of laser ripple in a plasma and its effect on plasma wave excitation. *Phys. Fluids* **24**, 914.

Sprangle, P. & Esarey, E. (1991). Stimulated backscattered harmonic generation from intense laser interactions with beams and plasmas. *Phys. Rev. Lett*. **67**, 2021.

Sun, G.Z., Ott, E., Lee, Y.C. & Guzdar, P. (1987). Self-focusing of short intense pulses in plasmas. *Phys. Fluids* **30**, 526.

Wang, X., Zgadzaj, R., Fazel, N., Li, Z., Yi, S.A., Zhang, X., Henderson, W., Chang, Y.Y., Korzekwa, R., Tsai, H.-E., Pai, C.-H., Quevedo, H., Dyer, G., Gaul, E., Martinez, M., Bernstein, A.C., Borger, T., Spinks, M., Donovan, M., Khudik, V., Shvets, G., Ditmire, T. & Downer, M.C. (2013). Quasi-monoenergetic laser-plasma acceleration of electrons to 2 GeV. *Nat. Commun*. **4**, 1988.

Zhou, J., Peatross, J., Murnane, M.M., Kapteyn, H.C. & Christov, I.P. (1996). Enhanced high harmonic generation using 25 fs laser pulses. *Phys. Rev. Lett*. **76**, 752–755.