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A piecewise smooth Fermi–Ulam pingpong with potential

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Abstract. In this paper we study a Fermi–Ulam model where a pingpong ball bounces elastically against a periodically oscillating platform in a gravity field. We assume that the platform motion $f(t)$ is 1-periodic and piecewise C^3 with a singularity, $\dot{f}(0+) \neq \dot{f}(1-)$. If the second derivative $\ddot{f}(t)$ of the platform motion is either always positive or always less than $-g$, where g is the gravitational constant, then the escaping orbits constitute a null set and the system is recurrent. However, under these assumptions, escaping orbits co-exist with bounded orbits at arbitrarily high energy levels.

Key words: Fermi acceleration, Fermi–Ulam, ergodic theory, bouncing ball, escaping orbit
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1. *Introduction*

There has been an extensive study of Fermi–Ulam pingpong models since Fermi [15] and Ulam [28] proposed the bouncing ball mechanism as an explanation for the existence of high-energy particles in cosmic rays. The original Fermi–Ulam model describes a point particle bouncing elastically between two infinitely heavy walls, one fixed and the other oscillating periodically [28]. Ulam conjectured [28], based on his numerical experiment with a piecewise linearly oscillating wall, the existence of escaping orbits, that is, orbits whose energy grows to infinity in time. In addition, bounded orbits (that is, those whose energy always stays bounded) and oscillatory orbits (that is, those whose energy has a finite liminf but infinite limsup) might also exist in Fermi–Ulam models and various attempts have been made to examine the existence and prevalence of each of these three types of orbits (we refer to [12, 16, 19] for surveys).

Later, Kolmogorov–Arnold–Moser (KAM) theory has negated the existence of accelerating orbits with sufficiently smooth wall motions as the prevalence of invariant curves

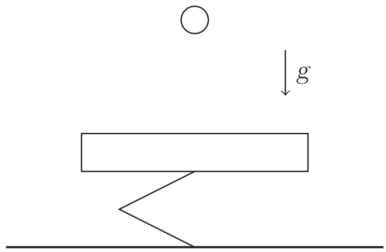

FIGURE 1. Bouncing pingpong in gravity field.

prevents energy diffusion [17, 25, 26]. In non-smooth cases, Zharnitsky [29] found linearly escaping orbits in a piecewise linear model. In a piecewise smooth model with one singularity, de Simoi and Dolgopyat [8] showed that there exists a parameter determining whether the linear part of the limiting system at infinity is elliptic or hyperbolic (that is, whether the absolute value of the trace is less or greater than two) and that bounded orbits co-exist with escaping ones in elliptic regimes while escaping orbits have zero measure but full Hausdorff dimension in hyperbolic regimes.

When background potential is introduced, Arnold and Zharnitsky [2] found unbounded orbits in a pinball system with switching potentials. If the fixed wall is removed and gravity is present, Pustylnikov [24] showed that there exists an open set of wall motions in the space of analytic periodic functions admitting analytic extension to a fixed strip $|\Im t| < \varepsilon$ which produce infinite measure of escaping orbits. In a Duffing equation with a time-dependent polynomial potential with one discontinuity, Levi and You [18] proved the existence of oscillatory orbits. Ortega provided conditions for the existence of escaping orbits in piecewise linear oscillators [21, 22]. For intermediate cases where the potential takes the form $U = x^\alpha$ and the wall motion is sinusoidal, Dolgopyat [11] proved that the escaping orbits do not exist for $\alpha > 1, \alpha \neq 2$ and constitute a null set for $\alpha < 1/3$, while de Simoi [7] showed in the same setting that the escaping orbits possess full Hausdorff dimension for $\alpha < 1$.

In this paper we study a Fermi–Ulam pingpong model with a potential. The model describes a point mass bouncing elastically against an infinitely heavy moving wall in a gravity field. The motion (height) of the wall is a piecewise smooth periodic function $f(t)$ and the gravitational constant is given by g (cf. Figure 1).

We are interested in the case when the motion $f(t)$ of the wall is continuous, 1-periodic and piecewise C^3, that is, $f \in C^3(0, 1)$ and $\dot{f}(0+) \neq \dot{f}(1-)$. We record the time t of each collision and the velocity v immediately after each collision. We exclude from our discussion the singular collisions at integer times, which form a null set in the (t, v)-phase cylinder. We investigate the dynamics of the model by looking at the collision map F, which sends one collision (t, v) to the next one $(\bar{t}, \bar{v})$. Our main result is that if the second derivative of the wall motion behaves, that is, the second derivative is either always positive ($\ddot{f}(t) > 0$) or always less than the negative of the gravitational constant ($\ddot{f}(t) < -g$), then the escaping orbits have zero measure and F is recurrent. We also show that under these assumptions, escaping and bounded orbits exist at arbitrarily high energy levels. The argument in our proof requires piecewise C^2 regularity of F and thus piecewise C^3

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regularity of f. Systems of lower regularities might exhibit similar or other behaviors, but they require other machinery and are beyond the scope of discussion in this paper.

2. *Main results*

In this section we state the main results of the paper.

We denote the second derivative of the wall motion as $k(t) = \ddot{f}(t)$. The collision map F preserves an absolutely continuous measure $\mu = w\, dt\, dv$, where $w = v - \dot{f}$ is the relative velocity after collision (cf. §3.1).

For large velocities, the dynamics can be approximated by

$$F(t, v) = F_\infty(t, v) + \mathcal{O}\left(\frac{1}{v}\right),$$

where

$$F_\infty(t, v) = \left(t + \frac{2v}{g}, v + 2\dot{f}\left(t + \frac{2v}{g}\right)\right).$$

It is easy to verify that the limit map F_∞ is area preserving and it covers a map $\tilde{F}_\infty$ on the torus $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/g\mathbb{Z}$:

$$\tilde{F}_\infty : \begin{cases} \tilde{t}_1 = \tilde{t}_0 + \dfrac{2\tilde{v}_0}{g}, \\ \tilde{v}_1 = \tilde{v}_0 + 2\dot{f}(\tilde{t}_1), \end{cases}$$

where $\tilde{t} = t \pmod 1$, $\tilde{v} = v$ (mod g).

If the second derivative $\ddot{f}$ of the wall motion is either always positive or always less than $-g$, then the limit map $\tilde{F}_\infty$ is ergodic.

THEOREM 1. *Suppose that $\ddot{f}(t) > 0$ for any $t > 0$. Then the map $\tilde{F}_\infty$ is ergodic.*

THEOREM 2. *Suppose that for any $t > 0$, $\ddot{f}(t) < -g$, where g is the gravitational constant. Then the map $\tilde{F}_\infty$ is ergodic.*

We shall call a wall motion $f(t)$ *admissible* if either for all t, $\ddot{f} > 0$ or for all t, $\ddot{f} < -g$.

Remark 2.1. The map $\tilde{F}_\infty$ might not be ergodic if the assumptions in Theorems 1 and 2 fail. For example, when the wall motion is analytic, Pustylnikov [24] found a KAM island for the limit map for an open set of analytic periodic wall motions which admit analytic extension to a strip $|\Im t| < \varepsilon$. Note that for analytic motions $\int_0^1 \ddot{f}(t)\, dt = 0$, so analytic motions are not admissible.

Besides the above ergodic properties, we also obtain stronger statistical properties of $\tilde{F}_\infty$ under the same assumptions.

For every $x, y \in \mathbb{T}$, we define their *forward separation time* $s_+(x, y)$ to be the smallest non-negative integer n such that x, y belongs to distinct continuity components of $\tilde{F}^n_\infty$. We can define similarly their *backward separation time* $s_-(x, y)$ for the inverse iterates. A function $\varphi : \mathbb{T} \to \mathbb{R}$ is said to be *dynamically Hölder continuous* if there exists $\vartheta =$

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$\vartheta(\tilde{F}_\infty) \in (0, 1]$ such that

$$|\varphi|^+_\vartheta := \sup\left\{\frac{|\varphi(x) - \varphi(y)|}{\vartheta^{s_+(x,y)}} : x \neq y \text{ on the same unstable manifold}\right\} < \infty$$

and that

$$|\varphi|^-_\vartheta := \sup\left\{\frac{\varphi(x) - \varphi(y)|}{\vartheta^{s_-(x,y)}} : x \neq y \text{ on the same stable manifold}\right\} < \infty.$$

THEOREM 3. Exponential decay of correlations *Suppose that the wall motion is admissible. Then the map $\tilde{F}_\infty$ enjoys exponential decay of correlations for dynamically Hölder continuous observables: there exists $b > 0$ such that for any pair of dynamically Hölder continuous observables φ, ϕ, there exists $C_{\varphi,\phi}$ such that*

$$\left|\int_{\mathbb{T}} (\varphi \circ \tilde{F}^n_\infty)\phi \, d\tilde{\mu} - \int_{\mathbb{T}} \varphi \, d\tilde{\mu} \int_{\mathbb{T}} \phi \, d\tilde{\mu}\right| \leq C_{\varphi,\phi} e^{-bn}, \quad n \in \mathbb{N}.$$

We observe that for any dynamically Hölder observable φ, the following quantity is finite due to Theorem 3:

$$\sigma^2_\varphi := \sum_{n=-\infty}^{\infty} \int_{\mathbb{T}} \varphi \cdot (\varphi \circ \tilde{F}^n_\infty) \, d\tilde{\mu} < \infty.$$

THEOREM 4. (CLT) *Suppose that the wall motion is admissible. Then the map $\tilde{F}_\infty$ satisfies the central limit theorem (CLT) for dynamically Hölder observables, that is,*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ \tilde{F}^i_\infty \overset{dist}{\rightharpoonup} \mathcal{N}(0, \sigma^2_\varphi),$$

where φ is dynamically Hölder with zero average, $\int_{\mathbb{T}} \varphi \, d\tilde{\mu} = 0$.

As for the original system, under the admissible assumption the escaping orbit of the collision map F constitutes a null set.

THEOREM 5. (Null escaping set) *Suppose that the wall motion is admissible. Then the set E of escaping orbits of F has zero measure.*

It turns out that the escaping set is exactly the dissipative part of the system and consequently under the admissible assumption the system is recurrent.

COROLLARY 6. (Recurrence) *Suppose that the wall motion is admissible. Then F is recurrent, that is, almost every orbit comes arbitrarily close to its initial point.*

However, under the admissible assumption, escaping and bounded orbits still exist.

THEOREM 7. *Suppose that $f(t)$ is admissible. Then F possesses escaping and bounded orbits with arbitrarily high energy.*

Moreover, F satisfies the following *global global mixing* property for *global functions*. We say that a function Φ is *global* if it is bounded, uniformly continuous and has a finite average $\bar{\Phi}$ in the following sense: for any ϵ, there exists N so large that for any rectangle $V = [0, 1) \times [a, b]$ with $b - a > N$, we have

$$\left| \frac{1}{\mu(V)} \int_V \Phi \, d\mu - \bar{\Phi} \right| \leq \epsilon.$$

We denote by $\mathbb{G}_U$ the space of all such global functions.

THEOREM 8. (Global global mixing) *Suppose that the wall motion is admissible. Then F is global global mixing with respect to $\mathbb{G}_U$, that is, for any $\Phi_1, \Phi_2 \in \mathbb{G}_U$, the following holds:*

$$\lim_{n\to\infty} \limsup_{\mu(V)\to\infty} \frac{1}{\mu(V)} \int_V \Phi_1 \cdot (\Phi_2 \circ F^n) \, d\mu$$
$$= \lim_{n\to\infty} \liminf_{\mu(V)\to\infty} \frac{1}{\mu(V)} \int_V \Phi_1 \cdot (\Phi_2 \circ F^n) \, d\mu = \bar{\Phi}_1 \bar{\Phi}_2,$$

where $V = [0, 1) \times [a, b]$ and $\mu(V) = b - a$ is the area of the rectangle V.

3. *Preliminaries*

In this section we discuss the collision map. The study of the collision map relies substantially on the behavior of its limiting map, that is, the approximated collision map for large velocities. We also discuss the singularity lines/curves of the limit map as they will play a very important role in the proofs later.

3.1. *The collision map.*

We denote by $s_n = t_{n+1} - t_n$ the flight time between two consecutive collisions.

Two consecutive collisions satisfy the following equations:

$$\begin{cases} -(v_n - g s_n - \dot{f}(t_{n+1})) = v_{n+1} - \dot{f}(t_{n+1}), \\ f(t_n) + v_n s_n - \frac{1}{2} g s_n^2 = f(t_{n+1}). \end{cases} \tag{1}$$

We compute the derivative of the collision map F by differentiating these equations:

$$dF = \begin{pmatrix} 1 + \dfrac{\dot{f}(t_n) - \dot{f}(t_{n+1})}{w_{n+1}} & \dfrac{s_n}{w_{n+1}} \\ 2\ddot{f}(t_{n+1}) + (2\ddot{f}(t_{n+1}) + g) \dfrac{\dot{f}(t_n) - \dot{f}(t_{n+1})}{w_{n+1}} & (2\ddot{f}(t_{n+1}) + g) \dfrac{s_n}{w_{n+1}} - 1 \end{pmatrix}.$$

We observe that $\det dF = w_n/(w_{n+1})$ and hence F preserves the measure $\mu = w \, dt dv$ on the phase cylinder.

3.2. *The limit map.*

If we only consider collisions with large velocities, the dynamics can be approximated by

$$F(t, v) = F_\infty(t, v) + \mathcal{O}\left(\frac{1}{v}\right),$$

where

$$F_\infty(t, v) = \left(t + \frac{2v}{g}, v + 2\dot{f}\left(t + \frac{2v}{g}\right) \right).$$

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With an abuse of notation we denote $(t_1, v_1) = F_\infty(t_0, v_0)$; then

$$t_1 = t_0 + \left(\frac{2v_0}{g}\right), \qquad v_1 = v_0 + 2\dot{f}(t_1).$$

As mentioned in §2, the limit map F_∞ covers a map $\tilde{F}_\infty$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/g\mathbb{Z}$:

$$\tilde{t}_1 = \tilde{t}_0 + \frac{2\tilde{v}_0}{g}, \qquad \tilde{v}_1 = \tilde{v}_0 + 2\dot{f}(\tilde{t}_1),$$

where $\tilde{t} = t \pmod 1$, $\tilde{v} = v \pmod g$.

Denote $k(t) = \ddot{f}(t)$. The dynamics of $\tilde{F}_\infty$ can be decomposed as

$$\tilde{t}_1 = \tilde{t}_0 + \frac{2\tilde{v}_0}{g} \pmod 1, \qquad \tilde{v}_0 = \tilde{v}_0$$

and

$$\tilde{t}_1 = \tilde{t}_1, \qquad \tilde{v}_1 = \tilde{v}_0 + 2\dot{f}(\tilde{t}_1).$$

Hence, the derivative of $\tilde{F}_\infty$ at $(\tilde{t}_0, \tilde{v}_0)$ is

$$d_{(\tilde{t}_0,\tilde{v}_0)}\tilde{F}_\infty = \begin{pmatrix} 1 & \dfrac{2}{g} \\ 2k_1 & \dfrac{4k_1}{g} + 1 \end{pmatrix},$$

where $k_1 = k(t_1)$.

We observe that $\det d\tilde{F}_\infty = 1$, so $\tilde{F}_\infty$ preserves the Lebesgue measure $\tilde{\mu} = d\tilde{t}d\tilde{v}$ on the torus.

3.3. *The singularity lines of the limit map.* A singularity occurs when the ball collides with the wall at the singularities of the wall motion, that is, $t \in \mathbb{N}$; hence, the singularity line $\mathcal{S}^+$ of $\tilde{F}_\infty$ consists of the points whose next collisions happen at integer times, that is,

$$\mathcal{S}^+ = \{\tilde{t}_1 = 0\} = \left\{\tilde{t}_0 + \frac{2\tilde{v}_0}{g} = 0 \pmod 1\right\}.$$

Similarly, the singularity line $\mathcal{S}^-$ of $\tilde{F}_\infty^{-1}$ consists of the points whose preimages land on integer times, that is,

$$\mathcal{S}^- = \{\tilde{t}_{-1} = 0\} = \left\{\tilde{t}_0 + \frac{4}{g}\dot{f}(\tilde{t}_0) - \frac{2\tilde{v}_0}{g} = 0 \pmod 1\right\}.$$

We observe that $\mathcal{S}^\pm$ consists of finitely many line/curve segments.

4. *Ergodicity of the limit map*

In this section we establish the ergodicity of the limit map $\tilde{F}_\infty$ under the assumptions in Theorems 1 and 2. We use the result by Liverani and Wojtkowski in [20], where they proved ergodicity for a large class of Hamiltonian systems with invariant cones. We first describe the class of symplectic maps (X, T) considered in [20] and then show that $\tilde{F}_\infty$ satisfies the conditions of [20].

Those conditions involve strictly invariant cones and the least coefficient of expansion, which are defined as follows.

Suppose that (X, ω) is a compact symplectic manifold and $(T, \mu) : X$ is a symplectic map preserving the measure μ. For a point $p \in X$, let V_1^p and V_2^p be two transverse Lagrangian subspaces of T_pX; then each vector $v \in T_pX$ has a unique decomposition $v = v_1 + v_2$ with $v_i \in V_i^p$. For any $p \in X$, we define the following quadratic form $\mathcal{Q}_p(v) = \omega(v_1, v_2)$, where $v = v_1 + v_2 \in T_pX$ is the decomposition mentioned before. For any $p \in X$, we consider the two complementary cones

$$\mathcal{C}(p) = \{v \in T_pX : \mathcal{Q}_p(v) \geq 0\}, \quad \mathcal{C}'(p) = \{v \in T_pX : \mathcal{Q}_p(v) \leq 0\}.$$

We say that p possesses *strictly monotone cones* if d_pT strictly preserves $\mathcal{C}(p)$ and d_pT^{-1} strictly preserves $\mathcal{C}'(p)$.

For $p \in X$ with strictly monotone cones $\mathcal{C}(p), \mathcal{C}'(p)$, the *coefficient β of expansion at $v \in T_pX$* is defined as

$$\beta(v, d_pT) = \sqrt{\frac{\mathcal{Q}_p(d_pTv)}{\mathcal{Q}_p(v)}}$$

and the *least coefficient σ of expansion* is defined as

$$\sigma(d_pT) = \inf_{v \in int\ \mathcal{C}(p)} \beta(v, d_pT).$$

Now we list here the six conditions of [20] in the two-dimensional case.

(1) The phase space X is a finite disjoint union of compact subsets of a linear symplectic space $\mathbb{R}^2$ with dense and connected interior and *regular* boundaries, that is, they are finite unions of curves which intersect each other at at most finitely many points.

(2) For every $n \geq 1$, the singularity sets $\mathcal{S}_n^+$ and $\mathcal{S}_n^-$ of T^n and T^{-n} respectively are regular.

(3) Almost every point $p \in X$ possesses strictly monotone cones $\mathcal{C}(p)$ and their complementary cones $\mathcal{C}'(p)$.

(4) The singularity sets $\mathcal{S}^+$ and $\mathcal{S}^-$ are *properly aligned*, that is, the tangent line of $\mathcal{S}^-$ at any $p \in \mathcal{S}^-$ is contained strictly in the cone $\mathcal{C}(p)$ and the tangent line of $\mathcal{S}^+$ at any $p \in \mathcal{S}^+$ is contained strictly in the complementary cone $\mathcal{C}'(p)$. In fact, it is sufficient to assume that there exists N such that $T^N\mathcal{S}^-$ and $T^{-N}\mathcal{S}^+$ are properly aligned.

(5) *Non-contraction*: There is a constant $a \in (0, 1]$ such that for every $n \geq 1$ and for every $p \in X \backslash \mathcal{S}_n^+$,

$$\|d_pT^nv\| \geq a\|v\|$$

for every vector $v \in \mathcal{C}(p)$.

(6) *Sinai–Chernov ansatz*: For almost every $p \in \mathcal{S}^-$ with respect to the measure $\mu_{\mathcal{S}}$ (the measure μ restricted to $\mathcal{S}$), its least coefficient of expansion satisfies

$$\lim_{n\to\infty} \sigma(d_pT^n) = \infty.$$

We note from [20] that σ is supermultiplicative, that is, $\sigma(d_{T_p}Td_pT) \geq \sigma(d_{T_p}T)\sigma(d_pT)$, and that if the coordinates are such that the cone $\mathcal{C}(p)$ is the positive

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cone (that is, $\mathcal{C}(p) = \{\delta x \delta y \geq 0\}$) and $d_p T$ takes the form

$$d_p T = \begin{pmatrix} A_p & B_p \\ C_p & D_p \end{pmatrix},$$

then σ can be computed as $\sigma(d_p T) = \sqrt{1 + t_p} + \sqrt{t_p}$, where $t_p = B_p C_p$.

Liverani and Wojtkowski have proved local ergodicity for symplectic maps satisfying the above six conditions.

THEOREM 9. [20] *Suppose that (X, T) satisfies the above conditions. For any $n \geq 1$ and for any $p \in X \backslash \mathcal{S}_n^+$ such that $\sigma(d_p T^n) > 3$, there is a neighborhood of p which is contained in one ergodic component of T.*

Now we prove Theorem 1.

Proof of Theorem 1. Suppose that $\ddot{f} > 0$.

First we prove local ergodicity by verifying the above six conditions for $\tilde{F}_\infty$.

The singularity lines $\mathcal{S}^\pm$ are finite unions of short lines/curves and they cut our phase space X, which is a torus, into finitely many pieces.

The strict monotonicity follows easily from the fact that $d\tilde{F}_\infty$ is positive when $\ddot{f} > 0$ and hence $d_p \tilde{F}_\infty$ preserves strictly the positive cone $\mathcal{C}^+(p) = \{\delta\tilde{t}\delta\tilde{v} \geq 0\}$ and $d_p \tilde{F}_\infty^{-1}$ preserves strictly the complementary negative cone $\mathcal{C}^-(p) = \{\delta\tilde{t}\delta\tilde{v} \leq 0\}$.

It is straightforward from the previous discussion that $\mathcal{S}^\pm$ are properly aligned since the slope of the tangent line to $\{\tilde{t}_1 = 0\}$ at $(\tilde{t}_0, \tilde{v}_0)$ is $-(g/2) < 0$, and the slope of the tangent line to $\{\tilde{t}_{-1} = 0\}$ is $(g/2)(1 + (4k_0/g)) > 0$.

Next, we verify the non-contraction property. For any non-singular point $p = (\tilde{t}, \tilde{v})$ and any vector $\boldsymbol{v} = (\delta\tilde{t}, \delta\tilde{v}) \in \mathcal{C}^+(p)$ with $\delta\tilde{t}\delta\tilde{v} \geq 0$,

$$\|d_p \tilde{F}_\infty \boldsymbol{v}\|^2 = (1 + 4k_1^2)\delta\tilde{t}^2 + \left(\frac{4}{g^2} + \left(\frac{4k_1}{g} + 1\right)^2\right)\delta\tilde{v}^2 + \left(\frac{4}{g} + 4k_1\left(\frac{4k_1}{g} + 1\right)\right)\delta\tilde{t}\delta\tilde{v} \tag{2}$$

$$\geq \delta\tilde{t}^2 + \delta\tilde{v}^2 = \|\boldsymbol{v}\|^2;$$

thus, the non-contraction property follows for $a = 1$.

Now we verify the Sinai–Chernov ansatz.

We denote $\mathcal{S}_0 = \{\tilde{t}_0 = 0\}$. For any point $p \in \mathcal{S}^- \backslash \cup_{n \geq 0} \mathcal{S}_n$, which excludes a $\tilde{\mu}_\mathcal{S}$-null set, where $\tilde{\mu}_\mathcal{S}$ is the measure $\tilde{\mu}$ restricted to $\mathcal{S}$, since $\mathcal{S}^-$ intersects each $\mathcal{S}_n$ at at most finitely many points,

$$\sigma(d_p \tilde{F}_\infty) = \sqrt{1 + \frac{4k_1}{g}} + \sqrt{\frac{4k_1}{g}} \geq \sqrt{1 + \frac{4k_{\min}}{g}} + \sqrt{\frac{4k_{\min}}{g}} > 1,$$

where $k_{\min} = \min_{t \in (0,1)} \ddot{f}(t) > 0$ by our assumption; then the supermultiplicativity of σ implies that $\lim_{n \to \infty} \sigma(d_p \tilde{F}_\infty^n) = \infty$.

Finally, it remains to check that the singularity sets $\mathcal{S}_n^-$ and $\mathcal{S}_n^+$ of $\tilde{F}_\infty^n$ and $\tilde{F}_\infty^{-n}$ respectively are regular. We claim that for every $n > 0$, $\mathcal{S}_n^-$ $(\mathcal{S}_n^+)$ is a finite union of

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increasing (decreasing) curves, that is, curves with bounded positive (negative) slope. This can be proved by an inductive argument. Firstly the claim holds for $n = 1$ as already shown before. Now suppose that $\mathcal{S}_n^-$ is a finite union of increasing curves. Since $\mathcal{S}_{n+1}^- = \mathcal{S}_n^- \cup \tilde{F}_\infty \mathcal{S}_n^-$ and $d\tilde{F}_\infty$ is a positive matrix and the second derivative $\ddot{f}$ is bounded, $\mathcal{S}_{n+1}^-$ is a finite union of increasing curves. The claim for $\mathcal{S}_n^+$ can be proved similarly.

Observe that $k(t) = \ddot{f}(t) > 0$ is uniformly bounded and hence there exists $N > 0$ such that $\sigma(d_p \tilde{F}_\infty^N) > 3$ for any $p \notin \mathcal{S}_N^+$. Therefore, we have obtained local ergodicity for $\tilde{F}_\infty$ by Theorem 9.

Now we argue for global ergodicity by contradiction.

Suppose that there exists some non-trivial ergodic component M of $\tilde{F}_\infty$; then its boundary ∂M lies on $\mathcal{S}_N^+$. But

$$\mathcal{S}_N^+ = \bigcup_{n=0}^{N-1} \tilde{F}_\infty^{-n} \mathcal{S}^+$$

and hence there exists a smallest integer $N_0 \geq 1$ such that $\partial M \in \mathcal{S}_{N_0}^+$.

Observe that $\tilde{F}_\infty(\partial M) = \partial M$ by the invariance of M. However, $\tilde{F}_\infty(\mathcal{S}_{N_0}^+) = \mathcal{S}_{N_0-1}^+ \cup \mathcal{S}_0$, which contradicts the minimality of N_0. Note that although $\tilde{F}_\infty$ is multivalued at $\mathcal{S}^+$, we have $\tilde{F}_\infty \mathcal{S}^+ = \mathcal{S}_0$ anyway.

Therefore, we conclude that there cannot be any non-trivial ergodic component and hence $\tilde{F}_\infty$ is ergodic. □

The proof of Theorem 2 follows a similar strategy. The main difficulty arises from finding invariant cones as the derivative matrix is no longer positive. However, we still manage to construct invariant cones out of the 'eigenvectors' of the derivative matrix.

Proof of Theorem 2. Suppose that $\ddot{f} < -g$.

First of all, we recall that the derivative of $\tilde{F}_\infty$ at $(\tilde{t}_0, \tilde{v}_0)$ is

$$d_{(\tilde{t}_0,\tilde{v}_0)} \tilde{F}_\infty = \begin{pmatrix} 1 & \dfrac{2}{g} \\ 2k_1 & \dfrac{4k_1}{g} + 1 \end{pmatrix}.$$

Now we consider the following two cones:

$$\mathcal{C}^u(\tilde{t}_0, \tilde{v}_0) = \left\{ \frac{\delta v}{\delta t} \leq k_0 \right\}, \quad \mathcal{C}^s(\tilde{t}_0, \tilde{v}_0) = \left\{ \frac{\delta v}{\delta t} \geq k_0 \right\}.$$

We verify that they are invariant under $d_{(\tilde{t}_0,\tilde{v}_0)} \tilde{F}_\infty$ and $d_{(\tilde{t}_0,\tilde{v}_0)} \tilde{F}_\infty^{-1}$, respectively. For any $(\delta t, \delta v) \in \mathcal{C}^u(\tilde{t}_0, \tilde{v}_0)$,

$$\begin{pmatrix} \bar{\delta t} \\ \bar{\delta v} \end{pmatrix} = \begin{pmatrix} 1 & \dfrac{2}{g} \\ 2k_1 & \dfrac{4k_1}{g} + 1 \end{pmatrix} \begin{pmatrix} \delta t \\ \delta v \end{pmatrix} = \begin{pmatrix} \delta t + \dfrac{2}{g} \delta v \\ 2k_1 \left(\delta t + \dfrac{2}{g} \delta v\right) + \delta v \end{pmatrix};$$

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thus,

$$\begin{aligned}\frac{\bar{\delta v}}{\bar{\delta t}} &= 2k_1 + \frac{\delta v/\delta t}{1+(2/g)\delta v/\delta t} \\ &\leq 2k_1 + \frac{k_0}{1+(2k_0/g)} \\ &< 2k_1 + g < k_1,\end{aligned} \tag{3}$$

where the first inequality follows from $(\delta v/\delta t) \leq k_0$ and the last two inequalities from $\ddot{f} < -g$. Thus, $(\bar{\delta t}, \bar{\delta v}) \in \mathcal{C}^u(\tilde{t}_1, \tilde{v}_1)$.

For any $(\delta t, \delta v) \in \mathcal{C}^s(\tilde{t}_0, \tilde{v}_0)$,

$$\begin{pmatrix}\tilde{\delta t}\\ \tilde{\delta v}\end{pmatrix} = \begin{pmatrix}\dfrac{4k_0}{g}+1 & -\dfrac{2}{g}\\ -2k_0 & 1\end{pmatrix}\begin{pmatrix}\delta t\\ \delta v\end{pmatrix} = \begin{pmatrix}\delta t - \dfrac{2}{g}(-2k_0\delta t + \delta v)\\ -2k_0\delta t + \delta v\end{pmatrix};$$

thus,

$$\begin{aligned}\frac{\tilde{\delta v}}{\tilde{\delta t}} &= -\frac{g}{2} + \frac{g}{2}\frac{1}{1+(2/g)(2k_0-(\delta v/\delta t))} \\ &> -\frac{g}{2} + \frac{g}{2}\frac{1}{1+(2k_0/g)} \\ &> -g > k_{-1},\end{aligned} \tag{4}$$

where the first inequality follows from $(\delta v/\delta t) \geq k_0$ and the last two inequalities from $\ddot{f} < -g$. Thus, $(\tilde{\delta t}, \tilde{\delta v}) \in \mathcal{C}^u(\tilde{t}_{-1}, \tilde{v}_{-1})$.

If we can verify that $\mathcal{S}^{\pm}$ are properly aligned, then the regularity of the singularity curves $\mathcal{S}_n^{\pm}$ follows automatically from the strict invariance of the cones $\mathcal{C}^{u/s}$ since $\mathcal{S}_n^{\pm}$ consist of finitely many transverse short curves.

We claim that $\mathcal{S}^{\pm}$ are properly aligned. Indeed, the slope of the tangent line to $\{\tilde{t}_1 = 0\}$ at $(\tilde{t}_0, \tilde{v}_0)$ is $-(g/2) > k_0$, which is properly contained in $\mathcal{C}^s$. Also, the slope of the tangent line to $\{\tilde{t}_{-1} = 0\}$ is $2k_0 + (g/2) < k_0$, which is properly contained in $\mathcal{C}^u$.

The non-contraction property still holds with $a = 1$.

The unstable cone $\mathcal{C}^u$ is not canonical, that is, it is not the positive cone; hence, we need to switch to the new basis $((0, 1), (1, k_0))$ and $d_p\tilde{F}_\infty$ takes the form

$$\begin{pmatrix}\dfrac{2k_1}{g}+1 & k_0+k_1+\dfrac{2k_0k_1}{g}\\ 2/g & 1+\dfrac{2k_0}{g}\end{pmatrix}.$$

Then the Sinai–Chernov ansatz holds since

$$\begin{aligned}\sigma(d_p\tilde{F}_\infty) &= \sqrt{1+\frac{2}{g}\bigg(\frac{2k_0k_1}{g}+k_0+k_1\bigg)} + \sqrt{\frac{2}{g}\bigg(\frac{2k_0k_1}{g}+k_0+k_1\bigg)} \\ &\geq \sqrt{1+\frac{2}{g}\bigg(\frac{2k_{\min}^2}{g}+2k_{\min}\bigg)} + \sqrt{\frac{2}{g}\bigg(\frac{2k_{\min}^2}{g}+2k_{\min}\bigg)} > 1.\end{aligned}$$

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By Theorem 9, the local ergodicity of $\tilde{F}_\infty$ follows for the case when $\ddot{f} < -g$. Finally, the global ergodicity can be obtained by a similar argument as in the proof of Theorem 1. □

5. *Recurrence of the collision map*

In this section we prove Theorem 5 and Corollary 6 as they are direct consequences of the ergodicity of the limit map $\tilde{F}_\infty$ on the torus.

The proof of Theorem 5 uses a result from [8], which shows that for an asymptotically periodic map with an ergodic limiting map, if the energy change in the limiting map has zero average, then the escaping orbits of the original dynamics have zero measure.

We state this result for our case specifically. First we decompose the velocity v into an integer part and a fractional part, that is, there exists some $m \in \mathbb{Z}$ such that

$$v = \tilde{v} + mg \quad \text{where } \tilde{v} \in [0, g).$$

Then we decompose the limit map F_∞ on the cylinder into its projection $\tilde{F}_\infty$ on the torus and a map γ on integers $\mathbb{Z}$, that is,

$$(\tilde{t}_1, \tilde{v}_1, m_1) = F_\infty(\tilde{t}_0, \tilde{v}_0, m_0) = (\tilde{F}_\infty(\tilde{t}_0, \tilde{v}_0), m_0 + \gamma(\tilde{t}_0, \tilde{v}_0)).$$

LEMMA 5.1. [8] *Suppose that $\tilde{F}_\infty$ is ergodic with respect to the measure $\tilde{\mu} = d\tilde{t}d\tilde{v}$ on the torus. If the energy change of F_∞ has zero average, that is, $\int_{\mathbb{T}} \gamma(\tilde{t}_0, \tilde{v}_0) d\tilde{\mu} = 0$, then the escaping set of F has zero measure.*

Now we prove Theorem 5.

Proof of Theorem 5. If $f(t)$ is admissible, then, by Theorems 1 and 2, $\tilde{F}_\infty$ is ergodic. Thus, by Lemma 5.1, it suffices to check that the energy change γ of F_∞ has zero average. With an abuse of notation, let us denote $(t_1, v_1) = F_\infty(t_0, v_0)$ and $(\tilde{t}_1, \tilde{v}_1) = \tilde{F}_\infty(\tilde{t}_0, \tilde{v}_0)$, respectively. Observe that

$$\int \tilde{v}_0 \, d\tilde{\mu} = \int \tilde{v}_1 \, d\tilde{\mu}$$

since $\tilde{F}_\infty$ preserves the measure $\tilde{\mu}$. Thus,

$$\int \gamma d\tilde{\mu} = \frac{1}{g} \int (v_1 - v_0) \, d\tilde{\mu}.$$

But $v_1 - v_0 = 2\dot{f}(\tilde{t}_1)$ and hence

$$\int (v_1 - v_0) \, d\tilde{\mu} = \int 2\dot{f}(\tilde{t}_1) \, d\tilde{\mu} = \int 2\dot{f}(\tilde{t}_0) \, d\tilde{\mu} = 0,$$

where the last two equalities follow from the facts that $\tilde{F}_\infty$ preserves $\tilde{\mu}$ and that f is 1-periodic. Therefore, by Lemma 5.1, the escaping orbits of F have zero measure. □

The set E of escaping orbits is in fact the transient part of F and hence Theorem 5 implies that F is recurrent, in the spirit of [10].

Proof of Corollary 6. We claim that the set E of escaping orbits is the transient component of the system and hence Theorem 5 implies Corollary 6.

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Indeed, the complement of E is $\cup_N E_N$, where

$$E_N = \{(t_0, v_0) : \liminf v_n \leq N\}.$$

For any $N \in \mathbb{N}$, E_N is invariant and all points in E_N will visit the set $V_N = \{v \leq N + 1\}$. Now we prove that E_N is recurrent, that is, for any subset A of E_N with finite measure, almost every point in A returns to A.

Now suppose that A is a subset of E_N with finite measure. For any $x \in A$, we denote the first hitting time into V_N as

$$r(x) = \min\{k \geq 0 : F^k x \in V_N\}.$$

Now, for any $K \in \mathbb{N}$, we define

$$A_K := \bigcup_{x \in A : r(x) \leq K} F^{r(x)} x.$$

To show recurrence in A, it suffices to show recurrence in A_K for all $K \in \mathbb{N}$, that is, almost every point in A_K visits itself infinitely often. Indeed, for any $x \in A$, $r(x) = K$ for some $K \in \mathbb{N}$, so $F^K x \in A_K$. If $F^K x$ is a recurrent point of A_K, $F^{K+n} x \in A_K$ for some large $n > K$; then there exists some $x' \in A$ such that $F^{K+n} x = F^{r(x')} x'$ and thus $F^{K+n-r(x')} x = x' \in A$. Now we show recurrence in A_K for any $K \in \mathbb{N}$. We note that $A_K \subseteq E_N \cap V_N$ by definition of A_K and the invariance of E_N. All points in E_N visit V_N; thus, the first return map P_N on $E_N \cap V_N$ is well defined. Now our goal is achieved by applying the Poincaré recurrence theorem to $(E_N \cap V_N, P_N)$. □

6. *Statistical properties of the limit map*

In this section we prove Theorems 3, 4 and 8. Throughout this section we assume that the wall motion is admissible.

6.1. *Background.*

The proof of Theorem 3 uses a result of Chernov and Zhang in [6] and the proof of Theorem 4 uses a result of Chernov in [4]. We first describe the class of hyperbolic symplectic maps considered in [4, 6] and then show that our map $\tilde{F}_\infty$ belongs to this class.

Let $T : M \to M$ be a C^2 diffeomorphism of a two-dimensional Riemannian manifold M with singularities $\mathcal{S}$. Suppose that T satisfies the following conditions.

(1) *Uniform hyperbolicity of T.* There exist two continuous families of unstable cones $\mathcal{C}^u_x$ and stable cones $\mathcal{C}^s_x$ in the tangent spaces $\mathcal{T}_x M$ for all $x \in M$, and there exists a constant $\Lambda > 1$ such that:
   - (a) $DT(\mathcal{C}^u_x) \subset \mathcal{C}^u_{Tx}$, and $DT(\mathcal{C}^s_x) \supset \mathcal{C}^s_{Tx}$ whenever DT exists;
   - (b) $\|D_x T \mathrm{v}\| \geq \Lambda \|\mathrm{v}\|$ for all $\mathrm{v} \in \mathcal{C}^{\mathrm{u}}_{\mathrm{x}}$, and $\|D_x T^{-1} \mathrm{v}\| \geq \Lambda \|\mathrm{v}\|$ for all $\mathrm{v} \in \mathcal{C}^{\mathrm{s}}_{\mathrm{x}}$;
   - (c) the angle between $\mathcal{C}^u_x$ and $\mathcal{C}^s_x$ is uniformly bounded away from zero.

(2) *Singularities $\mathcal{S}^\pm$ of T and T^{-1}.* The singularities $\mathcal{S}^\pm$ have the following properties:
   - (a) $T : M \backslash \mathcal{S}^+ \to M \backslash \mathcal{S}^-$ is a C^2 diffeomorphism;
   - (b) $\mathcal{S}_0 \cup \mathcal{S}^+$ is a finite or countable union of smooth compact curves in M;

(c) curves in $\mathcal{S}_0$ are transverse to the stable and unstable cones. Every smooth curve in $\mathcal{S}^+$ ($\mathcal{S}^-$) is a stable (unstable) curve. Every curve in $\mathcal{S}^+$ terminates either inside another curve of $\mathcal{S}^+$ or on $\mathcal{S}_0$;

(d) there exist $\beta \in (0, 1)$ and $c > 0$ such that for any $x \in M \backslash \mathcal{S}^+$, $\|D_x T\| \le c d(x, \mathcal{S}^+)^{-\beta}$.

(3) *Regularity of smooth unstable curves.* We assume that there exists a T-invariant class of unstable curves W such that:

(a) *bounded curvature.* The curvature of W is uniformly bounded from above;

(b) *distortion control.* There exist $\gamma \in (0, 1)$ and C such that for any regular unstable curve W and any $x, y \in W$,

$$|\log \mathcal{J}_W(x) - \log \mathcal{J}_W(y)| \le C d(x, y)^{\gamma},$$

where $\mathcal{J}_W(x) = |D_x T|_W|$ denotes the Jacobian of T at $x \in W$;

(c) *absolute continuity of the holonomy map.* Let $W, \bar{W}$ be two regular unstable curves that are close to each other. We denote

$$W' = \{x \in W : W^s(x) \cap \bar{W} \neq \emptyset\},$$
$$\bar{W}' = \{x \in \bar{W} : \bar{W}^s(x) \cap W \neq \emptyset\}.$$

The holonomy map $h : W' \to \bar{W}'$ is defined by sliding along the stable manifold. We assume that $h_\star \mu_{W'} \ll \mu_{\bar{W}'}$, where $\mu_{W'/\bar{W}'}$ is the measure μ restricted to $W'/\bar{W}'$, and that for some constants C and $\vartheta < 1$,

$$|\log \mathcal{J}h(x) - \log \mathcal{J}h(y)| \le C \vartheta^{s_+(x,y)}, \quad x, y \in W',$$

where $\mathcal{J}h$ is the Jacobian of h and $s_+(x, y)$ is the forward separation time of x, y (cf. the definition of *forward separation time* in §2).

(4) *SRB measure*. The measure $\tilde{\mu}$ is an Sinai–Ruelle–Bowen (SRB) measure, that is, the induced measure $\tilde{\mu}_{W^u}$ (the measure $\tilde{\mu}$ restricted to W^u) on any unstable manifold W^u is absolutely continuous with respect to Leb_{W^u}. We also assume that $\tilde{\mu}$ is mixing.

(5) *One-step expansion.* Let ξ^n denote the partition of M into connected components of $M \backslash \mathcal{S}_n^+$. Denote by V_α the connected component of TW with index $\alpha \in M/\xi^1$ and $W_\alpha = T^{-1} V_\alpha$. there exists $q \in (0, 1]$ such that

$$\liminf_{\delta \to 0} \sup_{W:|W|<\delta} \sum_{\alpha \in M/\xi^1} \left(\frac{|W|}{|V_\alpha|} \right)^q \frac{|W_\alpha|}{|W|} < 1,$$

where the supremum is taken over all unstable curves W.

THEOREM 10. [4, 6] *Under the assumptions 1–5 above, the system (T, M) above enjoys exponential decay of correlations and the central limit theorem for dynamically Hölder continuous observables.*

The verifications of the assumptions 1–5 are rather long. Moreover, their validity is of independent importance itself. So, we first state intermediary lemmas in §6.2 and then we prove, based on these lemmas, Theorems 3, 4 and 8 in §6.3. Finally, we prove all the lemmas in §6.4.

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6.2. *Intermediary lemmas.* In this section we list the intermediate lemmas. Their proofs are presented in §6.4.

Suppose that W is an unstable curve, that is, the tangent line of W lies in the unstable cone $\mathcal{C}^u$, with bounded curvature. We assume without loss of generality that $W \cap \mathcal{S}^+ = \emptyset$. Let $\mathcal{J}_W(x) = |D_x T|_W|$ denote the Jacobian of $\tilde{F}_\infty$ at $x \in W$. We have the following enhanced distortion control.

LEMMA 6.1. (Distortion control) *Suppose that W is an unstable curve with bounded curvature. Then, for any $x, y \in W$, there exists a constant C which depends only on $\tilde{F}_\infty$ and the curvature bound such that*

$$|\log \mathcal{J}_W(x) - \log \mathcal{J}_W(y)| \le C d(x, y).$$

Furthermore, for any $N \in \mathbb{N}$, if $W \cap \mathcal{S}_N^- = \emptyset$, then, for any $1 \le n \le N$, there exists a constant C' which depends only on $\tilde{F}_\infty$ and the curvature bound such that

$$|\log \mathcal{J}_W \tilde{F}_\infty^{-n}(x) - \log \mathcal{J}_W \tilde{F}_\infty^{-n}(y)| \le C'|W|.$$

In order to establish the N_0-step expansion, we need the following estimate of the speed of fragmentation of unstable curves.

LEMMA 6.2. (Complexity bound) *Suppose that z is a branching point of $\mathcal{S}_n^+$. Pick a small neighborhood of z and denote by $k_n(z)$ the number of sectors in the small neighborhood cut out by $\mathcal{S}_n^+$. Then $k_n(z) \le 6n + 4$.*

The linear complexity bound guarantees that a sufficiently short unstable curve W can break into at most $6n + 4$ connected components under $\tilde{F}_\infty^n$. Thus, there exists δ_0 so small that any unstable curve shorter than δ_0 satisfies the following expansion estimate.

LEMMA 6.3. (N_0-step expansion) *Suppose that W is an unstable curve with length $|W| \le \delta_0$ and that $\{W_i^n\}_i$ are the connected components of the image $\tilde{F}_\infty^n W$. Denote by $\Lambda_{i,n}$ the minimum rate of expansion on each preimage $\tilde{F}_\infty^{-n} W_i^n$. Then*

$$\sum_i \frac{1}{\Lambda_{i,N_0}} < 1,$$

where N_0 is the smallest integer such that $(6N_0 + 4/\Lambda^{N_0}) < 1$ and Λ is the expansion rate of $\tilde{F}_\infty$.

Next, we suppose that W and $\bar{W}$ are two unstable curves with bounded curvature. We define the following holonomy map h on

$$W' = \{x \in W : W^s(x) \cap \bar{W} \neq \emptyset\}$$

by sliding along the stable manifold from $x \in W$ to $\bar{x} \in \bar{W}$. Then $h : W' \to \bar{W}'$ is absolutely continuous with well-behaving density.

LEMMA 6.4. (Absolute continuity) *Suppose that W and $\bar{W}$ are two unstable curves with uniform bounded curvature. Then $h_\star \mu_W \ll \mu_{\bar{W}}$ and, for some constants C and $\Theta < 1$,* *which depend only on $\tilde{F}_\infty$ and the curvature bound,*

$$|\log \mathcal{J}h(x) - \log \mathcal{J}h(y)| \leq C\Theta^{s_+(x,y)}, \quad x, y \in W',$$

where $\mathcal{J}h$ is the Jacobian of h.

Finally, we provide an estimate of the measure of small unstable curves, which follows from Lemma 7 in [6].

Suppose that W is an unstable curve with length $|W| < \delta_0$. For any $x \in W$, we denote by $r_n(x)$ the distance from x to the nearest boundary of the connected component of $\tilde{F}_\infty^n W$ containing $\tilde{F}_\infty^n x$.

LEMMA 6.5. (Growth lemma) *Suppose that W is an unstable curve with length $|W| < \delta_0$. Then, for any $\epsilon > 0$,*

$$mes_W\{r_{nN_0}(x) < \epsilon\} \leq (\vartheta_1 \Lambda^{N_0})^n mes_W\left\{r_0(x) < \frac{\epsilon}{\Lambda^{nN_0}}\right\} + C\epsilon|W|,$$

where $\vartheta_1 = e^{C'\delta_0} \sum_i (1/\Lambda_{i,N_0})$, C' is the constant from Lemma 6.1, N_0 is the constant from Lemma 6.3 and Λ is the expansion rate of $\tilde{F}_\infty$.

Remark 6.1. We note that ϑ_1 can be made less than one by choosing δ_0 sufficiently small.

6.3. *Exponential decay of correlations, CLT and global global mixing.* In this section we present the proofs of Theorems 3, 4 and 8, based on the lemmas from §6.2.

We start with the proof for exponential decay of correlations and CLT.

Proofs of Theorems 3 and 4. Firstly we establish the exponential decay and CLT for $\tilde{F}_\infty^{N_0}$ by checking the conditions in Theorem 10 for $\tilde{F}_\infty^{N_0}$, where N_0 is the number from Lemma 6.3. Note that we gain from Theorem 10 the exponential decay and CLT for $\tilde{F}_\infty^{N_0}$ rather than for $\tilde{F}_\infty$ because we can only obtain N_0-step expansion on $\tilde{F}_\infty$.

The proof for the case $\ddot{f} > 0$ is very similar to that for $\ddot{f} < -g$ and hence we omit the latter. Although the positive/negative cones in the proof of Theorem 1 are strictly invariant, we cannot use them here since we require a positive angle between the unstable and stable cones. Instead, we consider their images, that is,

$$\mathcal{Q}^u(\tilde{t}_0, \tilde{v}_0) = \left\{2k_0 \leq \frac{\delta v}{\delta t} \leq 2k_0 + \frac{g}{2}\right\},$$

$$\mathcal{Q}^s(\tilde{t}_0, \tilde{v}_0) = \left\{-\frac{g}{2} \leq \frac{\delta v}{\delta t} \leq -\frac{2k_0}{(4k_0/g) + 1}\right\}.$$

It is easy to see that the angles between $\mathcal{Q}^u$ and $\mathcal{Q}^s$ are uniformly bounded away from zero since $k_0 > 0$ is bounded, and that these cones are strictly invariant.

We now compute the expansion rate Λ. With the same notation as above, for $(\delta t, \delta v) \in \mathcal{Q}^u(\tilde{t}_0, \tilde{v}_0)$, it follows from (2) that

$$\bar{\delta t}^2 + \bar{\delta v}^2 \geq \Lambda_1^2(\delta t^2 + \delta v^2),$$

where $\Lambda_1^2 = \min\{1 + 4k_{\min}, (4/g^2) + (1 + (4k_{\min}/g))^2\} > 1$.

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Similarly, for $(\delta t, \delta v) \in \mathcal{Q}^s(\tilde{t}_0, \tilde{v}_0)$,

$$\tilde{\delta t}^2 + \tilde{\delta v}^2 = \bigg(4k_1^2 + \bigg(1 + \frac{4k_1}{g}\bigg)^2\bigg)\delta t^2 + \bigg(\frac{4}{g^2} + 1\bigg)\delta v^2 - 2\bigg(2k_1 + \frac{2}{g}\bigg(1 + \frac{4k_1}{g}\bigg)\bigg)\delta t \delta v \\ \geq \Lambda_2^2(\delta t^2 + \delta v^2),$$

where $\Lambda_2^2 = \min\{4k_{\min}^2 + (1 + (4k_{\min}/g))^2, (4/g^2) + 1\} > 1$.

Take $\Lambda = \min\{\Lambda_1, \Lambda_2\}$; this gives our expansion rate.

Next, we examine the singularity curves $\mathcal{S}^{\pm}$.

Observe that $\tilde{F}_\infty$ is a C^2 diffeomorphism away from singularities if f is piecewise C^3. And $\mathcal{S}_0 \cup \mathcal{S}^+$ is a finite union of smooth compact curves on the torus $\mathbb{T}$; $\mathcal{S}_0$ is transverse to $\mathcal{Q}^u/\mathcal{Q}^s$. Moreover, the singularity curves are regular and properly aligned as shown in the proof of Theorem 1.

Assumption 2(d) is trivially satisfied since the norm of the derivative $d\tilde{F}_\infty$ is bounded and our phase space is compact.

As for the assumption 3, we have already obtained distortion control in Lemma 6.1 and absolute continuity of the holonomy map in Lemma 6.4. We note that by (6) the curvature of an unstable curve remains bounded after iterations.

Next, the invariant measure $\tilde{\mu} = d\tilde{t}d\tilde{v}$ is apparently an SRB measure.

Note that $\tilde{F}_\infty^n$ is ergodic with respect to $\tilde{\mu}$ for any $n > 0$ since $\tilde{F}_\infty^n$ also satisfies the conditions of Theorem 9. Now the results of [**23**] imply that $\tilde{\mu}$ is mixing (even Bernoulli).

Finally, since we already established N_0-step expansion from Lemma 6.3, we conclude from Theorem 10 that $\tilde{F}_\infty^{N_0}$ enjoys exponential decay of correlations and CLT for dynamically Hölder continuous observables.

The CLT for $\tilde{F}_\infty$ follows easily from that of $\tilde{F}_\infty^{N_0}$. Now we show that exponential mixing for $\tilde{F}_\infty^{N_0}$ implies exponential mixing for $\tilde{F}_\infty$.

Suppose that φ, ϕ are two dynamically Hölder continuous observables. For any integer $n \in \mathbb{N}$, $n = pN_0 + q$ for some integers $p > 0$ and $0 \leq q < N_0$. We denote $\tilde{\varphi}_q = \varphi \circ \tilde{F}_\infty^q$. For any x, y on the same unstable manifold W^u,

$$|\tilde{\varphi}_q(x) - \tilde{\varphi}_q(y)| = |\varphi(\tilde{F}_\infty^q x) - \varphi(\tilde{F}_\infty^q y)| \leq C\vartheta^{-q_+}\vartheta^{s_+(x,y)},$$

where $q_+ = \min\{q, s_+(x, y)\}$.

On the other hand, for any x, y on the same stable manifold W^s,

$$|\tilde{\varphi}_q(x) - \tilde{\varphi}_q(y)| = |\varphi(\tilde{F}_\infty^q x) - \varphi(\tilde{F}_\infty^q y)| \leq C\vartheta^{q}\vartheta^{s_-(x,y)}.$$

Therefore, $\tilde{\varphi}_q$ is also dynamically Hölder.

By applying the previous exponential decay result on $\tilde{F}_\infty^{N_0}$ with the observables $\tilde{\varphi}_q, \phi$, we know that there exist $C_{\tilde{\varphi}_q,\phi}$ and b such that

$$\bigg| \int_{\mathbb{T}} (\varphi \circ \tilde{F}_\infty^n)\phi \, d\tilde{\mu} - \int_{\mathbb{T}} \varphi \, d\tilde{\mu} \int_{\mathbb{T}} \phi \, d\tilde{\mu} \bigg| = \bigg| \int_{\mathbb{T}} (\tilde{\varphi}_q \circ \tilde{F}_\infty^{pN_0})\phi \, d\tilde{\mu} - \int_{\mathbb{T}} \tilde{\varphi}_q \, d\tilde{\mu} \int_{\mathbb{T}} \phi \, d\tilde{\mu} \bigg| \\ \leq C_{\tilde{\varphi}_q,\phi} e^{-bp} = C_{\tilde{\varphi}_q,\phi} e^{bq/N_0}(e^{-b/N_0})^n.$$

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If we take $C_{\varphi,\phi} = \max_q \{C_{\tilde{\varphi}_q,\phi} e^{bq/N_0}\}$ and replace b with b/N_0, then we have proved exponential decay of correlations in the case $\ddot{f} > 0$. □

Next, we prove the global global mixing property for the original collision map F.

Proof of Theorem 8. Under the assumptions 1–5, the limit map $\tilde{F}_\infty$ satisfies the conditions of [3] and thus it admits a Young tower with exponential tail. We recall from §3.2 that $\tilde{F}_\infty$ well approximates the original collision map F at infinity. Therefore, by Theorems 2.4 and 2.9 in [14], F is global global mixing. □

6.4. *Proof of intermediary lemmas.* In this section we prove the lemmas stated in §6.2. We start with the distortion control.

Proof of Lemma 6.1. We parametrize the unstable curve W as $v = \psi(t)$ for some smooth function ψ such that $\psi'(t) \in [2k, 2k + g/2]$ and ψ'' is bounded.

For $x, y \in W$, $|\log \mathcal{J}_W(x) - \log \mathcal{J}_W(y)| \leq \max_{z\in W} |(d/dz) \log \mathcal{J}_W(z)||x - y|$.

For $z \in W$, we take $\mathrm{w} = (1, \psi'(\mathrm{t})) \in \mathcal{T}_z\mathrm{W}$. Then

$$\mathcal{J}_W(z) = \frac{\|d_z\tilde{F}_\infty \mathrm{w}\|}{\|\mathrm{w}\|} = \frac{1}{\sqrt{1+\psi'(t)^2}}\bigg(1 + 4k_1^2 + \psi'(t)^2\bigg(\frac{4}{g^2} + \bigg(1 + \frac{4k_1}{g}\bigg)^2\bigg) + 2\psi'(t)\bigg(\frac{2}{g} + 2k_1\bigg(1 + \frac{4k_1}{g}\bigg)\bigg)\bigg)^{1/2},$$

where $k_1 = \ddot{f}(\tilde{F}_\infty z)$. Hence,

$$\log \mathcal{J}_W(z) = \frac{1}{2}\log\bigg(1 + 4k_1^2 + \psi'(t)^2\bigg(\frac{4}{g^2} + \bigg(1 + \frac{4k_1}{g}\bigg)^2\bigg) + 2\psi'(t)\bigg(\frac{2}{g} + 2k_1\bigg(1 + \frac{4k_1}{g}\bigg)\bigg)\bigg) - \frac{1}{2}\log(1 + \psi'(t)^2). \tag{5}$$

We note that each term inside the logarithms in (5) is greater than one and has bounded derivatives. Thus, $|(d/dz)\log \mathcal{J}_W(z)| \leq C$ for some constant C depending only on $\tilde{F}_\infty$ and the curvature bound.

Besides the above distortion bound, we have the following enhanced estimate. Now we assume further that $W \cap \mathcal{S}_n^- = \emptyset$.

We denote $x_n = \tilde{F}_\infty^{-n}x$, $y_n = \tilde{F}_\infty^{-n}y$ and $W_n = \tilde{F}_\infty^{-n}W$. Then

$$\begin{aligned} |\log \mathcal{J}_W\tilde{F}_\infty^{-n}(x) - \log \mathcal{J}_W\tilde{F}_\infty^{-n}(y)| &\leq \sum_{m=0}^{n-1} |\log \mathcal{J}_{W_m}\tilde{F}_\infty^{-1}(x_m) - \log \mathcal{J}_{W_m}\tilde{F}_\infty^{-1}(y_m)| \\ &\leq \sum_{m=0}^{n-1} |W_m| \max_{z_m\in W_m}\bigg|\frac{d}{dz_m}\log \mathcal{J}_{W_m}\tilde{F}_\infty^{-1}(z_m)\bigg|. \end{aligned}$$

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But

$$\frac{d}{dz_m}\log \mathcal{J}_{W_m}\tilde{F}_\infty^{-1}(z_m) = \frac{dz_{m+1}}{dz_m}\frac{d}{dz_{m+1}}\log\frac{1}{\mathcal{J}_{W_{m+1}}\tilde{F}_\infty(z_{m+1})}$$
$$= -\frac{1}{\mathcal{J}_{W_{m+1}}\tilde{F}_\infty(z_{m+1})}\frac{d}{dz_{m+1}}\log \mathcal{J}_{W_{m+1}}\tilde{F}_\infty(z_{m+1}).$$

Observe that $\mathcal{J}_{W_{m+1}}\tilde{F}_\infty(z_{m+1})$ is bounded. Next,

$$\frac{dv_m}{dt_m} = \frac{2k_m dt_{m-1} + (4k_m/g+1)\,dv_{m-1}}{dt_{m-1} + \dfrac{2}{g}dv_{m-1}}$$
$$= \frac{2k_m + (4k_m/g+1)(dv_{m-1}/dt_{m-1})}{1+(2/g)(dv_{m-1}dt_{m-1})} = 2k_m + g/2 - \frac{g/2}{1+(2/g)(dv_{m-1}/dt_{m-1})}.$$

Therefore,

$$\psi_m'' = 2\ddot{f}(t_m) - \frac{\psi_{m-1}''}{(1+(2/g)\psi_{m-1}')^3},$$

which implies that

$$|\psi_m''| \le 2\dddot{f}_{\max} + \theta|\psi_{m-1}''|,$$

where $\theta := 1/(1+4k_{\min}/g)^3 < 1$. Iterating, we obtain

$$|\psi_m''| \le \frac{2\dddot{f}_{\max}}{1-\theta} + \theta^m|\psi_0''|. \tag{6}$$

Hence, the curvature of the images W_m remains bounded and $|(d/dz_{m+1})\log \mathcal{J}_{W_{m+1}} \tilde{F}_\infty(z_{m+1})|$ is bounded. Thus,

$$|\log \mathcal{J}_W\tilde{F}_\infty^{-n}(x) - \log \mathcal{J}_W\tilde{F}_\infty^{-n}(y)| \le C''\sum_{m=0}^{n-1}|W_m| \le C''\sum_{m=0}^{n-1}\frac{|W|}{\Lambda^m} \le C'|W|,$$

where C' depends only on $\tilde{F}_\infty$ and the curvature bound. □

Next, we prove the complexity bound following an approach of [9].

Proof of Lemma 6.2. Suppose that z is a branching point of $\mathcal{S}_n^+$ (cf. Figure 2). We take a small neighborhood of z and cut it into four quadrants Q by vertical and horizontal lines through z. Denote by $k_n(z)|_Q$ the number of sectors cut out by $\mathcal{S}_n^+$ intersecting non-trivially with Q.

We are only interested in the active quadrants, that is, the quadrants in the northwest and southeast, because the tangent lines to the singularity curves $\mathcal{S}_n^+$ have negative slopes and the inactive quadrants (in the northeast and southwest) remain untouched by them and thus do not contribute to the complexity growth.

Denote by $\{V_i\}$ the sectors cut out by $\mathcal{S}^+$. Note that $\mathcal{S}^+ = \{\tilde{t}_1 = 0\}$ and hence there are at most two sectors cut out by $\mathcal{S}^+$ in a quadrant. By further cutting horizontally and vertically, we might assume that each $V_i \subseteq Q$.

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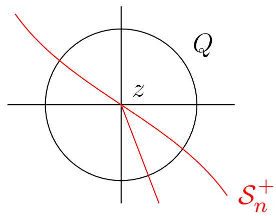

FIGURE 2. A branching point and its sectors.

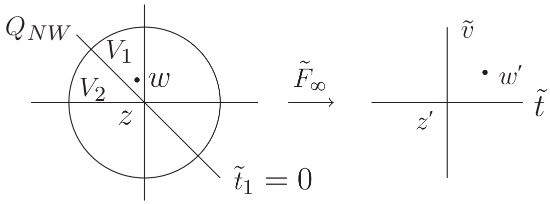

FIGURE 3. Northwest quadrant for $z \in \{\tilde{t}_1 = 0\}$.

We denote $V_i' = \tilde{F}_\infty(V_i)$, $z_i' = \tilde{F}_\infty(z_i)$ (this is defined by continuity, that is, the limit of $\tilde{F}_\infty w$ as $V_i \ni w \to z$) and $k_n(z)|_Q = \sum_i k_{n-1}(z_i')|V_i'$ (cf. Figure 3).

If $z \notin \mathcal{S}^+$, then $i = 1$ and $k_n(z)|_Q = k_{n-1}(z')|V'$.

If $z \in \mathcal{S}^+$, then $i = 2$ and we claim that at most one image V_i' of the sectors V_i remains active, so that in both cases we have

$$\begin{aligned} k_n(z)|_Q &= \sum_i k_{n-1}(z_i')|V_i' \\ &\leq 1 + k_{n-1}(z_i')|V_i' \quad (V_i' \text{ is the only active image}) \\ &\leq 3 + k_{n-1}(z_i')|Q_i' \quad (\text{by further cutting } V_i' \text{ horizontally and vertically}). \end{aligned}$$

Thus, $k_n(z)|_Q \leq 3n + 2$ implies that $k_n(z) \leq 6n + 4$, which is our desired complexity bound.

Now we prove our claim. Suppose that $z \in \{\tilde{t}_1 = 0\}$.

Recall that $\tilde{t}_1 = \tilde{t}_0 + (2\tilde{v}_0/g) \pmod 1$, $\tilde{v}_1 = \tilde{v}_0 + 2\dot{f}(\tilde{t}_1) \pmod g$.

Since $z \in \{\tilde{t}_1 = 0\}$, the $\tilde{t}$-coordinate of its images z' is zero, that is, $\tilde{t}(z_i') = 0$ $(i = 1, 2)$. We pick $w \in V_1$ sufficiently close to z; then the $\tilde{t}$-coordinate of its image w' is positive since w is at the right-hand side of the singularity line $\{\tilde{t}_1 = 0\}$. Also, since we assume that $\ddot{f} > 0$, $\dot{f}$ is increasing and hence the $\tilde{v}$-coordinate of its image w' is larger than that of z'. This means that the image $V_1' = \tilde{F}_\infty(V_1)$ is inactive.

Similarly, we can show that the lower one to the left of the singularity line $\mathcal{S}^+$ in the southeast quadrant becomes inactive after being mapped by $\tilde{F}_\infty$. □

Finally, we estimate the Jacobian of the holonomy map.

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Proof of Lemma 6.4. It follows from classical results in [1, 27] that the holonomy map is absolutely continuous and its Jacobian is given by

$$\mathcal{J}h(x)=\prod_{j=0}^{\infty}\frac{\mathcal{J}_{W_j}(x_j)}{\mathcal{J}_{\bar{W}_j}(\bar{x}_j)},$$

where $W_j/\bar{W}_j=\tilde{F}_{\infty}^{j}W/\tilde{F}_{\infty}^{j}\bar{W}$ and $x_j/\bar{x}_j=\tilde{F}_{\infty}^{j}x/\tilde{F}_{\infty}^{j}\bar{x}$, respectively. As a result,

$$\log\mathcal{J}h(x)=\sum_{j=0}^{\infty}\log\mathcal{J}_{W_j}(x_j)-\log\mathcal{J}_{\bar{W}_j}(\bar{x}_j).$$

For $i=1,2$, we parametrize the unstable curve W_i as $v=\psi_i(t)$ for some smooth function ψ_i such that $\psi_i'\in[2k,2k+(g/2)]$. We obtain by (5) that

$$\begin{aligned}
2|\log\mathcal{J}_{W_j}(x_j)-\log\mathcal{J}_{\bar{W}_j}(\bar{x}_j)| &\le \Bigg|\log\bigg(1+4k_{j+1}^2+\psi_j'^2\bigg(\frac{4}{g^2}+\bigg(1+\frac{4k_{j+1}}{g}\bigg)^2\bigg)\\
&\quad+2\psi_j'\bigg(\frac{2}{g}+2k_{j+1}\bigg(1+\frac{4k_{j+1}}{g}\bigg)\bigg)\bigg)-\log\bigg(1+4\bar{k}_{j+1}^2+\bar{\psi}_j'^2\bigg(\frac{4}{g^2}+\bigg(1+\frac{4\bar{k}_{j+1}}{g}\bigg)^2\bigg)\\
&\quad+2\bar{\psi}_j'\bigg(\frac{2}{g}+2\bar{k}_{j+1}\bigg(1+\frac{4\bar{k}_{j+1}}{g}\bigg)\bigg)\bigg)\Bigg|\\
&\quad+|\log(1+\psi_j'^2)+\log(1+\bar{\psi}_j'^2)|\\
&\le C\theta_1(|k_{j+1}-\bar{k}_{j+1}|+|\psi_j'-\bar{\psi}_j'|)+C\theta_2|\psi_j'-\bar{\psi}_j'|\\
&\le C\theta_1(|t_{j+1}-\bar{t}_{j+1}|+|\psi_j'-\bar{\psi}_j'|)+C\theta_2|\psi_j'-\bar{\psi}_j'|,
\end{aligned}$$

where

$$\theta_1^{-1/2}=1+4k_{\min}^2+4k_{\min}^2\bigg(\frac{4}{g^2}+\bigg(1+\frac{4k_{\min}}{g}\bigg)^2\bigg)+4k_{\min}\bigg(\frac{2}{g}+2k_{\min}\bigg(1+\frac{4k_{\min}}{g}\bigg)\bigg)>1,$$
$$\theta_2^{-1/2}=1+4k_{\min}^2>1.$$

It also follows from (3) that

$$\begin{aligned}
|\psi_j'-\bar{\psi}_j'| &\le C|t_j-\bar{t}_j|+C\theta_3|\psi_{j-1}'-\bar{\psi}_{j-1}'|\\
&\le C|t_j-\bar{t}_j|+C\theta_3|t_{j-1}-\bar{t}_{j-1}|+C\theta_3^2|\psi_{j-2}'-\bar{\psi}_{j-2}'|\\
&\cdots\\
&\le C|t_j-\bar{t}_j|+C\theta_3|t_{j-1}-\bar{t}_{j-1}|+\cdots+C\theta_3^{j-1}|t_1-\bar{t}_1|+C\theta_3^{j}|\psi_0'-\bar{\psi}_0'|\\
&\le C\frac{|t_0-\bar{t}_0|}{\Lambda^n}+C\theta_3\frac{|t_0-\bar{t}_0|}{\Lambda^{j-1}}+\cdots+C\theta_3^{j-1}\frac{|t_0-\bar{t}_0|}{\Lambda}+C\theta_3^{j}|\psi_0'-\bar{\psi}_0'|\\
&\le Cj\theta_4^{j}|t_0-\bar{t}_0|+C\theta_3^{j}|\psi_0'-\bar{\psi}_0'|,
\end{aligned}$$

where $\theta_3^{-1/2} = 1 + 4k_{\min}/g > 1, \theta_4 = \max\{\theta_3, \Lambda^{-1}\} < 1$ and Λ is the minimal expansion rate of unstable curves. Consequently,

$$|\log \mathcal{J}_{W_j}(x_j) - \log \mathcal{J}_{\bar{W}_j}(\bar{x}_j)| \le Cj\Theta^j |t_0 - \bar{t}_0| + C\Theta^j |\psi_0' - \bar{\psi}_0'|, \tag{7}$$

where $\Theta = \max\{\theta_1, \theta_2, \theta_3, \theta_4\} < 1$.

Finally, we are ready to estimate the Jacobian. Observe that $s_+(x, y) = s_+(\bar{x}, \bar{y})$ since each pair $(x, \bar{x})$, $(y, \bar{y})$ is connected by its corresponding stable manifold. Then

$$\begin{aligned}
&|\log \mathcal{J}h(x) - \log \mathcal{J}h(y)| \\
&\quad \le \sum_{j=0}^{\infty} |\log \mathcal{J}_{W_j}(x_j) - \log \mathcal{J}_{\bar{W}_j}(\bar{x}_j) - \log \mathcal{J}_{W_j}(y_j) + \log \mathcal{J}_{\bar{W}_j}(\bar{y}_j)| \\
&\quad \le \sum_{j<s_+(x,y)} (|\log \mathcal{J}_{W_j}(x_j) - \log \mathcal{J}_{W_j}(y_j)| + |\log \mathcal{J}_{\bar{W}_j}(\bar{x}_j) - \log \mathcal{J}_{\bar{W}_j}(\bar{y}_j)|) \\
&\qquad + \sum_{j\ge s_+(x,y)} (|\log \mathcal{J}_{W_j}(x_j) - \log \mathcal{J}_{\bar{W}_j}(\bar{x}_j)| + |\log \mathcal{J}_{W_j}(y_j) - \log \mathcal{J}_{\bar{W}_j}(\bar{y}_j)|) \\
&\quad \le C \sum_{j<s_+(x,y)} (|x_j - y_j| + |\bar{x}_j - \bar{y}_j|) + C \sum_{j\ge s_+(x,y)} j\Theta^j (|x_0 - \bar{x}_0| + |y_0 - \bar{y}_0|) \\
&\qquad + \Theta^j (|\psi'(x_0) - \bar{\psi}'(\bar{x}_0)| + |\psi'(y_0) - \bar{\psi}'(\bar{y}_0)|) \\
&\quad \le C \sum_{j<s_+(x,y)} \Lambda^{-j} (|x_{s_+(x,y)} - y_{s_+(x,y)}| + |\bar{x}_{s_+(x,y)} - \bar{y}_{s_+(x,y)}|) \\
&\quad \le C\Lambda^{-s_+(x,y)} (|x_{s_+(x,y)} - y_{s_+(x,y)}| + |\bar{x}_{s_+(x,y)} - \bar{y}_{s_+(x,y)}|) \\
&\qquad + C\Theta^{s_+(x,y)} (|x_0 - \bar{x}_0| + |y_0 - \bar{y}_0| + |\psi'(x_0) - \bar{\psi}'(\bar{x}_0)| + |\psi'(y_0) - \bar{\psi}'(\bar{y}_0)|) \\
&\quad \le C\Theta^{s_+(x,y)},
\end{aligned}$$

where the sum of small indices $j < s_+(x, y)$ is controlled by the distortion estimate from Lemma 6.1 and the sum of large indices $j \ge s_+(x, y)$ is controlled by (7). □

7. *Escaping and bounded orbits*

Theorem 5 shows that the escaping orbits take up a null set. However, in this section we show that the escaping orbits do exist and so do the bounded orbits.

We introduce the notion of *proper standard pair*. A *standard pair* (W, μ_W) consists of an unstable curve W and a *regular* probability measure μ_W supported on W, that is, μ_W is absolutely continuous and has a dynamically Hölder density. We say that a standard pair is *proper* if there exists a large constant C_p bounding the following quantity:

$$\mathcal{Z}_W := \sup_{\epsilon} \frac{\mu_W\{r_0 < \epsilon\}}{\epsilon}.$$

It is easy to see that in our case μ_W is the normalized Lebesgue measure on the unstable curve and that $\mathcal{Z}_W = 2/|W|$, so any unstable curve W longer than $\delta_2 = 2/C_p$ endowed with Lebesgue measure is a proper standard pair. We also observe that δ_2 can be made arbitrarily small by choosing C_p large. Therefore, by Theorem 3 and Lemmas 2.2 and 2.3 in [13], we have the following central limit theorem for all proper standard pairs.

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PROPOSITION 7.1. *There exists $\delta_2 \ll 1$ such that on any unstable curve W with $|W| > \delta_2$ we have the following central limit theorem for dynamically Hölder observables, that is,*

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ \tilde{F}_\infty^i \overset{\text{dist}}{\rightharpoonup} \mathcal{N}(0, \sigma_\varphi^2),$$

where φ is dynamically Hölder with zero average, $\int_{\mathbb{T}} \varphi \, d\tilde{\mu} = 0$.

Now we prove Theorem 7.

Proof of Theorem 7. Let us denote $(t_n^\infty, v_n^\infty) = F_\infty^n(t_0, v_0)$ and $(t_n, v_n) = F^n(t_0, v_0)$.

First we recall from §5 that the energy change γ in the limit map F_∞ on the cylinder has zero average. Moreover, γ is dynamically Hölder as it is piecewise C^1 and its discontinuities are located exactly on $\mathcal{S}^+$. Therefore, by Proposition 7.1, there exist n_1, A such that for every unstable curve W longer than δ_2 and any $n \geq n_1$,

$$\mathbb{P}_W(v_{nN_0}^\infty > v_0 + A\sqrt{nN_0}) > \tfrac{1}{3},$$

where N_0 is the constant from Lemma 6.3.

By Lemma 6.5, if δ_2 is sufficiently small, then there exists n_2 such that for any $n \geq n_2$,

$$\mathbb{P}_W(r_{nN_0} < 4\delta_2) < \tfrac{1}{15}.$$

We take $n_0 = \max\{n_1, n_2\}$.

We know from §3.2 that the limit map F_∞ well approximates the original collision map F for large velocity with an error of order $\mathcal{O}(v_0^{-1})$ on each continuity component of $F_\infty^{n_0 N_0} W$; thus, we can choose $v_* \gg 1$ so large that if $v_0 > v_*$ everywhere on W, then we have

$$\mathbb{P}_W(v_{n_0 N_0} > v_0 + A\sqrt{n_0 N_0}, r_{n_0 N_0} > 4\delta_2) > \tfrac{1}{4}.$$

By the estimate above, at least one component $W_1 \subset F_\infty^{n_0 N_0} W$ contains a segment $\bar{W}_1$ longer than δ_2 and $v_{n_0 N_0} > v_0 + A\sqrt{n_0 N_0}$ holds everywhere on $\bar{W}_1$. By repeating the argument on $\bar{W}_1$, we get another component $W_2 \subset F_\infty^{n_0 N_0} \bar{W}_1$ containing a segment $\bar{W}_2$ longer than δ_2 and the velocity increases by another $A\sqrt{n_0 N_0}$. Inductively, we construct an escaping orbit.

Similarly, we can find B, n_0', v_*', δ_2' such that if we start with an unstable curve W with $|W| > \delta_2'$ and initial velocity $v_0 > v_*'$ everywhere on W, then at least one component $W_1 \subset F_\infty^{n_0' N_0} W$ contains a segment $\bar{W}_1$ longer than $2\delta_2'$ and $v_0 < v_{n_0' N_0} < v_0 + B\sqrt{n_0' N_0}$ holds everywhere on $\bar{W}_1$. Repeating the argument in the energy-decreasing direction, we can show that at least one component $W_2 \subset F_\infty^{n_0' N_0} \bar{W}_2$ contains a segment $\bar{W}_2$ longer than $2\delta_2'$ and $v_{n_0' N_0} - B\sqrt{n_0' N_0} < v_{2n_0' N_0} < v_{n_0' N_0}$ holds everywhere on $\bar{W}_2$. We note that $v_0 - B\sqrt{n_0' N_0} < v_{2n_0' N_0} < v_0 + B\sqrt{n_0' N_0}$ holds everywhere on $\bar{W}_2$. One of the events

$$\{v_0 - B\sqrt{n_0' N_0} < v_{2n_0' N_0} < v_0\}, \; \{v_0 < v_{2n_0' N_0} < v_0 + B\sqrt{n_0' N_0}\}$$

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has probability greater than or equal to $1/2$; we might assume without loss of generality that the former is true. Then there exists a segment $\bar{W}_2' \subset \bar{W}_2$ longer than δ_2' such that $v_0 - B\sqrt{n_0' N_0} < v_{2n_0' N_0} < v_0$ holds everywhere on $\bar{W}_2'$. By repeating the argument with $\bar{W}_2'$ in the place of W, we construct a bounded orbit whose velocity remains in $[v_0 - B\sqrt{n_0' N_0}, v_0 + B\sqrt{n_0' N_0}]$. □

8. *Conclusions*

In this paper, we have studied a piecewise C^3-smooth Fermi–Ulam model in a constant potential field. The collision map F is well approximated by the limit map F_∞ for large velocities; F_∞ covers a map $\tilde{F}_\infty$ on a torus. For admissible wall motions we proved ergodicity, exponential decay of correlations and a central limit theorem for dynamically Hölder observables. When our assumptions fail, there are counterexamples in the class of analytic periodic platform motions by Pustylnikov [24] where KAM islands exist for the limit map and the original system possesses a positive-measure set of escaping orbits. The ergodic and statistical properties of the limit map $\tilde{F}_\infty$ established here in turn imply that the escaping set has zero measure and the typical behavior of the original collision dynamics F is recurrent, but escaping and bounded orbits still exist at arbitrarily high energy level.

It is also interesting to study long-time evolution of the energy distribution for typical high-velocity trajectories (cf. [5, 8] for similar results for other systems). Besides, we note that our results do not fully address the behavior of low-energy orbits. The problem becomes subtle when we come to the low-energy region as two consecutive collisions can happen in an arbitrarily short time interval, causing the system to be non-uniformly hyperbolic. In the future, we hope to establish ergodicity for the collision map F. This would imply in particular that almost every orbit is oscillatory, so that the energy eventually comes close to any given value.

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References

[1] D. V. Anosov and Y. G. Sinai. Some smooth ergodic systems. *Russian Math. Surveys* **22**(5) (1967), 103–167.

[2] M. Arnold and V. Zharnitsky. Pinball dynamics: unlimited energy growth in switching Hamiltonian systems. *Comm. Math. Phys.* **338**(2) (2015), 501–521.

[3] J. Chen, F. Wang and H.-K. Zhang. Markov partition and thermodynamic formalism for hyperbolic systems with singularities. *Preprint*, 2019, arXiv:1709.00527.

[4] N. Chernov. Decay of correlations and dispersing billiards. *J. Stat. Phys.* **94**(3–4) (1999), 513–556.

[5] N. Chernov and D. Dolgopyat. The Galton board: limit theorems and recurrence. *J. Amer. Math. Soc.* **22**(3) (2009), 821–858.

[6] N. Chernov and H.-K. Zhang. On statistical properties of hyperbolic systems with singularities. *J. Stat. Phys.* **136**(4) (2009), 615–642.

[7] J. de Simoi. Stability and instability results in a model of Fermi acceleration. *Discrete Contin. Dyn. Syst.* **25**(3) (2009), 719–750.
[8] J. de Simoi and D. Dolgopyat. Dynamics of some piecewise smooth Fermi–Ulam models. *Chaos* **22**(2) (2012), 026124.
[9] J. de Simoi and I. P. Toth. An expansion estimate for dispersing planar billiards with corner points. *Ann. Henri Poincaré* **15**(6) (2014), 1223–1243.
[10] D. Dolgopyat. Lectures on bouncing balls. Online notes at http://www.users.math.umd.edu/∼dolgop/BBNotes.pdf.
[11] D. Dolgopyat. Bouncing balls in non-linear potentials. *Discrete Contin. Dyn. Syst.* **22**(1–2) (2008), 165–182.
[12] D. Dolgopyat. Fermi acceleration. *Geometric and Probabilistic Structures in Dynamics (Contemporary Mathematics, 469)*. American Mathematical Society, Providence, RI, 2008, pp. 149–166.
[13] D. Dolgopyat and P. Nándori. Nonequilibrium density profiles in Lorentz tubes with thermostated boundaries. *Comm. Pure Appl. Math.* **69**(4) (2016), 649–692.
[14] D. Dolgopyat and P. Nándori. Infinite measure mixing for some mechanical systems. *Preprint*, 2018, arXiv:1812.01174.
[15] E. Fermi. On the origin of the cosmic radiation. *Phys. Rev.* **75** (1949), 1169–1174.
[16] V. Gelfreich, V. Rom-Kedar and D. Turaev. Fermi acceleration and adiabatic invariants for non-autonomous billiards. *Chaos* **22**(3) (2012), 033116, 21 pp.
[17] S. Laederich and M. Levi. Invariant curves and time-dependent potentials. *Ergod. Th. & Dynam. Sys.* **11**(2) (1991), 365–378.
[18] M. Levi and J.-G. You. Oscillatory escape in a Duffing equation with a polynomial potential. *J. Differential Equations* **140**(2) (1997), 415–426.
[19] A. J. Lichtenberg, M. A. Lieberman and R. H. Cohen. Fermi acceleration revisited. *Physica D* **1**(3) (1980), 291–305.
[20] C. Liverani and M. Wojtkowski. Ergodicity in Hamiltonian systems. *Dynamics Reported: Expositions in Dynamical Systems*. Springer, Berlin, Heidelberg, 1995, pp. 130–202.
[21] R. Ortega. Boundedness in a piecewise linear oscillator and a variant of the small twist theorem. *Proc. Lond. Math. Soc. (3)* **79**(2) (1999), 381–413.
[22] R. Ortega. Dynamics of a forced oscillator having an obstacle. *Variational and Topological Methods in the Study of Nonlinear Phenomena (Pisa, 2000) (Progress in Nonlinear Differential Equations and Their Applications, 49)*. Birkhäuser, Boston, MA, 2002, pp. 75–87.
[23] Y. B. Pesin. Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties. *Ergod. Th. & Dynam. Sys.* **12**(1) (1992), 123–151.
[24] L. D. Pustylnikov. Stable and oscillating motions in nonautonomous dynamical systems. II. *Tr. Mosk. Mat. Obš.* **34** (1977), 3–103.
[25] L. D. Pustylnikov. On Ulam's problem. *Theoret. Math. Phys.* **57** (1983), 1035–1038.
[26] L. D. Pustylnikov. Existence of invariant curves for maps close to degenerate maps, and a solution of the Fermi–Ulam problem. *Mat. Sb.* **185** (1994), 113–124.
[27] Y. G. Sinai. Dynamical systems with elastic reflections. *Russian Math. Surveys* **25**(2) (1970), 137–189.
[28] S. M. Ulam. On some statistical properties of dynamical systems. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability (Contributions to Astronomy, Meteorology, and Physics, 3)*. University of California Press, Berkeley, CA, 1961, pp. 315–320.
[29] V. Zharnitsky. Instability in Fermi–Ulam ping-pong problem. *Nonlinearity* **11**(6) (1998), 1481–1487.