# A CLASS OF POLYNOMIALS IN SELF-ADJOINT OPERATORS IN SPACES WITH AN INDEFINITE METRIC

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**1. Introduction.** Let $H$ be a Hilbert space with the usual product $[x, y]$ and with an indefinite inner product $(x, y)$ which, for some orthogonal decomposition

$$H = H_1 \oplus H_2$$

in $H$, is defined by

$$(x, y) = [x_1, y_1] - [x_2, y_2],$$

where

$$x = x_1 + x_2, \quad y = y_1 + y_2, \qquad x_1, y_1 \in H_1; \quad x_2, y_2 \in H_2,$$

and $\dim H_1 = \kappa$, a fixed positive integer. Such a space $H$ will be called a space $\Pi_\kappa$ with an indefinite metric. Another axiomatic definition of the space $\Pi_\kappa$ was given by I. S. Iohvidov and M. G. Kreĭn in **(1)**; we follow their terminology here and use the results of their paper.

A linear operator $A$ in $\Pi_\kappa$ is called symmetric if it maps a dense domain[1] $D(A)$ in $\Pi_\kappa$ into $\Pi_\kappa$ and has the property

$$(Ax, y) = (x, Ay) \quad \text{for all } x, y \in D(A).$$

A linear operator $A^*$ defined in $\Pi_\kappa$ is called the adjoint of a linear operator $A$ with a dense domain $D(A)$ in $\Pi_\kappa$ if $A^*$ is the maximum operator such that

$$(Ax, y) = (x, A^*y) \quad \text{for all } x \in D(A) \text{ and all } y \in D(A^*).$$

A symmetric operator is said to be self-adjoint if $A = A^*$.

L. S. Pontryagin **(2)** proved that for any self-adjoint operator $A$ there is a $\kappa$-dimensional invariant non-negative subspace $\mathscr{J}$. Let us consider the minimal polynomial $P_\mu(\lambda)$ (degree $P_\mu(\lambda) = \mu$) of the operator induced by $A$ in $\mathscr{J}$. Then the operator[2] $P_\mu(A)$ annihilates $\mathscr{J}$. Let $\bar{P}_\mu(\lambda)$ be the complex conjugate of the polynomial $P_\mu(\lambda)$. We have for any vector $x \in D(A^\mu)$, $y \in \mathscr{J}$,

$$(P_\mu(A)x, y) = (x, \bar{P}_\mu(A)y) = 0,$$

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[1]We shall always denote the domain of an operator $A$ by $D(A)$.

[2]We agree that $A^0 = I$, the identity operator for any operator $A$.



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that is, the linear manifold $\{P_\mu(A)y\colon y \in D(A^\mu)\}$ is orthogonal to $\mathscr{J}$. Hence, by Lemma 1.2 **(1)**, we can deduce that this linear manifold is non-positive, that is, for all $y \in D(A^\mu)$

$$(P_\mu(A)y, P_\mu(A)y) \leqslant 0.$$

It is, therefore, natural to ask whether for any polynomial $P(\lambda)$ which satisfies the above condition, the operator $P(A)$ annihilates a certain $\kappa$-dimensional non-negative invariant subspace $\mathscr{J}$ of the operator $A$. We shall show that the answer is affirmative (a similar assertion was proved by I. S. Iohvidov and M. G. Kreĭn **(1)** for unitary operators).

## 2. A class of polynomials.

*Definition* 2.1. A class $\mathscr{N}_A$ of polynomials $P_n(\lambda)$ is said to be a class of definitizing[3] polynomials with respect to a self-adjoint operator $A$ in $\Pi_\kappa$ if

$$(P_n(A)x, P_n(A)x) \leqslant 0 \text{ for all } x \in D(A^m),$$

where $m$ ($\geqslant n$ = degree $P_n(\lambda)$) is a natural number.

The class $\mathscr{N}_A$ is always not empty since $\bar{P}_\mu(\lambda) \in \mathscr{N}_A$ as shown in the Introduction.

Before we investigate the class $\mathscr{N}_A$, we shall prove a few lemmas for later use. A linear operator $U$ in $\Pi_\kappa$ is said to be unitary if

$$(Ux, Ux) = (x, x) \text{ for all } x \in \Pi_\kappa$$

and if $U$ maps $\Pi_\kappa$ onto $\Pi_\kappa$.

*Definition* 2.2. An operator $U$ is said to be $\zeta$-Cayley-Neumann connected with a self-adjoint operator $A$ if the non-real complex conjugate numbers $\zeta$, $\bar{\zeta}$ are not proper values of $A$ and if $U$ is defined by the following formulae:

$$y = (Ax - \zeta x),\ Uy = (Ax - \bar{\zeta}x) \qquad \text{for } x \in D(A).$$

I. S. Iohvidov and M. G. Kreĭn **(1**, §8**)** proved that such an operator $U$ is unitary. The definition is, therefore, well-defined.

LEMMA 2.3. *If a unitary operator $U$ is $\zeta$-Cayley–Neumann connected with a self-adjoint operator $A$, then $A^mU = UA^m$ and $D(A^m) = UD(A^m)$ for any natural number $m$.*

*Proof.* We shall prove that $(A - \zeta I)D(A^n) = D(A^{n-1})$ for any natural number $n$. For $n = 1$ the result is obvious since

$$(A - \zeta I)D(A) = D(U) = \Pi_\kappa = D(A^0).$$

For $n > 1$ we can prove the above assertion by induction.

[3]The word "definitizing" appeared in the translation of the paper by I. S. Iohvidov and M. G. Kreĭn; see *Spectral theory of operators in spaces with an indefinite metric.* II, Transl. Amer. Math. Soc. (2), *34* (1963), 283–374.

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By Definition 2.2 we have $U = (A - \bar{\zeta}I)(A - \zeta I)^{-1}$. It follows that $U(A - \zeta I) = (A - \bar{\zeta}I)$. Hence we have

$$A^m U(A - \zeta I)x = U(A - \zeta I)A^m x = UA^m(A - \zeta I)x \quad \text{for all } x \in D(A^{m+1}).$$

Since $(A - \zeta I)D(A^{m+1}) = D(A^m)$, we know, for any $y \in D(A^m)$, that there exists an $x \in D(A^{m+1})$ such that $y = (A - \zeta)x$. Therefore we have $A^m Uy = UA^m y$ for all $y \in D(A^m)$; that is $UA^m = A^m U$.

Since $(A - \zeta I)D(A^{m+1}) = D(A^m)$ and, similarly, $(A - \bar{\zeta}I)D(A^{m+1}) = D(A^m)$, we have

$$UD(A^m) = U(A - \zeta I)D(A^{m+1}) = (A - \zeta I)D(A^{m+1}) = D(A^m).$$

The lemma is proved.

LEMMA 2.4. *If $A$ is a self-adjoint operator in $\Pi_\kappa$, then $D(A^m)$, the domain of the operator $A^m$, is dense in $\Pi_\kappa$ for any natural number $m$.*

*Proof.* By definition we have $D(A) = \Pi_\kappa$. For $m > 1$ we prove this lemma by induction. As in Lemma 2.3 there exist non-real complex conjugate numbers $\zeta$ and $\bar{\zeta}$ which are not proper values of $A$ such that $(A - \zeta I)D(A^{n+1}) = D(A^n)$ and $(A - \bar{\zeta}I)D(A) = \Pi_\kappa$. Now let $x = (A - \bar{\zeta})x'$ be any vector in $\Pi_\kappa$ such that $(x, y) = 0$, for all $y \in D(A^{n+1})$. It thus follows that for any $y \in D(A^{n+1})$ we have

$$0 = ((A - \bar{\zeta})x', y) = (x', (A - \zeta)y),$$

that is,

$$(x', z) = 0 \quad \text{for all} \quad z \in D(A^n).$$

Since $D(A^n)$ is dense in $\Pi_\kappa$, we have $x' = \theta$, the zero vector. Hence $x = (A - \bar{\zeta}I)x' = \theta$. It thus follows that $D(A^{n+1})$ is dense in $\Pi_\kappa$.

THEOREM 2.5. *Let $A$ be a self-adjoint operator in $\Pi_\kappa$ and let $P_n(\lambda)$ be a polynomial of degree $n$. Then the polynomial $P_n(A)$ belongs to the class $\mathscr{N}_A$ of definitizing polynomials if and only if there exists a $\kappa$-dimensional non-negative invariant subspace $\mathscr{J}$ of the operator $A$ such that $P_n(A)$ annihilates $\mathscr{J}$.*

*Proof.* The sufficiency was shown in the Introduction. It remains to prove the necessity.

Let $\bar{P}_n(\lambda)$ be the complex conjugate polynomial of $P_n(\lambda)$ and consider the subspace $N = \overline{\bar{P}_n(A)D(A^m)}$ $(m \geqslant n)$ and its orthogonal complement $M$. It is easy to see that $N$ is a non-positive subspace. By Theorem 4.1 (**1**, § 15) we have

$$\Pi_\kappa = M_1 \oplus N_1 \oplus (G \dot{+} F),$$

where $G$ is the common isotropic subspace, $\dim G = \dim F = q$, and $F$ is skew-connected with $G$; $M = M_1 \oplus G$, $N = N_1 \oplus G$, the subspace $N_1$ being a negative subspace. Hence it is easy to see that $M_1$ is a space of $\Pi_{\kappa'}$ type, where $\kappa' = \kappa - q$.

Let $U$ be a unitary operator which is $\zeta$-Cayley–Neumann connected to

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the operator $A$. We shall prove that $UM = M$. In fact, if $x \in M$ we have, by Lemma 2.3,

$$(x, \bar{P}_n(A)y) = (Ux, U\bar{P}_n(A)y) = (Ux, \bar{P}_n(A)Uy) = 0$$

for all $y \in D(A^m)$. Since $UD(A^m) = D(A^m)$, we have $Ux \perp N$, and hence $UM \subset M$. Similarly, we have $U^{-1}M \subset M$. It thus follows that $UM = M$. Therefore by Theorem 4.4 **(1**, §16**)** the operator $U$ has a $\kappa$-dimensional non-negative invariant subspace $\mathscr{J}$ $(\subset M)$. Hence by Theorem 2.7 **(1**, §8**)** it is also an invariant subspace of $A$. For any $x \in \mathscr{J}$ and all $y \in D(A^m)$, we have

$$(x, \bar{P}_n(A)y) = (P_n(A)x, y) = 0.$$

Since $D(A^m)$ is dense in $\Pi_\kappa$, by Lemma 2.4 we have $P_n(A)x = \theta$, for any $x \in \mathscr{J}$. The theorem is proved.

We shall show that all the polynomials $P_n(\lambda)\bar{P}_n(\lambda)$ have a common factor if $P_n(\lambda) \in \mathscr{N}_A$.

Let $\mathscr{J}_+$ be a $\kappa$-dimensional non-negative invariant subspace of a self-adjoint operator $A$, let $\lambda_i$ $(\operatorname{Im} \lambda_j > 0)$, $i = 1, 2, \ldots, r$ $(0 \leqslant r \leqslant \kappa)$ and $\mu_j$ $(\mu_j = \bar{\mu}_j)$, $j = 1, 2, \ldots, s$ $(0 \leqslant s \leqslant \kappa)$, be all the proper values of the operator induced by $A$ in $\mathscr{J}_+$, and let $\sigma_i$ and $r_j$ be the multiplicities corresponding to $\lambda_i$ and $\mu_j$, respectively. The collection of the pairs $(\lambda_i, \sigma_i)$, $i = 1, 2, \ldots, r$, and $(\mu_i, r_j)$, $j = 1, 2, \ldots, s$, is invariant with respect to $\mathscr{J}_+$ **(1**, §16, Theorem 4.5 and §8, Theorem 6**)**. We define the characteristic polynomial of $\mathscr{J}_+$ by the formulae:

$$P_\kappa(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{\sigma_i} \prod_{j=1}^{s} (\lambda - \mu_j)^{r_j}, \qquad \sum_{i=1}^{r} \sigma_i + \sum_{j=1}^{s} r_j = \kappa.$$

Each fixed real proper value $\mu_j$ $(j = 1, 2, \ldots, s)$ has corresponding to it a definite selection of elementary divisors **(**cf. **1**, §4, Theorem 4**)**

$$(\lambda - \mu_j)^{\rho_{j1}}, \ (\lambda - \mu_j)^{\rho_{j2}}, \ldots, \ (\lambda - \mu_j)^{\rho_{jk_j}},$$

where $\rho_{j1} \geqslant \rho_{j2} \geqslant \ldots \geqslant \rho_{jk_j} \geqslant 1$ and $\rho_{j1} + \rho_{j2} + \ldots + \rho_{jk_j} = r_j$. The number $k_j$ of the elementary divisors, unlike the number $r_j$, is, in general, not an invariant of $A$ but depends on the choice of the subspace $\mathscr{J}$. The last equation shows that only a finite number of different choices of $\rho_{j1}, \rho_{j2}, \ldots, \rho_{jk_j}$ is possible. In particular, we can select those of the subspaces for which the first exponent $\rho_{j1}$ is a minimum. Let this minimum be $\rho(\mu_j)$. If we make a similar selection for each of the proper values $\mu_j$ $(j = 1, 2, \ldots, s)$, we obtain an invariant subspace $\mathscr{J}_+$ in which the operator $A$ corresponds not only to a unique characteristic polynomial $P_\kappa(\lambda)$ but also to a unique minimal polynomial

$$P_\mu(\lambda) = \prod_{i=1}^{r} (\lambda - \lambda_i)^{p_i} \prod_{j=1}^{s} (\lambda - \mu_j)^{\rho(\mu_j)},$$

where the number $p_i$ $(i = 1, 2, \ldots, r)$, $\rho(\mu_j)$ $(j = 1, 2, \ldots, r_j)$, and

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$$\mu = \sum_{i=1}^{r} p_i + \sum_{j=1}^{s} \rho(\mu_j)$$

are also invariant of $A$.

We can carry these arguments further. From the subspaces $\mathscr{J}_+$ with $\rho_{j1} = \rho(\mu_j)$ we can select those with minimal $\rho_{j2}$, then those with minimal $\rho_{j3}$, and so on. In this way we can prescribe those subspaces $\mathscr{J}_+$ in which $A$ corresponds to elementary divisors of minimal degree (in the sense that exponents are selected on the dictionary principle, beginning with the first; thereby obtaining the "shortest" Jordan chains).

*Definition* 2.6. A $\kappa$-dimensional non-negative subspace $\mathscr{J}_+$ invariant with respect to a self-adjoint operator $A$ is called *regular* if the Jordan chain for operator $A$ in $\mathscr{J}_+$ is the shortest in the sense explained above. The polynomial $P_{2\mu}(\lambda) = P_\mu(\lambda)\bar{P}_\mu(\lambda)$ (where $P_\mu(\lambda)$ is defined above) is called the *characteristic polynomial of the self-adjoint operator* $A$.

It is obvious that $\bar{P}_\mu(\lambda)$ and $P_\mu(\lambda)$ belong to $\mathscr{N}_A$ and that any minimal polynomial $P_m(\lambda)$ of the operator induced by $A$ in $\mathscr{J}_+$ is divisible in $P_\mu(\lambda)$.

THEOREM 2.6. *Let $A$ be a self-adjoint operator in $\Pi_\kappa$ and let $P_n(\lambda)$ and $\bar{P}_n(\lambda)$ be complex conjugate polynomials of degree n. Then the polynomial $P_n(\lambda)$ belongs to the class $\mathscr{N}_A$ of definitizing polynomials if and only if $P_n(\lambda)\bar{P}_n(\lambda)$ is divisible by the characteristic polynomial of the operator $A$.*

*Proof*. The sufficiency is obvious. Now let us show the necessity. Let us define, for a polynomial

$$P_n(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i),$$

the polynomial

$$P_n'(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i'),$$

where $\lambda_i' = \lambda_i$ if $\operatorname{Im} \lambda_i \geqslant 0$ and $\lambda_i' = \bar{\lambda}_i$ if $\operatorname{Im} \lambda_i < 0$. Clearly, $P_n'(\lambda) \in \mathscr{N}_A$ if $P_n(\lambda) \in \mathscr{N}_A$. By Theorem 2.5 there exists a $\kappa$-dimensional non-negative invariant subspace $\mathscr{J}$ of the operator $A$ such that $P_n'(A)$ annihilates $\mathscr{J}$. Hence $P_n'(\lambda)$ is divisible by the minimal polynomial $P_m(\lambda)$ of the operator induced by $A$ in $\mathscr{J}$. It thus follows that the characteristic polynomial of $\mathscr{J}$ has no root which has a negative imaginary part. Hence $P_m(\lambda)$ is divisible by $P_\mu(\lambda)$. Since $P_n(\lambda)\overline{P_n}(\lambda) = P_n'(\lambda)\overline{P_n'}(\lambda)$, we have $P_n(\lambda)\overline{P_n}(\lambda)$ is divisible by $P_\mu(\lambda)\overline{P_\mu}(\lambda)$. The theorem is proved.

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