SAMPLING FROM AN ISOTROPIC GAUSSIAN PROCESS

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1. *Introduction.* Isotropic Gaussian processes, of which we shall give a formal definition presently, arise in various practical problems. The present inquiry arose from the consideration of the variability found in the yields of plots in agricultural field experiments. Samples of such patterns of variability can be obtained from uniformity trials, whereby a piece of land is treated uniformly throughout and the crop is harvested in small units or plots. The results from such trials have been widely used to determine optimum plot sizes for future experiments with the crop concerned, but it has never been clear how valid is the generalization from a uniformity trial to future experiments on other sites. One difficulty arises from the lack of a suitable model to express the variability; another difficulty arises from the formidable analytical problems besetting any attempts to apply deductive reasoning to even simple models. We may mention two approaches that have been made to the uniformity trial problem: Fairfield Smith (3) has supplied an empirical law connecting the variance of contiguous groups of plots with the size of the group, and Quenouille (6) and Whittle (8) have considered the fitting of two-dimensional isotropic Gaussian processes to uniformity trial data. Whittle has pointed out that a satisfactory model may have to incorporate an additional random element at each point, and has outlined the difficulties of estimation which arise when such an element is introduced. The connexion between Fairfield Smith's law and the approach via two-dimensional stochastic processes, if any, seems to be quite unknown.

The complexity of the question suggests that an approach based on sampling from certain plausible isotropic Gaussian processes might be valuable. The problem then arises of drawing a sample from such a process. This is the problem considered in this paper.

The Gaussian process is a random function. Little previous work has been done on sampling for random functions; and, to the best of our knowledge, what has been done (e.g. D. G. Kendall's construction of a sample birth-and-death process (5)) has always been by way of an artificial realization of some *given* physical process. In our inquiry, the data consist of a given distribution (of a random function), and we have first to discover a suitable physical process whose realization will then provide a sample function.†

† Theoretically we may derive a finite set of n values of such a sample function directly as follows: Let $\boldsymbol{\xi}$ be a vector of n independent random variates from a normal distribution of mean zero. Then, if $\mathbf{A}$ be any symmetric $n \times n$ matrix, $\mathbf{A}\boldsymbol{\xi}$ is a random vector with correlation matrix $\mathbf{A}^2$. Thus to produce a sample of n values from a process with given correlation matrix $\mathbf{R}$, we could solve the equation $\mathbf{A}^2 = \mathbf{R}$ for $\mathbf{A}$, and form $\mathbf{A}\boldsymbol{\xi}$. However, in practice, n is likely to be at least 100 or more, and the labour involved in the calculation of $\mathbf{A}$ would be formidable.

The agricultural example, quoted above, concerns a Gaussian process in the plane. Gaussian processes also arise in three-dimensional space, for example, in studying turbulence of fluids. We may as well, therefore, deal directly with the isotropic Gaussian process in n-dimensional Euclidean space. At the end of the paper, however, we shall discuss the particular case $n = 2$ in detail.

2. *Definition of the isotropic Gaussian process.* Let E_n denote n-dimensional Euclidean space, in which $\mathbf{x}, \mathbf{y}, \ldots$ (perhaps with suffixes) denote typical points. Let W be the space of all functions $w(\mathbf{x})$ with domain E_n and range E_1. An isotropic Gaussian process is a probability measure $P(w)$ on W such that

(i) for every given finite set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, the joint distribution of $w(\mathbf{x}_1)$, $w(\mathbf{x}_2), \ldots, w(\mathbf{x}_k)$ is a k-variate normal distribution;

(ii) for every given pair of points $\mathbf{x}$, $\mathbf{y}$, the correlation coefficient between $w(\mathbf{x})$ and $w(\mathbf{y})$ depends only on $r = |\mathbf{x}-\mathbf{y}|$, the distance between $\mathbf{x}$ and $\mathbf{y}$.

By introducing a linear transformation $w^*(\mathbf{x}) = \alpha(\mathbf{x}) + \beta(\mathbf{x})\, w(\mathbf{x})$, where α and β are deterministic functions, we may suppose that the mean and variance of $w(\mathbf{x})$ are 0 and 1 respectively, independently of $\mathbf{x}$. After (ii), we can then write $\rho(r)$ for the covariance of $w(\mathbf{x})$ and $w(\mathbf{y})$.

Not every function ρ is admissible in this role. We suppose that we are given some admissible ρ, and that we have to discover a physical process which will yield a sample from $P(w)$ with this prescribed ρ. A necessary and sufficient condition for admissibility of ρ is that, for every finite set $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, the matrix (ρ_{ij}) shall be positive semi-definite, where $\rho_{ij} = \rho(|\mathbf{x}_i - \mathbf{x}_j|)$ for $i, j = 1, 2, \ldots, k$. Our analysis will provide an alternative condition, which is sufficient (see §4) but not necessary (see §11).

3. *Proposed procedure.* In terms of the prescribed function $\rho(r)$, we shall determine two functions $f(z)$ and $v(z)$, where $f(z)$ is a frequency function of a non-negative random scalar and $v(z)$ is a non-negative function of a non-negative real variable. For the moment, however, suppose that f and v are known functions.

We shall also require a fixed family of cumulative distribution functions $F_y(s)$, the family being generated by y varying over all non-negative real numbers and the variable s in each such distribution being a real scalar. The members of this family are to satisfy two conditions

$$\int_{-\infty}^{\infty} s\,dF_y(s) = 0, \quad \int_{-\infty}^{\infty} s^2\,dF_y(s) = v(y), \tag{1}$$

but may otherwise be completely arbitrary. For instance, we might take F_y to be the distribution having the two discrete values $\pm[v(y)]^{\frac{1}{2}}$, each with probability $\frac{1}{2}$; or, alternatively, F_y might be rectangular for some values of y and normal for other values, subject to the conditions (1) that the mean and variance of F_y are always to be zero and $v(y)$ respectively.

In principle the procedure to be adopted is this: We throw on to E_n an infinite number of (n-dimensional) spheres at random. The centres of these spheres are to be uniformly and independently distributed over E_n, and their radii are to be independently distributed with frequency function $f(z)$. To each sphere of radius z we attach a random score s, which is to be a random observation from the distribution $F_z(s)$. The scores from the various spheres are to be mutually independent. Let $w(\mathbf{x})$ be the sum of the scores of all spheres which contain $\mathbf{x}$. We assert that, if f and v are properly defined in terms of ρ, $w(\mathbf{x})$ will have the prescribed distribution $P(w)$; and we shall prove this assertion in §4.

We notice first, however, that for a practical realization of this procedure, we must modify it slightly, since we can (in practice) only throw down a finite number of spheres and we can only cover with them some bounded region of E_n. In fact, we shall only want to observe how $w(\mathbf{x})$ behaves in some large sphere S, centred at the origin and of radius R. If R is large enough, $w(\mathbf{x})$ for $\mathbf{x} \in S$ will give an adequate picture of the behaviour of $w(\mathbf{x})$ in the whole of E_n. In this case then we need only consider those random spheres which meet S; for all other spheres will not influence $w(\mathbf{x})$ in S. To compensate for working within S only, we must replace $f(z)$ by a suitably chosen frequency function $f_R(z)$ depending on the fixed number R. The procedure for obtaining a random sphere consequently becomes: Choose a radius z at random from $f_R(z)$; then choose a centre uniformly at random in the sphere of radius $R+z$ centred at the origin; and finally attach a score s to the sphere by drawing an observation from $F_z(s)$. This is to be repeated a large number of times, say N, so that the number of scores contributing to $w(\mathbf{x})$ for any particular $\mathbf{x}$ is large. We shall then invoke the central limit theorem to claim that the sum of scores at each $\mathbf{x}$ is asymptotically normal.

Because N is finite, the final result will be an approximation to $w(\mathbf{x})$ in two respects. First, $w(\mathbf{x})$, for each particular $\mathbf{x}$, will be only approximately normal. Secondly, $w(\mathbf{x}) = w(\mathbf{y})$ whenever $\mathbf{x}$ and $\mathbf{y}$ are sufficiently close together, because $\mathbf{x}$ and $\mathbf{y}$ will then be covered by the same set of spheres. Thus $w(\mathbf{x})$ will be a step-function approximation to the limiting random function. Clearly, however, as N increases, the sizes of the various regions in which w remains constant will decrease; and therefore, when we are only interested in observing w at some finite set of points in S, we can choose N large enough to make the approximation adequate.

4. *Justification of the process.* Consider any given finite set of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ in S. In accordance with the modified procedure above, we throw N random spheres on to E_n, each one to meet S; but, as we have already seen, so far as the scores within S are concerned, the procedure is equivalent to throwing spheres with centres uniformly at random over E_n. Thus, if s_j is the score allotted to the jth sphere ($j = 1, 2, \ldots, N$) and $s_{ij} = s_j$ or 0 according as $\mathbf{x}_i$ falls within the jth sphere or not, then the vectors $\mathbf{s}_j = (s_{1j}, s_{2j}, \ldots, s_{kj})$ are independently and identically distributed, and

$$\mathscr{E}(\mathbf{s}_j) = \mathbf{0}$$

by (1), and

$$\mathscr{E}(s_{ij}^2) = \sigma^2, \tag{2}$$

independently of i, j. Let us suppose that we can satisfy

$$\sigma^2 = 1. \tag{3}$$

Then, by the multivariate central limit theorem for identically distributed random vectors (Cramér (2), Theorem 20a), the normalized total score $N^{-\frac{1}{2}} \sum_{j=1}^{N} \mathbf{s}_j$ will be asymptotically normal as $N \to \infty$. (It is worth noticing in passing, that the distribution of $\mathbf{s}_j$ will *never* be a multivariate normal distribution, except in the trivial case when $F_y(s) = 1$ for $s \geqslant 0$ and $F_y(s) = 0$ for $s < 0$.) Hence the procedure will be fully justified if we can satisfy

$$\mathscr{E}(s_{11} s_{21}) = \rho(r) \quad (r \geqslant 0), \tag{4}$$

where $r = |\mathbf{x}_1 - \mathbf{x}_2|$, since (3) amounts to the special case $r = 0$ in (4).

Now $s_{11} s_{21} = s_1^2$ or 0 according as the sphere does or does not cover both $\mathbf{x}_1$ and $\mathbf{x}_2$. The probability that z will be drawn from f_R is $f_R(z)\, dz$; and, conditional upon z having been drawn, the expected value of s_1^2 is $v(z)$, by (1). Hence

$$\mathscr{E}(s_{11} s_{21}) = \int_0^\infty Q(z)\, v(z) f_R(z)\, dz, \tag{5}$$

where $Q(z)$ is the probability that a sphere of radius z will cover both $\mathbf{x}_1$ and $\mathbf{x}_2$.

Let $k_n z^n$ denote the volume of an n-dimensional sphere of radius z. Here k_n is a constant, expressible in terms of gamma-functions of n. Now $Q(z) = 0$ if $z < \frac{1}{2}r$. Otherwise write

$$z \sin\theta = \tfrac{1}{2}r, \tag{6}$$

so that $\frac{1}{2}\pi - \theta$ is the semi-vertical angle of the cone (with vertex at $\mathbf{x}_1$) through the intersection of the surfaces of two spheres of radius z centred at $\mathbf{x}_1$ and $\mathbf{x}_2$. The random sphere will cover both $\mathbf{x}_1$ and $\mathbf{x}_2$ if and only if its centre lies in the region common to these two spheres; this region is made up of elementary $(n-1)$-dimensional disks (perpendicular to the axis of the cone) of radius $z\cos\chi$ and thickness $d(z\sin\chi)$, and its total volume is therefore

$$2\int_{\chi=\theta}^{\frac{1}{2}\pi} k_{n-1}(z\cos\chi)^{n-1}\, d(z\sin\chi) = 2k_{n-1} z^n \int_\theta^{\frac{1}{2}\pi} \cos^n\chi\, d\chi. \tag{7}$$

By momentarily taking $r = 0$ we deduce the well-known relation

$$k_n = 2k_{n-1}\int_0^{\frac{1}{2}\pi} \cos^n\chi\, d\chi = 2k_{n-1}\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}{2\Gamma(\frac{1}{2}n+1)}. \tag{8}$$

The available positions for the centre of the random sphere of radius z fill the interior of a sphere of radius $R+z$. Hence, from (7),

$$Q(z) = \frac{2k_{n-1} z^n}{k_n (R+z)^n}\int_\theta^{\frac{1}{2}\pi} \cos^n\chi\, d\chi. \tag{9}$$

Define functions ϕ and G by means of

$$G(\zeta) = \int_\zeta^\infty \frac{z^n}{(R+z)^n} v(z) f_R(z)\, dz = \phi\left(\frac{1}{\zeta}\right). \tag{10}$$

Since $Q(z) = 0$ for $z < \frac{1}{2}r$, we get from (4), (5), (6), (9) and (10)

$$\begin{aligned}\frac{k_n\rho(r)}{2k_{n-1}} &= \frac{k_n}{2k_{n-1}}\int_{\frac{1}{2}r}^{\infty} Q(z)\,v(z) f_R(z)\,dz \\ &= \int_{\frac{1}{2}r}^{\infty}\left\{\int_{\theta}^{\frac{1}{2}\pi}\cos^n\chi\,d\chi\right\}\frac{z^n}{(R+z)^n}v(z)f_R(z)\,dz \\ &= -\int_{\frac{1}{2}r}^{\infty}\left\{\int_{\theta}^{\frac{1}{2}\pi}\cos^n\chi\,d\chi\right\}dG(z) \\ &= -\left[G(z)\int_{\theta}^{\frac{1}{2}\pi}\cos^n\chi\,d\chi\right]_{z=\frac{1}{2}r}^{\infty} - \int_{z=\frac{1}{2}r}^{\infty} G(z)\cos^n\theta\,d\theta \\ &= -\int_{\frac{1}{2}\pi}^{0} G\left(\frac{r}{2\sin\theta}\right)\cos^n\theta\,d\theta \\ &= \int_0^{\frac{1}{2}\pi}\phi\left(\frac{2}{r}\sin\theta\right)\cos^n\theta\,d\theta.\end{aligned}$$

Hence, substituting $t = 2/r$, and using (8), we get

$$\frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\phi(t\sin\theta)\cos^n\theta\,d\theta = \rho\left(\frac{2}{t}\right) \quad (0 \leqslant t \leqslant \infty). \tag{11}$$

The integral equation (11) is a generalized form of Schlömilch's equation; and we shall obtain its solution in §5. It will then follow from (10) that we shall have solved the problem provided we take f_R and v to be functions satisfying

$$v(z) f_R(z) = \left(1+\frac{R}{z}\right)^n \frac{\phi'(1/z)}{z^2}, \tag{12}$$

where $\phi'(u) = d\phi(u)/du$. Since f_R is a frequency function, it must also satisfy

$$\int_0^{\infty} f_R(z)\,dz = 1; \tag{13}$$

and since v is a variance we must have $\phi' \geqslant 0$. Hence a sufficient condition for the admissibility of ρ is that the solution ϕ of (11) shall be non-decreasing.†

5. *A generalization of Schlömilch's integral equation.* The integral equation to be considered is

$$\frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^{\frac{1}{2}\pi}\phi_n(t\sin\theta)\cos^n\theta\,d\theta = \psi_n(t) \quad (0 \leqslant t \leqslant \infty). \tag{14}$$

Here ψ_n is a given function, and we want to solve the equation for the unknown function ϕ_n. We assume throughout that n is a non-negative integer. Schlömilch's

† This conclusion is slightly more general than that actually proved in the text; it follows easily enough, however, by replacing $f_R(z)\,dz$ by $dF_R(z)$ and allowing discontinuities in F_R. Here F_R is simply the cumulative distribution function associated wiih f_R, and is unconnected with F_v defined in §3. When $n = 1$, it may be inferred from equation (29) below that any convex ρ is admissiible. Pólya (*Proc.* 1*st Berkeley Symp. on Math. Statist. and Prob.* p. 116) has already noted that convexity is a sufficient condition for a symmetric characteristic function. Symmetric characteristic functions and isotropic correlation functions are formally equivalent. Hence (29) confirms Pólya's result, while (30) and (31) generalize it for $n = 2$ and $n = 3$.

equation is the particular case $n = 0$; and (Whittaker and Watson (7), p. 229) its solution is

$$\phi_0(t) = \psi_0(0) + t\int_0^{\frac{1}{2}\pi} \psi_0'(t\sin\theta)\,d\theta, \tag{15}$$

where $\psi_0'(t) = d\psi_0(t)/dt$. The case $n = 1$ is easily solved. We have

$$t\psi_1(t) = \int_0^{\frac{1}{2}\pi} \phi_1(t\sin\theta)\,t\cos\theta\,d\theta = \int_0^t \phi_1(u)\,du,$$

whence

$$\phi_1(t) = \psi_1(t) + t\psi_1'(t). \tag{16}$$

We recall Cauchy's integration formulae: namely, that the system of equations

$$\phi_n^{\{0\}}(t) = \phi_n(t), \quad \phi_n^{\{m\}}(t) = \int_0^t \phi_n^{\{m-1\}}(x)\,dx \quad (m = 1, 2, \ldots) \tag{17}$$

is equivalent to

$$\phi_n^{\{m\}}(t) = \int_0^t \frac{(t-x)^{m-1}}{(m-1)!}\phi_n(x)\,dx \quad (m = 1, 2, \ldots). \tag{18}$$

The analogous system of equations

$$\psi_n^{[0]}(t) = \psi_n(t), \quad \psi_n^{[m]}(t) = \int_0^t x\psi_n^{[m-1]}(x)\,dx \quad (m = 1, 2, \ldots) \tag{19}$$

is equivalent to

$$\psi_n^{[m]}(t) = \int_0^t \frac{(t^2-x^2)^{m-1}}{2^{m-1}(m-1)!}x\psi_n(x)\,dx \quad (m = 1, 2, \ldots). \tag{20}$$

These results are true for arbitrary functions ϕ_n, ψ_n (i.e. not necessarily connected by (14)), and (20) can easily be deduced from (18) by writing $\frac{1}{2}x^2$ for x, $\frac{1}{2}t^2$ for t, and $\psi_n^{[m]}(x) = \phi_n^{\{m\}}(\frac{1}{2}x^2)$. Alternatively, (19) and (20) can be proved from first principles in the same way as Cauchy's formulae.

Now (14), (17) and (19) yield

$$\begin{aligned}
\psi_n^{[1]}(t) &= \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^t x\left\{\int_0^{\frac{1}{2}\pi} \phi_n^{\{0\}}(x\sin\theta)\cos^n\theta\,d\theta\right\}dx \\
&= \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^t dx\int_{\theta=0}^{\frac{1}{2}\pi} \cos^{n-1}\theta\,d\phi_n^{\{1\}}(x\sin\theta) \\
&= \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}\int_0^t dx\int_0^{\frac{1}{2}\pi} \phi_n^{\{1\}}(x\sin\theta)\,(n-1)\cos^{n-2}\theta\sin\theta\,d\theta \\
&= \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}(n-1)\int_0^{\frac{1}{2}\pi} \phi_n^{\{2\}}(t\sin\theta)\cos^{n-2}\theta\,d\theta.
\end{aligned}$$

Repeating this process m times, where $2m \leqslant n$, we get

$$\psi_n^{[m]}(t) = \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}n+\frac{1}{2})}(n-1)(n-3)\ldots(n-2m+1)\int_0^{\frac{1}{2}\pi} \phi_n^{\{2m\}}(t\sin\theta)\cos^{n-2m}\theta\,d\theta. \tag{21}$$

We can thus reduce the general case of (14) to one or other of the cases $n = 0$ or $n = 1$. Thus if n is even and positive, we take $2m = n$ in (21) and get

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \phi_n^{\{n\}}(t\sin\theta)\,d\theta = \Psi_n(t),$$

where
$$\Psi_n(t) = \frac{1}{2^{n-1}(\frac{1}{2}n)!\,(\frac{1}{2}n-1)!}\int_0^t (t^2-x^2)^{\frac{1}{2}n-1}\,x\psi_n(x)\,dx. \tag{22}$$

Then, from (15) and (17),
$$\phi_n(t) = \frac{d^n}{dt^n}\left\{t\int_0^{\frac{1}{2}\pi}\Psi_n'(t\sin\theta)\,d\theta\right\} \quad (n \geqslant 2 \text{ and even}). \tag{23}$$

Alternatively, if n is odd, we take $2m = n-1$ and deduce
$$\phi_n(t) = \frac{d^n}{dt^n}\left\{\frac{(n-1)}{n!}t\int_0^t (t^2-x^2)^{\frac{1}{2}n-\frac{3}{2}}x\psi_n(x)\,dx\right\} \quad (n \geqslant 2 \text{ and odd}). \tag{24}$$

6. *Remarks on the nature of the general solution.* It follows from (11) and (12) that $v(\bullet)f_R(\bullet)$ is a linear functional of $\rho(\bullet)$; and this fact can be exploited in the usual way. For instance, suppose that
$$v(z)f_R(z) = u_\lambda(z) \tag{25}$$
is the solution when
$$\rho(r) = e^{-\lambda r}. \tag{26}$$
Then
$$v(z)f_R(z) = \int u_\lambda(z)\,dH(\lambda) \tag{27}$$
is the solution when
$$\rho(r) = \int e^{-\lambda r}\,dH(\lambda). \tag{28}$$

We shall give explicit expressions for $u_\lambda(z)$ in case $n = 1, 2$ or 3; so that a general solution (27) will be available in these cases whenever the correlation function can be represented as a Laplace–Stieltjes transform. The solution (27) is formal to the extent that it is only meaningful if never negative. Thus as a particular case of (27) we can obtain a formal but meaningless solution for $\rho(r) = e^{-\lambda r}\cos\mu r$ by picking out the real part of $u_{\lambda+i\mu}(z)$.

In applying the theory to a physical realization, one has considerable freedom of choice, since it is only the product of f_R and v which is specified, and, moreover, the distributions F_y are arbitrary apart from (1). We are uncertain how best to use this freedom of choice. If F_y is normal, $w(\mathbf{x})$ for any fixed value of $\mathbf{x}$ will not be normal when a finite number N spheres are thrown; for $w(\mathbf{x})$ will be the sum of N numbers, of which n (say) are zero and $N-n$ are drawn from normal populations with a *random* variance $v(z)$, since z is random. Also n is a random variate, being binomially distributed with parameter
$$p = \mathscr{E}\left(\frac{z}{R+z}\right)^n = \int_0^\infty \left(\frac{z}{R+z}\right)^n f_R(z)\,dz.$$

It follows that the characteristic function of $w(\mathbf{x})$, for fixed $\mathbf{x}$, is
$$\left[p + (1-p)\int_0^\infty e^{-\frac{1}{2}t^2 v(z)} f_R(z)\,dz\right]^N,$$
which is not the characteristic function of a normal distribution. When $v(z)$ is constant, it might be possible to get a fairly good approximation to normality in $w(\mathbf{x})$ by taking F_y as a discrete distribution, normal in shape apart from the fact that the frequency at zero is reduced by a sufficient amount to allow for the factor p. When $v(z)$ is not

constant, the situation becomes more difficult. In any case adjustments to F_y of the foregoing type will not ensure that $w(\mathbf{x}_1), \ldots, w(\mathbf{x}_k)$ are jointly in multivariate normal form for a finite number of spheres. The whole problem of the rapidity of convergence to a Gaussian process, as the number of spheres becomes large, seems rather formidable, since it is intimately bound up with the behaviour of $w(\mathbf{x})$ and $w(\mathbf{y})$ for points $\mathbf{x}$ and $\mathbf{y}$ close together.†

7. *Explicit solutions in three or fewer dimensions.* By a straightforward application of the methods described in §§4 and 5, we find

$$v(z) f_R(z) = 4(R+z)\,\rho''(2z), \quad \text{when } n = 1; \tag{29}$$

$$v(z) f_R(z) = -4(R+z)^2 \int_0^{\frac{1}{2}\pi} \rho'''\left(\frac{2z}{\sin\theta}\right) \frac{d\theta}{\sin^2\theta}, \quad \text{when } n = 2; \tag{30}$$

$$v(z) f_R(z) = \frac{4(R+z)^3}{3z^2} \{\rho''(2z) - 2z\rho'''(2z)\}, \quad \text{when } n = 3. \tag{31}$$

In (29), (30) and (31)

$$\rho''(r) = d^2\rho(r)/dr^2, \quad \rho'''(r) = d^3\rho(r)/dr^3.$$

For higher values of n, $v(z) f_R(z)$ can be expressed similarly in terms of various derivatives of ρ; but the expressions become increasingly complicated as n increases, and the cases $n \geqslant 4$ are unlikely to be wanted in practice. A useful alternative form to (30) springs from the substitution $\operatorname{cosec}\theta = \cosh u$:

$$v(z) f_R(z) = -4(R+z)^2 \int_0^\infty \rho'''(2z\cosh u) \cosh u\, du, \quad \text{when } n = 2. \tag{32}$$

8. *Markovian linear process.* This is the special but important case $n = 1$, $\rho(r) = e^{-\lambda r}$, where λ is a constant. We find from (29)

$$v(z) f_R(z) = 4\lambda^2 (R+z)\, e^{-2\lambda z}; \tag{33}$$

and a convenient method will be to sample from the exponential distribution‡

$$\left.\begin{aligned} f_R(z) &= 2\lambda\, e^{-2\lambda z} \\ \text{with} \qquad v(z) &= 2\lambda R + 2\lambda z. \end{aligned}\right\} \tag{34}$$

As a generalization of (33), the case $n = 1$, $\rho(r) = \exp(-\lambda r^\alpha)$ leads to

$$v(z) f_R(z) = 2^\alpha \alpha\lambda z^{\alpha-2}\{1 - \alpha + \alpha\lambda(2z)^\alpha\}(R+z) \exp\{-\lambda(2z)^\alpha\},$$

provided $0 < \alpha \leqslant 1$.

† *Note added* 26 *March* 1955. Since preparing this paper, one of us (J. A. N.) has carried out some numerical sampling for the case $n = 2$, $\rho(r) = e^{-0.6r}$. With the choices of f_R and v used so far, the rate of convergence is disappointingly slow. We are at present investigating devices for improving the rate of convergence, and we hope to publish information on this in a subsequent paper.

‡ Since only the product vf_R has to depend on R, it is immaterial whether we make f_R or v depend on R; the latter choice might be more convenient.

9. *Pseudo-Markovian planar process.* This is the case $n = 2$, $\rho(r) = e^{-\lambda r}$, and seems the simplest model for the agricultural example mentioned in §1. We find from (32)

$$v(z)f_R(z) = 4\lambda^3(R+z)^2\int_0^\infty e^{-2\lambda z\cosh u}\cosh u\,du$$
$$= 4\lambda^3(R+z)^2 K_1(2\lambda z), \tag{35}$$

where K_1 is the modified Bessel function of the second kind in Macdonald's notation, namely,

$$K_1(z) = \lim_{\epsilon\to 0}\tfrac{1}{2}\pi\{I_{-1-\epsilon}(z) - I_{1+\epsilon}(z)\}\operatorname{cosec}\pi\epsilon.$$

This form of K_1 is the one usually tabulated. For further remarks on notation, see the footnotes to Whittaker and Watson ((7), p. 373), and Jeffreys and Jeffreys ((4), p. 579). In Heaviside's notation (27) reads

$$v(z)f_R(z) = 2\pi\lambda^3(R+z)^2\,\mathrm{Kh}_1(2\lambda z). \tag{36}$$

The main difference between (33) and (35) is that, in (33),

$$\int_0^\infty v(z)f_R(z)\,dz \tag{37}$$

is finite; so that we can, if we wish, arrange that the scores attached to the random spheres are all bounded. On the other hand, with (35), (37) is infinite; so that we *must* use an unboundedly large variance $v(z)$ and *a fortiori* unboundedly large scores for very small spheres. This, of course, does not mean that the variance of $w(\mathbf{x})$ will be unbounded; in fact our analysis has already ensured that it shall be unity, which may be otherwise verified from (5) and (9), thus:

$$\operatorname{var}[w(\mathbf{x})] = \int_0^\infty 4\lambda^3 z^2 K_1(2\lambda z)\,dz = \frac{1}{2}\int_0^\infty x^2 K_1(x)\,dx$$
$$= \frac{1}{2}\int_0^\infty x^2\int_0^\infty e^{-x\cosh u}\cosh u\,du\,dx = \int_0^\infty \frac{du}{\cosh^2 u} = 1.$$

Since the function $e^x K_1(x)$ is tabulated (1), it might be convenient to satisfy (35) by taking

$$f_R(z) = 2\lambda\, e^{-2\lambda z}, \quad v(z) = \tfrac{1}{2}(2\lambda R + 2\lambda z)^2 e^{2\lambda z} K_1(2\lambda z). \tag{38}$$

This form also has the advantage that we shall not have to deal as a rule with very large random circles carrying very small scores.

The more general case $n = 2$, $\rho(r) = \exp(-\lambda r^\alpha)$ is also admissible if $0 < \alpha \leqslant 1$; and $f_R(z)\,v(z)$ is expressible in terms of a rather complicated definite integral, which we do not quote.

10. *Pseudo-Markovian spatial process.* The three-dimensional case of $\rho(r) = e^{-\lambda r}$ gives

$$v(z)f_R(z) = 4\lambda^2(R+z)^3(1+2\lambda z)\,e^{-2\lambda z}/3z^2. \tag{39}$$

11. *Whittle's planar process.* Whittle (8) has given reasons for regarding the process with correlation function

$$\rho(r) = \lambda r K_1(\lambda r) \tag{40}$$

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(where K_1 is the Bessel function used in § 9) as the 'elementary' process in two dimensions corresponding to $\rho(r) = e^{-\lambda r}$ in one dimension. We now show that Whittle's process cannot be realized by the methods of this paper, thus substantiating the remark at the end of § 2. Since λ is only a scale factor, it will be convenient to take $\lambda = 1$; so that (40) becomes

$$\rho(r) = rK_1(r).$$

Now ((4), p. 582) we have the relations

$$\frac{d}{dr}[rK_1(r)] = -rK_0(r), \quad \frac{d}{dr}[K_0(r)] = -K_1(r).$$

Hence

$$\rho''(r) = rK_1(r) - K_0(r).$$

From (32) we have $\quad v(z)f_R(z) = -2(R+z)^2 \dfrac{d}{dz}\displaystyle\int_0^\infty \rho''(2z\cosh u)\,du,$

and so it is enough to show that

$$M(z) = \frac{d}{dz}\int_0^\infty \{2z\cosh u\, K_1(2z\cosh u) - K_0(2z\cosh u)\}\,du$$

can take positive values. Since, by definition of K_n,

$$K_n(x) = \int_0^\infty \cosh nu\, e^{-x\cosh u}\,du,$$

we have

$$\begin{aligned} M(z) &= \frac{d}{dz}\int_0^\infty\int_0^\infty (2z\cosh u\cosh v - 1)\,e^{-2z\cosh u\cosh v}\,du\,dv \\ &= -\frac{d}{dz}\left(1 + z\frac{d}{dz}\right)\int_0^\infty\int_0^\infty e^{-2z\cosh u\cosh v}\,du\,dv \\ &= -\frac{1}{4}\frac{d}{dz}\left(1 + z\frac{d}{dz}\right)\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-2z\cosh u\cosh v}\,du\,dv. \end{aligned}$$

In the integral, put $U = u+v$ and $V = u-v$. Since

$$2\cosh u\cosh v = \cosh U + \cosh V,$$

we get

$$\begin{aligned} M(z) &= -\frac{1}{8}\frac{d}{dz}\left(1 + z\frac{d}{dz}\right)\left\{\int_{-\infty}^\infty e^{-z\cosh U}\,dU\right\}^2 \\ &= -\frac{1}{2}\frac{d}{dz}\left(1 + z\frac{d}{dz}\right)[K_0(z)]^2 \\ &= K_0(z)\,K_1(z) - z[K_0(z)]^2 - z[K_1(z)]^2. \end{aligned}$$

Since $K_0(z)$ is logarithmically infinite at $z = 0$, while $K_1(z)$ has a simple pole at $z = 0$, we see that $M(z)$ is positive for all sufficiently small positive values of z.

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