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On infinitesimal τ-isospectrality of locally symmetric spaces

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Abstract. Let (τ, V_τ) be a finite dimensional representation of a maximal compact subgroup K of a connected non-compact semisimple Lie group G, and let Γ be a uniform torsion-free lattice in G. We obtain an infinitesimal version of the celebrated Matsushima–Murakami formula, which relates the dimension of the space of automorphic forms associated to τ and multiplicities of irreducible $\tau^\vee$-spherical spectra in $L^2(\Gamma\backslash G)$. This result gives a promising tool to study the joint spectra of all central operators on the homogenous bundle associated to the locally symmetric space and hence its infinitesimal τ-isospectrality. Along with this, we prove that the almost equality of τ-spherical spectra of two lattices assures the equality of their τ-spherical spectra.

1 Introduction

1.1 τ-isospectrality and representation equivalence

Let G be a connected non-compact semisimple Lie group with finite centre. Let K be a maximal compact subgroup in G. Then G/K is a symmetric space which carries a G-invariant Riemannian metric induced by the $\mathrm{Ad}(G)$-invariant inner product on the Lie algebra $\mathfrak{g}$ of G. For any finite dimensional complex representation (τ, V_τ) of K, one has the homogeneous vector bundle E_τ on G/K (see Section 2.3 for the details) whose smooth sections are given by the space

$$\mathcal{A}^\infty(G/K,\tau) := \{\phi : G \to V_\tau \mid \text{smooth}, \ \phi(gk) = \tau(k^{-1})(\phi(g)) \text{ for all } k \in K, g \in G\}.$$

This space $\mathcal{A}^\infty(G/K,\tau)$ is $\mathfrak{U}(\mathfrak{g})^K$-stable. In particular, the centre $\mathfrak{Z}(\mathfrak{g})$ of the universal enveloping algebra acts on $\mathcal{A}^\infty(G/K,\tau)$, and the Casimir element in the centre induces a second order elliptic differential operator Δ_τ on $\mathcal{A}^\infty(G/K,\tau)$. When τ is the trivial representation of K, Δ_τ coincides with the Laplace–Beltrami operator acting on the smooth functions on G/K.

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Let Γ be a uniform lattice in G. Then $X_\Gamma := \Gamma\backslash G/K$ is a compact locally symmetric space which is manifold if Γ acts freely on G/K. The space X_Γ has a Riemannian metric induced from G/K. We denote by $V_{\Gamma,\tau}$ to be the space of all Γ-invariant smooth sections in $\mathcal{A}^\infty(G/K,\tau)$. Then $V_{\Gamma,\tau}$ is stable under the action of $\mathfrak{U}(\mathfrak{g})^K$, in particular of the centre $\mathfrak{Z}(\mathfrak{g})$. The Casimir element C, induces the second order self-adjoint elliptic operator $\Delta_{\tau,\Gamma}$ on $V_{\Gamma,\tau}$, which has non-negative discrete spectrum of eigenvalues with finite multiplicities. The multiset of eigenvalues with multiplicity is denoted by $\mathrm{Spec}(\Delta_{\tau,\Gamma})$. Such spectrum of a locally symmetric space is closely related to the multiplicity of irreducible representations occurring in the right regular representation $L^2(\Gamma\backslash G)$ of G. Here we mention the famous Matsushima–Murakami formula (see [9], see also [7, Proposition 2.4]):

$$\mathrm{Mult}_{\Delta_{\tau,\Gamma}}(\lambda) = \sum_{\pi\in\widehat{G},\pi(C)=\lambda} m(\pi,\Gamma)\dim(\mathrm{Hom}_K(\tau^\vee,\pi|_K)), \tag{1.1}$$

where, $\tau^\vee$ is the dual representation of τ and $m(\pi,\Gamma)$ is the multiplicity of π in the right regular representation $L^2(\Gamma\backslash G)$.

Let $\widehat{G}_\tau = \{\pi\in\widehat{G} : \mathrm{Hom}_K(\tau,\pi|_K)\neq 0\}$. Two uniform lattices Γ_1 and Γ_2 are called *τ-representation equivalent* if $m(\pi,\Gamma_1) = m(\pi,\Gamma_2)$ for all $\pi\in\widehat{G}_\tau$. From Eq. 1.1, it is easily observed that if $m(\pi,\Gamma_1) = m(\pi,\Gamma_2)$ for all $\pi\in\widehat{G}_{\tau^\vee}$, then $\mathrm{Spec}(\Delta_{\tau,\Gamma_1}) = \mathrm{Spec}(\Delta_{\tau,\Gamma_2})$ for X_{Γ_1} and X_{Γ_2}, respectively. Therefore, if two lattices Γ_1 and Γ_2 are $\tau^\vee$-representation equivalent, then the corresponding elliptic operators have the same spectrum for the locally symmetric spaces X_{Γ_1} and X_{Γ_2}, respectively i.e., they are *τ-isospectral* (see [7, Proposition 2.5]).

Definition 1.1.1 Two co-compact lattices Γ_1 and Γ_2 are called almost-τ-representation equivalent if $m(\pi,\Gamma_1)$ is equal to $m(\pi,\Gamma_2)$ for all but finitely many $\pi\in\widehat{G}_\tau$.

In the first half of this paper, we give an affirmative answer of the question: whether the almost-τ-representation equivalence implies τ-representation equivalence between two uniform torsion free lattices Γ_1 and Γ_2 in non-compact symmetric space G/K with arbitrary rank, and for any finite dimensional representation (τ,V_τ) of K.

Theorem 1.1.2 *Let G be a non-compact connected semisimple Lie group with finite center. If for two uniform torsion free lattices Γ_1 and Γ_2,*

$$m(\pi,\Gamma_1) = m(\pi,\Gamma_2)$$

for all but finitely many $\pi\in\widehat{G}_\tau$, then Γ_1 and Γ_2 are τ-representation equivalent lattices.

Remark 1.1.3 Consequently, the above hypothesis implies that X_{Γ_1} and X_{Γ_2} are $\tau^\vee$-isospectral. When τ is trivial, Theorem 1.1.2 reduces to [1, Theorem 1.2]. A special case of Theorem 1.1.2 is [2, Theorem 4.1] for the group $\mathrm{PSL}(2,\mathbb{R})$.

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1.2 Infinitesimal τ-isospectrality

For an infinitesimal character χ of $\mathfrak{Z}(\mathfrak{g})$, let $V_{\chi,\Gamma,\tau} = \{\phi \in V_{\Gamma,\tau} \mid z.\phi = \chi(z)\phi$ for all $z \in \mathfrak{Z}(\mathfrak{g})\}$.

We define a refinement of the notion of τ-isospectrality as follows.

Definition 1.2.1 Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} of non-compact type are *infinitesimally τ-isospectral* if

$$\dim V_{\chi,\Gamma_1,\tau} = \dim V_{\chi,\Gamma_2,\tau} \text{ for all } \chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}.$$

Let $[\chi]$ be the set of all irreducible representations of G which has infinitesimal character χ. It is known that $[\chi]$ is a finite subset of $\widehat{G}$ (see [4, Corollary 10.37]). We have obtained the following variant of Matsushima–Murakami formula.

Theorem 1.2.2 *Let G be a connected non-compact semisimple Lie group. Assume that Γ is a uniform lattice in G. Then for any $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$ and for any finite dimensional representation τ of K,*

$$\dim V_{\chi,\Gamma,\tau} = \sum_{\pi\in[\chi]} m(\pi,\Gamma)\dim(\mathrm{Hom}_K(\tau^\vee, \pi|_K)).$$

Remark 1.2.3 For the rank one semisimple Lie group G, an infinitesimal character $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$ is completely determined by its value on the Casimir element. Therefore the above result becomes the earlier mentioned Matsushima–Murakami formula Eq. 1.1.

Remark 1.2.4 Let $\tau = \tau_p$ be the p-th exterior power of the adjoint representation of K on $\mathfrak{p}^*_{\mathbb{C}}$. The associated homogeneous vector bundle is identified with the p-th exterior product of the co-tangent bundle on G/K. In [8], Matsushima proved the relationship between the dimension of harmonic p-forms on X_Γ and the multiplicity of irreducible representations in $L^2(\Gamma\backslash G)$ that occur with a nonzero τ_p-isotypic component. Therefore, Theorem 1.2.2 can be seen as an infinitesimal version of Eq. 1.1.

Corollary 1.2.5 *If Γ_1 and Γ_2 are $\tau^\vee$-representation equivalent then the spaces X_{Γ_1} and X_{Γ_2} are infinitesimally τ-isospectral.*

Remark 1.2.6 If τ is trivial, the above formula in Theorem 1.2.2 reduces to [1, Theorem 1.3].

Remark 1.2.7 The finite set $[\chi]$ of all irreducible representations of a real reductive group (informally a χ-packet) is quite close to Langlands L-packet that consists of irreducible admissible representations having same L-parameter. It is known that the irreducible representations from a fixed L-packet have same infinitesimal character i.e., they cannot be distinguished by the spectral data. According to our knowledge, the converse of this is not known in general. It is interesting to formulate a suitable variant of Matsushima–Murakami formula in terms of L-packet instead, and that might reflect some analogous relationship between the multiplicities of irreducible representations in a given L-packet and isospectrality.

1.3 Literature review

We briefly review some literature related to the main results in this paper. The question: *Can the space X_Γ (up to isometry) be determined by its spectrum?* has been of great interest over last few decades. The works of Milnor (see [10]) and Vignéras (see [14]) are in the frontline to answer this negatively. Later, the construction of non-isometric isospectral manifolds by Sunada brought significant arithmetic flavour in this context (see [13]). In another important work in this context, C. S. Rajan studied the appropriate arithmetic properties which are determined by the spectrum (see [12]). Other than the ambitious inverse spectral problem there is another converse of the above discussion: *Given two τ-isospectral spaces X_{Γ_1} and X_{Γ_2}, are Γ_1 and Γ_2 $\tau^\vee$-representation equivalent?* This is called the representation-spectral converse for the pair (G, K). This has gotten attention for quite some time. There is a slightly weaker version of this, defined as follows.

Definition 1.3.1 Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} with the same universal cover G/K are said to be almost-τ-isospectral if $\mathrm{Mult}_{\Delta_{\tau,\Gamma_1}}(\lambda) = \mathrm{Mult}_{\Delta_{\tau,\Gamma_2}}(\lambda)$ for all but finitely many λ.

In the direction of '*almost-τ-representation equivalence implies isospectrality*', several approaches have been made. In [5], E. Lauret and R. Miatello answered this question for the case where K is a closed subgroup of a compact group G, i.e., for the case of compact homogeneous space G/K for compact group G. They further proved that the multiplicity of an appropriate finite subset of $\widehat{G}_\tau$ determines all multiplicities (see [5, Theorem 1.1]). Later, they studied the representation-spectral converse in the context of simply connected compact Riemannian symmetric space G/K of rank one and proved there are infinitely many $\tau \in \widehat{K}$ such that almost-τ-isospectrality implies $\tau^\vee$-representation equivalence and hence τ-isospectral (see [6, Theorem 1.1]).

For many non-compact Riemannian symmetric spaces, Pesce proved the validity of representation-spectral converse where τ is trivial (see [11]). When τ is trivial the τ-spectra are called spherical spectra. Bhagwat and Rajan answered that almost-spherical representation equivalence implies spherical representation equivalence (see [1, Theorem 1.2]). In fact, they described that for a spherical irreducible representation $\pi \in \widehat{G}_1$, there exists a character λ_π of the algebra of G-invariant differential operators on X_Γ such that $\mathrm{Mult}(\lambda_\pi) = m(\pi, \Gamma)$ and conversely. Later, Kelmer obtained several density results which relates the isospectrality and representation equivalence via the notion of length equivalence for the case of X_Γ with real rank one and of non-compact type (having non-compact universal cover), in particular for compact hyperbolic manifolds (see [3]).

1.4 Methodology

Let us briefly outline our methods for obtaining Theorem 1.1.2. We employ the well-established and effective tool, namely the Selberg Trace Formula to study the representation spectra of $L^2(\Gamma\backslash G)$ with respect to a convolution operator. Since our focus is on the τ-spherical irreducible representations in $L^2(\Gamma\backslash G)$, we need to annihilate the non

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τ-spherical component using appropriate test functions. To achieve this, we instead consider the right regular representation $L^2(\Gamma\backslash G, V_\tau)$ (see 2.1) of G. We also utilize an algebra, denoted by $C_c^\infty(G, K, V_\tau)$ consisting of compactly supported smooth $\mathrm{End}(V_\tau)$-valued test functions on G which are τ-equivariant (see 2.3). One can suitably define the convolution operator on $L^2(\Gamma\backslash G, V_\tau)$ for such test functions. It is straightforward to compute both the spectral and geometric expansions of the trace of these convolution operators, which leads to the required Selberg Trace Formula for $L^2(\Gamma\backslash G, V_\tau)$ (see 2.12). The advantage of using such operator valued τ-equivariant test functions is the corresponding convolution operator annihilates the non τ-spherical representation spectra (see 3.1.3). Moreover, due to our hypothesis, taking the difference between the trace formula for Γ_1 and Γ_2 yields a finite linear combination of Harish Chandra character distribution associated with finitely many τ-spherical irreducible representations. In Proposition 3.1.5, we construct a left K-saturated open set in G that avoids all conjugacy classes $[\gamma]_G$ for $\gamma \in \Gamma_1 \cup \Gamma_2$. For the aforementioned test functions supported on this K-stable open set, the orbital integrals, and thus the entire geometric side vanish. Finally, the remainder of the argument relies on the analyticity of locally integrable character functions and the linear independence of character distributions for inequivalent irreducible representations.

The organization of the article is as follows: In Section 2, we set up the preliminaries and recall the Harish Chandra character distributions, Isospectrality and Representation equivalence. In Section 2.5, we explicitly calculate the Selberg Trace Formula for $L^2(\Gamma\backslash G), V_\tau)$. In Section 3, we discuss the required lemmas and propositions to prove Theorem 1.1.2. In Section 4, we provide some observations and complete the proof of Theorem 1.2.2. In Section 5, we consider the case where the group G has discrete series representations and using the results from [16] and Theorem 1.2.2, we show that the dimension of χ-eigenspace of the automorphic forms of type $\tau^\vee$ is equal to the q_λ-th L^2-cohomology of the automorphic line bundle $\Gamma\backslash\mathcal{L}_\lambda$ associated to the discrete series representation π with minimal K-type τ and infinitesimal character χ.

2 Preliminaries

2.1 Basic setup

Let G be a connected non-compact semisimple Lie group and let K be a maximal compact subgroup of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Then the homogeneous space G/K is a symmetric space with a G-invariant metric. Let (τ, V_τ) be a finite dimensional representation of K. Let Γ be a uniform torsion-free lattice in G.

We consider the right regular representation ρ of G on the space

$$L^2(\Gamma\backslash G, V_\tau) := \{\phi : G \to V_\tau \mid \phi \text{ is measurable},$$

$$\phi(\gamma g) = \phi(g) \text{ for all } \gamma \in \Gamma, \int_{\Gamma\backslash G} |\phi(g)|^2\, dg < \infty\}. \tag{2.1}$$

Since $\Gamma\backslash G$ is compact, the right regular representation $L^2(\Gamma\backslash G)$ of G is completely reducible and each irreducible subrepresentation occurs with finite multiplicity. In other words,

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$$L^2(\Gamma\backslash G) \cong \widehat{\bigoplus_{\pi\in\widehat{G}}} m(\pi,\Gamma)\ W_\pi$$

with $m(\pi,\Gamma) < \infty$ for all $\pi \in \widehat{G}$.

We choose an orthonormal basis for each copy of π. Taking union of those we get an orthonormal basis $\mathcal{W}$ of $L^2(\Gamma\backslash G)$. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis of V_τ. For any $\psi \in \mathcal{W}$, we let $\psi_i(x) = \psi(x)v_i$. Then $\{\psi_i \mid \psi \in \mathcal{W}, 1 \le i \le n\}$ forms an orthonormal basis of $L^2(\Gamma\backslash G, V_\tau)$. Note that $L^2(\Gamma\backslash G, V_\tau) = L^2(\Gamma\backslash G) \otimes V_\tau$.

Therefore, we get

$$L^2(\Gamma\backslash G, V_\tau) \cong \widehat{\bigoplus_{\pi\in\widehat{G}}} m(\pi,\Gamma)\ (W_\pi \otimes V_\tau). \tag{2.2}$$

For each copy of $W_\pi \otimes V_\tau$, we identify a subspace V_π in $L^2(\Gamma\backslash G, V_\tau)$. The subspace V_π is not an irreducible G-subspace, rather it is direct sum of dim V_τ many copies of W_π.

We introduce a space $C_c^\infty(G, K, V_\tau)$ defined by

$$\begin{aligned}&\{f : G \to \mathrm{End}(V_\tau) \mid f \text{ is compactly supported and smooth such that}\\ &f(kx) = f(x)\tau(k^{-1}) \text{ for all } x \in G, k \in K\}.\end{aligned} \tag{2.3}$$

For $f \in C_c^\infty(G, K, V_\tau)$, let $h_f(x) = \mathrm{Trace}\,(f(x))$ for all $x \in G$. Now for any $f \in C_c^\infty(G, K, V_\tau)$, we define the convolution operator $\rho(f) : L^2(\Gamma\backslash G, V_\tau) \to L^2(\Gamma\backslash G, V_\tau)$ by

$$\rho(f)\phi(x) = \int_G f(y)(\phi(gy))\,dy \quad \text{for all } \phi \in L^2(\Gamma\backslash G, V_\tau). \tag{2.4}$$

Proposition 2.1.1 *Each V_π is $\rho(f)$-stable subspace for all $f \in C_c^\infty(G, K, V_\tau)$.*

Proof Let $\mathcal{U} \subset \mathcal{W}$ be the subset such that $\mathcal{U}$ is an orthonormal basis of W_π. Then $\{\phi_i : \phi \in \mathcal{U}, 1 \le i \le n\}$ is an orthonormal basis of V_π. Thus it suffices to show that

$$\langle \rho(f)\phi_i, \psi_j\rangle = 0 \quad \text{for any } \phi \in \mathcal{U}, \psi \in \mathcal{W}\setminus\mathcal{U} \text{ and any } 1 \le i, j \le n.$$

We compute,

$$\begin{aligned}\langle \rho(f)\phi_i, \psi_j\rangle &= \int_{\Gamma\backslash G} \langle \rho(f)\phi_i(g), \psi_j(g)\rangle\,dg\\ &= \int_{\Gamma\backslash G} \langle \int_G \phi(gy)f(y)v_i, \psi(g)v_j\rangle\,dy\,dg\\ &= \int_{\Gamma\backslash G}\int_G \phi(gy)\overline{\psi(g)}\langle f(y)v_i, v_j\rangle\,dy\,dg\\ &= \int_G \langle f(y)v_i, v_j\rangle \int_{\Gamma\backslash G} \phi(gy)\overline{\psi(g)}\,dg\,dy\\ &= 0.\end{aligned} \tag{2.5}$$

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2.2 Results on Harish–Chandra characters

We will be using the following two important results on *Harish–Chandra characters.*

Harish–Chandra character distribution: Let (π, W_π) be an irreducible unitary representation of G. Let $C_c^\infty(G)$ be the space of all compactly supported smooth functions on G. For any $f \in C_c^\infty(G)$, the convolution operator $\pi(f)$ on W_π is defined by $\pi(f)v = \int_G f(g)\pi(g)v\, dg$ for all $v \in W_\pi$. It is a trace class operator (see [4, Theorem 10.2]) and

$$\chi_\pi(f) = \mathrm{Trace}\,(\pi(f)) \text{ for all } f \in C_c^\infty(G).$$

Theorem 2.2.1 *Let $\{\pi_i\}$ be a finite collection of mutually inequivalent unitary irreducible representations of G. Then their characters $\{\chi_{\pi_i}\}$ are linearly independent distributions on G.*

Proof Reader is referred to [4, Theorem 10.6]. ∎

Theorem 2.2.2 *Let π be an irreducible unitary representation of G. Then the distribution character χ_π is given by a locally integrable function ϕ_π on G i.e., for $f \in C_c^\infty(G)$,*

$$\chi_\pi(f) = \int_G f(g)\phi_\pi(g)\, dg.$$

Moreover, the restriction of ϕ_π to the regular set of G is a real analytic function invariant under conjugation.

Proof Reader is referred to [4, Theorem 10.25]. ∎

2.3 Isospectrality and representation equivalence

There is a natural homogeneous vector bundle E_τ on G/K that is associated with the representation (τ, V_τ) of K. The space of all smooth global sections of E_τ can be realized as $\mathcal{A}^\infty(G/K,\tau) = \{\phi : G \to V_\tau \mid \phi \text{ is smooth}, \phi(xk) = \tau(k^{-1})(\phi(x)) \text{ for all } x \in G, k \in K\}$. Note that $\mathcal{A}^\infty(G/K,\tau)$ is a $\mathfrak{U}(\mathfrak{g})^K$ module. In particular, $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})^K$ acts on $\mathcal{A}^\infty(G/K,\tau)$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$. We choose a basis $\{X_i\}$ and $\{Y_j\}$ of $\mathfrak{k}$ and $\mathfrak{p}$, respectively with respect to a bilinear form B on $\mathfrak{g}$ induced from the killing form such that $B(X_k, X_l) = -\delta_{kl}$ and $B(Y_m, Y_n) = \delta_{mn}$. Let C be the Casimir element given by $C = -\sum X_i^2 + \sum Y_j^2$. The Casimir element C induces a second order symmetric elliptic operator Δ_τ on $\mathcal{A}^\infty(G/K,\tau)$.

Let Γ be a uniform torsion-free lattice in G, and let $X_\Gamma = \Gamma\backslash G/K$ be the associated compact locally symmetric space of non-compact type. We consider the vector bundle $E_{\tau,\Gamma}$ on X_Γ defined by the relation $[\gamma g, w] \sim [g, w]$ for all $\gamma \in \Gamma$ and $[g, w] \in E_\tau$. The space of all smooth global sections of $E_{\tau,\Gamma}$ can be realised as the space $V_{\Gamma,\tau} = \{\phi \in \mathcal{A}^\infty(G/K,\tau) \mid \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma\}$. Hence, the centre $\mathfrak{Z}(\mathfrak{g})$ acts on $V_{\Gamma,\tau}$.

Let $\Delta_{\tau,\Gamma} = \Delta_\tau|_{V_{\Gamma,\tau}}$. This is again a second order symmetric elliptic operator on X_Γ. Its spectrum $\mathrm{Spec}(\Delta_{\tau,\Gamma})$ is a discrete subset of the non-negative real numbers. Let $\mathrm{Mult}_{\Delta_{\tau,\Gamma}}(\lambda)$ = *the multiplicity of* $\lambda \in \mathrm{Spec}(\Delta_{\tau,\Gamma})$. Recall Eq. 1.1

$$\mathrm{Mult}_{\Delta_{\tau,\Gamma}}(\lambda) = \sum_{\pi\in\widehat{G},\pi(C)=\lambda} m(\pi,\Gamma)\dim(\mathrm{Hom}_K(\tau^\vee,\pi|_K)),$$

where $\pi(C)$ is the scalar by which the Casimir element C acts on W_π.

Definition 2.3.1 X_{Γ_1} and X_{Γ_2} are called *τ-isospectral* if the operators Δ_{τ,Γ_1} and Δ_{τ,Γ_2} have same spectrum (with multiplicity).

We denote $\widehat{G}_\tau = \{\pi\in\widehat{G} : \mathrm{Hom}_K(\tau,\pi)\neq 0\}$.

Definition 2.3.2 Γ_1 and Γ_2 are called *τ-representation equivalent* if $m(\pi,\Gamma_1) = m(\pi,\Gamma_2)$ for all $\pi\in\widehat{G}_\tau$.

It is clear from Eq. 1.1 that, if Γ_1 and Γ_2 are $\tau^\vee$-representation equivalent, then X_{Γ_1} and X_{Γ_2} are τ-isospectral.

2.4 A further refinement

We have seen that $\mathfrak{Z}(\mathfrak{g})$ acts on $V_{\Gamma,\tau}$ (see Section 1.1). For any character $\chi\in\widehat{\mathfrak{Z}(\mathfrak{g})}$, we have the χ-eigenspace defined as

$$V_{\chi,\Gamma,\tau} = \{\phi\in V_{\Gamma,\tau} \mid z.\phi = \chi(z)\phi \text{ for all } z\in\mathfrak{Z}(\mathfrak{g})\}.$$

In fact, $V_{\Gamma,\tau}$ decomposes as $V_{\Gamma,\tau} = \bigoplus_{\chi\in\widehat{\mathfrak{Z}(\mathfrak{g})}} V_{\chi,\Gamma,\tau}$.

We here introduce a notion of two locally symmetric spaces being *infinitesimally τ-isospectral*, defined as follows.

Definition 2.4.1 Two locally symmetric spaces X_{Γ_1} and X_{Γ_2} are infinitesimally τ-isospectral if for all characters $\chi\in\widehat{\mathfrak{Z}(\mathfrak{g})}$,

$$\dim V_{\chi,\Gamma_1,\tau} = \dim V_{\chi,\Gamma_2,\tau}.$$

Remark 2.4.2 If G is a real rank one Lie group, the centre $\mathfrak{Z}(\mathfrak{g})$ is the polynomial algebra over $\mathbb{C}$ in the Casimir element C. Therefore, the χ-eigenspace is simply the λ-eigenspace where λ is an eigenvalue of the Laplace-Beltrami operator $\Delta_{\tau,\Gamma}$ on $E_{\tau,\Gamma}$. In this case, infinitesimal τ-isospectrality reduces to τ-isospectrality as defined above.

As in [1] and [2], we can expect that the dim $V_{\chi,\Gamma,\tau}$ is related with the multiplicities $m(\pi,\Gamma)$ of $\pi\in\widehat{G}_{\tau^\vee}$ occurring in $L^2(\Gamma\backslash G)$ with infinitesimal character χ. This is the content of Theorem 1.2.2.

2.5 Selberg trace formula for $L^2(\Gamma\backslash G, V_\tau)$

Recall $\{\psi_j : \psi\in\mathcal{W}, j\in 1,\dots,n\}$ is an orthonormal basis of $L^2(\Gamma\backslash G, V_\tau)$ (recall 2.1). For any $f\in C_c^\infty(G,K,V_\tau)$ (recall 2.3), let $[f_{ij}]_{n\times n}$ be the matrix representation of f with respect to the basis $\{v_i\}_{i=1}^n$ of V_τ. For $f\in C_c^\infty(G,K,V_\tau)$, observe that,

$$\begin{aligned}\rho(f)\psi_j(x) &= \int_G f(x^{-1}y)(\psi_j(y))\,dy\\ &= \int_G f(x^{-1}y)(\psi(y)v_j)\,dy \qquad (2.6)\\ &= \sum_i \int_G \psi(y) f_{ij}(x^{-1}y)v_i\,dy.\end{aligned}$$

Then, we have

$$\begin{aligned}\text{Trace}\,(\rho(f)) &= \sum_{\psi\in\mathcal{W}}\sum_j \langle \rho(f)\psi_j, \psi_j\rangle\\ &= \sum_{\psi\in\mathcal{W}}\sum_j \int_{\Gamma\backslash G}\langle \sum_i \int_G \psi(y) f_{ij}(x^{-1}y)v_i\,dy, \psi(x)v_j\rangle\,dx\\ &= \sum_{\psi\in\mathcal{W}}\sum_j \int_{\Gamma\backslash G}\int_{\Gamma\backslash G} \psi(y)\,\psi(x)\,\sum_{\gamma\in\Gamma}\langle \sum_i f_{ij}(x^{-1}\gamma y)v_i,\, v_j\rangle\,dy\,dx\\ &= \sum_{\psi\in\mathcal{W}}\int_{\Gamma\backslash G\times\Gamma\backslash G}\sum_{\gamma\in\Gamma}\text{Trace}\,(f(x^{-1}\gamma y))\overline{\psi(x)}\psi(y)\,dy\,dx\\ &= \sum_{\psi\in\mathcal{W}}\int_{\Gamma\backslash G\times\Gamma\backslash G}\sum_{\gamma\in\Gamma} h_f(x^{-1}\gamma y)\overline{\psi(x)}\psi(y)\,dy\,dx. \qquad (2.7)\end{aligned}$$

Put $K_f(x,y) = \sum_{\gamma\in\Gamma} h_f(x^{-1}\gamma y)$. Above equals,

$$\sum_{\psi\in\mathcal{W}}\int_{\Gamma\backslash G\times\Gamma\backslash G} K_f(x,y)\overline{\psi(x)}\psi(y)\,dy\,dx. \tag{2.8}$$

Consider the integral operator T on $L^2(\Gamma\backslash G)$ defined by

$$T\psi(x) = \int_{\Gamma\backslash G} K_f(x,y)\psi(y)\,dy. \tag{2.9}$$

Then T is a trace class operator and its trace is given by

$$\text{Trace}\,(T) = \sum_{\psi\in\mathcal{W}}\int_{\Gamma\backslash G\times\Gamma\backslash G} K_f(x,y)\overline{\psi(x)}\psi(y)\,dy\,dx. \tag{2.10}$$

On the other hand the trace of such an operator T equals $\int_{\Gamma\backslash G} K_f(x,x)\,dx$. Therefore,

$$\text{Trace}\,(\rho(f)) = \int_{\Gamma\backslash G} K_f(x,x)\,dx = \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} h_f(x^{-1}\gamma x)\,dx.$$

From Proposition 2.1.1, we have

$$\text{Trace}\,(\rho(f)) = \sum_{\pi\in\widehat{G}} m(\pi,\Gamma)\,\text{Trace}\,(\rho(f)|_{V_\pi}).$$

Now,

$$\begin{aligned}\text{Trace}\,(\rho(f)|_{V_\pi}) &= \sum_j \sum_{\psi\in\mathcal{W}\cap W_\pi} \langle \rho(f)\psi_j, \psi_j\rangle \\ &= \sum_{\psi\in\mathcal{W}\cap W_\pi} \int_{\Gamma\backslash G}\int_G \psi(xy)\,\overline{\psi(x)}\,\text{Trace}\,(f(y))\,dy\,dx \qquad (2.11)\\ &= \text{Trace}\,(\pi(h_f)).\end{aligned}$$

Therefore, we have the following *Selberg trace formula*:

$$\begin{aligned}\sum_{\pi\in\widehat{G}} m(\pi,\Gamma)\,\text{Trace}\,(\pi(h_f)) &= \int_{\Gamma\backslash G}\sum_{\gamma\in\Gamma} h_f(x^{-1}\gamma x)\,dx \\ &= \sum_{[\gamma]\in[\Gamma]_G} \text{vol}\,(\Gamma_\gamma\backslash G_\gamma)\int_{G_\gamma\backslash G} h_f(x^{-1}\gamma x)\,dx \qquad (2.12)\\ &= \sum_{[\gamma]\in[\Gamma]_G} a(\gamma,\Gamma)\,O_\gamma(h_f),\end{aligned}$$

where, $a(\gamma,\Gamma)$ and $O_\gamma(h_f)$ denote vol $(\Gamma_\gamma\backslash G_\gamma)$ and $\int_{G_\gamma\backslash G} h_f(x^{-1}\gamma x)\,dx$, respectively.

Remark 2.5.1 This trace formula is a generalisation of the well-known Selberg trace formula. For more explicit description about the "geometric side", see [15].

3 Proof of the first main result Theorem 1.1.2

3.1 Some preliminary results

We will describe the lemmas and propositions required to prove Theorem 1.1.2.

Lemma 3.1.1 *Let Kx be a left coset of K in G for some $x\in G$. Let U be an open set containing the left coset Kx. Then there exists $f\in C_c^\infty(G)$ with Supp $(f)\subset U$ such that $\int_K f(kx)\chi_\tau(k)\,dk \neq 0$. (Here χ_τ is the character function of τ.)*

Proof For any $f\in C_c^\infty(G)$ with $\text{Supp}(f)\subset U$, we define $\psi\in C^\infty(K)$ by $\psi(k) = f(kx)$. Conversely, for any $\psi\in C^\infty(K)$, define $f(kx)=\psi(k)$, and extend f smoothly to U.

Now, if $\int_K f(kx)\chi_\tau(k)\,dk = 0$ for all $f\in C_c^\infty(G)$ such that $\text{Supp}(f)\subset U$, then

$$\int_K \psi(k)\,\chi_\tau(k)\,dk = 0 \text{ for all } \psi\in C^\infty(K).$$

But that implies χ_τ is identically zero leading to a contradiction. ∎

Lemma 3.1.2 *Let $Kx\neq Ky$ be two distinct left cosets of K in G. Then there exists $F\in C_c^\infty(G,K,V_\tau)$ such that $h_F(x)\neq 0$ and $h_F|_{Ky}=0$.*

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Proof $Kx \neq Ky$ are disjoint compact sets in G. Choose disjoint open sets U_1, U_2 with $Kx \subset U_1, Ky \subset U_2$, and $U_1 \cap U_2 = \emptyset$. From Lemma 3.1.1, there exists $f \in C_c^\infty(G)$ with Supp $(f) \subset U_1$ such that $\int_K f(kx)\chi_\tau(k)\, dk \neq 0$.

Let $F(g) = \int_K f(kg)\tau(k)\, dk$. Then, $F \in C_c^\infty(G, K, V_\tau)$.

Furthermore, $h_F(g) = \int_K f(kg)\chi_\tau(k)\, dk$. Hence $h_F(x) \neq 0$ and $h_F(y) = 0$ and hence $h_F|_{Ky} = 0$. ∎

Proposition 3.1.3 *Let π be an irreducible representation of G occurring as a subrepresentation of ρ. Assume that π is not τ-spherical i.e.,* $\mathrm{Hom}_K(V_\tau, V_\pi) = 0$. *Then $\rho(f)$ is zero on V_π.*

Proof It is enough to show that $\rho(f)$ is zero at each element of the orthonormal basis $\{\psi_j : \psi \in \mathcal{U}, 1 \leq j \leq n\}$. Here, $\mathcal{U} = \mathcal{W} \cap W_\pi$. Recall that (see Eq. 2.4)

$$\rho(f)\psi_j(g) = \int_G \psi(gy)f(y)(v_j)\, dy.$$

For a fixed $\psi \in \mathcal{U}$, consider the subspace $U_\psi = \text{span } \{\psi_j : j \in \{1, \ldots, n\}\}$. If $v = \sum_{j=1}^{n} a_j v_j$, we write $\psi_v = \sum_{j=1}^n a_j \psi_j$. Then the subspace U_ψ is same as $\{\psi_v : v \in V_\tau\}$. There is an action of K on U_ψ by $\tau(k)\psi_v = \psi_{\tau(k)v}$.

Clearly, U_ψ is a representation of K isomorphic to τ. Note that $\widehat{\bigoplus_{\psi \in \mathcal{W}}} U_\psi = V_\pi$. We show that $\rho(f)|_{U_\psi} = 0$. In fact, we show that $\rho(f) \in \mathrm{Hom}_K(U_\psi, V_\pi)$.

Let $\psi_v \in U_\psi$ and $k_0 \in K$. Then for every $g \in G$, we have

$$\begin{aligned}
(\rho(k_0)\rho(f)\psi_v)(g) &= (\rho(f)\psi_v)(gk_0) \\
&= \int_G f(y)\, (\psi(gk_0y)v)\, dy \\
&= \int_G \psi(gy)\, f(k_0^{-1}y)v\, dy \qquad (3.1) \\
&= \int_G \psi(gy)\, f(y)\tau(k_0)v\, dy \\
&= \rho(f)\, \tau(k_0)\, \psi_v(g).
\end{aligned}$$

Therefore, $\rho(f)$ is zero on U_ψ for all $\psi \in \mathcal{W}$. Consequently, $\rho(f)$ is zero on V_π. ∎

Lemma 3.1.4 *Let Γ be a torsion-free uniform lattice in G. For a non-trivial element $\gamma \in \Gamma$, the conjugacy class $[\gamma]_G$ of γ in G is disjoint from K.*

Proof Reader is referred to [1, Lemma 4.2]. ∎

Proposition 3.1.5 *Let Γ_1 and Γ_2 be two uniform lattices in G. Then there exists an open set B in G such that $[\gamma]_G \cap B$ is empty for all $\gamma \in \Gamma_1 \cup \Gamma_2$, and B is stable under left K action on G.*

Proof Let U' be a relatively compact open set containing the identity element e in G. Let $U = KU'$. Then U is relatively compact and therefore it intersects at most finitely many conjugacy classes $[\gamma]_G$. Since, the natural map $G \to K\backslash G$ is proper, $K[\gamma]_G$ is closed in G. Since U is K-stable, $K[\gamma]_G \cap U \neq \emptyset$ if and only if $[\gamma]_G \cap U \neq \emptyset$. Let $E = \bigcup_{\gamma \neq e} (K[\gamma]_G \cap U)$. Since E is a finite union of closed sets, it is closed and K-stable subset of U. If there is $k \in K$, $\gamma \neq e$ and $x \in G$ such that $kx^{-1}\gamma x = e$, then $x^{-1}\gamma x \in K$; which contradicts the previous lemma. Therefore $e \notin K[\gamma]_G$ for any $\gamma \neq e$ and hence $e \notin E$. Choose an open set V containing e such that $E \cap V = \emptyset$. Now, let $B = KV \cap K^c$. Then B is the desired open set in G. ∎

3.2 Proof of Theorem 1.1.2

We have two uniform torsion free lattices Γ_1 and Γ_2 in G. So by Eq. 2.2,

$$(\rho_{\Gamma_i}, L^2(\Gamma_i\backslash G, V_\tau)) = \widehat{\bigoplus_{\pi \in \widehat{G}}} m(\pi, \Gamma_i) V_\pi$$

for $i = 1, 2$. Let $t_\pi = m(\pi, \Gamma_1) - m(\pi, \Gamma_2)$. By hypothesis there exists a finite set $\mathcal{S} \subset \widehat{G}_\tau$ such that $t_\pi = 0$ for all $\pi \in \widehat{G}_\tau \backslash \mathcal{S}$. For $f \in C_c^\infty(G, K, V_\tau)$, $\rho_{\Gamma_i}(f)$ is zero on V_π if $\pi \notin \widehat{G}_\tau$ by Proposition 3.1.3. Recall that $h_f(y) = \text{Trace}\,(f(y))$. Therefore from the Selberg trace formula (2.12), we have

$$\begin{aligned}
\sum_{[\gamma] \in [\Gamma_1]_G \cup [\Gamma_2]_G} (a(\gamma, \Gamma_1) - a(\gamma, \Gamma_2)) O_\gamma(h_f) &= \sum_{\pi \in \mathcal{S}} t_\pi \,\text{Trace}\,(\pi(h_f)) \\
&= \sum_{\pi \in \mathcal{S}} t_\pi \int_G h_f(y)\phi_\pi(y)\, dy \\
&= \int_G h_f(y)\phi(y)\, dy,
\end{aligned}$$

where, $\phi = \sum_{\pi \in \mathcal{S}} t_\pi \phi_\pi$.

Let B be the open set from Proposition 3.1.5. For any $f \in C_c^\infty(G, K, V_\tau)$ supported in B, the orbital integrals on the right-hand side is zero. For such functions *f*, we have

$$\int_B h_f(y)\phi(y)\, dy = 0.$$

From Lemma 3.1.2, the functions h_f separates points in B. Hence ϕ must vanish on the open subset B of G. Since ϕ is real analytic (see Theorem 2.2.2), it vanishes on all of G. By the linear independence of functions ϕ_π (see Theorem 2.2.1), we conclude that $m(\pi, \Gamma_1) = m(\pi, \Gamma_2)$ for all $\pi \in \widehat{G}_\tau$.

4 Proof of the second main result Theorem 1.2.2

4.1 Some observations

In this subsection, we make some observations that will be useful in the proof of Theorem 1.2.2.

Recall from Section 2.3 that $\mathcal{A}^\infty(G/K,\tau)$ is the space of smooth sections of the vector bundle E_τ and $V_{\Gamma,\tau}$ is the subspace defined by

$$V_{\Gamma,\tau} = \{\phi \in \mathcal{A}^\infty(G/K,\tau) \mid \phi(\gamma x) = \phi(x) \text{ for all } \gamma \in \Gamma, x \in G\}.$$

Let $n = \dim(V_\tau)$. Recall the choice of an orthonormal basis $\{v_i\}_{i=1}^n$ of V_τ from Section 2.1. For a fixed $\phi \in V_{\chi,\Gamma,\tau}$, write $\phi(x) = \sum_{i=1}^n \phi_i(x) v_i$ for all $x \in G$, where each ϕ_i is smooth complex valued function on $\Gamma\backslash G$. Recall from Section 2.1 that $V_{\chi,\Gamma,\tau}$ is the χ-eigenspace of $V_{\Gamma,\tau}$ with respect to the action of $\mathfrak{Z}(\mathfrak{g})$. For any $X \in \mathfrak{Z}(\mathfrak{g})$ and for all $x \in G$, we can see that

$$X \cdot \phi(x) = \sum_{i=1}^n X \cdot \phi_i(x)\, v_i. \tag{4.1}$$

Therefore, we conclude that $X \cdot \phi_i = \chi(X)\phi_i$ for all i.

Each $\tau(k)$ has a matrix representation with respect to the chosen orthonormal basis of V_τ. Let the (i,j)-th entry of $\tau(k)$ be denoted by $a_{ij}(k)$.

Note that $\phi(xk) = \tau(k^{-1})(\phi(x))$ for all $x \in G$ and $k \in K$. Thus we have

$$\begin{aligned}\sum_i \phi_i(xk)\, v_i &= \tau(k^{-1})\left(\sum_j \phi_j(x)\, v_j\right)\\ &= \sum_l \left(\sum_j a_{lj}(k^{-1})\phi_j(x)\right) v_l.\end{aligned} \tag{4.2}$$

Therefore, $\phi_i(xk) = \sum_{j=1}^n a_{ij}(k^{-1})\, \phi_j(x)$ for all $x \in G,\ k \in K$.

4.2 Proof of Theorem 1.2.2

We denote $[\chi] = \{\pi \in \widehat{G} : \text{ the infinitesimal character of } \pi \text{ is } \chi\}$. Recall that

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi\in\widehat{G}}} m(\pi,\Gamma) W_\pi.$$

There are $m(\pi,\Gamma)$ copies of W_π inside $L^2(\Gamma\backslash G)$ say, $\{W_{\pi_t} : 1 \le t \le m(\pi,\Gamma)\}$. Let P_{π_t} be the projection onto W_{π_t}.

Clearly, $\phi_i \in L^2(\Gamma\backslash G)$ for all $\phi \in V_{\chi,\Gamma,\tau}$. We have

$$\phi_i = \sum_{\pi\in[\chi]} \sum_{t=1}^{m(\pi,\Gamma)} P_{\pi_t}\phi_i.$$

For any $1 \le t \le m(\pi,\Gamma)$, clearly $P_{\pi_t} \in \mathrm{Hom}_G(L^2(\Gamma\backslash G), W_{\pi_t})$. Using Eq. 4.2, we get

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$$\pi(k)P_{\pi_t}\phi_i = P_{\pi_t}\rho(k)\phi_i = P_{\pi_t}\sum_{j=1}^{n} a_{ij}(k^{-1})\phi_j = \sum_{j=1}^{n} a_{ij}(k^{-1})P_{\pi_t}\phi_j. \tag{4.3}$$

Let F_{π_t} = Span of $\{P_{\pi_t}\phi_j : 1 \le j \le n\}$. From Eq. 4.3, it follows that F_{π_t} is a finite dimensional representation of K. Let $\tau^\vee$ be the dual representation of τ of K on the dual space V_τ^*. Let $\{v_i^*\}_{i=1}^n$ be the dual basis of V_τ^*. Then there is K-isomorphism between F_{π_t} and V_τ^* which maps each $P_{\pi_t}\phi_i$ to v_i^*.

Now we assume that τ is irreducible (and hence $\tau^\vee$ as well). For a fixed π_t, let $W_{\pi_t}(\tau^\vee)$ be the isotypic component of $\tau^\vee$ in W_{π_t}. Then $F_{\pi_t} \subset W_{\pi_t}(\tau^\vee)$. We have a decomposition

$$W_{\pi_t}(\tau^\vee) = \bigoplus_{p=1}^{M_{\tau^\vee}} H_p,$$

where, each H_p is an irreducible representation of K isomorphic to $\tau^\vee$. Note that $M_{\tau^\vee} = \dim(\mathrm{Hom}_K(\tau^\vee, \pi))$.

For each p, let $\{\phi_{t,p,s}\}_{s=1}^n$ be a basis of H_p satisfying $\phi_{p,t,s}(xk) = \sum_{u=1}^{n} a_{us}(k^{-1})$ $\phi_{t,p,u}(x)$. Note that this is possible because each H_p is isomorphic to F_{π_t}.

Now, $P_{\pi_t}\phi_i = \sum_p \sum_s \alpha_{p,s,i}\ \phi_{t,p,s}$. For each p, the matrix $(\alpha_{p,s,i})_{s,i}$ represents a K-homomorphism between F_{π_t} and H_p. Therefore, $\alpha_{p,s,i} = \alpha_p \delta_{s,i}$ for some scalar α_p by Schur's lemma.

Therefore, $P_{\pi_t}\phi_i = \sum_p \alpha_p\ \phi_{t,p,i}$. We define $\phi_{t,p}(x) = \sum_{i=1}^{n} \phi_{t,p,i}(x)\ v_i$ for all t and p. Hence, we get

$$\begin{aligned}\phi &= \sum_{i=1}^{n}\phi_i v_i \\ &= \sum_{i=1}^{n}\left(\sum_{\pi\in[\chi]}\sum_{t=1}^{m(\pi,\Gamma)} P_{\pi_t}\ \phi_i\right) v_i \\ &= \sum_{i=1}^{n}\sum_{\pi\in[\chi]}\sum_{t=1}^{m(\pi,\Gamma)}\sum_{p=1}^{M_{\tau^\vee}} \alpha_p\ \phi_{t,p,i}\ v_i \\ &= \sum_{\pi\in[\chi]}\sum_{t=1}^{m(\pi,\Gamma)}\sum_{p=1}^{M_{\tau^\vee}} \alpha_p\ \phi_{t,p}.\end{aligned} \tag{4.4}$$

We conclude that $\dim V_{\chi,\Gamma,\tau} \le \sum_{\pi\in[\chi]} m(\pi,\Gamma)\ dim(\mathrm{Hom}_K(\tau^\vee,\pi))$.

Conversely, for every t and for every p, choose a basis $\{\phi_{t,p,i}\}_{i=1}^n$ of H_p such that

$$\phi_{t,p,i}(xk) = \sum_{j=1}^{n} a_{ji}(k^{-1})\ \phi_{t,p,j}(x).$$

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Let $\phi_{t,p} = \sum_{i=1}^{n} \phi_{t,p,i} \quad v_i$. Then $\phi_{t,p}(xk) = \tau(k^{-1})(\phi_{t,p}(x))$. Since $X \cdot \phi_{t,p,i} = \chi(X)\ \phi_{t,p,i}$ for all $X \in \mathfrak{Z}(\mathfrak{g})$, it follows that $\phi_{t,p} \in V_{\chi,\Gamma,\tau}$.

We conclude that $\sum_{\pi \in [\chi]} m(\pi,\Gamma) \dim(\mathrm{Hom}_K(\tau^\vee,\pi)) \le \dim V_{\chi,\Gamma,\tau}$. Therefore, we have the equality

$$\dim V_{\chi,\Gamma,\tau} = \sum_{\pi \in [\chi]} m(\pi,\Gamma) \dim(\mathrm{Hom}_K(\tau^\vee,\pi)) \text{ for all } \tau \in \widehat{K}. \tag{4.5}$$

Now, let us consider the general case that τ is a finite dimensional representation of K (possibly reducible). Let $\tau \cong \bigoplus_{i=1}^{q} m_i\ \tau_i$ be a decomposition of τ into irreducible representations of K. Hence $\tau^\vee \cong \bigoplus_{i=1}^{q} m_i\ (\tau_i)^\vee$.

Let $V_{\Gamma,\tau}$ be as in Section 2.3. The center $\mathfrak{Z}(\mathfrak{g})$ acts on this space. Therefore for any character $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$, we can consider the χ-eigenspace $V_{\chi,\Gamma,\tau} \subset V_{\Gamma,\tau}$. We observe that

$$\dim V_{\chi,\Gamma,\tau} = \sum_{i=1}^{q} m_i \dim V_{\chi,\Gamma,\tau_i}.$$

We can use the argument culminating into Eq. 4.5 for each τ_i and the additivity properties of Hom spaces with respect to decomposition of τ to complete the proof of Theorem 1.2.2.

5 Discrete series representations and cohomology

In this section, we comment about the case when G is non-compact connected semisimple group that admits a discrete series representation. (This is same as saying rank G = rank T, where T is a "*compact*" Cartan subgroup.) It is well-known that for a given $\chi \in \widehat{\mathfrak{Z}(\mathfrak{g})}$, and a K-type τ there exists at most one (up to infinitesimal equivalence) discrete series representation $\pi_{\lambda+\rho}$ with the infinitesimal character χ and minimal K-type τ where $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ such that $\lambda + \rho$ is regular and integral linear form. Also, $\mathrm{Hom}_K(\tau, \pi_{\lambda+\rho}) = 1$.

Therefore Theorem. 1.2.2 implies the dimension of $V_{\chi,\Gamma,\tau^\vee}$ is equal to the multiplicity of the above discrete series $\pi_{\lambda+\rho}$ in $L^2(\Gamma\backslash G)$.

For $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$, there exists a G-equivariant holomorphic line bundle $\mathcal{L}_\lambda$ on G/T. Under a mild condition on λ (see [16, Equation (7.66)]), [16, Theorem 7.65] implies the L^2-cohomology $H^*_2(\Gamma\backslash\mathcal{L}_\lambda)$ is non-vanishing in exactly one degree q_λ (depends on λ) and

$$\dim H^{q_\lambda}_2(\Gamma\backslash\mathcal{L}_\lambda) = m(\pi_{\lambda+\rho},\Gamma).$$

Hence, the dimension of the χ-eigenspace of the automorphic forms of type $\tau^\vee$ is equal to the q_λ-th L^2-cohomology of the automorphic line bundle $\Gamma\backslash\mathcal{L}_\lambda$.

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