Coincidence of the barycentre and the geometric centre of weighted points

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1. *Introduction*

Recently, Gerhard J. Woeginger [1] gave a survey on the interesting history of results on equiangular n-vertex polygons with edge lengths in arithmetic progression. Such a polygon exists if, and only if, n has at least two distinct prime factors.

An equivalent formulation of the problem is as follows: Suppose you have n objects whose weights $m_1, \ldots, m_n$ are in arithmetic progression. For which values of n is it possible to place these objects evenly spaced around the circumference of a disc so that the disc will exactly balance on the centre point?

In algebraic terms, is there a permutation τ of the integers $1, \ldots, n$ such that

$$\frac{1}{M}\sum_{k=1}^{n} m_{\tau(k)} e^{2\pi ik/n} = 0$$

is valid? Such a permutation exists if, and only if, n has at least two distinct prime factors. For references see the list contained in [1].

Though Woeginger closed his article with the statement "And that's the end of this story" we want to continue it by the following generalisation.

Problem 1: Let $n \in \mathbb{N}$. Consider a real linear space V and select $x_1, \ldots, x_n \in V$. Let $m_k \in \mathbb{R}$ be a point mass at the point x_k ($k = 1, \ldots, n$). Suppose that the total mass $M = \sum_{k=1}^{n} m_k$ is positive. For which positive integers is there a permutation τ on $\{1, \ldots, n\}$ such that the barycentre coincides with the geometric centre point? In algebraic terms, this means that

$$\frac{1}{M}\sum_{k=1}^{n} m_{\tau(k)} x_k = \frac{1}{n}\sum_{k=1}^{n} x_k$$

is valid.

The problem presented in [1] is the solution in the case $V = \mathbb{C}$ regarded as $\mathbb{R}^2$,

$$x_k = e^{2\pi ik/n} \qquad (k = 1, \ldots, n),$$

and m_k is an arithmetic sequence of real numbers.

In this Article, we consider the case of equispaced points $x_1, \ldots, x_n$ on the real line $V = \mathbb{R}$, and derive a complete solution of the problem if the masses m_k are an arithmetic sequence of real numbers.

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2. *Point masses on the line*

We consider $V = \mathbb{R}$, equispaced points

$$x_k = k \qquad (k = 1, \dots, n)$$

and a non-constant arithmetic sequence of numbers m_k. Without loss of generality we can assume that $m_k = k$ $(k = 1, \dots, n)$. Then the geometric centre point is $\frac{1}{2}(n + 1)$ and the total mass is $M = \frac{1}{2}n(n + 1)$. Hence, we study the following problem which is a special case of Problem 1:

Problem 2: Is there a permutation τ on $\{1, \dots, n\}$ such that

$$\sum_{k=1}^{n} k \cdot \tau(k) = \frac{n(n + 1)^2}{4}. \tag{1}$$

Obviously, this is valid for $n = 1$, but there is no such permutation for $n \in \{2, 3\}$. For $n = 4$, we have exactly 2 solutions:

Position:	1	2	3	4
Mass:	2	4	1	3
	3	1	4	2

For $n = 5$, we list the exactly 6 permutations satisfying (1):

Position:	1	2	3	4	5	written as a product of cycles
Mass:	1	5	4	3	2	(1) (2, 5) (3, 4)
	2	3	4	5	1	(1, 2, 3, 4, 5)
	2	5	3	1	4	(1, 2, 5, 4)(3)
	4	1	3	5	2	(1, 4, 5, 2)(3)
	4	3	2	1	5	(1, 4)(2, 3)(5)
	5	1	2	3	4	(1, 5, 4, 3, 2)

Furthermore, we list the number of solutions of Problem 2 and their percentage of the total number of possible permutations, for small values of n:

n:	1	2	3	4	5	6	7	8	9	10
Number of solutions:	1	0	0	2	6	0	184	936	6688	0
Number of permutations:	1	2	6	24	120	720	5040	40320	362880	3628800
Percentage:	100	0	0	8.33	5	0	3.65	2.32	1.843	0

These calculations were achieved by brute force using the computer algebra system *Mathematica*.

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For each of the following values of n we list one arbitrary solution:

n	permutation written as a product of cycles
11	(1, 9, 6, 3, 7, 11, 8, 5) (2) (4, 10)
12	(1, 2, 7, 11, 8, 4, 9, 5) (3, 12) (6) (10)
13	(1, 2, 12, 7, 4, 3, 11) (5) (6, 13, 10, 9, 8)
14	no solution
15	(1, 8, 9, 13, 11, 14, 3, 2, 10, 4, 15, 5, 6, 7) (12)
16	(1, 9, 12, 10, 11, 4, 3, 7, 8, 5) (2, 16) (6, 13, 15, 14)
17	(1, 11) (2, 16, 17, 12) (3, 8, 4, 7, 14, 6, 5, 13, 9, 10) (15)
18	no solution

For $n \geqslant 2$, the number of solutions is even, because with τ also the permutation $k \rightarrow \tau(n + 1 - k)$ is a solution of (1). The following theorem gives a complete answer to the Problem 2.

Theorem 3: Let S be the set of all positive integers for which Problem 2 has a solution. Then

$$S = \{n \in \mathbb{N} : n \neq 4k + 2 \text{ for all integers } k \geqslant 0\} \setminus \{3\}.$$

Proof: For $n = 3$ one can easily check that there is no solution. When $n \equiv 2 \bmod 4$, we have $n(n + 1)^2 \equiv 2 \bmod 4$. Hence, the right-hand side of (1) is not an integer. The proof for all other values of n is divided into several cases.

Case 1: n is an integral multiple of 4, i.e. $n = 4r$ with $r \in \mathbb{N}$,

We provide a proof in a constructive way by describing a permutation which solves (1). First, we place the masses of odd values in ascending order and, subsequently, the masses of even value in descending order:

Position:	1	2	3	4	…	$2r$	$2r + 1$	…	$4r - 2$	$4r - 1$	$4r$
Mass:	1	3	5	7	…	$4r - 1$	$4r$	…	6	4	2

The associated permutation is given by

$$\tau_1(k) = \begin{cases} 2k - 1 & (1 \leqslant k \leqslant 2r), \\ 2(4r + 1 - k) & (2r + 1 \leqslant k \leqslant 4r), \end{cases}$$

and we obtain

$$\sum_{k=1}^{n} k \cdot \tau_1(k) = \sum_{k=1}^{2r} k \cdot (2k - 1) + \sum_{k=2r+1}^{4r} (4r + 1 - k) \cdot 2k$$

$$= \sum_{k=1}^{2r} \left[k \cdot (2k - 1) + (2r + 1 - k) \cdot 2(k + 2r)\right]$$

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$$= 8r^2(2r + 1) + \sum_{k=1}^{2r} k = r(2r + 1)(8r + 1)$$

$$= \frac{1}{8}n(n + 2)(2n + 1) = \frac{n(n + 1)^2}{4} + \sigma,$$

where $\sigma = \frac{1}{8}n^2 = 2r^2$. Swapping, for $1 \leqslant k \leqslant 2r$, both the masses $m_{4r-2k+1}$ and $m_{4r-2k+2}$ at the positions $2r + 1 - k$ and $2r + k$, respectively, diminishes the sum $\sum_{k=1}^{n} k \cdot \tau_1(k)$ by a total amount of $2k - 1$. Note that

$$\sum_{k=1}^{2r} (2k - 1) = 4r^2 > \sigma.$$

Subcase 1(a): r is even, $n = 8, 16, 24, \ldots$.

If $r = 2\rho$, say, then

$$\sum_{k=1}^{\rho} (2k - 1) + \sum_{k=3\rho+1}^{4\rho} (2k - 1) = 8\rho^2 = 2r^2 = \sigma.$$

Doing the relevant interchanges leads to the desired permutation τ.

For the sake of a better understanding we illustrate the general construction by a concrete example:

Take $n = 8, \rho = 1$.

Position:	1	2	3	4	5	6	7	8
Mass:	2	3	5	8	7	6	4	1

The permutation given as a product of cycles is $(1, 2, 3, 5, 7, 4, 8)(6)$ and $1 \cdot 2 + 2 \cdot 3 + \ldots + 8 \cdot 1 = 162 = \frac{1}{4}(8 \cdot 9^2)$.

Subcase 1(b): r is odd, $n = 12, 20, 28, \ldots$.

The case $r = 1$, i.e., $n = 4r = 4 \in S$ is already shown above. Suppose $r \geqslant 3$ is odd, $r = 2\rho + 1$, say, then swapping both the masses m_{2r+1} and m_{2r} at the positions $r + 1$ and $3r + 1$, respectively, increases the sum $\sum_{k=1}^{n} k \cdot \tau_1(k)$ by a total amount of $2r$. Furthermore,

$$\sum_{k=1}^{\rho+1} (2k - 1) + \sum_{k=3\rho+2}^{4\rho+2} (2k - 1) - 2r = 2(2\rho + 1)^2 = 2r^2 = \sigma.$$

Doing the relevant interchanges leads to the desired permutation τ. Note that in the range from position 1 up to $2r$, the second sum and the first sum involve the positions from 1 up to $\rho + 1$ and from $3\rho + 1$ up to $4\rho + 2$ while the last term $-2r$ involves the position $2\rho + 2$. The analoguous situation is valid in the range from position $2r + 1$ up to $4r$.

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As an example take $n = 12, r = 3, \rho = 1$.

Position:	1	2	3	4	5	6	7	8	9	10	11	12
Mass:	2	4	5	6	10	12	11	9	8	7	3	1

The permutation given as a product of cycles is (1, 2, 4, 6, 12) (3, 5, 10, 7, 11) (8, 9) and $1 \cdot 2 + 2 \cdot 4 + \dots + 12 \cdot 1 = 507 = \frac{1}{4}(12 \cdot 13^2)$.

Case 2: $n \geqslant 5$ is an odd integer.

First, we place the mass 1 at the last position n. Then we distribute the masses 2, 3, 4, 5, 6, 7, 8, 9, … pairwise on the free positions alternating furthest to the left and furthest to the right:

Position:	1	2	3	4	…	$n-4$	$n-3$	$n-2$	$n-1$	n
Mass:	2	3	6	7	…	8	9	4	5	1

The associated permutation is given by $\tau_2(k) = 2k$, $\tau_2(k+1) = 2k+1$, for 'small' odd values of k and $\tau_2(n+1-k) = 2k+1$, $\tau_2(n-k) = 2k$, for "small" even values of $k \geqslant 1$, and finally $\tau_2(n) = 1$. In order to study the situation in more detail we distinguish two subcases.

Subcase 2(a): $n \equiv 1 \bmod 4$, $n = 5, 9, 13, 17, \dots$.

Put $n = 4r + 1$ with $r \in \mathbb{N}$. Then, the permutation τ_2 can be defined by

$$\begin{cases} \tau_2(2i-1) = 2(2i-1) & (1 \leqslant i \leqslant r), \\ \tau_2(2i) = 2(2i-1)+1 & (1 \leqslant i \leqslant r), \\ \tau_2(2r+2i-1) = 4(r-i+1) & (1 \leqslant i \leqslant r), \\ \tau_2(2r+2i) = 4(r-i+1)+1 & (1 \leqslant i \leqslant r), \\ \tau_2(4r+1) = 1 \end{cases}$$

and we obtain

$$\sum_{k=1}^{n} k \cdot \tau_2(k)$$

$$= \sum_{i=1}^{r}(2i-1)(4i-2) + \sum_{i=1}^{r}(2i)(4i-1) + \sum_{i=1}^{r}(2r+2i-1) \cdot 4(r-i+1)$$

$$+ \sum_{i=1}^{r}(2r+2i)\big(4(r-i+1)+1\big) + (4r+1) \cdot 1$$

$$= \sum_{i=1}^{r}\big(16r^2 + 14r + 12i - 2\big) + 4r + 1$$

$$= 16r^3 + 20r^2 + 8r + 1 = (4r+1)(2r+1)^2 = \frac{n(n+1)^2}{4},$$

which is condition (1).

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Example: $n = 9, r = 2$.

Position:	1	2	3	4	5	6	7	8	9
Mass:	2	3	6	7	8	9	4	5	1

The permutation given as a product of cycles is (1, 2, 3, 6, 9) (4, 7) (5, 8) and $1 \cdot 2 + 2 \cdot 3 + \ldots + 9 \cdot 1 = 225 = \frac{1}{4}\left(9 \cdot 10^2\right)$.

Subcase 2(b): $n \equiv 3 \bmod 4$, $n = 7, 11, 15, 19, \ldots$. Put $n = 4r + 3$ with $r \in \mathbb{N}$. Then, the permutation τ_2 can be defined by

$$\begin{cases} \tau_2(2i - 1) = 2(2i - 1) & (1 \leqslant i \leqslant r + 1), \\ \tau_2(2i) = 2(2i - 1) + 1 & (1 \leqslant i \leqslant r + 1), \\ \tau_2(2r + 2i + 1) = 4(r - i + 1) & (1 \leqslant i \leqslant r), \\ \tau_2(2r + 2i + 2) = 4(r - i + 1) + 1 & (1 \leqslant i \leqslant r), \\ \tau_2(4r + 3) = 1 \end{cases}$$

and we obtain

$$\begin{aligned} &\sum_{k=1}^{n} k \cdot \tau_2(k) \\ &= \sum_{i=1}^{r+1}(2i - 1)(4i - 2) + \sum_{i=1}^{r+1}(2i)(4i - 1) + \sum_{i=1}^{r}(2r + 2i + 1) \cdot 4(r - i + 1) \\ &\qquad + \sum_{i=1}^{r}(2r + 2i + 2)\left(4(r - i + 1) + 1\right) + (4r + 3) \cdot 1 \\ &= (2r + 1)(4r + 2) + (2r + 2)(4r + 3) \\ &\qquad + \sum_{i=1}^{r}\left(16r^2 + 30r + 16 - 4i\right) + (4r + 3) \cdot 1 \\ &= 16r^2 + 26r + 11 + \sum_{i=1}^{r}\left(16r^2 + 30r + 16 - 4i\right) \\ &= 16r^3 + 44r^2 + 40r + 11 \\ &= (4r + 3)(2r + 2)^2 - 1 = \frac{n(n + 1)^2}{4} - 1. \end{aligned}$$

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Now we make the following manipulations:

Position:	1	2	3	4	...	$n-4$	$n-3$	$n-2$	$n-1$	n
Mass (choice 1):	2	3	6	7	...	8	9	4	5	1
Mass (choice 2):	**3**	**2**	6	7	...	8	9	**5**	**1**	**4**

Swapping the masses 2 and 3 at the first two positions diminishes the sum $\sum_{k=1}^{n} k \cdot \tau_2(k)$ by an amount of 1. Interchanging the order of the last three masses 4, 5, 1 to 5, 1, 4 increases the sum by an amount of 2. Altogether the sum $\sum_{k=1}^{n} k \cdot \tau_2(k)$ increases by a total amount of 1. Thus, we have found a suitable permutation satisfying (1).

Example: $n = 11, r = 2$.

Position:	1	2	3	4	5	6	7	8	9	10	11
Mass (choice 1):	2	3	6	7	10	11	8	9	4	5	1
Mass (choice 2):	**3**	**2**	6	7	10	11	8	9	**5**	**1**	**4**

In the first choice we have $1 \cdot 2 + 2 \cdot 3 + \dots + 11 \cdot 1 = 395 = \frac{1}{4}\left(11 \cdot 12^2\right) - 1$. Doing the changes (choice 2) increases the sum $\sum_{k=1}^{n} k \cdot \tau_2(k)$ by an amount of 1.

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