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# Wave interaction with two-dimensional bodies floating in a two-layer fluid: uniqueness and trapped modes

By N. KUZNETSOV[1], M. McIVER[2] AND P. McIVER[2]

[1]Institute of Problems in Mechanical Engineering, Russian Academy of Sciences, V.O., Bol'shoy pr. 61, St Petersburg 199178, Russia

[2]Department of Mathematical Sciences, Loughborough University, Loughborough, Leics LE11 3TU, UK

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Existing results on the linearized water-wave problem for a homogeneous fluid are extended to the case of a two-layer fluid. In particular, the appropriate form of Maz'ya's identity is presented and used to obtain results on the uniqueness of the solutions to forcing problems for a structure in a two-layer fluid. Further, examples of geometries are constructed for which trapped modes occur; such modes are finite-energy solutions of the unforced problem and provide examples of non-uniqueness.

## 1. Introduction

This paper is concerned with the linearized problems of radiation and scattering of waves by bodies floating in and/or beneath the free surface of a fluid. It is important to know the conditions under which the solutions of these forcing problems are unique. The non-uniqueness of the solution to a forcing problem at a particular frequency is associated with the existence of a trapped mode at that frequency. Trapped modes are non-trivial solutions of the homogeneous boundary-value problem and represent fluid oscillations that are essentially confined to the vicinity of a structure. Their existence means that large-amplitude motions of the fluid and structure are possible when the system is forced at a frequency close to that of the trapped mode. For a homogeneous fluid, the questions of uniqueness and the existence of trapped modes have received much attention over several decades (see chapters 1–5 of the monograph by Kuznetsov, Maz'ya & Vainberg 2002), but much less is known about these questions for a two-layer fluid.

Retzler (2001) gives experimental evidence of trapped modes supported by a cylinder in a channel containing a homogeneous fluid; the frequencies of oscillation were found to be within 0.4% of the theoretical predictions of Callan, Linton & Evans (1991). To the authors' knowledge there have been no experimental studies of trapped modes supported by bodies in a two-layer fluid. However Teoh, Ivey & Imberger (1997) and Javam, Imberger & Armfield (2000) give experimental and numerical results for a stratified fluid that show how nonlinear interactions between waves produce trapped modes which lead to an overturning of the density field. Clearly the existence of any type of trapped mode could have important consequences for the wave forces on a structure such as, for example, an underwater pipe bridge in a fjord where fresh water overlies salt water.

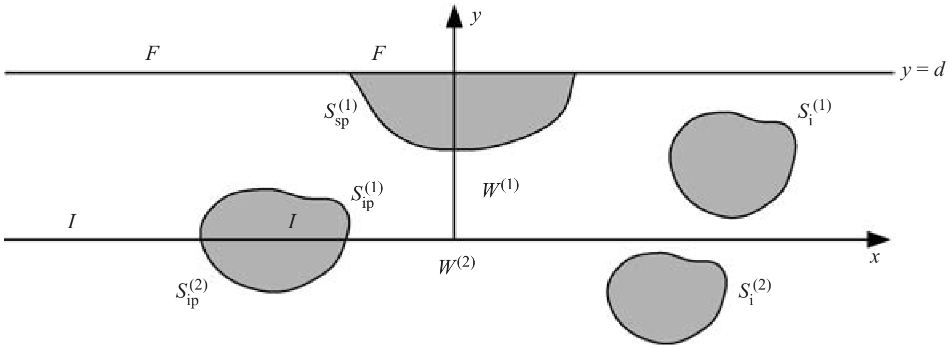


FIGURE 1. Definition sketch for bodies in a two-layer fluid.

The aim of the present note is to demonstrate that in many, but not all, respects the problem of bodies floating in a two-layer fluid is similar to that for a homogeneous fluid. For the sake of simplicity we restrict ourselves to the two-dimensional problem. The so-called 'Maz'ya's integral identity' for a two-layer fluid is presented in § 3 and used to obtain results on uniqueness in § 4. These results are in the form of conditions on the geometries of structures that guarantee uniqueness of the solution to a forcing problem. A general proof is not possible as examples of particular structures that support trapped modes are presented in § 5.

## 2. Statement of the problem and the energy lemma

A sketch of the geometry is shown in figure 1 in which $W^{(1)}$ and $W^{(2)}$ are domains occupied by fluids having respectively densities $\rho_1$ and $\rho_2$, with $\rho_2 > \rho_1 > 0$. The superscripts (1) and (2) indicate body contours confined to $L = \{-\infty < x < \infty,\ 0 < y < d\}$ and $\mathbb{R}^2_- = \{-\infty < x < \infty,\ y < 0\}$, respectively, and it is assumed that the lower fluid is unbounded from below. The subscript i indicates immersed contours within either $L$ or $\mathbb{R}^2_-$, whereas sp (ip) denotes surface-piercing (interface-piercing) contours. Smooth curves $S_{sp}^{(1)}$ and $S_{ip}^{(j)}$ $(j = 1, 2)$ are assumed not to be tangent to $\{y = d\}$ and $\{y = 0\}$ respectively; some of the curves $S_{sp}^{(1)}$ may pass through the interface. The parts of the free surface of the upper fluid and the interface outside all bodies are denoted by $F$ and $I$ respectively. Both fluids are assumed to be inviscid and incompressible and their motion to be irrotational so that it may be described by velocity potentials $\phi^{(1)}$ in $\overline{W^{(1)}}$ and $\phi^{(2)}$ in $\overline{W^{(2)}}$, respectively. The corresponding coupled boundary-value problem is as follows (a time dependence $e^{-i\omega t}$ having been extracted):

$$\nabla^2\phi^{(j)} = 0 \quad \text{in} \quad W^{(j)}, \quad \partial_n\phi^{(j)} = 0 \quad \text{on} \quad S^{(j)}, \quad j = 1, 2, \tag{2.1}$$

$$\partial_y\phi^{(1)} - \nu\phi^{(1)} = 0 \quad \text{on} \quad F, \tag{2.2}$$

$$\rho\left(\partial_y\phi^{(1)} - \nu\phi^{(1)}\right) = \partial_y\phi^{(2)} - \nu\phi^{(2)} \quad \text{and} \quad \partial_y\phi^{(1)} = \partial_y\phi^{(2)} \quad \text{on} \quad I. \tag{2.3}$$

Here $\partial_n$ indicates the normal derivative on $S^{(1)} = S_{sp}^{(1)} \cup S_i^{(1)} \cup S_{ip}^{(1)}$ and $S^{(2)} = S_{ip}^{(1)} \cup S_i^{(2)}$, $\nu = \omega^2/g > 0$, where $g$ is the acceleration due to gravity, and $\rho = \rho^{(1)}/\rho^{(2)}$ is the non-dimensional measure of stratification. The kinematic and dynamic boundary conditions have been combined on each of the free surface and interface and then linearized about the mean position of each surface. The homogeneous Neumann condition indicates an absence of any forcing and is used in investigations of uniqueness and trapped modes.

Usually problem (2.1)–(2.3) is supplemented by radiation conditions (formulae (2.8) and (2.9) in Linton & McIver 1995). However, these formulae are not given explicitly as the following assertion holds in the present case.

THE ENERGY LEMMA: *Let* $\phi^{(1)}$ *and* $\phi^{(2)}$ *satisfy* (2.1)–(2.3), *radiation conditions, and the conditions that the Dirichlet integrals over certain neighbourhoods of the points of intersection of* $S_{sp}^{(1)}$ *and* $S_{ip}^{(j)}$ ($j=1,2$) *with respectively* $F$ *and* $I$ *are finite. Then*

$$\sum_{j=1}^{2} \int_{W^{(j)}} \left|\nabla\phi^{(j)}\right|^2 \mathrm{d}x\,\mathrm{d}y + \nu \int_F \left|\phi^{(1)}\right|^2 \mathrm{d}x + \nu^{-1} \int_I \left|\partial_y \phi^{(2)}\right|^2 \mathrm{d}x < \infty. \tag{2.4}$$

This lemma means that trapped modes (if they exist) have both finite kinetic and finite potential energy. To the authors' knowledge, this assertion has not previously been formulated explicitly. Its proof is based on the asymptotic formulae for the Green's functions (see Linton & McIver 1995) and on the Green's representation formulae involving certain cut-off functions (see Kuznetsov *et al.* 2002, § 2.1).

We assume $\phi^{(1)}$ and $\phi^{(2)}$ to be real functions which is admissible in view of (2.4).

## 3. Maz'ya's identity

In Kuznetsov *et al.* (2002, chaps. 2 and 3), it is demonstrated that Maz'ya's integral identity is a powerful tool for finding configurations of bodies immersed either totally or partially in a homogeneous fluid such that the uniqueness theorem is true for those geometries. In this section we generalize that identity for solutions of problem (2.1)–(2.3) satisfying (2.4). The starting point is Maz'ya's differential identity

$$2[(\boldsymbol{V}\cdot\nabla u + Hu)\nabla^2 u] = 2\nabla\cdot[(\boldsymbol{V}\cdot\nabla u + Hu)\nabla u] + (\boldsymbol{Q}\,\nabla u)\cdot\nabla u + u^2\nabla^2 H - \nabla\cdot[|\nabla u|^2 \boldsymbol{V} + u^2\nabla H], \tag{3.1}$$

which can easily be verified by direct calculation. Here $u$ is an arbitrary twice-differentiable function, $\boldsymbol{V} = (V_x, V_y)$ is a real vector field with components that are uniformly Lipschitz in a certain fluid domain, $H$ is a real function with uniformly Lipschitz first derivatives in the same domain, and

$$\boldsymbol{Q} = \begin{bmatrix} -\partial_x V_x + \partial_y V_y - 2H & -(\partial_y V_x + \partial_x V_y) \\ -(\partial_y V_x + \partial_x V_y) & \partial_x V_x - \partial_y V_y - 2H \end{bmatrix}.$$

Lemma (2.4) means that provided $\boldsymbol{V}^{(j)}$ and $H^{(j)}$ ($j = 1, 2$) grow as described above as $x^2 + y^2 \to \infty$, (3.1) with $u$ replaced by $\phi^{(j)}$ may be integrated over $W^{(j)}$. The resulting equality for $j = 1$ is multiplied by $\rho$ and added to that for $j = 2$, and the conditions

$$V_x^{(1)} = V_x^{(2)}, \quad V_y^{(1)} = V_y^{(2)} = 0, \quad H^{(1)} = H^{(2)}, \quad \partial_y H^{(1)} = \partial_y H^{(2)} = 0, \tag{3.2}$$

are imposed on $I$. After some algebra one arrives at *Maz'ya's integral identity for a* *two-layer fluid*, namely

$$\rho \left\{ \int_F \left[ V_y^{(1)} \nu^2 + \left(2H^{(1)} - \partial_x V_x^{(1)}\right)\nu - \partial_y H^{(1)} \right] \left|\phi^{(1)}\right|^2 \mathrm{d}x \right.$$

$$\left. - \int_F V_y^{(1)} \left|\partial_x \phi^{(1)}\right|^2 \mathrm{d}x - \nu \sum_{k=1}^{M^{(1)}} \left[ \left|\phi^{(1)}(x,d)\right|^2 V_x^{(1)}(x,d) \right]_{x=a_k^{(1)}}^{x=b_k^{(1)}} \right\}$$

$$+ \frac{1-\rho}{\nu} \left\{ \int_I \left(2H^{(2)} - \partial_x V_x^{(2)}\right) \left|\partial_y \phi^{(2)}\right|^2 \mathrm{d}x - \sum_{k=1}^{M^{(2)}} \left[ \left|\partial_y \phi^{(2)}(x,0)\right|^2 V_x^{(2)}(x,0) \right]_{x=a_k^{(2)}}^{x=b_k^{(2)}} \right\}$$

$$+ \sum_{j=1}^{2} \rho^{2-j} \left\{ \int_{W^{(j)}} \left[ \left(\boldsymbol{Q}^{(j)} \nabla \phi^{(j)}\right) \cdot \nabla \phi^{(j)} + \left|\phi^{(j)}\right|^2 \nabla^2 H^{(j)} \right] \mathrm{d}x\, \mathrm{d}y \right.$$

$$\left. + \int_{S^{(j)}} \left( \left|\nabla \phi^{(j)}\right|^2 \boldsymbol{V}^{(j)} \cdot \boldsymbol{n} + \left|\phi^{(j)}\right|^2 \partial_n H^{(j)} \right) \mathrm{d}S \right\} = 0. \tag{3.3}$$

Here $\boldsymbol{n}$ is the unit normal to $S^{(1)} \cup S^{(2)}$ directed into the fluid and $M^{(1)}$ ($M^{(2)}$) is the number of surface-piercing (interface-piercing) bodies. The left and right endpoints of the contour of the $k$th surface-piercing body are denoted, respectively, by $(a_k^{(1)}, d)$ and $(b_k^{(1)}, d)$ and of the $k$th interface-piercing body by $(a_k^{(2)}, 0)$ and $(b_k^{(2)}, 0)$.

Next we apply the Maz'ya integral identity and obtain several sets of conditions that guarantee that (2.1)–(2.3) have only a trivial solution.

## 4. Geometries providing uniqueness

To obtain geometric conditions that guarantee uniqueness, $H^{(j)}$ and $\boldsymbol{V}^{(j)}$ ($j = 1, 2$) must be chosen so that all left-hand-side terms of (3.3) are non-negative and at least one of them is strictly positive for non-trivial $\phi^{(1)}$ and $\phi^{(2)}$; this leads to a contradiction thus proving the uniqueness theorem. Throughout this section we take $H^{(1)} = H^{(2)} = -1/2$. Moreover, it is easy to show that *$\boldsymbol{Q}$ is non-negative definite when $H \leqslant 0$ and* $\det \boldsymbol{Q} \geqslant 0$.

Example 1: Let $\boldsymbol{V}^{(1)} = \boldsymbol{V}^{(2)} = (-x, 0)$, so that conditions (3.2) hold, and let $S^{(1)} \cup S^{(2)}$ be an arbitrary set of finite segments on the $y$-axis. Then all terms on the left-hand side of (3.3) vanish except for the area integrals. The latter are strictly positive for non-trivial $\phi^{(1)}$ and $\phi^{(2)}$ because

$$\boldsymbol{Q}^{(1)} = \boldsymbol{Q}^{(2)} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

for our choice of $H^{(j)}$ and $\boldsymbol{V}^{(j)}$ ($j = 1, 2$). Thus *the uniqueness theorem holds for a vertical barrier with gaps* (*intersecting or not the free surface and the interface*). The same is true when, apart from finite segments, $S^{(1)} \cup S^{(2)}$ includes a semi-infinite ray extending downwards from a point $(0, c)$, where $c < d$.

Example 2: Let

$$\boldsymbol{V}^{(1)} = \boldsymbol{V}^{(2)} = \begin{cases} \pm(b - |x|), & \pm x > b, \\ 0, & |x| < b, \end{cases}$$

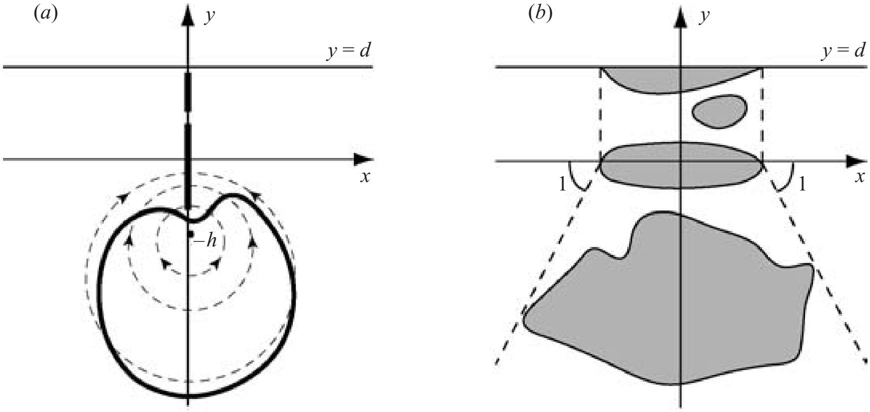


FIGURE 2. Bodies for which uniqueness is established in (*a*) example 3 and (*b*) example 4.

and again all of conditions (3.2) hold (example 1 is the degenerate case $b=0$). Further suppose that there is both a surface-piercing and an interface-piercing body with horizontal extremes at $|x|=b$ on $\{y=d\}$ and $\{y=0\}$, then the uniqueness theorem holds irrespective of the presence or absence of further fully immersed bodies within $|x| \leqslant b$.

This result extends the uniqueness theorem of John (1950) for the two-dimensional water-wave problem to the case of a two-layer fluid. John's result states that no trapped modes can be supported by a body which has the property that vertical lines drawn from every point on the free surface do not intersect the body. For the proof, considerations from example 1 must be applied when $|x| > b$. For $|x| < b$, it is sufficient to note that $\boldsymbol{Q}^{(1)}$ and $\boldsymbol{Q}^{(2)}$ are each the $2 \times 2$ identity matrix.

EXAMPLE 3: Let $\boldsymbol{V}^{(1)} = (-x, 0)$ and $\boldsymbol{V}^{(2)} = (x(y^2-x^2-h^2)/N, -2x^2y/N)$ where $N^2 = (y^2-x^2-h^2)^2 + 4x^2y^2$ and $h$ is a non-negative constant. Then conditions (3.2) are satisfied and a direct but tedious calculation gives $\det \boldsymbol{Q} = 4x^2h^2/N^2$ and so $\boldsymbol{Q}$ is a non-negative definite matrix as noted in § 3. Let $\overline{W^{(1)}}$ either be free of bodies or contain a vertical barrier as described in example 1. Then all terms on the left-hand side of (3.3) are non-negative except for the integral over $S^{(2)}$. The latter is also non-negative when

$$x(y^2-x^2-h^2)n_x - 2x^2yn_y \geqslant 0 \quad \text{on} \quad S^{(2)}, \tag{4.1}$$

where $n_x$ and $n_y$ are the components of $\boldsymbol{n}$ (see equation (3.3)), and so *this inequality is a sufficient condition for uniqueness*. Geometrically, inequality (4.1) means that the vector field makes angles not exceeding $\pi/2$ with the normals on $S^{(2)}$ directed into the fluid. The integral curves of the vector field are circles belonging to one of the coordinate lines of the bipolar system with poles at $(0, \pm h)$. A geometry satisfying (4.1) and hence guaranteeing uniqueness is shown in figure 2(*a*).

EXAMPLE 4: Let $\boldsymbol{V}^{(1)}$ be the same as in example 2 and

$$\boldsymbol{V}^{(2)} = \begin{cases} -[1-\pi-\theta_-](x+b,\, y), & -\pi \leqslant \theta_- \leqslant -\pi+1, \\ 0, & \text{in } W_0, \\ -[1+\theta_+](x-b,\, y), & -1 \leqslant \theta_+ \leqslant 0, \end{cases}$$

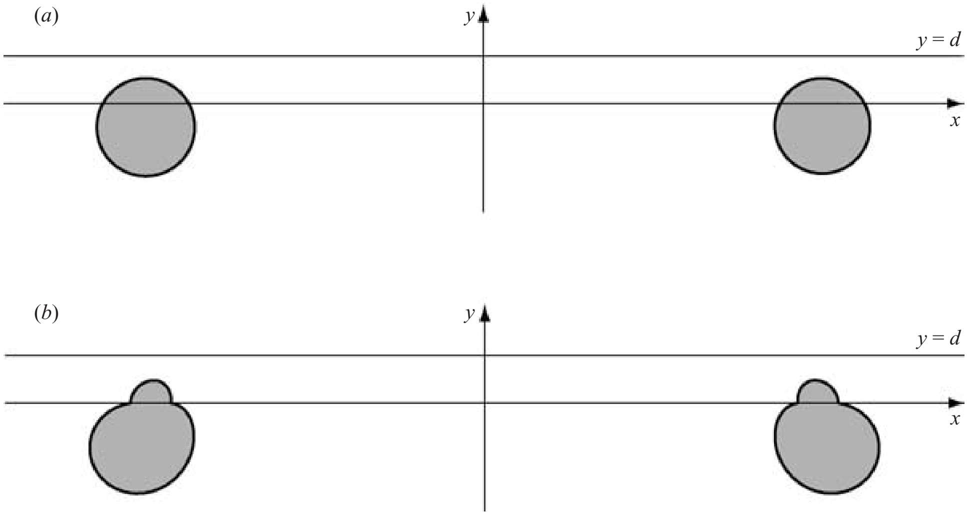


FIGURE 3. Geometries for which Maz'ya's identity has not been applied. (*a*) A pair of circles for which no suitable scalar and vector fields have not been found. (*b*) Trapping structures for which it is impossible to find suitable fields.

where $\theta_{\pm} = \arctan(y/(x \mp b)$ and $W_0$ is the subdomain of $\mathbb{R}^2_-$ between the lines $\theta_+ = 1$ and $\theta_- = 1 - \pi$. Again all required properties are fulfilled and it is simple to show that $\det \boldsymbol{Q}^{(2)} \geqslant 0$. This example generalizes example 2 by allowing bodies fully immersed in the lower fluid to be within the region $W_0$ that is wider than the strip $\{|x| \leqslant b\}$. It is also an extension of the example of Weck (1990) for a homogeneous fluid. Figure 2(*b*) illustrates a geometry for which uniqueness is guaranteed. However, unlike example 2 which extends to the three-dimensional case, example 4 has a straightforward generalization only to geometries confined within the dashed lines of figure 2(*b*) revolved about the $y$-axis.

The power of Maz'ya's identity lies in the identification of suitable scalar and vector fields, $H^{(j)}$ and $\boldsymbol{V}^{(j)}$, $(j = 1, 2)$ respectively. For a given system of bodies this is a non-trivial task and figure 3 illustrates two different configurations of bodies for which no such fields have been found. It is not known whether the velocity potential for the bodies in figure 3(*a*) is unique at all frequencies. However Maz'ya's identity may definitely not be applied to the configuration in figure 3(*b*) because, as will be demonstrated in the next section, this system of bodies supports a trapped mode.

## 5. Trapped modes

In this section examples of geometries that support trapped modes are constructed by the inverse method of McIver (1996) in which non-trivial solutions to the homogeneous problem are found from singular solutions of the governing equations. Individual singular solutions radiate waves to infinity, but two solutions may be combined in such a way as to cancel these waves. The streamline pattern reveals lines which isolate the singularities and hence some streamlines may represent structures that support trapped modes.

First we summarize some of the properties of the dispersion equation for waves in two-layer fluid, which is $(k - \nu)[\nu(\sigma + e^{-2kd}) - k(1 - e^{-2kd})] = 0$, where $k$ is the wavenumber and $\sigma = (1 + \rho)/(1 - \rho) > 1$; the dispersion equation has two positive roots, $k = \nu$ and $k = \nu_0$ (see Linton & McIver 1995) such that

$$\nu\sigma < \nu_0 < \nu(\sigma + 1)/(1 - \mathrm{e}^{-2\sigma\nu d}). \tag{5.1}$$

## 5.1. *Sources*

One of the features of sources in the presence of an interface is that it is not possible to construct an isolated source for which the potential does not grow logarithmically as the distance from the source point tends to infinity (see, for example, equations (27) and (28) of Gorgui & Kassem 1978). Thus, to obtain from source potentials a trapped-mode potential that is bounded at infinity it is necessary to combine at least two singularities. Here attention will be restricted to the case when there are singularities on either side of the interface so that solutions singular at $(x, y) = (\xi, \pm\eta)$ are sought and the limit $\eta \to 0$ taken. The derivation follows closely that used for other singularities by Linton & McIver (1995) and hence is omitted. The resulting singular solution is

$$G_0^{(j)}(x, y; \xi) = \begin{cases} \displaystyle ⨍_0^\infty [A(k)\,\mathrm{e}^{ky} + B(k)\,\mathrm{e}^{-ky}] \cos k(x - \xi)\,\mathrm{d}k, & j = 1, \\ \displaystyle ⨍_0^\infty C(k)\,\mathrm{e}^{ky} \cos k(x - \xi)\,\mathrm{d}k, & j = 2, \end{cases}$$

where $G_0^{(j)}$ is the potential in layer $j$, $⨍$ denotes a principal-value integral, and

$$A(k) = -\frac{(k + \nu)\,\mathrm{e}^{-2kd}}{(k - \nu)h(k)}, \quad B(k) = -\frac{1}{h(k)}, \quad C(k) = \frac{[k - \nu - (k + \nu)\,\mathrm{e}^{-2kd}]}{(k - \nu)h(k)},$$

with $h(k) = (k + \nu)\,\mathrm{e}^{-2kd} - k + \sigma\nu$.

It is readily shown (see Linton & McIver 1995) that for $j = 1, 2$ as $\nu|x| \to \infty$

$$G_0^{(j)}(x, y; \xi) \sim -\pi\Bigg[\mathrm{R}(C : \nu)\,\mathrm{e}^{\nu y} \sin|\nu(x - \xi)| \\ + \mathrm{R}(C : \nu_0)\,\mathrm{e}^{\nu_0 y}\left[\frac{\nu\sigma - \nu_0}{\nu(\sigma - 1)} + \frac{\nu - \nu_0}{\nu(\sigma - 1)}\,\mathrm{e}^{-2\nu_0 y}\right]^{2-j} \sin|\nu_0(x - \xi)|\Bigg], \tag{5.2}$$

where $\mathrm{R}(C : \mu)$ denotes the residue of $C(k)$ at $k = \mu$. To construct trapped-mode solutions the wave terms at infinity are annulled by combining two singularities as

$$\mathscr{U}_\pm^{(j)}(x, y; \xi) = G_0^{(j)}(x, y; -\xi) \pm G_0^{(j)}(x, y; \xi), \quad j = 1, 2. \tag{5.3}$$

It is a simple matter to verify that the waves at infinity are annulled in $\mathscr{U}_+^{(j)}$ by choosing

$$\nu_0\xi = (2m + 1)\pi/2 \quad \text{and} \quad \nu\xi = (2n + 1)\pi/2 \tag{5.4}$$

and in $\mathscr{U}_-^{(j)}$ by choosing

$$\nu_0\xi = m\pi \quad \text{and} \quad \nu\xi = n\pi, \tag{5.5}$$

where in each case $m$ and $n$ are integers. For a given $\sigma$, $\nu$ and $\nu_0$ are chosen in the form of either (5.4) or (5.5) such that $\nu_0/\nu > \sigma$, see (5.1); for given $\nu_0/\nu$, $\nu_0 d$ follows from the dispersion equation. As noted above, structures that support trapped modes are found by identifying suitable streamlines which are the level contours of the stream function $\mathscr{V}_\pm^{(j)}$ corresponding to $\mathscr{U}_\pm^{(j)}$, $j = 1, 2$.

A wide variety of trapping structures may be generated. One example of the streamline pattern that may be obtained by the above construction (with the sources

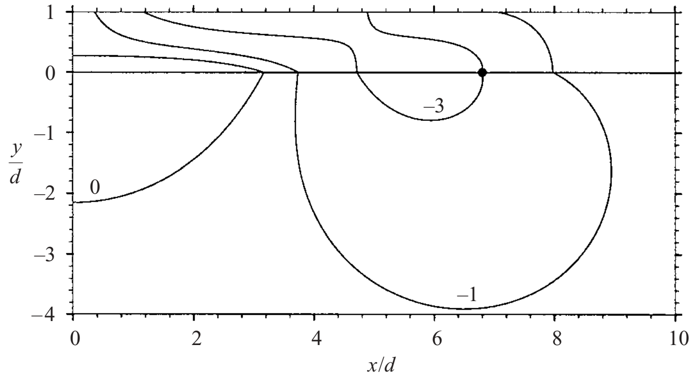


FIGURE 4. Streamline pattern for two sources on either side of an interface; $m = 1$, $n = 0$, $\sigma = 2$.

added) is shown in figure 4 where the position of the singularity is shown by a solid circle; the numbers on the streamlines denote the corresponding value of the stream function $\mathscr{V}_+$. The pattern is symmetric about $x = 0$ so that only the pattern for $x \geqslant 0$ is shown. The streamline $\mathscr{V}_+ = -1$ excludes the singularity from the flow field and is one example of the surface of a trapping structure. The part of this streamline in the lower fluid has a similar shape to those found in the corresponding construction for a homogeneous fluid by McIver (1996). Other closed streamlines, such as that corresponding to $\mathscr{V}_+ = 0$, may also be included in an arrangement of trapping structures. On the other hand the streamline $\mathscr{V}_+ = -3$ passes through the singularity and does not correspond to the surface of a trapping structure.

### 5.2. *Dipoles*

Unlike the source potential considered above, it is possible for a dipole singularity that is bounded at infinity to exist in isolation. Dipoles have been used for constructing examples of trapped modes in a homogeneous fluid by Motygin and Kuznetsov; see chapter 4 in Kuznetsov *et al.* (2002). For a two-layer fluid, dipoles have already been obtained by Linton & McIver (1995). For simplicity, attention is again restricted here to the limits in which the singular point approaches the interface between the two fluids. For a horizontal dipole on the upper side of the interface at $(x, y) = (\xi, 0_+)$, the potentials are

$$G_1^{(1,j)}(x, y; \xi) = \begin{cases} \dfrac{1}{\nu} \displaystyle\int_0^\infty \left(A^{(1)}(k)\,\mathrm{e}^{ky} + \left[1 + B^{(1)}(k)\right]\mathrm{e}^{-ky}\right) \sin k(x-\xi)\,\mathrm{d}k, & j = 1 \\ \dfrac{1}{\nu} \displaystyle\int_0^\infty C^{(1)}(k)\,\mathrm{e}^{ky} \sin k(x-\xi)\,\mathrm{d}k, & j = 2. \end{cases}$$

The first index in the $G$ superscript refers to the layer in which the singularity lies, and the second index to the layer in which the particular expression is valid. Here

$$A^{(1)}(k) = \frac{[(\sigma+1)\nu - 2k](k+\nu)\,\mathrm{e}^{-2kd}}{(k-\nu)h(k)}, \quad B^{(1)}(k) = -\frac{(k+\nu)\,\mathrm{e}^{-2kd} + k - \nu}{h(k)},$$

$$C^{(1)}(k) = -\frac{(\sigma-1)\nu B^{(1)}(k)}{k-\nu}.$$

The corresponding stream functions are denoted by $H_1^{(1,j)}(x,y;\xi)$, $j=1,2$, and as $r=[(x-\xi)^2+y^2]^{1/2}\to 0$,

$$H_1^{(1,j)}(x,y;\xi)\sim\begin{cases}-2\cos\theta/(\nu r)+(\sigma-1)\log r+O(1), & j=1,\\ (\sigma-1)\log r+O(1), & j=2.\end{cases}\tag{5.6}$$

Thus, the solution is dipole-like on only the upper side of the interface, although it is still singular on the lower side of the interface with a vortex singularity.

For a horizontal dipole on the lower side of the interface at $(x,y)=(\xi,0_-)$, the potentials in the upper and lower fluid layers are respectively

$$G_1^{(2,j)}(x,y;\xi)=\begin{cases}\dfrac{1}{\nu}\displaystyle⨍_0^\infty\left(A^{(2)}(k)\,\mathrm{e}^{ky}+B^{(2)}(k)\,\mathrm{e}^{-ky}\right)\sin k(x-\xi)\,\mathrm{d}k, & j=1,\\ \dfrac{1}{\nu}\displaystyle⨍_0^\infty\left[1+C^{(2)}(k)\right]\mathrm{e}^{ky}\sin k(x-\xi)\,\mathrm{d}k, & j=2,\end{cases}\tag{5.7}$$

where

$$A^{(2)}(k)=\frac{\nu(\sigma+1)(k+\nu)\,\mathrm{e}^{-2kd}}{(k-\nu)h(k)},\qquad B^{(2)}(k)=\frac{\nu(\sigma+1)}{h(k)},$$

and

$$C^{(2)}(k)=-\frac{[(k+\sigma\nu)\,\mathrm{e}^{-2kd}-k+\nu](k+\nu)}{(k-\nu)h(k)}.$$

The corresponding stream functions are denoted by $H_1^{(2,j)}(x,y;\xi)$, $j=1,2$, and as $r\to 0$,

$$H_1^{(2,j)}(x,y;\xi)\sim\begin{cases}-(\sigma+1)\log r+O(1), & j=1,\\ -2\cos\theta/(\nu r)-(\sigma+1)\log r+O(1), & j=2,\end{cases}\tag{5.8}$$

Now, the solution is dipole-like on only the lower side of the interface while, to leading order, there is a vortex singularity on the upper side of the interface.

As $\nu|x|\to\infty$

$$G_1^{(l,j)}(x,y;\xi)\sim \pi\,\mathrm{sgn}\,x\Bigg[\mathrm{R}\left(C^{(l)}:\nu\right)\mathrm{e}^{\nu y}\cos\nu(x-\xi)$$
$$+\mathrm{R}\left(C^{(l)}:\nu_0\right)\mathrm{e}^{\nu_0 y}\left[\frac{\nu\sigma-\nu_0}{\nu(\sigma-1)}+\frac{\nu-\nu_0}{\nu(\sigma-1)}\,\mathrm{e}^{-2\nu_0 y}\right]^{2-j}\cos\nu_0(x-\xi)\Bigg].$$

Trapped-mode solutions can be found from potentials

$$\mathscr{U}_\pm^{(j)}(x,y;\xi)=\alpha\left[G_1^{(1,j)}(x,y;-\xi)\pm G_1^{(1,j)}(x,y;\xi)\right]$$
$$+\beta\left[G_1^{(2,j)}(x,y;-\xi)\pm G_1^{(2,j)}(x,y;\xi)\right],\quad j=1,2,$$

where $\alpha$ and $\beta$ are constants and, to annul the waves at infinity, $\xi$ is chosen in exactly the same way as described after equation (5.3). The corresponding stream functions $\mathscr{V}_\pm^{(j)}$, $j=1,2$, are easily found. A local analysis based on the asymptotic forms in (5.6) and (5.8) indicates that streamlines enclosing the singularity are possible only if $\alpha$ and $\beta$ do not have the same sign and this has been confirmed in numerical calculations.

An example of a streamline pattern that may be obtained by the above dipole construction (with the dipoles added in each pair) is shown in figure 5. The general

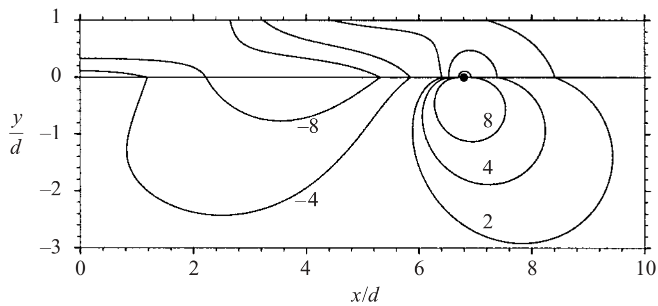


FIGURE 5. Streamline pattern for a dipole; $\alpha = 0$, $\beta = 1$, $m = 1$, $n = 0$, $\sigma = 2$.

comments about the source construction made above apply here also. It can be seen in figure 5 that there are streamlines that completely enclose the singularity and correspond to submerged bodies that straddle the interface. Trapped modes for submerged bodies in a homogeneous fluid have been constructed by McIver (2000) using dipole potentials.

## 6. Conclusion

In this paper the question of uniqueness of solution of the radiation and scattering problems in two-layer fluids has been studied and examples of bodies which support trapped modes have been constructed. By virtue of a new form of Maz'ya's identity, uniqueness of solution was established for several classes of obstacles floating in a two-layer fluid. These classes correspond to those for which uniqueness was demonstrated for a homogeneous fluid and include totally immersed obstacles as well as obstacles intersecting the free surface and the interface. In the latter case the classical result of John (1950) for a homogeneous fluid was extended to the configurations of bodies in a two-layer fluid illustrated in example 2.

However a general proof of uniqueness is not possible as it has been shown that trapped modes do indeed exist for certain configurations of bodies in two-layer fluids. Such bodies were found using an extension of the inverse procedure of McIver (1996) where bodies are formed from the streamlines of flow fields associated with singular solutions of the boundary value problem. All of the bodies constructed here have discontinuous gradients at the points of intersection of the body and the interface between the fluids. This is in contrast to the situation for a homogeneous fluid, where bodies with smooth boundaries have been found which support trapped modes.

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