



Nearly Parallel G_2 -structures with Large Symmetry Group

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Abstract. We prove the existence of a one-parameter family of nearly parallel G_2 -structures on the manifold $S^3 \times \mathbb{R}^4$, which are mutually non-isomorphic and invariant under the cohomogeneity one action of the group $SU(2)^3$. This family connects the two locally homogeneous nearly parallel G_2 -structures that are induced by the homogeneous ones on the sphere S^7 .

1 Introduction

A nearly parallel G_2 -structure (NP-structure for brevity) on a 7-dimensional manifold M is given by a positive 3-form $\varphi \in \Omega^3(M)$ such that $d\varphi = \lambda *_{\varphi} \varphi$ for some (non-zero) $\lambda \in \mathbb{R}$, where $*_{\varphi}$ denotes the Hodge star operator relative to the associated Riemannian metric g . The name “nearly parallel” comes from the fact that only a 1-dimensional component of $\nabla \varphi$ is different from zero (see [15]), where ∇ is the Levi Civita connection of g , and these structures are also said to have weak holonomy G_2 , where this terminology goes back to Gray ([20]). The Riemannian manifold (M, g) is irreducible Einstein with scalar curvature given by $\frac{21}{8}\lambda^2$ and the existence of an NP-structure is equivalent to the existence of a spin structure with a non-zero Killing spinor as well as to the existence of a torsion-free $\text{Spin}(7)$ -structure on the cone $C(M) := \mathbb{R}^+ \times M$ inducing the cone metric $dr^2 + r^2g$ (see [5]). More precisely, an NP-structure on a compact simply connected manifold M will be called *proper* if the cone metric on $C(M)$ has full holonomy $\mathcal{H} = \text{Spin}(7)$, or equivalently if the space of Killing spinors is one-dimensional. When the NP-structure is not proper and the metric g has not constant curvature, the holonomy \mathcal{H} reduces either to $SU(4)$ or further to $\text{Sp}(2)$, corresponding to the existence of a Sasakian (but not 3-Sasakian) and a 3-Sasakian structure on M , respectively. It is known (see [19]) that any 3-Sasakian manifold admits a second NP-structure that is proper, and the squashed sphere S^7 is an example of this situation.

In some sense, NP-structures are a seven-dimensional analogue of nearly Kähler structures in six dimensions, which are automatically Einstein and admit a Killing spinor. Actually, the cone metric on the cone over a six-dimensional strict nearly Kähler manifold N has holonomy inside G_2 , and, moreover, for both nearly Kähler and NP-structures their canonical metric connections $\bar{\nabla}$ have totally skew-symmetric torsion, which is also $\bar{\nabla}$ -parallel. It is also known that given a six-dimensional strict nearly Kähler manifold N , the cone $C(N)$ endowed with the sine-cone metric has an NP-structure (see e.g., [7]).

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In order to find possibly new examples, it is very natural to investigate manifolds endowed with special structures, such as nearly Kähler or NP-structures, whose full automorphism group acts with orbits of low codimension (see also [1] for another possible construction of NP-structures). The classification of compact homogeneous NP-structures was achieved in [19], where many examples were found, and many useful results were also proved on the full automorphism group (to be compared to other G_2 -structures such as the closed ones ([25])), while later in [12], the classification of compact, homogeneous, nearly Kähler, six-dimensional manifolds was obtained. In [26, 27], the study of compact six-dimensional nearly Kähler manifolds that admit a compact Lie group of automorphisms with generic orbits of codimension one was initiated, and more recently, Foscolo and Haskins [17] proved the existence of completely new, inhomogeneous, nearly Kähler structures on the sphere S^6 and on $S^3 \times S^3$, invariant under the cohomogeneity one action of the group $SU(2) \times SU(2)$. As for NP-structures, Cleyton and Swann [13] classified all manifolds that carry such a structure with a simple Lie group of automorphisms acting by cohomogeneity one; in strong contrast to the homogeneous case, they found that the standard sphere S^7 and $\mathbb{R}P^7$ acted on by the exceptional Lie group G_2 are the only complete examples.

In this work, we investigate the existence of G -invariant NP-structures on the manifold $M \cong S^3 \times \mathbb{R}^4$, which admit a cohomogeneity one (almost effective) action of the group $G = SU(2)^3$. The manifold M can be realized as the complement $M = S^7 \setminus \Sigma$, where $\Sigma \cong S^3$ is one these two singular orbits for a cohomogeneity one action of G on S^7 , and it is special in the sense that it already admits a complete G -invariant metric with full holonomy G_2 , namely, the well-known example constructed by Bryant and Salamon [11] on the spin bundle over S^3 . The group G appears in the list of possible groups that can act by cohomogeneity one preserving a G_2 -structure ([13]) and actually it is (locally) isomorphic to the full isometry group of the Bryant-Salamon metric. Moreover, in view of the results in [19], the automorphism group of an NP-structure on a compact manifold acts transitively on it whenever its dimension is at least 10, so that the group G has the highest possible dimension to allow non-homogeneous examples. Principal G -orbits are diffeomorphic to $Y := S^3 \times S^3$, and the non-trivial isotropy representation of a principal isotropy subgroup allows us to easily determine the space of invariant 2- and 3-forms on M . A G -invariant NP-structure on $M_o \cong \mathbb{R}^+ \times Y$ given by a 3-form φ induces a family of so-called nearly half-flat G -invariant $SU(3)$ -structures (ω, ψ_+, ψ_-) on Y (see [16]); the 2-form ω is forced to lie in a one-dimensional subspace of invariant 2-forms on Y and when these $SU(3)$ -structures are all nearly Kähler structures on Y , we obtain the well-known example of the sine-cone over the homogeneous nearly Kähler manifold Y (see [6, 16]).

In our main result Theorem 4.1 we prove the existence of a one-parameter family \mathcal{F}_a ($a \in \mathbb{R}^+$) of G -invariant NP-structures on M , mutually non-isomorphic, connecting the two locally homogeneous NP-structures on M induced by the known homogeneous NP-structures on S^7 ; the parameter $a \in \mathbb{R}^+$ gives a measure of the size of the singular orbit S^3 . The problem of understanding which of these structures extends over a G -equivariant compactification \overline{M} is unsolved, although there is some numerical evidence that no such structure exists besides the homogeneous ones. In case a global G -invariant NP-structure on S^7 should exist, we prove that it would be proper and distinct from any of the Einstein metrics of cohomogeneity one on S^7

found by Böhm [8]. One might expect to find more invariant NP-structures by reducing the group to $SU(2)^2 \times U(1)$, by analogy with what happens for G_2 -holonomy metrics on M (see the recent results in [18]), or further to $SU(2)^2$ and this can be the object of further investigations.

The work is structured as follows. In the second section, we describe the manifold M together with the G -action as well as all the G -invariant G_2 -structures. In Section 3 we write down the equations defining the G -invariant NP-structures. We continue describing the special solutions to the system (3.2) given by the sine-cone construction over the nearly Kähler homogeneous manifold $S^3 \times S^3$ and by the two well-known homogeneous NP-structures on S^7 . We then analyze the symmetries of the system (3.2), proving (Prop. 3.5) the existence of a two-dimensional family of mutually non-isomorphic and not locally homogeneous NP-structures on an open tubular neighborhood of a G -principal orbit. In Subsection 3.4, we give sufficient and necessary conditions on the solutions of the system (3.2) on the regular part so that the corresponding NP-structures extend smoothly to an NP-structure on the whole M . In the last section, we prove our main Theorem 4.1 and the main properties of a global solution in Proposition 4.5.

Notation Lie groups and their Lie algebras will be indicated by capital and gothic letters, respectively. Given a Lie group L acting on a manifold N , for every $X \in \mathfrak{l}$, we will denote by \widehat{X} the corresponding vector field on N induced by the one-parameter subgroup $\exp(tX)$.

2 Preliminaries

In this section, we first consider the non-compact 7-dimensional manifold M together with the action of the group $G \cong SU(2)^3$ with generic orbits of codimension one. We will then describe the space of all G -invariant G_2 -structures on M .

2.1 The Manifold M and the Group Action of G

We start with the standard (almost effective) action of the compact group $U = Sp(2) \times Sp(1)$ on \mathbb{H}^2 given by $(A, q) \cdot v = Av\bar{q}$, where $(A, q) \in U$ and $v \in \mathbb{H}^2$. The sphere $S^7 \subset \mathbb{H}^2$ can be written as the quotient space U/K^+ with $K^+ = \{(\text{diag}(q, q'), q) \in U\} \cong Sp(1) \times Sp(1)$ being the isotropy subgroup at the point $e_1 = (1, 0) \in \mathbb{H}^2$.

We consider the action of $G := \{(\text{diag}(q_1, q_2), q_3) \in U \mid q_1, q_2, q_3 \in Sp(1)\} \cong Sp(1)^3$ on S^7 . The curve $\gamma: t \mapsto (\cos t, \sin t) \in S^7$ is transverse to the G -orbits, and we easily see that

$$G_{\gamma(t)} = Sp(1)_{\text{diag}} =: H \quad t \in (0, \pi/2),$$

$$G_{\gamma(0)} = K^+, \quad G_{\gamma(\pi/2)} = \{(q, q', q') \in G\} =: K^-.$$

It then follows that G acts on S^7 by cohomogeneity one with principal orbits diffeomorphic to $S^3 \times S^3$. We also fix an $\text{Ad}(H)$ -invariant decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus V^+ \oplus V^-, \quad \mathfrak{m} := V^+ \oplus V^-$$

where

$$V^+ := \{(X, -2X, X) \mid X \in \mathfrak{sp}(1)\}, \quad V^- := \{(-2X, X, X) \mid X \in \mathfrak{sp}(1)\}.$$

Note that $\mathfrak{k}^\pm = \mathfrak{h} \oplus V^\pm$. We fix the standard basis of $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$ given by

$$h := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

with

$$[h, e] = 2v, \quad [h, v] = -2e, \quad [e, v] = 2h,$$

and we consider the maps $j_\pm: \mathfrak{sp}(1) \rightarrow V^\pm$ given by $j_+(X) = (X, -2X, X)$ and $j_-(X) = (-2X, X, X)$. We then define a basis of \mathfrak{m} as follows:

$$\begin{aligned} e_2 &:= j_+(h), & e_3 &:= j_+(e), & e_4 &:= j_+(v), \\ e_5 &:= j_-(h), & e_6 &:= j_-(e), & e_7 &:= j_-(v). \end{aligned}$$

We consider the manifold $M := G \times_{K^+} \mathbb{H}$, where the K^+ -action on \mathbb{H} is induced by the standard action of U . Then M can be identified with $S^7 \setminus (G \cdot \gamma(\frac{\pi}{2}))$, and it is an \mathbb{R}^4 -bundle over the singular orbit $G \cdot \gamma(0) = G/K^+ \cong S^3$; namely, it is diffeomorphic to $S^3 \times \mathbb{R}^4$. The regular open subset M_o of M is G -equivariantly diffeomorphic to $(0, \frac{\pi}{2}) \times G/H$. In the open manifold M_o , we can identify the tangent spaces $T_{\gamma(t)}M_o = \mathbb{R}\gamma'(t) \oplus \widehat{\mathfrak{m}}_t$, where $\widehat{\mathfrak{m}}_t = \{\widehat{v}_{\gamma(t)} \mid v \in \mathfrak{m}\}$. Along the curve γ , we have a frame, again denoted by $\{e_1, \dots, e_7\}$, that is given by $\mathcal{B}_t := \{\gamma'(t), \widehat{e}_2|_{\gamma(t)}, \dots, \widehat{e}_7|_{\gamma(t)}\}$, and its dual coframe will be denoted by $\{e^1, \dots, e^7\}$. For basic information on cohomogeneity one manifolds, we refer the reader to [2, 3].

2.2 Invariant G_2 -structures

We start by recalling some basic facts about G_2 -structures. Given a 7-dimensional manifold M and its frame bundle $L(M) \rightarrow M$, a G_2 -structure is a reduction of $L(M)$ to a subbundle P with structure group $G_2 \subset SO(7)$. It is known that G_2 -structures are in one to one correspondence with smooth sections of the associated bundle $\Lambda^3_+(M) := L(M) \times_{GL(7, \mathbb{R})} \Lambda^3_+(\mathbb{R}^7) \subset \Lambda^3(M)$, where $\Lambda^3_+(\mathbb{R}^7) \subset \Lambda^3(\mathbb{R}^7)$ is the open orbit $GL(7, \mathbb{R}) \cdot \varphi_o$ through a 3-form φ_o with stabilizer $GL(7, \mathbb{R})_{\varphi_o} = G_2$ (see e.g., [9,19]). A smooth section φ of $\Lambda^3_+(M)$ (hence a G_2 -structure on M) determines a Riemannian metric g_φ as follows: at each point $p \in M$, we consider the non-degenerate symmetric bilinear map

$$b_\varphi: T_pM \times T_pM \longrightarrow \Lambda^7(T_pM^*), \quad (v, w) \longmapsto \frac{1}{6} \iota_v \varphi \wedge \iota_w \varphi \wedge \varphi,$$

and if $\{v_1, \dots, v_7\}$ is any basis of T_pM with dual basis $\{v^1, \dots, v^7\}$, then for $i, j = 1, \dots, 7$,

$$b_\varphi(v_i, v_j) = \beta_\varphi(v_i, v_j) v^1 \wedge \dots \wedge v^7$$

for some non-degenerate matrix $B_\varphi := (\beta_\varphi(v_i, v_j))_{i,j=1,\dots,7}$; the Riemannian metric g_φ is then given by (see e.g., [21])

$$(2.1) \quad g_\varphi(v_i, v_j) = (\det(B_\varphi))^{-1/9} \beta_\varphi(v_i, v_j).$$

In order to investigate G -invariant G_2 -structures on $M = G \times_{K^+} \mathbb{H}$, we start considering invariant G_2 -structures on the open dense submanifold M_o .

The description of G -invariant 3-forms on M_o is reduced to the study of the space of H -invariant 3-forms $\Lambda^3(V^*)$, where $V := \mathbb{R}e_1 + \mathfrak{m} \cong T_{\gamma(t)}M$ ($t \in (0, \frac{\pi}{2})$). We first

note that

$$\Lambda^3(V^*)^H \cong \Lambda^2(m^*)^H + \Lambda^3((V^+)^*) + \Lambda^3((V^-)^*) + [\Lambda^2((V^+)^*) \otimes (V^-)^*]^H + [(V^+)^* \otimes \Lambda^2((V^-)^*)]^H.$$

Using the standard notation $e^{i_1 i_2 \dots i_k} = e^{i_1} \wedge \dots \wedge e^{i_k}$, standard representation theory shows that the space $\Lambda^2(m^*)^H$ is generated by the form $\omega := e^{25} + e^{36} + e^{47}$ and that the space $\Lambda^3(V^*)^H$ is generated by the invariant 3-forms

$$e^1 \wedge \omega, \quad \varphi_1 := e^{234}, \quad \varphi_2 := e^{567}, \\ \varphi_3 := e^{237} - e^{246} + e^{345}, \quad \varphi_4 := e^{267} - e^{357} + e^{456}.$$

If we denote by φ a G -invariant 3-form on M_o , its restriction along γ can be written as

$$(2.2) \quad \varphi|_{\gamma(t)} = f_0(e^{125} + e^{136} + e^{147}) + f_1 e^{234} + f_2 e^{567} + f_3(e^{237} - e^{246} + e^{345}) + f_4(e^{267} - e^{357} + e^{456}),$$

for suitable $f_i \in \mathbb{C}^\infty((0, \frac{\pi}{2}))$. Let us fix the volume form $e^{1234567}$ along γ , so that we get an identification $\Lambda^7(V^*) \cong \mathbb{R}$. Then the matrix B_φ associated with the symmetric bilinear form b_φ with respect to \mathcal{B}_t is given by (here \mathbb{I} denotes the 3×3 -identity matrix)

$$B_\varphi = f_0 \begin{pmatrix} -f_0^2 & 0 & 0 \\ 0 & b_1 \mathbb{I} & b_3 \mathbb{I} \\ 0 & b_3 \mathbb{I} & b_2 \mathbb{I} \end{pmatrix},$$

where

$$b_1 := f_1 f_4 - f_3^2, \quad b_2 := f_2 f_3 - f_4^2, \quad b_3 := \frac{1}{2}(f_1 f_2 - f_3 f_4).$$

The 3-form φ defines a G_2 -structure if and only if B_φ is definite. In such a case, $g_\varphi = \det(B_\varphi)^{-1/9} B_\varphi$ is positive definite and

$$\det(B_\varphi) = \frac{1}{64} f_0^9 (f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2)^3 \neq 0.$$

In this case, we will suppose that the parameter t is the arc length parameter along the curve γ (hence, throughout the following the parameter, t will vary in some interval $I = (0, T)$), i.e., $g_\varphi(e_1, e_1) = 1$, so that, by (2.1), we have $\det(B_\varphi) = -f_0^{27}$, or equivalently,

$$(2.3) \quad f_0^2 = -\left(\frac{f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2}{4}\right)^{\frac{1}{3}}.$$

This implies that g_φ can be expressed as a block matrix

$$g_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & g_1 \mathbb{I} & g_3 \mathbb{I} \\ 0 & g_3 \mathbb{I} & g_2 \mathbb{I} \end{pmatrix},$$

where

$$g_1 := \frac{f_3^2 - f_1 f_4}{f_0^2}, \quad g_2 = \frac{f_4^2 - f_2 f_3}{f_0^2}, \quad g_3 := \frac{f_3 f_4 - f_1 f_2}{2 f_0^2}$$

together with the positivity condition, which in view of (2.3) can be written as

$$f_3^2 - f_1 f_4 > 0, \quad f_4^2 - f_2 f_3 > 0.$$

We can now compute the expression of the 4-form $*_\varphi \varphi$, where $*_\varphi$ denotes the Hodge operator with respect to g_φ . An easy but tedious computation shows that

$$\begin{aligned} *_\varphi \varphi = & A e^1 \wedge \left[(f_1^2 f_2 - 3 f_1 f_3 f_4 + 2 f_3^3) e^{234} - (f_1 f_2^2 - 3 f_2 f_3 f_4 + 2 f_4^3) e^{567} \right. \\ & + (f_1 f_2 f_3 - 2 f_1 f_4^2 + f_3^2 f_4) (e^{237} - e^{246} + e^{345}) \\ & \left. - (f_1 f_2 f_4 - 2 f_2 f_3^2 + f_3 f_4^2) (e^{267} - e^{357} + e^{456}) \right] \\ & + \left(\frac{f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2}{4} \right)^{\frac{1}{3}} (e^{2356} + e^{2457} + e^{3467}), \end{aligned}$$

where

$$A := f_0 2^{\frac{1}{3}} (f_1^2 f_2^2 - 6 f_1 f_2 f_3 f_4 + 4 f_1 f_4^3 + 4 f_2 f_3^3 - 3 f_3^2 f_4^2)^{-\frac{2}{3}}.$$

Using (2.3), we see that $A = \frac{1}{2} f_0^{-3}$. Consequently, the 4-form $*_\varphi \varphi$ can be rewritten as follows:

$$\begin{aligned} *_\varphi \varphi = & \frac{1}{2 f_0^3} e^1 \wedge \left[(f_1^2 f_2 - 3 f_1 f_3 f_4 + 2 f_3^3) e^{234} - (f_1 f_2^2 - 3 f_2 f_3 f_4 + 2 f_4^3) e^{567} \right. \\ & + (f_1 f_2 f_3 - 2 f_1 f_4^2 + f_3^2 f_4) (e^{237} - e^{246} + e^{345}) \\ & \left. - (f_1 f_2 f_4 - 2 f_2 f_3^2 + f_3 f_4^2) (e^{267} - e^{357} + e^{456}) \right] \\ & - f_0^2 (e^{2356} + e^{2457} + e^{3467}). \end{aligned}$$

In order to compute $d\varphi$, we need some preliminary remarks. First of all, a standard representation theory argument shows that

$$\Lambda^4(V^*)^H = \mathbb{R} e^1 \wedge \Lambda^3(\mathfrak{m}^*)^H + [\Lambda^2((V^+)^*) \otimes \Lambda^2((V^-)^*)]^H,$$

where the last summand is generated by the invariant form $\alpha := e^{2356} + e^{2457} + e^{3467}$. The next lemma follows by straightforward computations.

Lemma 2.1 *We have the following commutators for $x, y \in \mathfrak{sp}(1)$:*

$$\begin{aligned} [j_\pm(x), j_\pm(y)]_{\mathfrak{m}} &= -j_\pm([x, y]), \\ [j_+(x), j_-(y)]_{\mathfrak{m}} &= j_+([x, y]) + j_-([x, y]). \end{aligned}$$

Using the standard Koszul formula for the differential of an invariant form $\psi \in \Lambda^k(\mathfrak{m})^H$, namely for $X_0, X_1, \dots, X_k \in \mathfrak{m}$

$$d\psi(X_0, X_1, \dots, X_k) = \sum_{i < j} (-1)^{i+j} \psi([X_i, X_j]_{\mathfrak{m}}, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k),$$

(here the hat denotes omitted terms) we see that

$$d\varphi_1 = d\varphi_2 = 0, \quad d\omega = 6(\varphi_3 - \varphi_4), \quad d\varphi_3 = d\varphi_4 = 6\alpha.$$

Therefore, we obtain

$$\begin{aligned} d\varphi|_{\gamma(t)} = & f'_1 e^{1234} + f'_2 e^{1567} + (f'_3 - 6f_0)(e^{1237} - e^{1246} + e^{1345}) \\ & + (f'_4 + 6f_0)(e^{1267} - e^{1357} + e^{1456}) + 6(f_3 + f_4)(e^{2356} + e^{2457} + e^{3467}). \end{aligned}$$

3 Invariant Nearly Parallel G_2 -structures and their Equations

Recall that a G_2 -structure is *nearly parallel* if the defining 3-form φ satisfies the equation

$$(3.1) \quad d\varphi = \lambda *_{\varphi} \varphi,$$

for some non-zero real constant λ . In this case, the Riemannian metric g_{φ} induced by φ is Einstein with scalar curvature $\text{Scal}(g_{\varphi}) = \frac{21}{8} \lambda^2$. We now consider a G_2 -structure induced by a G -invariant 3-form φ , which can be described as in (2.2). Then φ defines an NP-structure if and only if f_0, f_1, f_2, f_3, f_4 satisfy the following equations:

$$\left\{ \begin{array}{l} f'_1 = \lambda \frac{1}{f_0^3} \left(f_1 \frac{f_1 f_2 - f_3 f_4}{2} - f_3 (f_1 f_4 - f_3^2) \right), \quad (a) \\ f'_2 = \lambda \frac{1}{f_0^3} \left(f_4 (f_2 f_3 - f_4^2) - f_2 \frac{f_1 f_2 - f_3 f_4}{2} \right), \quad (b) \\ f'_3 = 6f_0 + \lambda \frac{1}{2f_0^3} \left(f_1 (f_2 f_3 - f_4^2) - f_4 (f_1 f_4 - f_3^2) \right), \quad (c) \\ f'_4 = -6f_0 + \lambda \frac{1}{2f_0^3} \left(f_3 (f_2 f_3 - f_4^2) - f_2 (f_1 f_4 - f_3^2) \right), \quad (d) \\ f_4 + f_3 = -\frac{1}{6} \lambda f_0^2, \quad (e) \\ f_0^6 = (f_1 f_4 - f_3^2) (f_2 f_3 - f_4^2) - \frac{1}{4} (f_1 f_2 - f_3 f_4)^2 > 0, \quad (f) \\ 0 > f_1 f_4 - f_3^2, \quad 0 > f_2 f_3 - f_4^2. \end{array} \right.$$

We use equation (e) in equation (d) and compare it with equation (c). We then get the expression of f'_0 in terms of f_0, \dots, f_4 , and the system of equations can be written as follows:

$$(3.2) \quad \left\{ \begin{array}{l} f'_1 = \lambda \frac{1}{f_0^3} \left(f_1 \frac{f_1 f_2 - f_3 f_4}{2} - f_3 (f_1 f_4 - f_3^2) \right), \quad (a) \\ f'_2 = \lambda \frac{1}{f_0^3} \left(f_4 (f_2 f_3 - f_4^2) - f_2 \frac{f_1 f_2 - f_3 f_4}{2} \right), \quad (b) \\ f'_3 = 6f_0 + \lambda \frac{1}{2f_0^3} \left(f_1 (f_2 f_3 - f_4^2) - f_4 (f_1 f_4 - f_3^2) \right), \quad (c) \\ f'_4 = -6f_0 + \lambda \frac{1}{2f_0^3} \left(f_3 (f_2 f_3 - f_4^2) - f_2 (f_1 f_4 - f_3^2) \right), \quad (d) \\ f'_0 = -\frac{3}{2f_0^4} \left((f_1 + f_3) (f_2 f_3 - f_4^2) - (f_2 + f_4) (f_1 f_4 - f_3^2) \right), \quad (e) \\ f_4 + f_3 + \frac{1}{6} \lambda f_0^2 = 0, \quad (f) \\ f_0^6 - (f_1 f_4 - f_3^2) (f_2 f_3 - f_4^2) + \frac{1}{4} (f_1 f_2 - f_3 f_4)^2 = 0, \quad (g) \\ 0 > f_1 f_4 - f_3^2, \quad 0 > f_2 f_3 - f_4^2, \quad f_0 \neq 0. \quad (h) \end{array} \right.$$

The following lemma can be easily verified using a direct computation.

Lemma 3.1 Equations (f) and (g) in (3.2) hold for all $t \in I$ if and only if they hold at one point in I and equations (a)–(e) are satisfied for all $t \in I$.

As an immediate corollary, we note that the algebro-differential system (3.2) can be reduced to the system of ODE’s formed by equations (a)–(e) in (3.2) coupled with initial conditions at a fixed point $t_o \in I$ satisfying equations (f) and (g) at t_o , together with the inequalities (h). We will use this point of view when we will construct families of mutually non-isometric and not locally homogeneous NP-structures in a suitable neighbourhood of homogeneous solutions, which we describe in the following subsection.

Remark 3.2 Note that under the rescaling $\varphi \mapsto c \cdot \varphi$ ($c \neq 0$), we have $g_{c\varphi} = c^{2/3} \cdot g_\varphi$, and the constant in (3.1) $\lambda \mapsto c^{-1/3} \lambda$. This means that we can always fix λ to be any non-zero real number. Alternatively, one can consider new functions

$$\tilde{f}_0(t) := \lambda^2 f_0(t/\lambda), \quad \tilde{f}_i(t) := \lambda^3 f_i(t/\lambda), \quad i = 1, \dots, 4$$

which satisfy the system (3.2) with $\lambda = 1$.

Remark 3.3 It is well known that, given a manifold X endowed with an NP-structure with 3-form φ and associated Riemannian metric g , a hypersurface $f: S \rightarrow X$ inherits a so-called *nearly half-flat* $SU(3)$ -structure given by a 2-form ω and a 3-form ψ_+ so that

$$\omega := \iota_\nu \varphi, \quad \psi_+ := -\iota_\nu * \varphi, \quad \psi_- := J\psi_+ = -f^* \varphi,$$

where ν denotes the unit normal to S and J is the almost complex structure induced on S by the $SU(3)$ -structure (ω, ψ_+) (see [16]). This nearly half-flat structure (ω, ψ_+) satisfies the condition $d\psi_- = -2\omega \wedge \omega$, when the 3-form φ satisfies $d\varphi = 4 * \varphi$ (i.e., $\lambda = 4$). In our situation, the G -invariant nearly half-flat structures induced on the principal orbit $G/H = S^3 \times S^3$ have proportional 2-forms ω , as the isotropy representation of H forces the space of invariant 2-forms to be one-dimensional.

Vice-versa, given a smooth family of nearly half-flat structures $(\omega(t), \psi_+(t))_{t \in \mathbb{R}}$ on a 6-dimensional manifold S , the 3-form $\varphi := \omega \wedge dt - \psi_-$ on $\mathbb{R} \times S$ defines an NP-structure (with $\lambda = 4$) if and only if the following equations are fulfilled ([16, Prop. 5.2])

$$(3.3) \quad \begin{cases} \partial_t \psi_- = 4\psi_+ - d\omega, \\ d\psi_+ = -\frac{1}{2} \partial_t (\omega \wedge \omega). \end{cases}$$

In [29], it was proved that starting from a nearly half-flat structure on S , it is possible to extend it to a smooth one-parameter family of nearly half-flat structures satisfying (3.3), hence obtaining an NP-structure on $I \times S$ for some interval $I \subseteq \mathbb{R}$ (see also [14]). We will prove a local existence result in Proposition 3.5.

3.1 Particular Solutions

In this subsection, we will describe three special solutions to the system (3.2), corresponding to known NP-structures. More precisely, they are the sine-cone over the

homogeneous nearly Kähler manifold $S^3 \times S^3$ and the two homogeneous NP-structures on the sphere S^7 .

(a) It is known (see [6,7]) that the sine-cone $C_s(Y) = (0, \pi) \times Y$ over a nearly Kähler 6-dimensional manifold Y carries an NP-structure inducing the sine-cone metric $dt^2 + \sin^2 t \cdot g_Y$. The homogeneous nearly Kähler structure on $S^3 \times S^3$ is known to be invariant under the group $SU(2)^3$ (see [12]) and therefore it gives rise to a solution (f_0, \dots, f_4) of the system (3.2) for $t \in (0, \pi)$, namely,

$$\begin{aligned} \lambda &= 4, & f_0(t) &= -2\sqrt{3}(\sin t)^2, & f_1(t) &= f_2(t) = 8(\sin t)^4, \\ f_3(t) &= -4\sqrt{3}(\sin t)^3 \left(\cos t + \frac{1}{\sqrt{3}} \sin t \right), \\ f_4(t) &= -4\sqrt{3}(\sin t)^3 \left(-\cos t + \frac{1}{\sqrt{3}} \sin t \right). \end{aligned}$$

The metric g_Y is represented by the block matrix $\begin{pmatrix} 4\mathbb{I} & -2\mathbb{I} \\ -2\mathbb{I} & 4\mathbb{I} \end{pmatrix}$.

(b) We consider the standard NP-structure \mathcal{P}_1 on S^7 , inducing the standard constant curvature metric. Its full automorphism group is $\text{Aut}(\mathcal{P}_1) = \text{Spin}(7) \subset \text{SO}(8)$. We consider the octonions $\mathbb{O} = \{a + be, a, b \in \mathbb{H}\}$ together with the Cayley form $\Phi \in \Lambda^2 \mathbb{O}$ given by

$$\Phi(x, y, z, w) = \left\langle x, \frac{1}{2} [y(\bar{z}w) - w(\bar{z}y)] \right\rangle.$$

It is known that the group $\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1)$ acting almost faithfully on \mathbb{O} by

$$(q_1, q_2, q_3) \cdot (a + be) = q_1 a \bar{q}_3 + (q_2 b \bar{q}_3) e$$

preserves the form Φ (see [10, p. 11]) and therefore induces a cohomogeneity one action on the round sphere S^7 preserving the standard NP-structure φ given by $\varphi_x := \iota_x \Phi, x \in S^7$. If we consider the curve $\gamma(t) = \cos t + \sin t \cdot e \in \mathbb{O}$, we see that the corresponding functions $f_i(t)$ are given by

$$\begin{aligned} \lambda &= 4, & f_0(t) &= -9 \sin t \cos t, \\ f_1(t) &= 27 \sin^4 t, & f_2(t) &= 27 \cos^4 t, & f_3(t) &= f_4(t) = -27 \sin^2 t \cos^2 t. \end{aligned}$$

The metric can be represented by the block matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a\mathbb{I} & 0 \\ 0 & 0 & b\mathbb{I} \end{pmatrix}$ where $a = 9 \sin^2 t, b = 9 \cos^2 t$.

(c) We now consider the non-standard NP-structure \mathcal{P}_2 on the *squashed* S^7 , with full automorphism group given by $\text{Aut}(\mathcal{P}_2) = \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8)$. We refer to the exposition in [4, §8.2], where the authors describe this homogeneous structure using the presentation of S^7 as a normal homogeneous space of the group $\text{Sp}(2) \cdot \text{Sp}(1)$.

Along the geodesic $\gamma(t) = (\cos t, \sin t) \in S^7$, we have the following:

$$\begin{aligned} \lambda &= \frac{12}{\sqrt{5}}, & f_0(t) &= \frac{9}{\sqrt{5}} \sin t \cdot \cos t, \\ f_1(t) &= \frac{27}{\sqrt{5}} \left(3 \sin^4 t \cdot \cos^2 t - \frac{1}{5} \sin^6 t \right), \end{aligned}$$

$$\begin{aligned}
 f_2(t) &= \frac{27}{\sqrt{5}} \left(3 \cos^4 t \cdot \sin^2 t - \frac{1}{5} \cos^6 t \right), \\
 f_3(t) &= \frac{27}{\sqrt{5}} \sin^2 t \cdot \cos^2 t \cdot \left(\cos^2 t - \frac{11}{5} \sin^2 t \right), \\
 f_4(t) &= \frac{27}{\sqrt{5}} \sin^2 t \cdot \cos^2 t \cdot \left(\sin^2 t - \frac{11}{5} \cos^2 t \right).
 \end{aligned}$$

Note that the sign of the constant λ is opposite to that indicated in [4]. The metric can be represented by the block matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a\mathbb{I} & c\mathbb{I} \\ 0 & c\mathbb{I} & b\mathbb{I} \end{pmatrix}$, where

$$\begin{aligned}
 a &:= \frac{36}{5} \sin^2 t \cdot \left(\frac{5}{4} - \sin^2 t \right), & b &:= \frac{36}{5} \cos^2 t \cdot \left(\frac{5}{4} - \cos^2 t \right), \\
 c &:= -\frac{36}{5} \sin^2 t \cdot \cos^2 t.
 \end{aligned}$$

3.2 The Functions f_i 's and the Associated Nearly Parallel Structures

A solution $(f_0(t), \dots, f_4(t))$ of the system (3.2) defines an NP-structure on the open subset $J \times G/K$ for some subinterval $J \subseteq I$. In this subsection, we study the problem of when two such NP-structures are (locally) isomorphic. We start with the following proposition, which follows [27, Prop. 4.1] very closely and characterizes the local homogeneity of a G-manifold with a G-invariant NP-structure.

Proposition 3.4 *Let X be a 7-dimensional manifold endowed with an NP-structure given by a 3-form φ . Assume that the group $G \cong \text{SU}(2)^3$ acts on X by automorphisms of the G_2 -structure with cohomogeneity one and assume, moreover, that the NP-structure is locally homogeneous. Then (X, φ) is locally (isometrically) isomorphic to the standard sphere or to the squashed sphere endowed with their respective NP-structures.*

Proof We fix $p \in X$ and consider the Lie algebra \mathfrak{s} of germs of automorphisms of (X, φ) with isotropy subalgebra at p denoted by \mathfrak{u} . Note that \mathfrak{u} embeds into \mathfrak{g}_2 . Moreover, by local homogeneity, we can suppose that p is G-regular; hence, \mathfrak{u} contains the isotropy subalgebra $\mathfrak{g}_p = \mathfrak{u} \cap \mathfrak{g}$. Note that \mathfrak{g}_p is a compact algebra of dimension 3 and also embeds into \mathfrak{g}_2 , so that its rank is at most 2. This means that $\mathfrak{g}_p \cong \mathfrak{su}(2)$. Therefore, looking at the list of subalgebras of \mathfrak{g}_2 , we see that \mathfrak{u} can be isomorphic to $\mathfrak{su}(2)$, $\mathfrak{su}(3)$, $\mathbb{R} \oplus \mathfrak{su}(2)$, $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, \mathfrak{g}_2 . Let S be the simply connected Lie group with Lie algebra \mathfrak{s} and let U be the connected Lie subgroup of S with Lie algebra \mathfrak{u} . We claim that U is closed in S , whence X is locally isomorphic to a globally homogeneous space (see e.g., [28]). Suppose on the contrary that U is not closed in S . This implies that \mathfrak{u} is not semisimple (see [23, p. 615]), hence $\mathfrak{u} \cong \mathbb{R} \oplus \mathfrak{su}(2)$, and therefore $\mathfrak{u} = \mathfrak{g}_p \oplus \mathbb{R}$ and $\dim \mathfrak{s} = 11$. As \mathfrak{u} is reductive, we can write $\mathfrak{s} = \mathfrak{u} + V$, where $V \cong \mathbb{R}^7$ is $\text{ad}(\mathfrak{u})$ invariant and $\text{ad}(\mathfrak{u})|_V \subset \mathfrak{su}(3)$. This implies that $\text{ad}^{\mathfrak{s}}|_{\mathfrak{g}_p} = \rho_1 \oplus \text{ad} \oplus 4\mathbb{R}$, where ρ_1 is the standard representation of $\mathfrak{g}_p \cong \mathfrak{su}(2)$ on \mathbb{C}^2 , and therefore $\text{ad}^{\mathfrak{s}}|_{\mathfrak{g}_p}$ has a 4-dimensional trivial submodule. On the other hand, $\mathfrak{g}_p \subset \mathfrak{g} \subset \mathfrak{s}$ with $\text{codim}_{\mathfrak{s}} \mathfrak{g} = 2$, and therefore $\text{ad}^{\mathfrak{s}}|_{\mathfrak{g}_p} = \text{ad}^{\mathfrak{g}}|_{\mathfrak{g}_p} \oplus 2\mathbb{R}$. As \mathfrak{g}_p contains no ideal of \mathfrak{g} , we see that $\text{ad}^{\mathfrak{g}}|_{\mathfrak{g}_p}$ is either 3 ad or 2 ad \oplus 3 \mathbb{R} . This implies that the trivial submodule of $\text{ad}^{\mathfrak{s}}|_{\mathfrak{g}_p}$ has dimension 2 or 5, a contradiction.

A direct inspection of the globally homogeneous (hence compact) manifolds with G-invariant NP-structure (see [19]) and admitting a cohomogeneity one action of G proves our claim. ■

Let $\varphi = \sum_{i=0}^4 f_i \varphi_i$ and $\tilde{\varphi} = \sum_{i=0}^4 \tilde{f}_i \varphi_i$ be two G-invariant NP-structures, where the 5-tuples $(f_0, \dots, f_4), (\tilde{f}_0, \dots, \tilde{f}_4)$ satisfy the system (3.2) on some interval $J \subseteq I$. Assume then that these two structures are locally isomorphic; i.e., there are two open subsets $W, \tilde{W} \subseteq J \times G/H$ and a diffeomorphism $\psi: W \rightarrow \tilde{W}$ with $\psi^* \tilde{\varphi} = \varphi$ (and therefore inducing an isometry between the induced metrics g and \tilde{g} , respectively). We will first suppose that ψ does not map G-orbits onto G-orbits. As G acts with cohomogeneity one, this means that both W and \tilde{W} are locally homogeneous, and therefore, by Proposition 3.4, locally isometric to the standard or squashed sphere. As the full automorphisms of the two homogeneous NP-structures on S^7 , namely $\text{Spin}(7)$ and $\text{Sp}(2) \cdot \text{Sp}(1)$, contain precisely one copy of $\text{SU}(2)^3$ up to conjugation, we can find a local isometry $\tilde{\psi}: \tilde{W} \rightarrow W$ preserving $\tilde{\varphi}$ and with $\tilde{\psi}_*(\psi_*(\mathfrak{g})) = \mathfrak{g}$.

Therefore, we are reduced with the case where ψ preserves G-orbits. Up to some translation by an element of G and up to some reparameterization $t \mapsto t + c$, we can suppose $\psi(\gamma_t) = \gamma_{\pm t}$. If $\psi(\gamma_t) = \gamma_{-t}$, we can compose ψ with the transformation $(t, xH) \mapsto (-t, xH)$ reducing to $\psi(\gamma_t) = \gamma_t$; the corresponding transformation of the functions f_i 's reads

$$\tau_o: (f_0(t), f_1(t), f_2(t), f_3(t), f_4(t)) \mapsto (-f_0(-t), f_1(-t), f_2(-t), f_3(-t), f_4(-t)).$$

The map ψ induces an automorphism ψ_* of the Lie algebra \mathfrak{g} that preserves the regular isotropy \mathfrak{h} , as ψ preserves the curve γ_t . If ψ_* is inner, it is the conjugation by an element $n \in N_G(H)$, $n = \sigma \cdot h$ with $h \in H$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ where $\sigma_i = \pm \text{Id} \in \text{SU}(2)$. It then follows that $\text{Ad}(g)$ fixes ω as well as all $\varphi_1, \dots, \varphi_4$, so that the functions f_0, f_1, \dots, f_4 remain unchanged. We now examine the case where ψ_* is outer, namely, it permutes the simple factors $\mathfrak{f}_i \cong \mathfrak{su}(2)$, $i = 1, 2, 3$ of \mathfrak{g} . We now describe how the functions φ_i 's transform when ψ_* induces the generators (12) and (13) of the symmetric group S_3 .

Suppose ψ_* permutes the factors $\mathfrak{f}_1, \mathfrak{f}_2$ leaving \mathfrak{f}_3 fixed. As e^2, e^3, e^4 are mapped to e^5, e^6, e^7 , respectively, we see that $\omega \mapsto -\omega$, while φ_1, φ_2 are exchanged as well as φ_3, φ_4 . The functions f_i 's transform accordingly as follows

$$\tau_{(12)}: (f_0, f_1, f_2, f_3, f_4) \mapsto (-f_0, f_2, f_1, f_4, f_3).$$

When ψ_* induces the permutation (13) on the factors of \mathfrak{g} , the invariant forms are accordingly transformed as follows:

$$\begin{aligned} \omega &\mapsto -\omega, & \varphi_1 &\mapsto \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4, & \varphi_2 &\mapsto -\varphi_2 \\ \varphi_3 &\mapsto -\varphi_3 - 3\varphi_2 + 2\varphi_4, & \varphi_4 &\mapsto \varphi_4 - 3\varphi_2, \end{aligned}$$

so that the corresponding transformation of the functions f_i 's reads

$$\tau_{(13)}: (f_0, f_1, f_2, f_3, f_4) \mapsto (-f_0, f_1, -f_1 - f_2 - 3(f_3 + f_4), -f_1 - f_3, f_1 + 2f_3 + f_4).$$

Using these, we see that the remaining non-trivial permutations induce the following transformations

$$\begin{aligned} \tau_{(23)} : (f_0, f_1, f_2, f_3, f_4) &\longmapsto \\ &(-f_0, -f_1 - f_2 - 3(f_3 + f_4), f_2, f_2 + f_3 + 2f_4, -f_2 - f_4), \\ \tau_{(123)} : (f_0, f_1, f_2, f_3, f_4) &\longmapsto \\ &(f_0, -f_1 - f_2 - 3(f_3 + f_4), f_1, f_1 + 2f_3 + f_4, -f_1 - f_3), \\ \tau_{(132)} : (f_0, f_1, f_2, f_3, f_4) &\longmapsto \\ &(f_0, f_2, -f_1 - f_2 - 3(f_3 + f_4), -f_2 - f_4, f_2 + f_3 + 2f_4). \end{aligned}$$

3.3 Local Existence

We consider the regular ODE system given by equations (a)–(e) (for a fixed λ) in (3.2). Any solution of such a system is a curve in \mathbb{R}^5 lying in the subset

$$C := \left\{ (a_0, a_1, \dots, a_4) \in \mathbb{R}^5 \mid a_0 \neq 0, 0 > a_1 a_4 - a_3^2, \right. \\ \left. 0 > a_2 a_3 - a_4^2, R_1 = 0, R_2 = 0 \right\},$$

where

$$\begin{aligned} R_1(a_0, \dots, a_4) &:= a_3 + a_4 + \frac{1}{6} \lambda a_0^2, \\ R_2(a_0, \dots, a_4) &:= (a_1 a_4 - a_3^2) (a_2 a_3 - a_4^2) - \frac{1}{4} (a_1 a_2 - a_3 a_4)^2 - a_0^6. \end{aligned}$$

We fix $t_o = \frac{\pi}{4}$ and the initial condition

$$x_o := \left(-\frac{9}{2}, \frac{27}{4}, \frac{27}{4}, -\frac{27}{4}, -\frac{27}{4} \right)$$

that corresponds to the initial values at t_o of the homogeneous structure \mathcal{P}_1 on the sphere S^7 with constant $\lambda = 4$. In a suitable neighborhood W of x_o the set $C \cap W$ is a 3-dimensional submanifold, as can be easily verified. Moreover, if Σ is the group of transformations in \mathbb{R}^5 generated by $\tau_{(12)}$, $\tau_{(13)}$, we can shrink W so that $\tau(W) \cap W = \emptyset$ for every $\tau \in \Sigma$. Now, if $F(t)$ is the (homogeneous) solution starting from x_o , we can fix a 2-dimensional submanifold \mathcal{S} in $C \cap W$ that is transversal to the trace of F ; solutions starting from points in \mathcal{S} are all mutually not equivalent and not locally homogeneous. Therefore, we have proved the following proposition.

Proposition 3.5 *There exists a 2-dimensional family of mutually non-equivalent and not locally homogeneous G-invariant NP-structures on the space $J \times G/H$ for some open interval $J \subset \mathbb{R}$.*

3.4 Extendability Over the Singular Orbit $G/K^+ \cong S^3$

We aim at finding necessary and sufficient conditions on the functions f_i so that the 3-form φ and the corresponding metric g_φ extend smoothly over one singular orbit $G \cdot p = G/K^+ \cong S^3$, where $p = \gamma(0)$.

We first remark that when φ extends smoothly over the singular orbit, then $\varphi_{\gamma(t)}(\widehat{e}_i, \widehat{e}_j, \widehat{e}_k)$ is smooth in a neighborhood of the origin. In particular, $\varphi_{\gamma(0)}(\widehat{e}_i, \widehat{e}_j, \widehat{e}_k) = 0$ when $e_i, e_j, e_k \in V^+$, and therefore by (2.2), we have

$$f_0(0) = f_1(0) = f_3(0) = f_4(0) = 0.$$

Moreover, the element $h = ((\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}), 1) \in K^+$ reverses the curve γ , i.e., $h(\gamma(t)) = \gamma(-t)$ and its adjoint $\text{Ad}_G(h) = \text{Id}$, so that the functions f_i extend as follows:

$$f_0 \text{ odd}; \quad f_i \text{ even} \quad i = 1, 2, 3, 4.$$

We now consider the slice representation ρ of K^+ at p . If we write the $\text{ad}(\mathfrak{g})$ -invariant decomposition $\mathfrak{u} = \mathfrak{g} + \mathfrak{p}$, where $\mathfrak{u} = \mathfrak{sp}(2) + \mathfrak{su}(2)$ and $\mathfrak{p} \cong \mathbb{H}$, then ρ can be identified with $\text{Ad}^U|_{K^+}$ restricted to the invariant module \mathfrak{p} . If we choose standard coordinates $\{t = x_1, x_2, x_3, x_4\}$ on $\mathfrak{p} \cong \mathbb{H}$, we see that

$$\widehat{e}_2|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_2}, \quad \widehat{e}_3|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_3}, \quad \widehat{e}_4|_{(t,0,0,0)} = 3t \frac{\partial}{\partial x_4}.$$

We also need an $\text{ad}(\mathfrak{k}^+)$ -stable complement \mathfrak{s} in \mathfrak{g} , namely

$$\mathfrak{s} := \{(X, 0, -X) \mid X \in \mathfrak{su}(2)\},$$

and we fix the basis

$$w_1 := (h, 0, -h), \quad w_2 := (e, 0, -e), \quad w_3 := (v, 0, -v)$$

so that

$$w_i = -\frac{1}{3}e_{i+1} - \frac{2}{3}e_{4+i}, \quad i = 1, 2, 3.$$

We consider the local frame along the curve $t \mapsto (t, 0, 0, 0) \in \mathfrak{p}$ given by

$$e_1, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \widehat{w}_1|_{(t,0,0,0)}, \widehat{w}_2|_{(t,0,0,0)}, \widehat{w}_3|_{(t,0,0,0)}$$

with corresponding coframe $dt = dx_1, dx_2, dx_3, dx_4, w^1, w^2, w^3$ satisfying

$$\begin{aligned} e^1 = dt, \quad e^2 = \frac{1}{3t}dx_2 - \frac{1}{3}w^1, \quad e^3 = \frac{1}{3t}dx_3 - \frac{1}{3}w^2, \quad e^4 = \frac{1}{3t}dx_4 - \frac{1}{3}w^3, \\ e^5 = -\frac{2}{3}w^1, \quad e^6 = -\frac{2}{3}w^2, \quad e^7 = -\frac{2}{3}w^3. \end{aligned}$$

On the tubular neighborhood of the singular orbit G/K^+ given by $G \times_{K^+} \mathfrak{p}$, the 3-form φ defined by (2.2) on the complement of the zero section is completely determined by its restriction to \mathfrak{p} . Therefore, we can see φ as a K^+ -equivariant map $\widehat{\varphi}: \mathfrak{p} \setminus \{0\} \rightarrow \Lambda^3(\mathfrak{p}^* + \mathfrak{s}^*)$, which is fully determined by its restriction to the curve $\gamma(t) = (t, 0, 0, 0) \in \mathfrak{p}$. We can write down the components of $\widehat{\varphi}|_{\gamma(t)}$ ($t \neq 0$) along each of the four K^+ -summands in the decomposition

$$\Lambda^3(\mathfrak{p}^* + \mathfrak{s}^*) = \Lambda^3\mathfrak{p}^* \oplus \Lambda^3\mathfrak{s}^* \oplus (\Lambda^2\mathfrak{p}^* \otimes \mathfrak{s}^*) \oplus (\mathfrak{p}^* \otimes \Lambda^2\mathfrak{s}^*).$$

We will denote by π_i the K^+ -equivariant projection of $\Lambda^3(\mathfrak{p}^* + \mathfrak{s}^*)$ onto its i -th summand, $i = 1, \dots, 4$.

(a) Along $\Lambda^3\mathfrak{p}^* \cong \mathfrak{p}$, we have

$$\pi_1 \circ \widehat{\varphi}|_{\gamma(t)} = \frac{1}{27t^3} f_1(t) dx_2 \wedge dx_3 \wedge dx_4,$$

and therefore, we can consider the K^+ -equivariant map $\mathfrak{p} \rightarrow \mathfrak{p}^*$ whose restriction to γ is given by $\frac{1}{27t^3} f_1(t) dt$ for $t \neq 0$. This extends smoothly on the whole \mathfrak{p} if and only if $f_1(t)$ is an even smooth function of t with $f_1(0) = f_1''(0) = 0$ (indeed $\frac{f_1(t)}{t^4}$ must extend smoothly and we already know that f_1 is even and vanishes at $t = 0$).

(b) Along the trivial K^+ -module $\Lambda^3 \mathfrak{s}^*$, we obtain the component

$$\pi_2 \circ \widehat{\varphi}|_{\gamma(t)} = -\frac{1}{27}(f_1 + 8f_2 + 6f_3 + 12f_4) w^1 \wedge w^2 \wedge w^3.$$

Since $w^1 \wedge w^2 \wedge w^3$ is K^+ -invariant, the extendability condition in this case boils down to the condition that $f_1 + 8f_2 + 6f_3 + 12f_4$ must be even. This follows from the fact that each f_i is even for $i = 1, \dots, 4$.

(c) Along $\mathfrak{p}^* \otimes \Lambda^2 \mathfrak{s}^*$, we obtain the component

$$\pi_3 \circ \widehat{\varphi}|_{\gamma(t)} = \frac{1}{27t}(f_1 + 4f_3 + 4f_4) (dx_2 \wedge w^2 \wedge w^3 - dx_3 \wedge w^1 \wedge w^3 + dx_4 \wedge w^1 \wedge w^2).$$

This case can be handled in two different ways. First, we note that K^+ contains the normal subgroup $I = \{(1, q, 1) \in K^+ \mid q \in \text{Sp}(1)\}$, which still acts transitively on the unit sphere in \mathfrak{p} , but trivially on the orbit G/K^+ . Using the subgroup I , we can determine the full expression of $\widehat{\varphi}$ and check that its component along $\mathfrak{p}^* \otimes \Lambda^2 \mathfrak{s}^*$ extends smoothly over the whole \mathfrak{p} if and only if $\frac{1}{t^2}(f_1 + 4f_3 + 4f_4)$ extends smoothly over $t = 0$. This last condition is automatic, as we are supposing f_1, f_3, f_4 to be even functions vanishing at $t = 0$. The second approach considers the K^+ -module $\mathfrak{p}^* \otimes \Lambda^2 \mathfrak{s}^* \cong \mathfrak{p}^* \otimes \mathfrak{s} \cong Q_1 \oplus Q_2$, where $Q_1 \cong \mathfrak{p}$ and $Q_2 \cong \mathbb{R}^8$ is the real part of the complex K^+ -irreducible representation $\mathbb{C}^2 \otimes S^3(\mathbb{C}^2)$ (here $K^+ \cong \text{SU}(2)_1 \times \text{SU}(2)_2$ acts on \mathbb{C}^2 via $\text{SU}(2)_1$ and on $S^3(\mathbb{C}^2)$ via $\text{SU}(2)_2$). Since the space Q_2 does not contain any non-zero fixed point vector under the action of the subgroup $\text{Sp}(1) \cong H \subset K^+$, the element $dx_2 \wedge w^2 \wedge w^3 - dx_3 \wedge w^1 \wedge w^3 + dx_4 \wedge w^1 \wedge w^2$ belongs to the submodule $Q_1 \cong \mathfrak{p}$, and therefore, we can consider this component of $\widehat{\varphi}$ as a K^+ -equivariant map into \mathfrak{p} , leading to the same conclusion as above.

(d) Along the summand $\Lambda^2 \mathfrak{p}^* \otimes \mathfrak{s}^*$, we have

$$\begin{aligned} \pi_4 \circ \widehat{\varphi}|_{\gamma(t)} = & -\frac{2}{9t} f_0 \cdot (dx_0 \wedge dx_1 \wedge w^1 + dt \wedge dx_2 \wedge w^2 + dt \wedge dx_3 \wedge w^3) + \\ & -\frac{f_1 + 2f_3}{27t^2} \cdot (dx_1 \wedge dx_2 \wedge w^3 + dx_2 \wedge dx_3 \wedge w^1 - dx_1 \wedge dx_3 \wedge w^2). \end{aligned}$$

We can use the subgroup I to determine the full expression of $\pi_4 \circ \widehat{\varphi}$ on $\mathfrak{p} \setminus \{0\}$.

We only write here the component of $\pi_4 \circ \widehat{\varphi}$ along the 3-form $dx_0 \wedge dx_1 \wedge w^1$, namely,

$$-\frac{2}{9t} f_0 \cdot \frac{x_0^2 + x_1^2}{t^2} - \frac{f_1 + 2f_3}{27t^2} \cdot \frac{x_2^2 + x_3^2}{t^2},$$

where $t = \sum_{i=1}^4 x_i^2$. This can be clearly rewritten as

$$-\frac{2}{9t} f_0 + \left(\frac{2f_0}{9t} - \frac{f_1 + 2f_3}{27t^2} \right) \cdot \frac{x_2^2 + x_3^2}{t^2}.$$

Therefore, we see that the extendability condition reduces to $\frac{2f_0}{9t} - \frac{f_1+2f_3}{27t^2} = O(t^2)$. Since f_0 is odd and f_1, f_3 are even, the condition can be written as $\lim_{t \rightarrow 0} \frac{2f_0}{9t} - \frac{f_1+2f_3}{27t^2} = 0$ or equivalently, using the fact that $f_1 = O(t^4)$ by (a),

$$(3.4) \quad 6f_0'(0) = f_3''(0).$$

It can be easily checked that the conditions on the extendibility of the other components of $\pi_4 \circ \widehat{\varphi}$ are all equivalent to (3.4).

Summing up, the 3-form φ on the regular part M_0 extends on the whole tube $G \times_{K^+} \mathfrak{p}$ if and only if the functions f_i extend smoothly around $t = 0$ with the following properties:

$$(3.5) \quad f_0 \text{ is odd, } f_i \text{ are even, } i = 1, 2, 3, 4;$$

$$(3.6) \quad f_i(0) = 0, \quad i = 1, 3, 4;$$

$$(3.7) \quad f_1''(0) = 0; \quad 6f_0'(0) = f_3''(0).$$

When these conditions hold, the 3-form φ extends smoothly at the singular point with the expression

$$\begin{aligned} \varphi_p = & A w^1 \wedge w^2 \wedge w^3 + B (dx_0 \wedge dx_1 \wedge w^1 + dt \wedge dx_2 \wedge w^2 + dt \wedge dx_3 \wedge w^3 \\ & + dx_1 \wedge dx_2 \wedge w^3 + dx_2 \wedge dx_3 \wedge w^1 - dx_1 \wedge dx_3 \wedge w^2), \end{aligned}$$

where $A := -\frac{8}{27}f_2(0)$ and $B := -\frac{2}{9}f_0'(0)$. Now, it is not difficult to check that the 3-form φ_p is stable (i.e., the orbit $GL(T_pM) \cdot \varphi_p$ is open in $\wedge^3 T_pM^*$) if and only if $A \cdot B < 0$, and in this case, the induced metric is positive definite, coinciding with the limit metric g_φ at p . Therefore, we need to consider the non-degeneracy condition

$$f_2(0) \cdot f_0'(0) < 0.$$

Actually, if we use (3.5), (3.6), (3.7), and (g)–(h) in (3.2), we see that $f_2(0) \cdot f_0'(0) \leq 0$, so that the only condition we need to add is

$$(3.8) \quad f_2(0) \neq 0, \quad f_0'(0) \neq 0.$$

Therefore, we have proved the following proposition.

Proposition 3.6 *Let φ be a G -invariant 3-form on M_0 whose restriction to the curve $\gamma(t)$ ($t \neq 0$) has the expression (2.2). Assume that the form φ defines an NP-structure, so that the functions f_i 's satisfy the algebro-differential system (3.2). Then the form φ extends smoothly to a G -invariant 3-form on M defining an NP-structure on M if and only if the functions f_i 's extend smoothly around $t = 0$ fulfilling the conditions (3.5), (3.6), (3.7), and (3.8).*

4 The Main Theorem

In this section, we will prove the existence of a one-parameter family of NP-structures on M . In particular, we will prove our main theorem.

Theorem 4.1 *There exists a one parameter family \mathcal{F}_a , ($a \in \mathbb{R}^+$) of NP-structures on M . These structures are mutually non-isomorphic and not locally homogeneous,*

with the exception of two of them, which are locally isomorphic to the structures \mathcal{P}_1 and \mathcal{P}_2 on S^7 .

Remark 4.2 The parameter $a \in \mathbb{R}^+$ measures the size of the singular orbit S^3 . Moreover, the two (locally) homogeneous solutions (with $\lambda = 1$) correspond to the values $a = 36$ (round sphere) and $a = \frac{108}{5}$ (squashed sphere), as it can be easily seen using §3.1 and the Remark 3.2.

The proof of Theorem 4.1 will be achieved through several steps in this section. We know how to describe G-invariant NP-structures on the open dense subset M_o of G-regular points, which is identified with the complement of the zero section in the bundle $G \times_{K^+} \mathbb{H}$. Given a G-invariant 3-form φ on M_o , which defines an NP-structure, we considered its expression (2.2) along a transversal curve γ , and we could derive the algebro-differential system of equations (3.2) the functions f_i 's in (2.2) have to satisfy. Moreover, we found necessary and sufficient conditions in terms of the functions f_i 's so that the NP-structure on M_o extends smoothly to a global G_2 -structure on M . Now, instead of solving for the functions f_i 's, in view of Proposition 3.6, we can look for smooth even functions h_i defined on some interval $(-\varepsilon, \varepsilon)$, $\varepsilon \in \mathbb{R}^+$, so that

$$(4.1) \quad f_0 = t \cdot h_0, \quad f_1 = t^4 \cdot h_1, \quad f_2 = h_2, \quad f_3 = t^2 \cdot h_3, \quad f_4 = t^2 \cdot h_4.$$

Note that by (f) of (3.2),

$$h_4 = -h_3 - \frac{\lambda}{6} h_0^2.$$

If we set $a_i := h_i(0)$, $i = 0, \dots, 4$, then the extendability conditions (3.5), (3.6), (3.7), and (3.8) are then simply written as

$$(4.2) \quad a_3 = 3a_0, \quad a_0, a_2 \neq 0,$$

We now rewrite the system (3.2) using the above defined functions h_i as follows.

$$\begin{aligned} h'_0 &= \frac{1}{t}(f'_0 - h_0) \\ &= -\frac{1}{t}\left(h_0 + \frac{3h_2h_3^2}{h_0^4}\right) - \frac{3}{2h_0^4}(t(h_3 - h_4)(h_1h_2 + h_3h_4) - 2t^3h_1h_4^2); \\ h'_1 &= -\frac{4}{t}h_1 + \frac{1}{t^4}f'_1 \\ &= -\frac{4}{t}h_1 + \lambda \frac{1}{t^7h_0^3}\left(t^8h_1 \frac{h_1h_2 - h_3h_4}{2} - t^2h_3(t^6h_1h_4 - t^4h_3^2)\right) \\ &= \frac{1}{t}\left(-4h_1 + \frac{\lambda h_3^3}{h_0^3}\right) + \frac{\lambda t}{2h_0^3}(h_1^2h_2 - h_1h_3h_4 - 2h_1h_3h_4); \\ h'_2 &= f'_2 = \frac{\lambda t}{h_0^3}\left(h_4(h_2h_3 - t^2h_4^2) - \frac{1}{2}h_2(h_1h_2 - h_3h_4)\right); \\ h'_3 &= -\frac{2}{t}h_3 + \frac{1}{t^2}f'_3 \\ &= \frac{1}{t}(-2h_3 + 6h_0) + \lambda \frac{t}{2h_0^3}(h_1h_2h_3 + h_3^2h_4 - 2t^2h_1h_4^2). \end{aligned}$$

Therefore, if we put $h := (h_0, h_1, h_2, h_3)$, the system takes the form

$$(4.3) \quad h'(t) = \frac{1}{t}A(h) + B(h, t), \quad h(0) = \bar{h} = (a_0, a_1, a_2, a_3),$$

where $A: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is given by

$$A(h) = \left(-h_0 - \frac{3h_2h_3^2}{h_0^4}, -4h_1 + \frac{\lambda h_3^3}{h_0^3}, 0, -2h_3 + 6h_0\right),$$

and $B: \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4$ is defined by

$$\begin{aligned} B(h, t) = & \left(-\frac{3}{2h_0^4}(t(h_3 - h_4)(h_1h_2 + h_3h_4) - 2t^3h_1h_4^2), \right. \\ & \frac{\lambda t}{2h_0^3}(h_1^2h_2 - h_1h_3h_4 - 2h_1h_3h_4), \\ & \frac{\lambda t}{h_0^3}\left(h_4(h_2h_3 - t^2h_4^2) - \frac{1}{2}h_2(h_1h_2 - h_3h_4)\right), \\ & \left. \frac{\lambda t}{2h_0^3}(h_1h_2h_3 + h_3^2h_4 - 2t^2h_1h_4^2)\right). \end{aligned}$$

Clearly we must have $A(\bar{h}) = 0$, hence using (4.2)

$$a_3 = 3a_0, \quad a_2 = -\frac{1}{27}a_0^3, \quad a_1 = \frac{27}{4}\lambda.$$

Proposition 4.3 *The functions f_i 's as in (2.2) define a G -invariant NP-structure on $M = G \times_{\mathbb{K}^+} \mathbb{H}$ (with fixed constant λ) if and only if there exist smooth even functions h_0, h_1, h_2, h_3 defined on some interval $(-\varepsilon, \varepsilon)$, ($\varepsilon \in \mathbb{R}^+$), satisfying the equation (4.3) for $t \neq 0$ with initial condition at $t = 0$ given by $\bar{h} = (h_0(0), \dots, h_3(0))$ with*

$$(4.4) \quad h_0(0) = a, \quad h_1(0) = \frac{27}{4}\lambda, \quad h_2(0) = -\frac{1}{27}a^3, \quad h_3(0) = 3a,$$

for some $a \in \mathbb{R}$, $a \neq 0$.

Proof It remains to prove the “if” part. We clearly define the f_i 's in terms of the h_i 's using (4.1). Then the f_i 's satisfy the differential system (3.2) (a)–(e) on the open set M_o , while we have to prove that the algebraic conditions (3.2) (f)–(g) are also satisfied. By Lemma 3.1, the two quantities

$$\begin{aligned} F_1 &:= f_4 + f_3 + \frac{1}{6}\lambda f_0^2, \\ F_2 &:= f_0^6 - (f_1f_4 - f_3^2)(f_2f_3 - f_4^2) + \frac{1}{4}(f_1f_2 - f_3f_4)^2 \end{aligned}$$

are actually constant on the open set $t \neq 0$ and both vanish at $t = 0$; hence, they vanish everywhere. Moreover, condition (4.2) are also satisfied, so that the functions f_i 's define an NP-structure which extends to a G_2 -structure on the whole M . ■

We now prove the existence in the following lemma.

Lemma 4.4 *The equation (4.3) admits a unique solution $h = (h_0, \dots, h_3)$, which is smooth in an interval $(-\varepsilon, \varepsilon)$ for $(\varepsilon \in \mathbb{R}^+)$ and satisfies the initial conditions (4.4). Moreover, the functions h_0, \dots, h_3 are even.*

Proof We use [17, Theorem 4.7] (see also [8]), which asserts that the singular initial value problem we are considering has a unique smooth solution provided the following conditions are fulfilled:

- (a) $A(\bar{h}) = 0$;
- (b) $dA|_{\bar{h}} - l \cdot \text{Id}$ is invertible for all $l \in \mathbb{N}, l \geq 1$.

Condition (a) has already been fixed, while we can easily compute

$$dA|_{\bar{h}} = \begin{bmatrix} -5 & 0 & -27a^{-2} & 2/3 \\ -81\lambda a^{-1} & -4 & 0 & 27\lambda a^{-1} \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & -2 \end{bmatrix},$$

whence

$$\det(dA|_{\bar{h}} - l \cdot \text{Id}) = l(l + 4)(l^2 + 7l + 6) > 0, \quad l \geq 1.$$

Therefore, for any $a \neq 0$, we obtain a unique solution $(h_0(t), \dots, h_3(t))$ of the singular system (4.3) with initial data \bar{h} . The fact that the solutions h_i are even follows from uniqueness and the fact that $B(h, -t) = -B(h, t)$ for all $(h, t) \in \mathbb{R}^4 \times \mathbb{R}$. ■

We now investigate the question of when two NP-structures determined by two solutions h, \bar{h} are isomorphic. By the results obtained in Subsection 3.2, we see that the associated functions $f := (f_i)_{i=0, \dots, 4}$ and $\bar{f} := (\bar{f}_i)_{i=0, \dots, 4}$ are related by a transformation τ in the group generated by τ_{12}, τ_{13} . A simple check shows that $\tau(f) = \bar{f}$ satisfies the extendability conditions (3.5), (3.6), and (3.7), if and only if $\tau = \tau_{13}$. This transformation is equivalent to reversing $a \mapsto -a$ in (4.4), and therefore, we can restrict to $a > 0$. This concludes the proof of the main Theorem 4.1.

We conclude this section by pointing out some features of a possible G-invariant NP-structure in the family \mathcal{F}_a on M that extends to a global one on some G-equivariant compactification \bar{M} . Note that \bar{M} is diffeomorphic to either S^7 or to $S^3 \times S^4$.

Proposition 4.5 *Any NP-structure in the family $\mathcal{F}_a, a > 0$, which extends to a global NP-structure on some G-equivariant compactification \bar{M} and has not constant curvature is proper, namely the associated cone metric has full holonomy Spin(7).*

Proof Let $(\bar{M}, \bar{g}, \bar{\varphi})$ be a G-invariant NP-structure extending some element in \mathcal{F}_a , say for $a = \bar{a}$, and with \bar{g} of non-constant curvature. We will denote by \bar{G} the connected component of the full isometry group of (\bar{M}, \bar{g}) . We can also suppose that \bar{G} does not act transitively on \bar{M} . Indeed, if $\bar{M} \cong S^7$, it is well known that there are precisely two homogeneous Einstein metrics, which correspond to the round and the squashed sphere (see e.g., [31]). If $\bar{M} \cong S^3 \times S^4$, using a result by Kamberich (see [24, p. 274]), \bar{G} contains a transitive subgroup N locally isomorphic to $SU(2) \times SO(5)$, acting on \bar{M} in a standard way. Now any N-invariant Riemannian metric on \bar{M} is reducible, while \bar{g} is irreducible.

Now, as \overline{M} is compact and simply connected, our claim follows if we show that the holonomy \mathcal{H} of the cone metric on $\overline{M} \times \mathbb{R}^+$ is not $SU(4)$ nor $Sp(2)$.

(a) We prove that $\mathcal{H} \neq SU(4)$, i.e., that $(\overline{M}, \overline{g})$ does not carry any Sasakian structure that is not part of a 3-Sasakian structure. If ξ is the unit length Killing vector field giving the Sasakian structure, a classical theorem by Tanno [30] states that ξ belongs to the center of \overline{g} , since \overline{g} has non-constant curvature. On the other hand the isotropy representation of K^+ at p has no non-trivial fixed vector, forcing $\xi_p = 0$, a contradiction.

(b) We now suppose that $(\overline{M}, \overline{g})$ carries a 3 Sasakian structure given by a Lie algebra $\mathfrak{s} \cong \mathfrak{sp}(1)$ generated by three unit length Killing vector fields ξ_1, ξ_2, ξ_3 . Again by a result due to Tanno [30], we know that the Lie algebra \overline{g} splits as a sum of ideals $\overline{g} = \overline{g}_0 \oplus \mathfrak{s}$, where \overline{g}_0 is the centralizer of \mathfrak{s} in \overline{g} . As \overline{G} is supposed to act non-transitively on \overline{M} , it has the same orbits as its subgroup G . In particular, \overline{G} acts transitively on $S^3 \times S^3$.

Lemma 4.6 *The semisimple part \overline{g}_s of \overline{g} is isomorphic to $3\mathfrak{su}(2)$ or to $4\mathfrak{su}(2)$.*

Proof The isotropy subalgebra \mathfrak{f} of \overline{g}_s at a G -regular point q embeds as a compact subalgebra of $\mathfrak{so}(6)$. Looking at the list of maximal subalgebras in $\mathfrak{so}(6)$, we see that $\dim \mathfrak{f} \leq 7$, unless $\mathfrak{f} \cong \mathfrak{so}(5)$ or $\mathfrak{su}(3), \mathfrak{u}(3)$. If \mathfrak{f} contains a copy of $\mathfrak{su}(3)$, it acts transitively on the unit sphere of $T_q(Gq)$; hence, $Gq \cong S^3 \times S^3$ has constant curvature, a contradiction; the case $\mathfrak{f} \cong \mathfrak{so}(5)$ can be ruled out using a result about the gaps in the dimension of the isometry group of a Riemannian manifold ([22, Thm. 3.3]). Therefore, $\dim \mathfrak{f} \leq 7$ and $\dim \overline{g}_s \leq 13$. On the other hand, \overline{G} contains a subgroup isomorphic to $SU(2)^2$ that acts on Gq (almost) freely; therefore, by a result in [24, Cor. 3, p. 237], the algebra \overline{g}_s contains an ideal \mathfrak{a} isomorphic to $2\mathfrak{su}(2)$. Then $\overline{g}_s = \mathfrak{a} \oplus \mathfrak{b}$ for some other semisimple ideal \mathfrak{b} . As $\mathfrak{g} \subseteq \overline{g}_s$ and $1 \leq \dim \mathfrak{b} \leq 7$, we see that \mathfrak{b} is isomorphic to $2\mathfrak{su}(2)$ or to $\mathfrak{su}(2)$ and our claim follows. ■

Suppose now $\overline{g}_s = 4\mathfrak{su}(2)$. This implies that the semisimple part of \overline{g}_0 , say \mathfrak{l} , is isomorphic to $3\mathfrak{su}(2)$. As $[\mathfrak{l}, \mathfrak{s}] = 0$, the isotropy \mathfrak{l}_q leaves a 3-dimensional subspace fixed, hence $\dim \mathfrak{l}_q \leq 3$. This implies that $\dim Lq = 6$, where L is the subgroup with Lie algebra \mathfrak{l} . Then L has the same orbits as G and leaves each ξ_i fixed, a contradiction by the same arguments used in (a). Therefore, we are left with $\overline{g}_s = 3\mathfrak{su}(2) = \mathfrak{g}$, i.e., the ideal \mathfrak{s} is one of the three ideals, say $\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3$ of \mathfrak{g} . If we denote by $g_i(t), i = 1, 2, 3$ the functions

$$g_1(t) := \|\widehat{e}_2\|_{\gamma(t)}^2, \quad g_2(t) := \|\widehat{e}_5\|_{\gamma(t)}^2, \quad g_3(t) := \|\widehat{e}_2 + \widehat{e}_5\|_{\gamma(t)}^2,$$

we are reduced to considering the three possibilities when g_i are constant functions.

The condition $g_1(t) = f_3^2 - f_1 f_4 = \text{const}$ can be easily ruled out using Proposition 4.3 and (4.1). Moreover, under the admissible transformation τ_{13} , the function g_2 goes over to g_3 , so that we can confine ourselves to the case $g_2(t) = \text{const}$. Using Maple, we can write the series expansion of the solutions f_i 's as well as of the function $g_2(t)$ for $\lambda = 1$ obtaining

$$g_2(t) = \frac{1}{9}a^2 + \left(-\frac{5}{576}a^2 + \frac{1}{8}a + \frac{27}{4}\right)t^2 + o(t^2).$$

Therefore, if g_2 is constant, we immediately get $a = 36, -\frac{108}{5}$, which correspond to the known homogeneous solutions. This concludes the proof. ■

Remark 4.7 By the previous result, we can also show that none of the non-homogeneous Einstein metrics in the family \mathcal{F} , whenever extended to S^7 , is one of the metrics found by Böhm in [8]. Indeed, assume g is a metric in the family \mathcal{F} that is also in Böhm's family. The NP-structure associated with g is proper, hence its full isometry group preserves the NP-structure and therefore has dimension less or equal to 9 by [19, Thm. 7.1]. On the other hand, the metrics found in [8] are invariant under a bigger group, namely, $\text{SO}(4) \times \text{SO}(4)$.

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