On the Stereometric Generation of the De Jonquières Transformation.

By Dr J. F. Tinto.

(*Read 14th February 1919. Received 11th March 1919.*)

§1. In the geometry of the plane the logical interrelations of figures may often be rendered clearer by considering the plane to be a part of space of three dimensions. Thus, by taking the plane figure as part of a more extensive configuration in space of three dimensions, the elucidation of its properties, and in particular its relation with other figures, are often facilitated. Similarly, the figures of space of three dimensions can sometimes be treated more advantageously and compendiously by considering them as parts of figures in a space of four dimensions, and so on. As a single instance we may take Segre's elegant and powerful mode of treatment of the quartic surface which possesses a nodal conic. This surface he obtains as a projection in space of four dimensions of the quartic surface which constitutes the base of a pencil of quadratic varieties.* In the following paper this mode of treatment has been applied to the interesting variety of the Cremona transformation in the plane known as the De Jonquieres transformation,† a transformation which possesses some intrinsic interest in view of the fundamental rôle which it plays in the theory of Cremona Transformations. By the aid of a surface in space of three dimensions, a variety in space of four dimensions, etc., simple constructions are given for the De Jonquières transformation between two planes, between two spaces of three dimensions, etc., respectively.

It will be found that *all* possible species of the De Jonquières transformation, whether in the plane or in a space of higher dimensions, can be derived with equal facility.

---

* Segre: *Math. Ann.* XXIV., pp. 314–444.

† De Jonquières: *Nouv. Ann.*, Ser. 2, Tome 3, pp. 97–111.

§ 2. *Derivation of the Transformation in the plane by means of a surface in space of three dimensions.*

Consider a surface of order n having a line of multiplicity $n-2$ and two multiple points of order $n-1$, A and B, both lying on this multiple line.

Using homogeneous coordinates (x_1, x_2, x_3, x_4), the equation of such a surface is (omitting constants) of the form

$$u_n + x_3 u_{n-1} + x_4 v_{n-1} + x_3 x_4 u_{n-2} = 0, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

where u_n, u_{n-1}, etc., are homogeneous functions of x_1, x_2 of the degree indicated by the suffix.

The tetrahedron of reference has been so chosen that A is the point $(x_1, x_2, x_3 = 0)$, B the point $(x_1, x_2, x_4 = 0)$, and AB the line $(x_1, x_2 = 0)$.

A plane section of the surface through AB is a conic which passes through the points A and B.

Substituting $x_2 = \lambda x_1$ in (1), we get

$$u_n(\lambda) x_1^2 + u_{n-1}(\lambda) x_3 x_1 + v_{n-1}(\lambda) x_4 x_1 + u_{n-2}(\lambda) x_4 x_3 = 0, \qquad (2)$$

which is the equation of the conic determined by the plane $x_2 = \lambda x_1$.

For a finite number of positions of this plane the conics will be line-pairs, and the number of such will be obtained from the discriminant of (2). Forming the discriminant, we have

$$\Delta \equiv \begin{vmatrix} 2u_n(\lambda) & u_{n-1}(\lambda) & v_{n-1}(\lambda) \\ u_{n-1}(\lambda) & 0 & u_{n-2}(\lambda) \\ v_{n-1}(\lambda) & u_{n-2}(\lambda) & 0 \end{vmatrix} = 0.$$

It is evident that the left side contains a factor u_{n-2} and the remaining factor is of order $2(n-1)$.

The planes given by $u_{n-2}(\lambda) = 0$ meet the surface in the conics $u_n(\lambda) x_1^2 + u_{n-1}(\lambda) x_3 x_1 + v_{n-1}(\lambda) x_4 x_1 = 0$; *i.e.*, in $x_1 = 0$ and $u_n(\lambda) x_1 + u_{n-1}(\lambda) x_3 + v_{n-1}(\lambda) x_4 = 0$.

Hence each of the planes $u_{n-2}(\lambda) = 0$ meets the surface in a line coinciding with AB, and in another line which does not contain either A or B.

The other factor of order $2(n-1)$ gives rise to $2(n-1)$ proper tangent planes to the surface, each plane intersecting the surface in two lines, one of which passes through A and the other through B. The surface consequently possesses $2(n-1)$ lines through A and an equal number through B, and these lines evidently lie on the tangent cones at A and B respectively, as do also the $n-2$ lines which coincide with AB.

[The above type of surface is a modification of the general surface of order n, having a line of multiplicity $n-2$. The discriminant for this surface is of order $3n-4$, and the number of lines on the surface $6n-8$, none of which coincide with AB. If we move $n-2$ of these lines, one along each of the $n-2$ sheets into coincidence with AB, we obtain the type with two conical points of order $n-1$. When a line is indefinitely close to AB, the tangent plane to its sheet does not vary as we move along AB. The latter is consequently torsal for each sheet, and we infer the existence of the two conical points.]

Let π_A and π_B be any two planes intersecting the above surface, π_A being associated with A and π_B with B. Let the intersections of the lines through A with π_A be called A_r, $r = 1, 2, \ldots 2(n-1)$, with a similar notation B_r for the points on π_B. Lastly, let AB meet π_A in X_A and π_B in X_B. Then if P_A be any point on π_A, the line AP_A meets the surface in a single point p, and Bp meets π_B in a point P_A. Thus, to any point P_A on π_A there corresponds a unique point P_B on π_B, and *vice-versa*. The correspondence is consequently $(1, 1)$. Again, if l_A be any line on π_A, the plane Al_A intersects the surface in a curve of order n, having a node of order $n-1$ at A. The cone projecting this curve from B meets π_B in a curve of order n, having a node of order $n-1$ at X_B, and containing the $2(n-1)$ points B_r.

Similarly, a line l_B on π_B transforms into a curve of order n, having a node of $n-1$ at X_A, and containing the points A_r.

The transformation thus effected between the planes π_A and π_B is therefore a De Jonquières transformation.

§ 3. *The Fundamental Points.*

It is at once evident from the above construction that the correspondent of a point A_r is a line on π_B through X_B. Consequently A_r, $r = 1, 2, \ldots 2(n-1)$ are simple F-points on π_A, and a similar statement is true of the points B_r.

If we join a point indefinitely close to X_A to A, the joining line meets the surface in a point indefinitely close to B, and hence the correspondent of X_A is the section by π_B of the tangent cone at B, viz., a curve of order $n-1$, having a node of order $n-2$ at X_B and containing the $2(n-1)$ points B_r. X_A is thus a F-point of order $n-1$, and X_B is a similar point on π_B.

§ 4. *The Perspective Transformation in the Single Plane.*

If instead of π_A and π_B we take a single plane π, we get a correspondence between its points such that any two corresponding points lie on a line through X where π meets AB.

The F-points A_r and B_r for the same value of r also lie on lines through X. The curve of intersection of the surface with π, viz., a curve of order n with a node of order $n-2$ at X, is a curve of self-corresponding points for the transformation.

§ 5. *The Involutive Transformation in the Single Plane.*

In the perspective transformation the correspondent of any point P on π will be different, according as we join it first to A or to B.

Hence we must consider any point P on π as belonging to two systems, which we may typify as the "A" and the "B" systems. The transformation in which the correspondent of P is the same point, whether we consider it as belonging to the "A" system or to the "B" system, is called an involutive transformation. To obtain it we modify our construction as follows. Let C be any *ordinary* point on the multiple line AB. To find the correspondent of a point P on π, join PA, which we suppose as before, to meet the surface in p. Join Cp. The line Cp must meet the surface in a second point q, and Aq will determine on π a point Q, which we take as the correspondent of P. The correspondence is evidently (1, 1), and involutive. A line in the plane π determines with A, a plane section of the surface of order n, which possesses a node of order $n-1$ at A. The cone which projects this curve from C meets the surface in a curve of order $n^2-(n-1)(n-2)-n$, *i.e.*, $2(n-1)$. It is readily verified that the $n-2$ lines of the

https://doi.org/10.1017/S0013091500035392 Published online by Cambridge University Press

surface which coincide with AB form part of this curve, and hence the latter reduces to a proper curve of order n. It meets AB in $n-1$ points, one on each of the tangent planes through AB to the cone whose vertex is C. The curve of order n consequently projects from A into a curve of order n on π with a node of order $n-1$ at X. The points A_r, $r=1, 2, \ldots 2(n-1)$ are the simple F-points of the transformation, and X is the F-point of order $n-1$.

The self-corresponding points of the transformation lie, as in the perspective transformation, on a curve of order n, having a node of order $n-2$ at X and containing the $2(n-1)$ points A_r.

In the present case, however, this curve is not the curve of intersection of π with the surface. The tangents from C to the surface lie on a cone of order $2(n-1)$, which has AB as generator of order $2n-4$. This cone meets the surface in a curve of order $\frac{2n(n-1)-(2n-4)(n-2)}{2}$, *i.e.*, $3n-4$, and the $n-2$ lines which coincide with AB are to be reckoned twice as part of this curve. The remaining part, a proper curve of order n which meets AB in $n-2$ points, is the curve of contact of the tangent cone from C, and the projection of this curve from A, viz., a curve of order n with a node of order $n-2$ at X, is the locus of self-corresponding points on π. It intersects the curve of section of π with the surface in the $2(n-1)$ points A_r, and has the same tangents at its multiple point X as the latter.

It may be noted that the transformation, besides being involutive, is also perspective in character.

§ 6. *Specialised Transformations in which there are Coincidences amongst the F-points.*

Transformations in which two or more of the simple F-points coincide can be derived as follows.

The surface may possess other singularities besides the multiple line AB and the multiple points A and B. These additional singularities can only be double points.

The conditions for a double point are

$$u'_{(x_1)n} + x_3\, u'_{(x_1)n-1} + x_4\, v'_{(x_1)n-1} + x_3 x_4\, u'_{(x_1)n-2} = 0$$
$$u'_{(x_2)n} + x_3\, u'_{(x_2)n-1} + x_4\, v'_{(x_2)n-1} + x_3 x_4\, u'_{(x_1)n-2} = 0$$
$$u_{n-1} + x_4\, u_{n-2} = 0$$
$$v_{n-1} + x_3\, u_{n-2} = 0$$

where the dashes denote differentiation.

Eliminating $x_3\, x_4$, we get two equations of form

$$u'_n\, u^2_{n-2} - u_{n-2}\, v_{n-1}\, u'_{n-1} - u_{n-2}\, u_{n-1}\, v'_{n-1} + u'_{n-2}\, u_{n-1}\, v_{n-1} = 0 \quad \ldots(3)$$

where the dashes denote differentiation with respect to x_1 and x_2.

Multiplying these equations by x_1 and x_2 and adding, we get on simplifying

$$\Delta \equiv u_n\, u_{n-2} - u_{n-1}\, v_{n-1} = 0,$$

which is the simplified form of the discriminant used above.

The equation (3) is the condition that Δ should have a double factor, and hence we conclude that, when the surface has a double point, two of the $2(n-1)$ lines through A coincide and pass through the double point, as do also two of the lines through B.

In the transformation, therefore, two of the simple F-points coincide.

It is evident that since $2(n-1)$ lines in general pass through A and through B, the surface may have $n-1$ double points of type C_2.

If the double point is a binode of type B_k (k being the reduction in the class of a surface which it produces) k of the lines through A and k through B will coincide. The transformation has now k coincident simple F-points; k may have any value from 3 to $2(n-1)$.*

Coincidence of simple F-points with the multiple F-point of order $n-1$.

The tangent cone to the surface at A or B may degenerate. The equation of the cone at A is $v_{n-1} + x_3\, u_{n-2} = 0$, and if v_{n-1}, u_{n-2} have k common factors, we find at once by examining equation (2) that an additional number k of the lines through A coincide with AB. Consequently k of the simple F-points on π_A coincide with X_A, but if u_{n-2} has no factor in common with u_{n-1}, none of the F-points on π_B will coincide with X_B. The greatest number of such coincidences possible is evidently obtained by supposing u_{n-2} and v_{n-1} to have $n-2$ common factors. The tangent cone at A now consists of $n-2$ planes through AB and another plane through A. The remaining n F-points on π_A therefore lie on a straight line.

---

* For the required conditions see § 13.

If we wish more than $n-2$ of the lines through A to coincide with AB, we must suppose the surface to become a ruled surface with AB as multiple line of order $n-1$. The points A and B will then have no special peculiarity. Let the equation of the ruled surface be $u_n + x_3 u_{n-1} + x_4 v_{n-1} = 0$, where the letters have the same meaning as before. There will be $n-1$ generators through A and $n-1$ through B, and these give rise to $n-1$ simple F-points on each plane. A plane section of the surface through A projects from B into a curve of order n with a node of $n-2$ at X^B with $n-2$ fixed tangents. The remaining $n-1$ simple F-points therefore coincide with X_B, there being one on each tangent. The tangent planes to the various sheets of the surface at any point (x_3, x_4) on AB are given by $X_3 u_{n-1} + X_4 v_{n-1} = 0$.

If u_{n-1} and v_{n-1} have l simple factors, the tangent plane given by any of these will be a tangent plane for all points on AB, and it will meet the surface in n lines coinciding with AB. Hence there will be $n-l-1$ generators through A and through B, and the transformation will have the same number of simple F-points distinct from the multiple F-point. The number of points which coincide with the multiple point is now $n+l-1$.

If we suppose u_{n-1} and v_{n-1} to have $n-1$ factors in common, the tangent planes at $(x_3 x_4)$ reduce to the single plane $x_3 + x_4 = 0$, and the surface is clearly a cone of which AB is a multiple generator of order $n-1$.

All of the simple F-points now coincide with the multiple F-point.

With the ruled and conical surfaces two or more of the sheets may unite into a single cuspidal sheet, and this will cause coincidences to take place amongst the simple F-points.

§7. *The Analogue of the De Jonquières Transformation in space of three dimensions.**

The mode of obtaining this transformation is precisely analogous to the preceding case of the plane.

We consider a variety of V_n of order n in a space of four dimensions having a line AB of multiplicity $n-2$, with A and B two multiple points of order $n-1$ on it.

* First given by Noether: *Math. Ann.* III., pp. 547-580.

By means of this variety we may, as before, set up a (1, 1) correspondence between the points of two three-dimensional spaces π_A^3 and π_B^3 associated respectively with A and B. Any three-space through A intersects V_n in an ordinary surface of order n having a multiple point of order $n-1$ at A, and this surface is projected from B into a similar surface in π_B^3. The three-space containing A intersects π_A^3 in an ordinary plane, and hence we get for the correspondent of a plane in π_A^3 a "monoid" of order n in π_B^3, having its multiple point of order $n-1$ at X_B where AB intersects π_B^3. A plane in π_B^3 transforms in the same way into a monoid in π_A^3.

§8. *The Fundamental System.*

The tangent cone (α^2 lines) to the variety at A meets it in a singly infinite system of lines of order $n(n-1)$, which we shall call the cone of intersection.

Let the equation of the variety be

$$u_n + x_4 u_{n-1} + x_5 v_{n-1} + x_4 x_5 u_{n-2} = 0 \quad\quad (4)$$

where u_n, etc., are homogeneous functions of x_1, x_2, x_3. The multiple line is $(x_1, x_2, x_3) = 0$, and the points A and B $(x_1, x_2, x_3, x_4) = 0$, $(x_1, x_2, x_3, x_5) = 0$ respectively.

Let the equation of any line through A be

$$x_1 = X_1 t, \quad x_2 = X_2 t, \quad x_3 = X_3 t, \quad x_4 = X_4 t,$$

where t is a parameter.

Substituting these values in equation (4) we obtain

$$t\{u_n(X_1 X_2 X_3) + X_4 u_{n-1}(X_1 X_2 X_3)\}$$
$$+ x_5\{v_{n-1}(X_1 X_2 X_3) + X_4 u_{n-2}(X_1 X_2 X_3)\} = 0.$$

The conditions that the line should be entirely on the variety are therefore

$$u_n(X_1 X_2 X_3) + X_4 u_{n-1}(X_1 X_2 X_3) = 0 \quad\quad (5)$$

and

$$v_{n-1}(X_1 X_2 X_3) + X_4 u_{n-2}(X_1 X_2 X_3) = 0 \quad\quad (6)$$

(6) is obviously the tangent cone at A, and (5) represents another cone through A. The intersection of (5) and (6) is evidently the cone of intersection. It is of degree $n(n-1)$, and has $(x_1, x_2 x_3 = 0)$ as generator of multiplicity $(n-1)(n-2)$. Hence, it meets π_A^3 in a twisted curve of order $n(n-1)$, which possesses a node of order $(n-1)(n-2)$ at X_A.

https://doi.org/10.1017/S0013091500035392 Published online by Cambridge University Press

This curve is a curve of simple F-points in π_A^3, for corresponding to any point on it (except X_A) we get a line through X_B. Corresponding to X_A we get the section by the tangent cone at B made by π_B^3, viz., an ordinary surface of order $n-1$ having X_B as multiple point of order $n-2$, and containing the simple F-curve in π_A^3. X_A and X_B are F-points of order $n-1$.

Eliminating X_4 between (5) and (6) we get

$$u_n u_{n-2} - v_{n-1} u_{n-1} = 0,$$

which is the equation of α^2 three-spaces through AB, each of which contains a generator of the cone of intersection. This system of three-spaces meets π_B^3 in a system of planes through X_B, which envelop a cone of order $2(n-1)$. This cone is the F-surface corresponding to the F-curve in π_A^3, and it contains the F-curve in π_B^3.

§ 9. *The Perspective Transformation.*

This is obtained, as in the previous case, by taking, instead of π_A^3 and π_B^3, a single three-space π^3. The simple F-curves both lie on a cone of order $2(n-1)$ with vertex at x, where π^3 meets AB; while corresponding points of the transformation are collinear with X. The section of the variety by π^3, viz., a surface of order n with a point of order $n-2$ at X, is a surface of self-corresponding points.

§ 10. *The Involutive Transformation.*

This is also obtained, as in the previous case, by taking C an ordinary point on AB. There is a single F-curve of order $n(n-1)$ with a node of order $(n-1)(n-2)$ at X.

The surface of self-corresponding points is a surface of order n having a multiple point of order $n-2$ at X.

§ 11. *Specialised Transformations.*

The cone of intersection given by (5) and (6) may possess in addition to AB other multiple generators. The order of such generators can be at most *two*. When the cone of intersection possesses a double generator the F-curve in π_A^3 has a node of order 2. Again, the tangent cone to the variety at A may, as in the

https://doi.org/10.1017/S0013091500035392 Published online by Cambridge University Press

previous case, be degenerate and composed of three-spaces through AB together with a proper tangent cone. If it be composed of k three-spaces, the F-surface in π_A^3 will consist of k planes through X_A, and a monoid of order $n-k-1$. The F-curve will consequently be made up of k plane curves of order n, each with a node of order $n-1$ at X_A and a twisted curve of order $n(n-k-1)$, with a node of order $(n-1)(n-k-2)$ at the same point.

Finally, the F-curve may be made up of $n-2$ plane curves lying in planes through X_A and a curve whose plane does not contain X_A. If now we take the "ruled" variety whose equation is $u_n + x_4 u_{n-1} + x_5 v_{n-1} = 0$ and two points A and B on the multiple line, we obtain a transformation in which the F-surface corresponding to X_A or X_B is a cone of order $n-1$ containing the F-curve. If u_{n-1} and v_{n-1} have l common linear factors, this cone degenerates into l planes and a proper cone of order $n-l-1$.

When u_{n-1} and v_{n-1} have $n-1$ linear factors in common, the variety becomes a cone and the F-surface is entirely composed of planes.

§ 12. *The Analogues of the De Jonquières Transformation in hyper-spaces of four or more dimensions.*

The development of such transformations proceeds on precisely the same lines as in the case of three dimensions.

Starting with a hyper-surface

$$u_n + x_r u_{n-1} + x_{r+1} v_{n-1} + x_r x_{r+1} u_{n-2} = 0$$

in a space of r dimensions where u_n, u_{n-1}, etc., are homogeneous functions of $r-1$ coordinates $x_1 \ldots x_{r-1}$, we can obtain transformations between two spaces of $r-1$ dimensions which are in strict analogy with those given for ordinary spaces of three dimensions.

In conclusion, it may be remarked that if we take corresponding sections of the configuration in a space of r dimensions through X_A and X_B, the F-points of order $n-1$, by spaces of $r-1$ dimensions, the transformation between the latter will be also of the De Jonquières type, and the F-systems will be the intersection of the F-systems of the r-space with the intersecting spaces.

§ 13. *Conditions for binode of type B_k on the surface* (1).

Let $u_n(x_1, x_2) \equiv a_{n,0} x_1^n + a_{n-1,1} x_1^{n-1} x_2 + \dots$

$$\dots + a_{1,n-1} x_1 x_2^{n-1} + a_{0,n} x_2^n$$

$$u_{n-1}(x_1, x_2) \equiv b_{n-1,0} x_1^{n-1} + b_{n-2,1} x_1^{n-2} x_2 + \dots$$

$$\dots + b_{1,n-2} x_1 x_2^{n-2} + b_{0,n-1} x_2^{n-1}$$

$$v_{n-1}(x_1, x_2) \equiv c_{n-1,0} x_1^{n-1} + \dots$$

$$u_{n-2}(x_1, x_2) \equiv d_{n-2,0} x_1^{n-2} + \dots$$

The discriminant $\Delta \equiv u_n u_{n-2} - u_{n-1} v_{n-1} = 0$.

Substituting $x_2 = \lambda x_1$ in Δ we get

$$u_n(\lambda) u_{n-2}(\lambda) - u_{n-1}(\lambda) v_{n-1}(\lambda) = 0.$$

If $a_{n,0} = 0$, one of the roots is $\lambda = 0$, and the point $x_2, x_3, x_4 = 0$ is the point of contact of the corresponding plane.

If $a_{n-1,1} = b_{n-1,0} = c_{n-1,0} = 0$, $\lambda = 0$ is a repeated root, and the point $x_2, x_3, x_4 = 0$ is a conic node on the surface.

If $\lambda = 0$ is a thrice repeated root of $\Delta = 0$,

$$a_{n-2,2} d_{n-2,0} - b_{n-2,1} c_{n-2,1} = 0,$$

and this is the condition for a B_3.

The conditions for a B_4, B_5, etc., may be successively deduced in the same way.