Rational cohomology of $\mathcal{M}_{3,2}$

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Dedicated to Joseph Steenbrink on his 60th birthday

ABSTRACT

We compute the rational cohomology of the moduli space of non-singular complex plane quartic curves with two marked points. This allows us to calculate the rational cohomology of the moduli space of non-singular complex projective curves of genus three with two marked points.

1. Introduction

Let us denote by $\mathcal{M}_{3,2}$ the moduli space of non-singular complex projective curves with two marked points, and by $\mathcal{Q}_2$ the moduli space of plane quartic curves with two marked points. In this paper, we prove the following.

THEOREM 1.1. *The rational cohomology groups of $\mathcal{Q}_2$ and $\mathcal{M}_{3,2}$, with their mixed Hodge structures and their structures as $\mathfrak{S}_2$-representations, are as follows.*

(i) *We have*

$$
H^k(\mathcal{Q}_2; \mathbf{Q}) = \begin{cases}
\mathbf{S}_2 \otimes \mathbf{Q} & k = 0, \\
(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(-1) & k = 2, \\
\mathbf{S}_2 \otimes \mathbf{Q}(-3) & k = 5, \\
\mathbf{S}_2 \otimes \mathbf{Q}(-6) & k = 6, \\
(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(-7) + \mathbf{S}_{1,1} \otimes \mathbf{Q}(-8) & k = 8, \\
0 & otherwise.
\end{cases}
$$

(ii) *We have*

$$
H^k(\mathcal{M}_{3,2}; \mathbf{Q}) = \begin{cases}
\mathbf{S}_2 \otimes \mathbf{Q} & k = 0, \\
(\bigoplus^2 \mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(-1) & k = 2, \\
(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(-2) & k = 4, \\
\mathbf{S}_2 \otimes \mathbf{Q}(-3) & k = 5, \\
\mathbf{S}_2 \otimes \mathbf{Q}(-6) & k = 6, \\
(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(-7) & k = 8, \\
0 & otherwise.
\end{cases}
$$

Note that the Euler characteristic of $\mathcal{M}_{3,2}$ in the Grothendieck group of mixed Hodge structures was computed in [BT07], exploiting Bergström's count of the number of points of $\mathcal{M}_{3,2}$ defined over finite fields [Ber06]. Another computation of this Euler characteristic can be found in [Tom05b], ch. III]. All of these results agree with the topological Euler characteristic of $\mathcal{M}_{3,2}$ as calculated in [BH05].

It is well known that the canonical model of a non-hyperelliptic curve of genus three is a smooth quartic in the projective plane. Hence, $\mathcal{Q}_2$ is the complement of the hyperelliptic locus $\mathcal{H}_{3,2}$ inside $\mathcal{M}_{3,2}$. Let us start by considering the moduli space $\mathcal{Q}$ of smooth quartic curves in the projective plane. Quartic curves are defined by the vanishing of polynomials of degree four in three indeterminates, i.e. by elements of $S_4^2 := \mathbf{C}[x_0, x_1, x_2]_4$. Clearly, not every element of S_4^2 defines a non-singular curve, but we have to exclude the locus $\Sigma_4^2 \subset S_4^2$ of singular polynomials. The action of $G = \mathrm{GL}(3)$ on the coordinates x_0, x_1, x_2 induces an action on $S_4^2 \setminus \Sigma_4^2$, and $\mathcal{Q}$ is the geometric quotient of $S_4^2 \setminus \Sigma_4^2$ by the action of G.

The rational cohomology of $S_4^2 \setminus \Sigma_4^2$ was computed by Vassiliev in [Vas99]. Comparing this result with the rational cohomology of the moduli space $\mathcal{Q}$, as computed by Looijenga in [Loo93], one observes that the cohomology of the space of non-singular polynomials in S_4^2 is isomorphic (as a graded vector space) to the tensor product of the cohomology of the moduli space $\mathcal{Q}$ and that of $G = \mathrm{GL}(3)$. Indeed, Peters and Steenbrink [PS03] proved that this is always the case when comparing the rational cohomology of the space of non-singular homogeneous polynomials with the cohomology of the corresponding moduli space of smooth hypersurfaces.

As explained in [BT07, §5], Peters–Steenbrink's result can be adapted to moduli spaces of smooth hypersurfaces with m marked points, when m is small enough. This requires replacing the space S_d^n of homogeneous polynomials of degree d in $x_0, \ldots, x_n$ with a certain incidence correspondence. In our case ($n = 2, d = 4, m = 2$) we set

$$\mathcal{I}_2 := \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) \times (S_4^2 \setminus \Sigma_4^2) : f(\alpha) = f(\beta) = 0\},$$

where $F(\mathbf{P}^2, 2)$ denotes the complement of the diagonal in $\mathbf{P}^2 \times \mathbf{P}^2$. The action of $G = \mathrm{GL}(3)$ on $\mathbf{P}^2$ and S_4^2 can be extended to $\mathcal{I}_2$, and the geometric quotient $\mathcal{I}_2/G$ is isomorphic to $\mathcal{Q}_2$. Then the following isomorphism of graded vector spaces with mixed Hodge structures holds:

$$H^\bullet(\mathcal{I}_2; \mathbf{Q}) \cong H^\bullet(\mathcal{Q}_2; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3); \mathbf{Q}). \tag{1.1}$$

This follows from [PS03], in view of [BT07, Theorem 5.2]. As a consequence, we have that determining the rational cohomology of $\mathcal{I}_2$ immediately yields the rational cohomology of $\mathcal{Q}_2$. Note that the isomorphism (1.1) is compatible with the action of the symmetric group $\mathfrak{S}_2$ on the cohomology groups of $\mathcal{Q}_2$ and $\mathcal{I}_2$ induced by the involution interchanging the two marked points.

We compute $H^\bullet(\mathcal{I}_2; \mathbf{Q})$ by studying the natural projection $\pi_2 \colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. The map π_2 is a locally trivial fibration, whose fibre is the complement of Σ_4^2 in a linear subspace of S_4^2. Therefore, we can compute the cohomology of this fibre with Vassiliev–Gorinov's method for the cohomology of complements of discriminants. The study of the Leray spectral sequence associated to the fibration π_2 allows us to determine the cohomology of $\mathcal{I}_2$. In this last step, we will use the relation (1.1) very often.

The plan of the paper is as follows. In §§2 and 3 we compute the rational cohomology of $\mathcal{Q}_2$ and prove Theorem 1.1 by the methods explained above. We conclude the paper with a concise review of Vassiliev–Gorinov's method.

Notation

We use the following notation in this paper:

- $\mathfrak{S}_n$, the symmetric group in n letters;
- S_d^n, the vector space of homogeneous polynomials of degree d in $n+1$ indeterminates $x_0, \ldots, x_n$;
- Σ_d^n, the locus of singular polynomials in S_d^n;
- $K_0(\mathsf{HS_Q})$, the Grothendieck group of rational (mixed) Hodge structures over $\mathbf{Q}$;

- $K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$, the Grothendieck group of rational (mixed) Hodge structures endowed with an action of the symmetric group $\mathfrak{S}_n$;
- $\mathbf{Q}(m)$, the Tate Hodge structure of weight $-2m$;
- $\mathbf{L}$, the class of $\mathbf{Q}(-1)$ in $K_0(\mathsf{HS}_{\mathbf{Q}})$;
- $\mathbf{S}_\lambda$, the $\mathbf{Q}$-representation of $\mathfrak{S}_n$ indexed by the partition $\lambda \vdash n$;
- s_λ, the Schur polynomial indexed by the partition $\lambda \vdash n$;
- Δ_j, the j-dimensional closed simplex;
- $\mathring{\Delta}_j$, the interior of the j-dimensional closed simplex;
- $F(Z, k)$, the space of ordered configurations of k distinct points on the variety Z (see Definition 4.5);
- $B(Z, k)$, the space of unordered configurations of k distinct points on the variety Z (see Definition 4.5).

Throughout this paper we will make an extensive use of Borel–Moore homology, i.e. homology with locally finite support. An example reference for its definition and the properties we use is [Ful84, ch. 19].

To write the results on cohomology and Borel–Moore homology groups in a compact way, we will express them by means of polynomials in the following way. Let $T_\bullet$ denote a graded $\mathbf{Q}$-vector space with mixed Hodge structures. For every $i \in \mathbf{Z}$, we can consider the class $[T_i]$ in the Grothendieck group of rational Hodge structures. We define the Hodge–Grothendieck polynomial (HG polynomial) of $T_\bullet$ to be the polynomial

$$\wp(T_\bullet) := \sum_{i \in \mathbf{Z}} [T_i] t^i \in K_0(\mathsf{HS}_{\mathbf{Q}})[t].$$

If, moreover, a symmetric group $\mathfrak{S}_n$ acts on $T_\bullet$ respecting the grading and the mixed Hodge structures on $T_\bullet$, we define the $\mathfrak{S}_n$-*equivariant* Hodge–Grothendieck polynomial ($\mathfrak{S}_n$-HG polynomial) $\wp^{\mathfrak{S}_n}(T_\bullet)$ by replacing $K_0(\mathsf{HS}_{\mathbf{Q}})$ by $K_0(\mathsf{HS}_{\mathbf{Q}}^{\mathfrak{S}_n})$ in the definition of the HG polynomial.

2. Rational cohomology of $\mathcal{Q}_2$

Consider the space S_4^2 of homogeneous quartic polynomials in x_0, x_1, x_2, and denote by $\Sigma := \Sigma_4^2$ the discriminant, i.e. the locus of singular quartic polynomials. For every $p \in \mathbf{P}^2$, denote by V_p the linear subspace of S_4^2 of polynomials vanishing at p. The aim of this section is to calculate the rational cohomology of the incidence correspondence $\mathcal{I}_2 = \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) \times (S_4^2 \setminus \Sigma) : f(\alpha) = f(\beta) = 0\}$. Note that knowing the cohomology of $\mathcal{I}_2$ is equivalent to knowing the cohomology of its projectivization

$$\mathcal{P}_2 = \{(\alpha, \beta, [f]) \in F(\mathbf{P}^2, 2) \times \mathbf{P}(S_4^2 \setminus \Sigma) : f(\alpha) = f(\beta) = 0\},$$

as the rational cohomology of $\mathcal{I}_2$ is isomorphic to the tensor product of the cohomology of $\mathcal{P}_2$ and $H^\bullet(\mathbf{C}^*; \mathbf{Q})$.

We start by applying Vassiliev–Gorinov's method (see §4) to the calculation of the cohomology of $(V_p \cap V_q) \setminus \Sigma$, where p and q are two fixed distinct points in $\mathbf{P}^2$. Next, we consider the Leray spectral sequence for the natural projection $\pi_2 \colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. Note that the map π_2 is a locally trivial fibration with fibre isomorphic to $V_p \cap V_q \setminus \Sigma$.

By Alexander's duality between reduced cohomology and Borel–Moore homology, we have

$$\tilde{H}^\bullet((V_p \cap V_q) \setminus \Sigma; \mathbf{Q}) \cong \bar{H}_{25-\bullet}(V_p \cap V_q \cap \Sigma; \mathbf{Q})(-13). \tag{2.1}$$

To apply Vassiliev–Gorinov's method to $V_p \cap V_q \cap \Sigma$, we need an ordered list of all possible singular sets of the elements in $V_p \cap V_q \cap \Sigma$. We can easily obtain such a list by an adaptation of the list of possible singular configurations of quartic curves (such as that in [Vas99, Proposition 6]).

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For every configuration in the list, one has to distinguish further whether or not the singular points are in general position with respect to p and q (for instance, whether or not p or q are contained in the singular configuration). This procedure yields a complete list of singular sets of elements of $V_p \cap V_q$; let us denote by R the number of types of configurations in the list. As recalled in §4, Vassiliev–Gorinov's method gives a recipe to construct spaces $|\mathcal{X}|$, $|\Lambda|$ and a map

$$|\epsilon| \colon |\mathcal{X}| \longrightarrow V_p \cap V_q \cap \Sigma$$

inducing an isomorphism on Borel–Moore homology. The Borel–Moore homology of $|\mathcal{X}|$ (respectively, $|\Lambda|$) can be computed by considering the stratification $\{F_j\}_{j=1,\ldots,R}$ (respectively, $\{\Phi_j\}_{j=1,\ldots,R}$). The properties of F_j and Φ_j are explained in Proposition 4.3. Recall, in particular, that F_j is the total space of a vector bundle over Φ_j, and that for finite configurations the Borel–Moore homology of Φ_j coincides (after a shift in the indices) with the Borel–Moore homology of the space of configurations of type j with coefficients in a rank one local system changing its orientation every time two points in a configuration are interchanged.

In our case, for most indices $j \in \{1, \ldots, R\}$ the space of singular configurations of type j has trivial Borel–Moore homology in the appropriate system of coefficients. Hence, the strata F_j have trivial Borel–Moore homology. In view of Lemma 4.6, this is the case for configurations with too many points lying on the same rational curve. Furthermore, the same occurs for configurations containing rational curves as components (see [Tom05a, Lemma 2.17] and following remarks).

In Table 1 we list all remaining configurations, i.e. all singular configurations indexing strata that give a non-trivial contribution to the Borel–Moore homology of $\Sigma \cap V_p \cap V_q$. In the same table, we also give a description of the strata of $|\Lambda|$ and $|\mathcal{X}|$ corresponding to each configuration. From the descriptions, it is straightforward to compute the Borel–Moore homology of the strata Φ_j and F_j for $1 \leqslant j \leqslant 7$. The most difficult strata (corresponding to configurations of type 8, 9 and 10) are studied separately in §3. The results there, together with the description of the strata given in Table 1, allow us to compute the E^1 terms of the spectral sequences in Borel–Moore homology converging to $|\Lambda|$ and $|\mathcal{X}|$, induced by the filtrations associated, respectively, with $\{\Phi_j\}$ and $\{F_j\}$.

The columns of the spectral sequences converging to the Borel–Moore homology of $|\mathcal{X}|$ and $|\Lambda|$ can be divided into two blocks: one with the first seven columns, the other with columns 8, 9 and 10. Looking at Hodge weights, one can easily prove that all differentials in the spectral sequence between columns in the block 1–7 and in the block 8–10 are trivial. Furthermore, this behaviour carries on when one investigates the Leray spectral sequence associated to the fibration π_2. Therefore, we consider the two blocks separately. The contribution of columns 1–7 is computed below. The contribution of columns 8–10 to the rational cohomology of $\mathcal{I}_2$ is computed in §3.

In the spectral sequence converging to the Borel–Moore homology of $|\Lambda|$ all terms in the first seven columns are killed by differentials, with the exception of an $\mathfrak{S}_2$-invariant one-dimensional homology group in degree zero. This follows for dimensional reasons: if these classes were not killed, they would give rise to cohomology classes of degree at least 14 in the cohomology of $(V_p \cap V_q) \setminus \Sigma$, and this is impossible because the latter is affine of dimension 13. As a consequence, the strata 1–7 do not contribute to the Borel–Moore homology of the open cone F_{10}.

The first seven columns of the spectral sequence converging to the Borel–Moore homology of $\mathrm{Fil}_j |\mathcal{X}|$ are given in Table 2. Note that the description of the strata of the domain $|\mathcal{X}|$ of the geometric realization given in Table 1 allows us to study the behaviour of each Borel–Moore homology class with respect to the interchange of the points p, q. Table 2 also includes the information on the $\mathfrak{S}_2$-action generated by this involution.

In the spectral sequence in Table 2, the only possibly non-trivial differential is $d_2 \colon E^2_{5,15} \to E^2_{3,16}$. This is certainly zero, because otherwise we would get a contradiction with the isomorphism

$$H^\bullet((V_p \cap V_q) \setminus \Sigma; \mathbf{Q}) \cong H^\bullet(\mathbf{C}^*; \mathbf{Q}) \otimes H^\bullet(\mathbf{P}((V_p \cap V_q) \setminus \Sigma); \mathbf{Q}).$$

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TABLE 1. Singular configurations and their contribution.

1.	The point p or the point q. Stratum: F_1 is a $\mathbf{C}^{11}$-bundle over $\Phi_1 = \{p, q\}$.
2.	Any point different from p, q. Stratum: F_2 is a $\mathbf{C}^{10}$-bundle over $\Phi_2 \cong \mathbf{P}^2 \setminus \{p, q\}$.
3.	The pair $\{p, q\}$. Stratum: F_3 is a $\mathbf{C}^9$-bundle over $\Phi_3 \cong \mathring{\Delta}_1$.
4a.	Pairs of points on the line pq, different from $\{p, q\}$. Stratum: F_{4a} is a $\mathbf{C}^8$-bundle over Φ_{4a}, which is a non-orientable $\mathring{\Delta}_1$-bundle over a space which can be decomposed as the disjoint union of $\mathbf{C}^*$ and $B(\mathbf{C}, 2)$.
4b.	Pairs of points $\{a, b\}$ with $a \in \{p, q\}$, $b \notin pq$. Stratum: F_{4b} is a $\mathbf{C}^8$-bundle over Φ_{4b}, which is a non-orientable $\mathring{\Delta}_1$-bundle over the disjoint union of two copies of $\mathbf{C}^2$.
5.	Pairs of points $\{a, b\}$ with $a \in (pq \setminus \{p, q\})$, $b \notin pq$. Stratum: F_5 is a $\mathbf{C}^7$-bundle over Φ_5, which is a non-orientable $\mathring{\Delta}_1$-bundle over $\mathbf{C}^* \times \mathbf{C}^2$.
6.	Triplets consisting of p, q and another point outside pq. Stratum: F_6 is a $\mathbf{C}^6$-bundle over Φ_6, which is a $\mathring{\Delta}_2$-bundle over $\mathbf{C}^2$.
7.	Triplets with two points on pq (not both in $\{p, q\}$) and another point outside pq. Stratum: F_7 is a $\mathbf{C}^5$-bundle over Φ_7, which is a non-orientable $\mathring{\Delta}_2$-bundle over a space that can be decomposed as the disjoint union of $\mathbf{C}^* \times \mathbf{C}^2$ and $B(\mathbf{C}, 2) \times \mathbf{C}^2$.
8.	Five points $a, b, c, d, e \in \mathbf{P}^2$, such that a, b, d, e, p, q lie on a conic different from $ab \cup de$, $\{c\} = ab \cap de \not\subset \{p, q\}$ and $\{p, q\} \not\subset \{a, b, d, e\}$. Stratum: F_8 is a $\mathbf{C}$-bundle over Φ_8, which is a $\mathring{\Delta}_4$-bundle over the configuration space X_8 of § 3.
9.	Six points that are the pairwise intersection of four lines ℓ_i $(1 \leqslant i \leqslant 4)$ in general position, such that $\{p, q\} \subset \bigcup_i \ell_i$. Stratum: F_9 is a $\mathbf{C}$-bundle over Φ_9, which is a $\mathring{\Delta}_5$-bundle over the configuration spaces X_9 studied in § 3. The simplices bundle does not change its orientation when two lines ℓ_i, ℓ_j are interchanged.
10.	The entire $\mathbf{P}^2$. Stratum: F_{10} is an open cone over the space $|\Lambda|$, which is the union of all strata Φ_j with $j \leqslant 9$.

TABLE 2. First seven columns of the spectral sequence converging to the Borel–Moore homology of $V_p \cap V_q \cap \Sigma$.

	1	2	3	4	5	6	7
22	0	$\mathbf{S}_2 \otimes \mathbf{Q}(12)$	0	0	0	0	0
21	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(11)$	0	0	0	0	0	0
20	0	$\mathbf{S}_2 \otimes \mathbf{Q}(11)$	0	0	0	0	0
19	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(10)$	0	0	0	0	0
18	0	0	0	0	0	0	0
17	0	0	0	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(10)$	0	0	0
16	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(9)$	0	$\mathbf{S}_2 \otimes \mathbf{Q}(10)$	0	0
15	0	0	0	$\mathbf{S}_2 \otimes \mathbf{Q}(9)$	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(9)$	0	0
14	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(8)$	0	0	0
13	0	0	0	0	0	0	0
12	0	0	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(8)$	0
11	0	0	0	0	0	0	$\mathbf{S}_2 \otimes \mathbf{Q}(8)$
10	0	0	0	0	0	0	$\mathbf{S}_{1,1} \otimes \mathbf{Q}(7)$

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Table 3. First block of the Leray spectral sequence in cohomology associated to π_2'.

	0	1	2	3	4	5	6
7	$\mathbf{S}_{1,1}$ $\otimes$ $\mathbf{Q}(-5)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-6)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-7)$	0	$\mathbf{S}_2$ $\otimes$ $\mathbf{Q}(-8)$
6	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-4)$	0	$(\oplus^2\mathbf{S}_2+\oplus^2\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-5)$	0	$(\oplus^2\mathbf{S}_2+\oplus^2\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-6)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-7)$
5	0	0	0	0	0	0	0
4	$\mathbf{S}_{1,1}$ $\otimes$ $\mathbf{Q}(-3)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-4)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-5)$	0	$\mathbf{S}_2$ $\otimes$ $\mathbf{Q}(-6)$
3	$(\oplus^2\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-2)$	0	$(\oplus^3\mathbf{S}_2+\oplus^3\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-3)$	0	$(\oplus^3\mathbf{S}_2+\oplus^3\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-4)$	0	$(\mathbf{S}_2+\oplus^2\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-5)$
2	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	$\mathbf{S}_2$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-1)$	0	$(\mathbf{S}_2+\mathbf{S}_{1,1})$ $\otimes$ $\mathbf{Q}(-2)$	0	$\mathbf{S}_{1,1}$ $\otimes$ $\mathbf{Q}(-3)$

In view of Alexander's duality (2.1), we have that the part of the rational cohomology of $V_p \cap V_q \setminus \Sigma$ that comes from the first seven columns of Vassiliev–Gorinov's spectral sequence has $\mathfrak{S}_2$-HG polynomial

$$(1 + \mathbf{L}t)(s_2 + \mathbf{L}^2 t^3(2s_2 + s_{1,1}) + \mathbf{L}^3 t^4 s_{1,1} + \mathbf{L}^4 t^6(s_2 + s_{1,1}) + \mathbf{L}^5 t^7 s_{1,1}), \qquad (2.2)$$

where the $\mathfrak{S}_2$-action is generated by the involution interchanging p and q. Note that the second factor of (2.2) is the $\mathfrak{S}_2$-HG polynomial of the cohomology of the projectivization of $V_p \cap V_q \setminus \Sigma$.

Next, we study the contribution of this part of the cohomology of $V_p \cap V_q \setminus \Sigma$ to the Leray spectral sequence for the fibration $\pi_2 \colon \mathcal{I}_2 \to F(\mathbf{P}^2, 2)$. It is simpler to consider the $\mathbf{C}^*$-quotient and study the fibration $\pi_2' \colon \mathcal{P}_2 \to F(\mathbf{P}^2, 2)$, which is a locally trivial fibration with fibre $\mathbf{P}(V_p \cap V_q \setminus \Sigma)$. The E_2 terms of the Leray spectral sequence are written in Table 3. Note that the space $F(\mathbf{P}^2, 2)$ is simply connected and that its cohomology has $\mathfrak{S}_2$-HG polynomial $(s_2 + \mathbf{L}ts_{1,1})(1 + \mathbf{L}t^2 + \mathbf{L}^2 t^4)$ with respect to the natural action of $\mathfrak{S}_2$ generated by the involution $(\alpha, \beta) \leftrightarrow (\beta, \alpha)$.

The proof of the following lemma is based on a suggestion by Alexei Gorinov.

Lemma 2.1. *In the spectral sequence associated to π_2', the differential $d_4 \colon E_4^{0,3} \to E_4^{4,0}$ has rank two.*

Proof. Denote by $\mathcal{P}_1 \subset \mathbf{P}^2 \times \mathbf{P}(S_4^2 \setminus \Sigma)$ the variety of pairs $(\xi, [f])$ such that $f(\xi) = 0$. Consider the inclusion $i \colon \mathcal{P}_2 \to \mathcal{P}_1 \times \mathcal{P}_1$ defined by $i(\alpha, \beta, [f]) = ((\alpha, [f]), (\beta, [f]))$. There is a commutative diagram

$$
\begin{array}{ccc}
\mathcal{P}_2 & \xrightarrow{\;\;i\;\;} & \mathcal{P}_1 \times \mathcal{P}_1 \\
{\scriptstyle \pi_2'}\downarrow & & \downarrow{\scriptstyle \pi_1 \times \pi_1} \\
F(\mathbf{P}^2, 2) & \xrightarrow{\text{inclusion}} & \mathbf{P}^2 \times \mathbf{P}^2
\end{array}
\qquad (2.3)
$$

where $\pi_1 \colon \mathcal{P}_1 \to \mathbf{P}^2$ denotes the natural projection.

In particular, the differentials of the spectral sequences associated to π_2' and $\pi_1 \times \pi_1$ commute with the maps induced by i on the E_4 terms of the spectral sequences.

Recall from [BT07, Proposition 1] that the differential

$$E_4^{0,3}(\pi_1) \xrightarrow{d_4} E_4^{4,0}(\pi_1)$$

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has rank one. This implies that the differential $E_4^{0,3}(\pi_1 \times \pi_1) \xrightarrow{d_4} E_4^{4,0}(\pi_1 \times \pi_1)$ has rank two. Since

$$E_4^{4,0}(\pi_1 \times \pi_1) \cong E_2^{4,0}(\pi_1 \times \pi_1) \cong H^4(\mathbf{P}^2 \times \mathbf{P}^2; \mathbf{Q}) \otimes H^0(V_p \setminus \Sigma; \mathbf{Q})$$

and

$$E_4^{4,0}(\pi_2') \cong E_2^{4,0}(\pi_2') \cong H^4(F(\mathbf{P}^2, 2); \mathbf{Q}) \otimes H^0(V_p \cap V_q \setminus \Sigma; \mathbf{Q}),$$

one can verify directly that the composition of $d_4 \colon E_4^{0,3}(\pi_1 \times \pi_1) \to E_4^{4,0}(\pi_1 \times \pi_1)$ and the map $E_4^{4,0}(\pi_1 \times \pi_1) \to E_4^{4,0}(\pi_2')$ is surjective. Then the claim follows from the commutativity of the diagram (2.3). $\qquad\square$

Proof of Theorem 1.1. Comparing the Leray–Hirsch isomorphism (1.1) with Table 3 implies that the entire contribution of the first block to the cohomology of $\mathcal{I}_2$ is determined by the cohomology of $\mathcal{I}_2$ in degree at most five. This follows from the fact that the cohomology of GL(3) is trivial in degree $k \geqslant 10$. Then Lemma 2.1, together with the structure of $H^\bullet(\mathcal{I}_2; \mathbf{Q})$ as tensor product of GL(3), yields that the first block contributes

$$(1 + \mathbf{L}t)(1 + \mathbf{L}^2 t^3)(1 + \mathbf{L}^3 t^5)(s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3 t^5 s_2).$$

to the $\mathfrak{S}_2$-HG polynomial of $H^\bullet(\mathcal{I}_2; \mathbf{Q})$. This implies that the $\mathfrak{S}_2$-HG polynomial of the cohomology of the moduli space $\mathcal{Q}_2$ of smooth quartic curves with two marked points is

$$s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3 t^5 s_2, \tag{2.4}$$

plus the term coming from singular configurations of type 8–10. In the next section we will prove that this term equals $\mathbf{L}^6 t^6 s_2 + \mathbf{L}^7 t^8(s_2 + s_{1,1}) + \mathbf{L}^8 t^8 s_{1,1}$ (which is (3.9)).

Summing the contributions of the two block of columns, we get that the $\mathfrak{S}_2$-HG polynomial of the cohomology of $\mathcal{Q}_2$ is

$$s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^3 t^5 s_2 + \mathbf{L}^6 t^6 s_2 + \mathbf{L}^7 t^8(s_2 + s_{1,1}) + \mathbf{L}^8 t^8 s_{1,1}.$$

This establishes the first part of Theorem 1.1. To prove the second part of the theorem, recall from [Tom05b, Corollary III.2.2] that the $\mathfrak{S}_2$-HG polynomial of the cohomology of the hyperelliptic locus $\mathcal{H}_{3,2} \subset \mathcal{M}_{3,2}$ is $s_2 + \mathbf{L}t^2(s_2 + s_{1,1}) + \mathbf{L}^7 t^7 s_{1,1}$, and consider the long exact sequence associated to the inclusion $\mathcal{H}_{3,2} \hookrightarrow \mathcal{M}_{3,2}$:

$$\cdots \to H^k(\mathcal{M}_{3,2}; \mathbf{Q}) \to H^k(\mathcal{Q}_2; \mathbf{Q}) \to H^{k-1}(\mathcal{H}_{3,2}; \mathbf{Q}) \otimes \mathbf{Q}(-1) \to H^{k+1}(\mathcal{M}_{3,2}; \mathbf{Q}) \to \cdots,$$

which can be rephrased in Borel–Moore homology as

$$\cdots \to \bar{H}_k(\mathcal{M}_{3,2}; \mathbf{Q}) \to \bar{H}_k(\mathcal{Q}_2; \mathbf{Q}) \xrightarrow{d_k} \bar{H}_{k-1}(\mathcal{H}_{3,2}; \mathbf{Q}) \to \bar{H}_{k-1}(\mathcal{M}_{3,2}; \mathbf{Q}) \to \cdots.$$

If $k \neq 8$, the differentials d_k are always zero for Hodge-theoretic reasons. If $k = 8$, both $\bar{H}_8(\mathcal{Q}_2; \mathbf{Q})$ and $\bar{H}_7(\mathcal{H}_{3,2}; \mathbf{Q})$ have a one-dimensional summand of Hodge weight zero, on which $\mathfrak{S}_2$ acts as the sign representation. Hence, *a priori* d_8 can have either rank zero or one. To determine the rank of d_8, we observe that both the cohomology of $\mathcal{H}_{3,2}$ and $\mathcal{Q}_2$ were computed using Vassiliev–Gorinov's method. In particular, both $\bar{H}_8(\mathcal{Q}_2; \mathbf{Q})$ and $\bar{H}_7(\mathcal{H}_{3,2}; \mathbf{Q})$ are related to configurations of at least four singular points. Moreover, there configurations correspond to strata of the geometric realizations that have Borel–Moore homology which is a tensor product of that of the group acting. This means that both Borel–Moore homology groups can be interpreted as Borel–Moore homology groups of certain moduli spaces.

Specifically, consider the moduli space $\mathcal{N}$ whose elements are isomorphism classes of triples (C, p, q), where C is the union of two smooth rational curves intersecting transversally at four distinct points and p, q are any distinct (but possibly singular) points on C. Note that the arithmetic genus of such a curve C is three. Denote by S the rank one local system on $\mathcal{N}$ changing its orientation

every time a pair of nodes on C is interchanged, and denote by $\mathcal{N}_h$ the closed subset of $\mathcal{N}$ such that the four nodes have the same moduli on both rational components.

Observe that the problem with the determination of d_8 only concerns the Hodge weight zero summands of the Borel–Moore homology groups. For this reason, in the rest of the proof we will restrict to the Hodge weight zero summands of each homology group we consider.

The space $\mathcal{N}$ can be written as the disjoint union of locally closed strata, each of them isomorphic to the quotient by the action of a finite group, of a product of moduli spaces $\mathcal{M}_{0,n}$ with $4 \leqslant n \leqslant 6$, whose cohomology groups are completely known (see, e.g., [Get95]). Investigating this stratification, one obtains that the only Borel–Moore homology group of Hodge weight zero is $\bar{H}_4(\mathcal{N}; S) = \mathbf{S}_{1,1}$. Analogous considerations also apply to $\mathcal{N}_h$. In that case, one has $\bar{H}_3(\mathcal{N}_h; S) \cong \mathbf{S}_{1,1}$ as only Borel–Moore homology group with Hodge weight zero. By the constructions in [Tom05b, § III.2], there is a natural isomorphism $\bar{H}_3(\mathcal{N}_h; S) \cong \bar{H}_7(\mathcal{H}_{3,2}; \mathbf{Q})$.

The weight zero part of the Borel–Moore homology of $\mathcal{N} \setminus \mathcal{N}_h$ can also be computed directly with Vassiliev–Gorinov's method. This yields again that the only non-trivial Borel–More homology group with S-coefficient of $\mathcal{N} \setminus \mathcal{N}_h$ is $\bigoplus_2 \mathbf{S}_{1,1}$ in degree four. Moreover, the direct computation shows that $\bar{H}_4(\mathcal{N} \setminus \mathcal{N}_h; S)$ is generated by two classes, both related to configurations of type 9 in Table 1. This allows us to define a surjective map $\bar{H}_4(\mathcal{N} \setminus \mathcal{N}_h; S) \to \bar{H}_8(\mathcal{Q}_2; \mathbf{Q})$ making the following diagram commute.

$$
\begin{array}{ccccccccc}
& & \mathbf{S}_{1,1} & & \bigoplus_2 \mathbf{S}_{1,1} & & \mathbf{S}_{1,1} & & \\
& & \Vert \wr & & \Vert \wr & & \Vert \wr & & \\
0 & \longrightarrow & \bar{H}_4(\mathcal{N}; S) & \longrightarrow & \bar{H}_4(\mathcal{N} \setminus \mathcal{N}_h; S) & \xrightarrow{\gamma} & \bar{H}_3(\mathcal{N}; S) & \longrightarrow & 0 \\
& & & & \downarrow & & \downarrow & & \\
& & & & \bar{H}_8(\mathcal{Q}_2; \mathbf{Q}) & \xrightarrow{d_8} & \bar{H}_7(\mathcal{H}_{3,2}; \mathbf{Q}) & &
\end{array}
$$

The commutativity of the diagram immediately yields $\operatorname{rank} d_8 = \operatorname{rank} \gamma = 1$. $\qquad\square$

3. Configurations of five and six points

The aim of this section is to compute the contribution of singular configurations of type 8–10 (see Table 1) to the rational cohomology of $\mathcal{I}_2$ and $\mathcal{Q}_2$. For these configurations, it seems more natural to work directly with the cohomology of $\mathcal{I}_2$, without having to pass through the study of the fibre of π_2. This is indeed possible. Namely, consider the space

$$
\mathcal{D} := \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) \times \Sigma : f(\alpha) = f(\beta) = 0\}.
$$

Note that $\mathcal{D}$ is a closed subset of $\mathcal{V} := \{(\alpha, \beta, f) \in F(\mathbf{P}^2, 2) : f(\alpha) = f(\beta) = 0\}$. The space $\mathcal{V}$ is the total space of a vector bundle over $\mathbf{P}^2$, and $\mathcal{I}_2 = \mathcal{V} \setminus \mathcal{D}$. Vassiliev–Gorinov's method can be exploited to compute the Borel–Moore homology of $\mathcal{D}$. This is done by defining the singular locus of an element (α, β, f) in $\mathcal{D}$ as the subset $\{(\alpha, \beta)\} \times K_f$ of $F(\mathbf{P}^2, 2) \times \mathbf{P}^2$, where K_f denotes the singular locus of the polynomial f. In particular, the classification of singular sets of elements of $\mathcal{D}$ is obtained from the classification of singular sets of elements of $V_p \cap V_q \setminus \Sigma$ by allowing the pair (p, q) to move in $F(\mathbf{P}^2, 2)$.

Even though this is no longer the original setting of Vassiliev–Gorinov's method, one can mimic the construction of the cubical spaces Λ and $\mathcal{X}$ (see § 4), and obtain cubical spaces Λ' and $\mathcal{X}'$ that play an analogous role. In particular, the map $|\mathcal{X}'| \to \mathcal{D}$ induces an isomorphism on the Borel–Moore homology of these spaces, because it is a proper map with contractible fibres. Moreover, for the stratifications Φ' and F' obtained from the construction of Λ' and $\mathcal{X}'$, we have natural

maps $\Phi'_k \to F(\mathbf{P}^2, 2)$ and $F'_k \to F(\mathbf{P}^2, 2)$ which are locally trivial fibrations with fibre isomorphic to Φ_k, F_k, respectively.

In view of the considerations above, computing the rational Borel–Moore homology of the spaces Φ'_j and F'_j for $j \in \{8, 9, 10\}$ is enough to obtain the contribution of configurations of type 8–10 to the cohomology of $\mathcal{I}_2$. We start by determining the twisted Borel–Moore homology of the underlying families of configurations X'_8 and X'_9.

Define $Y_8 \subset F(\mathbf{P}^2, 2) \times F(\mathbf{P}^2, 5)$ to be the space of configurations $(p, q, e_1, e_2, e_3, e_4, c)$ such that:

(i) $\{c\} = e_1 e_2 \cap e_3 e_4 \not\subset \{p, q\}$;

(ii) p, q, e_1, e_2, e_3, e_4 lie on a conic different from the reducible conic $e_1 e_2 \cup e_3 e_4$;

(iii) $\{p, q\} \not\subset \{e_1, e_2, e_3, e_4\}$.

Then $X'_8 \subset F(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 5)$ is isomorphic to the quotient of Y_8 by the action of the subgroup G of $\mathfrak{S}_4$ generated by the permutations $(1, 2)$, $(3, 4)$ and $(1, 3)(2, 4)$. The action of G on Y_8 is given by permuting the four points (e_1, e_2, e_3, e_4) in the configurations. Since G is a subgroup of $\mathfrak{S}_4$, it makes sense to restrict the sign representation to it.

Furthermore, note that the conic passing through the points p, q, e_1, e_2, e_3, e_4 is uniquely determined for every configuration in Y_8. Therefore, Y_8 can be embedded in the space $W \subset F(\mathbf{P}^2, 2) \times F(\mathbf{P}^2, 5) \times \mathbf{P}(S_2^2)$ of configurations $(p, q, e_1, e_2, e_3, e_4, c, C)$, such that:

(i) $\{c\} = e_1 e_2 \cap e_3 e_4 \not\subset \{p, q\}$;

(ii) the points p, q, e_1, e_2, e_3, e_4 lie on the conic C;

(iii) the conic C is distinct from the reducible conic $e_1 e_2 \cup e_3 e_4$.

Hence, we have the chain of inclusions

$$Y_8 \hookrightarrow W \hookrightarrow F(\mathbf{P}^2, 2) \times F(\mathbf{P}^2, 5) \times \mathbf{P}(S_2^2).$$

LEMMA 3.1. *Denote by S the local system of coefficients induced on W/G and X'_8 by the sign representation on G. Consider the $\mathfrak{S}_2$-action generated by the involution interchanging the points $(p, q) \in F(\mathbf{P}^2, 2)$. Then one has*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X'_8; S)) = (\mathbf{L}^{-1} t^3 + \mathbf{L}^{-2} t^4) s_2 \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})).$$

To prove Lemma 3.1, we consider the quotient of Y_8 and W by the action of $\mathrm{PGL}(3)$. Since every configuration in W contains points e_1, e_2, e_3, e_4 which are in general position, the group $\mathrm{PGL}(3)$ acts freely and transitively on W, hence W is isomorphic to the product of $\mathrm{PGL}(3)$ and the quotient $W/\mathrm{PGL}(3)$. Recall that $\mathrm{PGL}(3)$ is isomorphic to the configuration space of four ordered points in general position in $\mathbf{P}^2$. This yields a natural identification between the quotient $W/\mathrm{PGL}(3)$ and the space

$$W_E := W \cap (F(\mathbf{P}^2, 2) \times \{(E_1, E_2, E_3, E_4, E_5)\}),$$

where

$$E_1 = [1, 0, 0], \quad E_2 = [0, 1, 0], \quad E_3 = [0, 0, 1], \quad E_4 = [1, 1, 1], \quad E_5 = [1, 1, 0].$$

In the following, we identify each element σ of G with the automorphism of $\mathbf{P}^2$ mapping E_i to $E_{\sigma(i)}$. This allows us to consider G as a subgroup of $\mathrm{Aut}(\mathbf{P}^2)$, and induces an action of G on W_E that makes the isomorphism $W \cong W_E \times \mathrm{PGL}(3)$ G-equivariant. Note that the action of G on $\mathrm{PGL}(3)$ is defined by restricting to G the natural action of the symmetric group $\mathfrak{S}_4$ on $\mathrm{PGL}(3) \hookrightarrow F(\mathbf{P}^2, 4)$ permuting the four points in the configuration. It is not difficult to prove that the rational Borel–Moore homology of $\mathrm{PGL}(3)$ is $\mathfrak{S}_4$-invariant and hence also G-invariant.

By applying the Künneth formula to the Borel–Moore homology of $W \cong W_E \times \mathrm{PGL}(3)$, and considering the part of the Borel–Moore homology which has the desired behaviour for the action

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of G, one obtains

$$\bar{H}_\bullet(W/G; S) \cong \bar{H}_\bullet(W_E/G; S) \otimes \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}), \qquad (3.1)$$

where S denotes the local system of rank one induced by the restriction of the sign representation to $G \subset \mathfrak{S}_4$.

The reasoning above applies to Y_8 as well as W. This yields the isomorphism

$$\bar{H}_\bullet(X'_8; S) \cong \bar{H}_\bullet((Y_8 \cap W_E)/G; S) \times \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}). \qquad (3.2)$$

The space W_E can be described in the following way. Denote by $\mathcal{L}$ the space of conics passing through the E_i and distinct from the reducible conic $(x_0 - x_1)x_2 = 0$. Note that $\mathcal{L}$ is isomorphic to an affine line. Then we have

$$W_E = \{(p, q, C) \in F(\mathbf{P}^2, 2) \times \mathcal{L} : p, q \in C\}.$$

LEMMA 3.2. *In the notation of Lemma 3.1, we have* $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(W_E/G; S)) = \mathbf{L}^{-2}t^4 s_2$.

Proof. The space $Q = W_E/G$ can be decomposed as the union of a closed locus K containing all equivalence classes of triples (p, q, C) such that C is a singular conic, and an open part U where the conic C is always non-singular.

We compute the Borel–Moore homology of K first. The locus K has two components, according to the position of the two points p, q. We denote by M the component of K such that p, q lie on the same irreducible component of C, and N the component in which p, q lie on two different components of C. The elements of the intersection $M \cap N$ are the configurations in which the singular point of C is either p or q.

Up to the G-action, the space M can be identified with the space of ordered configurations of two points on the projective line $x_1 = 0$, hence the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(M; S)$ is $(\mathbf{L}^{-1}t^2 + \mathbf{L}^{-2}t^4)s_2$.

Next, we identify the space $N \setminus M$ with a $\mathfrak{S}_2$-quotient of the space of pairs (p, q) where p lies on $x_1 = 0$, the point q lie on $x_0 - x_2 = 0$ and both points are distinct from the intersection point of these lines. The $\mathfrak{S}_2$-action interchanges E_1 and E_3, and E_2 and E_4, and we have to take invariant classes with respect to it. This implies that the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(N \setminus M; S)$ is $(\mathbf{L}^{-2}t^4)s_2$. Then, from the long exact sequence in Borel–Moore homology associated with the closed inclusion $M \hookrightarrow K$ we can conclude that the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(K; S)$ is $(\mathbf{L}^{-1}t^2 + 2\mathbf{L}^{-2}t^4)s_2$.

Subsequently, we compute the Borel–Moore homology of U by lifting U to a G-invariant subset $U' \subset W_E$, and looking for the part of the Borel–Moore homology of U' that has the desired behaviour with respect to the G-action. We have that U' projects to the locus of non-singular conics in $\mathcal{L}$, which is isomorphic to $\mathbf{C} \setminus \{\pm 1\}$. Note that the action of $(1, 2) \in G$ on $\mathcal{L} \cong \mathbf{C}$ interchanges the two singular conics. The projection $U' \to \mathbf{C} \setminus \{\pm 1\}$ is a locally trivial fibration with fibre isomorphic to the space $F(R, 2)$, where R is a chosen non-singular conic through the E_i. In order to study the action of G on the Borel–Moore homology of $F(R, 2)$, we assume that the conic R is fixed by all automorphisms in $G \subset \mathrm{Aut}(\mathbf{P}^2)$. If we fix an isomorphism $R \cong \mathbf{P}^1$, we have that taking the quotient by G gives finite maps $R \cong \mathbf{P}^1 \to \mathbf{P}^1$ and $F(R, 2) \cong F(\mathbf{P}^1, 2) \to F(\mathbf{P}^1, 2)$. In particular, the Borel–Moore homology with standard coefficients of $F(R, 2)$ is isomorphic to that of its quotient by G, hence all Borel–Moore homology classes of $F(R, 2)$ are G-invariant. Hence, the $\mathfrak{S}_2$-HG polynomial of $\bar{H}_\bullet(U; S)$ is the product of $\wp(F(\mathbf{P}^1, 2))s_2$ and the HG polynomial of the part of the Borel–Moore homology of $\mathbf{C} \setminus \{\pm 1\}$ which is anti-invariant for the involution $\xi \leftrightarrow -\xi$, which equals t.

To compute $\bar{H}_\bullet(Q; S)$, we can now use the long exact sequence in Borel–Moore homology associated to the closed inclusion $K \hookrightarrow Q$:

$$\cdots \to \bar{H}_k(K; S) \to \bar{H}_k(Q; S) \to \bar{H}_k(U; S) \to \bar{H}_{k-1}(K; S) \to \cdots.$$

This yields immediately $\bar{H}_k(Q; S) = 0$ if $k > 5$ or $k < 2$. Moreover, we have

$$0 \to \bar{H}_5(Q; S) \to \mathbf{Q}(2) \xrightarrow{\delta_5} \mathbf{Q}(2)^2 \xrightarrow{\delta_5'} \bar{H}_4(Q; S) \to 0,$$

$$0 \to \bar{H}_3(Q; S) \to \mathbf{Q}(1) \xrightarrow{\delta_3} \mathbf{Q}(1) \xrightarrow{\delta_3'} \bar{H}_2(Q; S) \to 0.$$

Then the claim follows from the fact that both δ_5 and δ_3 are injections. As we will see, the subset $M \cup U$ has trivial Borel–Moore homology with S-coefficients, hence $\bar{H}_k(M; S)$ is contained in the kernel of δ_k' for every k.

To compute the Borel–Moore homology of $M \cup U$, consider the surjective map $\pi \colon M \cup U \to \mathcal{L}/G$ obtained by restricting the natural projection $W_E \to \mathcal{L}$. The map π is clearly locally trivial on U. We claim that π is also locally trivial in a neighbourhood of the point w_0 in $\mathcal{L}/G$ parametrizing singular conics. Up to the G-action, and possibly the choice of a sufficiently small neighbourhood U_0, we can assume that this singular conic is $Y \colon x_1(x_0 - x_2) = 0$, and identify $\pi^{-1}(w)$ ($w \in U_0$) with the locus of triples (Y, α, β) such that α and β lie on the line $x_1 = 0$. Then the fibre of π near w_0 can be identified with $F(\{x_1 = 0\}, 2)$ by considering the projection from the point E_4, which maps every non-singular conic in $\mathcal{L}$ onto the line $x_1 = 0$. This construction yields a map from the preimage in π of a neighbourhood of w_0 to $F(\mathbf{P}^1, 2) \cong F(\{x_1 = 0\}, 2)$, which admits a section. Hence, the map π is a locally trivial fibration over $\mathcal{L}/G$.

Note that this implies that the Borel–Moore homology of $M \cup U$ in the local system S is trivial. The Borel–Moore homology of $\mathcal{L}$ is clearly G-invariant, hence the elements of $H_\bullet(M \cup U; S)$ have to come from Borel–Moore homology classes of the fibre of π, in a local system different from the standard system. The fibre of π is isomorphic to a G-quotient of $F(\mathbf{P}^1, 2)$, and the whole Borel–Moore homology of $F(\mathbf{P}^1, 2)$ is G-invariant. For this reason, the fibre of π has trivial Borel–Moore homology in all local systems different from the standard system. $\quad\square$

Proof of Lemma 3.1. We start by investigating the space $W_E \setminus Y_8$ and its quotient Q' by the action of G. Recall that a configuration $(p, q, C) \in Q$ lies in Q' if and only if $\{p, q\} \subset \{E_1, E_2, E_3, E_4\}$. It is easy to see that Q' has two components, according to whether p and q lie both on the same component of the reducible conic $x_2(x_0 - x_1) = 0$, or not. Denote by Q_a the component corresponding to the first case and by Q_b the component corresponding to the second case. Up to the action of G, we may assume that for every configuration in Q_a we have $p = E_1$, $q = E_2$. Hence, the space Q_a is isomorphic to the quotient $\mathcal{L}/\iota$, where the involution ι is $(3, 4) \in G \subset \mathrm{Aut}(\mathbf{P}^2)$. Since $\mathcal{L}/\iota$ is isomorphic to $\mathbf{C}$, the Borel–Moore homology of Q_a with S-coefficients is isomorphic to the Borel–Moore homology of $\mathbf{C}$ induced by the sign representation on $\langle \iota \rangle = \mathfrak{S}_2$, which is trivial.

Analogously, up to the G-action one can assume that $p = E_1$, $q = E_3$ hold for every configuration in Q_b. In particular, Q_b is isomorphic to $\mathcal{L} \cong \mathbf{C}$ and is invariant for the involution interchanging p and q. This proves that $\bar{H}_\bullet(Q'; S)$ is isomorphic to $\bar{H}_\bullet(\mathbf{C})$ and is invariant for the involution $p \leftrightarrow q$. Then the claim follows from the long exact sequence in Borel–Moore homology associated to the closed inclusion $Q' \hookrightarrow Q$ and isomorphism (3.2). $\quad\square$

Recall from Table 1 that X_9' is the locus

$$X_9' := \left\{ (p, q, S) \in F(\mathbf{P}^2, 2) \times B(\mathbf{P}^2, 6) : \exists \{r_i\}_{1 \leqslant i \leqslant 4} \in B(\mathbf{P}^{2\vee}, 4) \left(S = \mathrm{Sing}\left(\bigcup_i r_i \right), p, q \in \bigcup_i r_i \right) \right\}.$$

Observe that giving six points that are the pairwise intersection of four lines in general position is equivalent to giving the configuration of four lines. Denote by $\tilde{F}(\mathbf{P}^{2\vee}, 4)$ the space of ordered configurations of lines in general position (i.e. such that no three of them pass through the same point),

and by $\tilde{B}(\mathbf{P}^{2\vee}, 4)$ the analogous space of unordered configurations. Then we have

$$X_9' \cong \left\{ (p, q, \{r_1, r_2, r_3, r_4\}) \in F(\mathbf{P}^2, 2) \times \tilde{B}(\mathbf{P}^{2\vee}, 4) : p, q \in \bigcup_i r_i \right\}.$$

We start by investigating the closed subset X_{9a}' of configurations $(p, q, \{r_i\}_i) \in X_9'$ such that p and q lie on the same line r_j for some index j.

LEMMA 3.3. *We have*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X_{9a}'; \pm \mathbf{Q})) = \wp^{\mathfrak{S}_2}(\bar{H}_\bullet(F(\mathbf{P}^1, 2); \mathbf{Q})) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})).$$

Proof. Consider the variety

$$A := \{ (p, q, r_1, r_2, r_3, r_4) \in F(\mathbf{P}^2, 2) \times \tilde{F}(\mathbf{P}^{2\vee}, 4) : p, q \in r_4 \}.$$

Note that X_{9a}' is the quotient of A by the action of $\mathfrak{S}_3$ interchanging r_1, r_2 and r_3. On the other hand, we have $A \cong F(\mathbf{P}^1, 2) \times \mathrm{PGL}(3)$, where we used the fact that $\mathrm{PGL}(3)$ is isomorphic to the space of four lines in general position, and chosen an isomorphism $\mathbf{P}^1 \cong r_4$ (for instance, the one mapping 0 to $r_1 \cap r_4$, 1 to $r_2 \cap r_4$ and ∞ to $r_3 \cap r_4$). Hence, we can obtain the Borel–Moore homology of X_{9a}' by taking the $\mathfrak{S}_3$-invariant part of the Borel–Moore homology of A. This establishes the claim. $\square$

Next, we consider $X_{9b}' := X_9' \setminus X_{9a}'$.

LEMMA 3.4. *We have*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X_{9b}'; \pm \mathbf{Q})) = (t^2 s_{1,1} + \mathbf{L}^{-2} t^4 s_2) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})).$$

Proof. Consider the space

$$Y_9 := \left\{ (p, q, r_1, r_2, r_3, r_4) \in F(\mathbf{P}^2, 2) \times \tilde{F}(\mathbf{P}^{2\vee}, 4) : p, q \in \bigcup_i r_i \right\}.$$

Observe that $\tilde{F}(\mathbf{P}^{2\vee}, 4)$ is isomorphic to $\mathrm{PGL}(3)$, and that the group $\mathrm{PGL}(3)$ acts freely and transitively on Y_9. The quotient of this action is isomorphic to the fibre of the projection $Y_9 \to \tilde{F}(\mathbf{P}^{2\vee}, 4)$ at the configuration (l_1, l_2, l_3, l_4), where

$$l_1 : x_0 = 0, \quad l_2 : x_1 = 0, \quad l_3 : x_2 = 0, \quad l_4 : x_0 + x_1 + x_2 = 0.$$

If we pose $L : x_0 x_1 x_2 (x_0 + x_1 + x_2) = 0$, this implies that $Y_9/\mathrm{PGL}(3)$ is isomorphic to $F(L, 2)$, and we have an isomorphism

$$Y_9 \cong F(L, 2) \times \mathrm{PGL}(3). \tag{3.3}$$

Consider the action of $\mathfrak{S}_4$ on L and $F(L, 2)$ defined by identifying every permutation $\sigma \in \mathfrak{S}_4$ with the automorphism of $\mathbf{P}^2$ sending the line l_i to l_{σ_i} for all i, $1 \leqslant i \leqslant 4$. The natural action of $\mathfrak{S}_4$ on $\tilde{F}(\mathbf{P}^2, 4)$ defines an action on $\mathrm{PGL}(3)$ via the isomorphism $\mathrm{PGL}(3) \cong \tilde{F}(\mathbf{P}^{2\vee}, 4)$, making isomorphism (3.3) $\mathfrak{S}_4$-equivariant. Applying the Künneth formula and taking the $\mathfrak{S}_4$-invariant part of the Borel–Moore homology of Y_9 yields

$$\bar{H}_\bullet(X_9'; \mathbf{Q}) \cong \bar{H}_\bullet(F(L, 2); \mathbf{Q})^{\mathfrak{S}_4} \otimes \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q}),$$

where we used the fact that X_9' is the quotient of Y_9 by the action of $\mathfrak{S}_4$, and that the whole Borel–Moore homology of $\mathrm{PGL}(3)$ is $\mathfrak{S}_4$-invariant.

Let us see what these considerations tell us about the Borel–Moore homology of X_{9b}'. The space X_{9b}' is isomorphic to the $\mathfrak{S}_4$-quotient of the product of $\mathrm{PGL}(3)$ and the locus of configurations of two points $(a, b) \in F(L, 2)$ not lying on the same component of L. This locus can be decomposed

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TABLE 4. Spectral sequence in Borel–Moore homology associated to the stratification S_i.

	1	2	3
1	0	0	$\mathbf{S}_2 \otimes \mathbf{Q}(2)$
0	0	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(1)$	$(\mathbf{S}_2 + \mathbf{S}_{1,1}) \otimes \mathbf{Q}(1)$
−1	$\mathbf{S}_2$	$\mathbf{S}_2 + \mathbf{S}_{1,1}$	$\bigoplus^2 \mathbf{S}_{1,1}$

according to whether a and b are or are not singular points of L into the loci

$$S_1 := \{(a,b) \in F(L,2) : a \text{ and } b \text{ are both singular points}\},$$
$$S_2 := \{(a,b) \in F(L,2) : \text{only one of the points } a \text{ and } b \text{ is a singular point of } L\},$$
$$S_3 := \{(a,b) \in F(L,2) : a \text{ and } b \text{ are non-singular points of } L\}.$$

The quotient $S_1/\mathfrak{S}_4$ consists of only one point, the class of the pair $([1,0,0], [0,1,-1])$. The quotient $S_2/\mathfrak{S}_4$ has two isomorphic components, according to which point $(a$ or $b)$ is a singular point of L. Consider the case in which a is singular. Up to the action of $\mathfrak{S}_4$, we can assume that a is the point $[1,0,0]$ and b lies on $x_0 = 0$. By the definition of S_2 we know that b is different from the points $[0,1,-1]$, $[0,0,1]$ and $[0,1,0]$. Note that, since we are working modulo $\mathfrak{S}_4$, the coordinates of b are defined up to the involution interchanging x_1 and x_2. This proves that both components of $S_2/\mathfrak{S}_4$ are isomorphic to $\mathbf{C}^*$.

Finally, we determine the Borel–Moore homology of the quotient of S_3 by the action of $\mathfrak{S}_4$. Up to the action of the group, we can assume that a lies on the line l_3 and b on l_4. The position of both points is determined up to the involution interchanging the lines l_1 and l_2. If we identify l_3 and l_4 with $\mathbf{P}^1$, and $l_3 \cap l_4$ with the point at infinity of the projective line, we have that $S_3/\mathfrak{S}_4$ can be embedded into the quotient of $(\mathbf{C} \setminus \{\pm 1\})^2$ by the relation $(t,s) \sim (-t,-s)$. The complement of $S_3/\mathfrak{S}_4$ in this quotient is the locus such that either t or s is equal to ± 1. We can study $(\mathbf{C}\setminus\{\pm1\})^2/\sim$ as follows:

$$
\begin{array}{ccccc}
\mathbf{C}^2 & \xrightarrow{\text{mod} \sim} & \{(x,y,z) \in \mathbf{C}^3 : y^2 = xz\} & \xrightarrow{\text{mod } \mathfrak{S}_2} & \mathbf{C}^2 \\
(t,s) & \longmapsto & (t^2, ts, s^2) & \longmapsto & (t^2+s^2, ts) \\
(1,s) & \longmapsto & (1, s, s^2) & \longmapsto & (s^2+1, s) \\
(t,1) & \longmapsto & (t^2, t, 1) & \longmapsto & (t^2+1, t),
\end{array}
$$

where the second map denotes the quotient by the action of $\mathfrak{S}_2$ interchanging t and s. Concluding, the spectral sequence associated to this stratification has E^1 term as in Table 4 (where we have taken into account the $\mathfrak{S}_2$-action interchanging a and b).

We can use the geometric description of S_1, S_2 and S_3 to determine all differentials of the spectral sequence above. In particular, the zeroth row is exact, and both differentials in the row of index -1 have rank one. Then the claim follows from the fact that $\bar{H}_\bullet(X'_{9b}; \mathbf{Q})$ is isomorphic to $\bar{H}_\bullet(S_1 \cup S_2 \cup S_3; \mathbf{Q})^{\mathfrak{S}_4} \otimes \bar{H}_\bullet(\mathrm{PGL}(3))$. □

PROPOSITION 3.5. *The Borel–Moore homology groups of the unions of strata* $\Phi'_8 \cup \Phi'_9 \subset |\Lambda'|$ *and* $F'_8 \cup F'_9 \cup F'_{10} \subset |\mathcal{X}'|$ *are as follows:*

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(\Phi'_8 \cup \Phi'_9; \mathbf{Q})) = (\mathbf{L}^{-2}t^9 s_2 + (\mathbf{L}^{-1}s_2 + \mathbf{L}^{-1}s_{1,1} + s_{1,1})t^7) \cdot \wp(\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})), \quad (3.4)$$

$$\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(F'_8 \cup F'_9 \cup F'_{10}; \mathbf{Q})) = (\mathbf{L}^{-2}t^9 s_2 + (\mathbf{L}^{-1}s_2 + \mathbf{L}^{-1}s_{1,1} + s_{1,1})t^7) \cdot \wp(\bar{H}_\bullet(\mathrm{GL}(3); \mathbf{Q})). \quad (3.5)$$

Proof. Lemmas 3.3 and 3.4 imply that $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(X'_9; \pm\mathbf{Q}))$ equals $(1 + \mathbf{L}^2 t^{-3})(1 + \mathbf{L}^3 t^{-5})\mathbf{L}^{-8}t^{16} \cdot (2\mathbf{L}^{-2}t^4 s_2 + (\mathbf{L}^{-1}+1)t^2 s_{1,1})$. Recall from Table 1 that Φ'_9 is a simplices bundle with five-dimensional fibre. Hence, the $\mathfrak{S}_2$-HG polynomial of the Borel–Moore homology of Φ'_9 equals that of X'_9 multiplied by t^5. Analogously, Lemma 3.1 and Table 1 yield that $\wp^{\mathfrak{S}_2}(\bar{H}_\bullet(\Phi'_8; \mathbf{Q}))$ is $(1 + \mathbf{L}^2 t^{-3})(1 + \mathbf{L}^3 t^{-5})\mathbf{L}^{-8}t^{16} \cdot (\mathbf{L}^{-2}t^8 + \mathbf{L}^{-1}t^7)s_2$.

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We compute the Borel–Moore homology of $\Psi := \Phi_8' \cup \Phi_9'$ by exploiting the long exact sequence

$$\cdots \to \bar{H}_k(\Phi_8'; \mathbf{Q}) \to \bar{H}_k(\Psi; \mathbf{Q}) \to \bar{H}_k(\Phi_9'; \mathbf{Q}) \xrightarrow{\delta_k} \bar{H}_{k-1}(\Phi_8'; \mathbf{Q}) \to \cdots. \qquad (3.6)$$

Both the Borel–Moore homology of Φ_8' and Φ_9' are tensor products of the Borel–Moore homology of $\mathrm{PGL}(3)$. The question is whether their structure as tensor products of $\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})$ is respected by the maps in (3.6) or not.

We computed in §2 that the strata 1–7 do not contribute to the Borel–Moore homology of F_{10}'. Recall that $F_8' \cup F_9'$ is a vector bundle of rank one over Ψ. By comparing the geometry of $\mathcal{D}$ and its projectivization, we can conclude that

$$\bar{H}_\bullet\left(\bigcup_{i=8}^{10} F_i'; \mathbf{Q}\right) \cong \bar{H}_\bullet(\Psi; \mathbf{Q}) \otimes \bar{H}_\bullet(\mathbf{C}^*; \mathbf{Q}). \qquad (3.7)$$

Isomorphism (1.1), together with the computation of the Borel–Moore homology of $\bigcup_{i=1}^{7} F_i'$ in §2, yields that the Borel–Moore homology of $\bigcup_{i=8}^{10} F_i'$ is a tensor product of $\bar{H}_\bullet(\mathrm{GL}(3); \mathbf{Q}) \cong \bar{H}_\bullet(\mathbf{C}^*; \mathbf{Q}) \otimes \bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})$. In view of (3.7), this property implies that the Borel–Moore homology of Ψ is a tensor product of $\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})$. The only possibility for this is that the maps of the exact sequence (3.6) respect the structure of Φ_8' and Φ_9' as tensor products of $\bar{H}_\bullet(\mathrm{PGL}(3); \mathbf{Q})$. This is important, because it implies that all differentials δ_k in (3.6) are determined, once one knows the rank of

$$\delta_{25} \colon \bar{H}_{25}(\Phi_9'; \mathbf{Q}) \cong \bigoplus^2 \mathbf{S}_2 \otimes \mathbf{Q}(2) \to \bar{H}_{24}(\Phi_8'; \mathbf{Q}) \cong \mathbf{S}_2 \otimes \mathbf{Q}(2).$$

We claim that δ_{25} has rank one. This would yield part (3.4) in the claim. Note that, in view of (3.7), equality (3.4) implies (3.5).

Define $B \subset X_9'$ as the locus of configurations $(p, q, \{r_i\})$ such that

$$p, q \notin \mathrm{Sing}\left(\bigcup_i r_i\right), \quad pq \notin \{r_1, r_2, r_3, r_4\}.$$

Denote by $\mathcal{B} \to B$ the restriction of the bundle $\Phi_9' \to X_9'$ to B. Next, consider the locus $A \subset X_8'$ of configurations $(p, q, \{e_1, e_2, e_3, e_4\})$ such that $p \in e_1 e_3$, $q \in e_2 e_4$, $\{p, q\} \cap (\{e_1, e_2, e_3, e_4\} \cup (e_1 e_2 \cap e_3 e_4)) = \emptyset$. Denote by $\mathcal{A} \to A$ the restriction of the bundle $\Phi_8' \to X_8'$ to A. Note that for every element $a = (p, q, \{e_i\})$ of A, the configuration $C_a := (p, q, \{e_1 e_2, e_1 e_3, e_2 e_4, e_3 e_4\})$ is an element of B. This means that the face of the four-dimensional open simplex lying above a is identified (in $|\Lambda'|$) with one of the external faces of the five-dimensional simplex contained in $\mathcal{B}$ which lies above $C_a \in B$. Moreover, this four-dimensional open simplex is the only face of C_a which lies in $\mathcal{A}$. Recall that the Borel–Moore homology of the union of an open simplex and one of its open faces is trivial, because of the characterization of Borel–Moore homology as the relative homology of the one-point compactification of a space modulo the added point. This implies that the Borel–Moore homology of $\mathcal{A} \cup \mathcal{B}$ is trivial.

We have the following chains of inclusions.

$$\mathcal{A} \underset{\mathrm{open}}{\hookrightarrow} \Phi_8'$$
$$\mathrm{closed} \cap \qquad \cap \, \mathrm{closed}$$
$$\mathcal{A} \cup \mathcal{B} \underset{\mathrm{open}}{\hookrightarrow} \Psi$$
$$\mathrm{open} \cup \qquad \cup \, \mathrm{open}$$
$$\mathcal{B} \underset{\mathrm{open}}{\hookrightarrow} \Phi_9'$$

In particular, if we consider the long exact sequence in Borel–Moore homology associated to the closed inclusion $\mathcal{A} \hookrightarrow \mathcal{A} \cup \mathcal{B}$, we have that the map $\bar{H}_{25}(\mathcal{B}; \mathbf{Q}) \to \bar{H}_{24}(\mathcal{A}; \mathbf{Q})$ is an isomorphism.

By the computation of the Borel–Moore homology of X'_8 in Lemmas 3.1 and 3.2 we have that $\bar{H}_{24}(\mathcal{A}; \mathbf{Q}) \cong \bar{H}_{24}(\Phi'_8; \mathbf{Q})$ and $\bar{H}_{25}(\mathcal{B}; \mathbf{Q}) \subset \bar{H}_{25}(\Phi'_9; \mathbf{Q})$ are both one-dimensional. □

Now that the contribution of strata 8–10 to the Borel–Moore homology of $\mathcal{D}$ is known, we want to deduce their contribution to the rational cohomology of $\mathcal{I}_2$. Then the closed inclusion $\mathcal{D} \to \mathcal{V}$ (for the definition, see the beginning of the present section) induces a long exact sequence

$$\cdots \to H^k(\mathcal{V}; \mathbf{Q}) \to H^k(\mathcal{I}_2; \mathbf{Q}) \to \bar{H}_{33-k}(\mathcal{D}; \mathbf{Q})(-k) \to H^{k+1}(\mathcal{V}; \mathbf{Q}) \to \cdots,$$

which in the case of the part of Borel–Moore homology of $\mathcal{D}$ that comes from strata 8–10, gives the following contribution to the $\mathfrak{S}_2$-HG polynomial of the rational cohomology of $\mathcal{I}_2$:

$$(1 + \mathbf{L}t)(1 + \mathbf{L}^2 t^3)(1 + \mathbf{L}^3 t^5)\big(\mathbf{L}^6 t^6 s_2 + \mathbf{L}^7 t^8(s_2 + s_{1,1}) + \mathbf{L}^8 t^8 s_{1,1}\big), \tag{3.8}$$

hence the contribution of these strata to the HG polynomial of the cohomology of $\mathcal{Q}_2$ is

$$\mathbf{L}^6 t^6 s_2 + \mathbf{L}^7 t^8(s_2 + s_{1,1}) + \mathbf{L}^8 t^8 s_{1,1}. \tag{3.9}$$

4. Vassiliev–Gorinov's method

In order to make the article as self-contained as possible, we include here an introduction to Vassiliev–Gorinov's method for computing the cohomology of complements of discriminants, following [Tom05a] and [Tom05b]. This review of the method is by no means complete, and we encourage the interested reader to consult [Gor05, Tom05a, Vas99].

Let Z be a projective variety, $\mathcal{F}$ a vector bundle on Z and V the space of global sections of $\mathcal{F}$. Define the discriminant $\Sigma \subset V$ as the locus of sections with a vanishing locus which is either singular or not of the expected dimension. Assume that Σ is a subvariety of V of pure codimension one. Our aim is to compute the rational cohomology of the complement of the discriminant, $X = V \setminus \Sigma$. This is equivalent to determining the Borel–Moore homology of the discriminant, because there is an isomorphism between the reduced cohomology of X and Borel–Moore homology of Σ. If we denote by M the dimension of V, this isomorphism can be formulated as

$$\tilde{H}^\bullet(X; \mathbf{Q}) \cong \bar{H}_{2M-\bullet-1}(\Sigma; \mathbf{Q})(-M).$$

DEFINITION 4.1. A subset $S \subset Z$ is called a *configuration* in Z if it is compact and non-empty. The space of all configurations in Z is denoted by $\mathrm{Conf}(Z)$.

PROPOSITION 4.2 (Gorinov [Gor05]). *The Fubini–Study metric on Z induces in a natural way on* $\mathrm{Conf}(Z)$ *the structure of a compact complete metric space.*

To every element in $v \in V$, we can associate its singular locus $K_v \in \mathrm{Conf}(Z) \cup \{\emptyset\}$. We have that K_0 equals Z, and that $L(K) := \{v \in V : K \subset K_v\}$ is a linear space for all $K \in \mathrm{Conf}(Z)$.

Vassiliev–Gorinov's method is based on the choice of a collection of families of configurations $X_1, \ldots, X_R \subset \mathrm{Conf}(Z)$, satisfying some axioms ([Gor05, § 2] and [Tom05a, List 2.1]). Intuitively, we have to start by classifying all possible singular loci of elements of V. Note that singular loci of the same type have a space $L(K)$ of the same dimension. We can put all singular configurations of the same type in a family. Then we order all families we obtain according to the inclusion of configurations. In this way we obtain a collection of families of configurations which may already satisfy Gorinov's axioms. If this is not the case, the problem can be solved by adding new families to the collection. Typically, the elements of these new families will be degenerations of configurations already considered. For instance, configurations with three points on the same projective line and a point outside it can degenerate into configurations with four points on the same line, even if there is no $v \in V$ which is only singular at four collinear points.

Once the existence of a collection $X_1, \ldots, X_R$ satisfying Gorinov's axioms is established, Vassiliev–Gorinov's method gives a recipe for constructing a space $|\mathcal{X}|$ and a map

$$|\epsilon| \colon |\mathcal{X}| \longrightarrow \Sigma,$$

called *geometric realization*, which is a homotopy equivalence and induces an isomorphism on Borel–Moore homology. The original construction by Vassiliev and Gorinov uses topological joins to construct $|\mathcal{X}|$. This construction was reformulated in [Tom05a] by using the language of cubical spaces. This ensures, in particular, that the map induced by $|\epsilon|$ on Borel–Moore homology respects mixed Hodge structures.

Vassiliev–Gorinov's method provides also a stratification $\{F_j\}_{j=1,\ldots,N}$ on $|\mathcal{X}|$. Each F_j is locally closed in $|\mathcal{X}|$, hence one gets a spectral sequence converging to $\bar{H}_\bullet(\Sigma; \mathbf{Q}) \cong \bar{H}_\bullet(|\mathcal{X}|; \mathbf{Q})$, with $E^1_{p,q} \cong \bar{H}_{p+q}(F_p)$. To compute the Borel–Moore homology of F_j for all $j = 1, \ldots, R$, it is helpful to use an auxiliary space $|\Lambda|$, whose construction depends only on the geometry of the families $X_1 \ldots, X_R$, and which is covered by locally closed subsets $\{\Phi_j\}_{j=1,\ldots,N}$.

PROPOSITION 4.3 (Gorinov [Gor05]). *We have the following.*

(1) *For every $j = 1, \ldots, R$, the stratum F_j is a complex vector bundle over Φ_j. The space Φ_j is, in turn, a fiber bundle over the configuration space X_j.*

(2) *If X_j consists of configurations of m points, the fiber of Φ_j over any $x \in X_j$ is an $(m-1)$-dimensional open simplex, which changes its orientation under the homotopy class of a loop in X_j interchanging a pair of points in x_j.*

(3) *If $X_R = \{Z\}$, F_R is the open cone with vertex a point (corresponding to the configuration Z), over $|\Lambda| \setminus \Phi_R$.*

We recall here the topological definition of an open cone.

DEFINITION 4.4. Let B be a topological space. Then a space is said to be an *open cone* over B with vertex a point if it is homeomorphic to the space $B \times [0,1)/R$, where the equivalence relation is $R = (B \times \{0\})^2$.

The fiber bundle $\Phi_j \to X_j$ of Proposition 4.3 is, in general, non-orientable. As a consequence, we have to consider the homology of X_j with coefficients not in $\mathbf{Q}$, but in some local system of rank one. Therefore, we recall some constructions concerning Borel–Moore homology of configuration spaces with twisted coefficients.

DEFINITION 4.5. Let Z be a topological space. Then for every $k \geqslant 1$ we have the space of ordered configurations of k points in Z,

$$F(Z, k) = Z^k \setminus \bigcup_{1 \leqslant i < j \leqslant k} \{(z_1, \ldots, z_k) \in Z^k : z_i = z_j\}.$$

There is a natural action of the symmetric group $\mathfrak{S}_k$ on $F(k, Z)$. The quotient is called the space of unordered configurations of k points in Z,

$$B(Z, k) = F(Z, k)/\mathfrak{S}_k.$$

The *sign representation* $\pi_1(B(Z, k)) \to \mathrm{Aut}(\mathbf{Z})$ maps the paths in $B(Z, k)$ defining odd (respectively, even) permutations of k points to multiplication by -1 (respectively, 1). The local system $\pm\mathbf{Q}$ over $B(Z, k)$ is the one locally isomorphic to $\mathbf{Q}$, but with monodromy representation equal to the sign representation of $\pi_1(B(Z, k))$. We often call $\bar{H}_\bullet(B(Z, k), \pm\mathbf{Q})$ the *Borel–Moore homology of $B(Z, k)$ with twisted coefficients* or, simply, the *twisted Borel–Moore homology of $B(Z, k)$*.

The following is Lemma 2 in [Vas99].

Lemma 4.6. *We have the following.*

(i) *If $N \geqslant 1$, $k \geqslant 2$, the twisted Borel–Moore homology of $B(\mathbf{C}^N, k)$ is trivial.*

(ii) *If $N \geqslant 1$, we have isomorphisms*

$$\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) \cong \bar{H}_{\bullet-k(k-1)}(\mathbf{G}(k-1, \mathbf{P}^N); \mathbf{Q})$$

for every $k \geqslant 1$, where $\mathbf{G}(k-1, \mathbf{P}^N)$ denotes the Grassmann variety of $(k-1)$-dimensional linear subspaces in $\mathbf{P}^N$. In particular, $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) = 0$ if $k \geqslant N + 2$.

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