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Setoids and universes

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Setoids commonly take the place of sets when formalising mathematics inside type theory. In this note, the category of setoids is studied in type theory with universes that are as small as possible (and thus, the type theory is as weak as possible). In particular, we will consider epimorphisms and disjoint sums. We show that, given the minimal type universe, all epimorphisms are surjections, and disjoint sums exist. Further, without universes, there are countermodels for these statements, and if we use the Logical Framework formulation of type theory, these statements are provably non-derivable.

1. Introduction

Type theory is intended as a foundation for constructive mathematics. However, it is intensional, and so direct formalisations of mathematics in it frequently become too restrictive. The usual approach is to work instead with *setoids*, which are simply a type together with an equivalence relation on that type (sometimes known as the 'book equality', which is terminology originating in the Automath project (Nederpelt *et al.* 1994)). This is essentially a formalisation of Bishop's notion of a (constructive) set (Bishop 1967; Bishop and Bridges 1985).

In this paper, I investigate some properties of setoids, particularly those related to how much strength is needed to prove that all epimorphisms of setoids are surjective and that there are disjoint unions of setoids. It turns out that the minimal universe L, which only contains names for the empty set and the standard singleton set, is sufficient. We also show that type theory without universes is insufficient, building on earlier work (Fridlender 2002; Smith 1988) showing that type theory without universes is insufficient to show $0 \neq 1$.

The treatment in this paper is, as far as possible, purely type-theoretical, with particular attention paid to type theory in the Logical Framework, but it should be noted that similar results have been obtained by other means: there are corresponding category-theoretical results (for example, disjointness of sums in Carboni (1995)), which can be translated by means of the results in Maietti (2005; 2007). Some type-theoretical results on the disjointness of sums were also announced in Maietti (1998; 2007), with more details in Maietti (2009).

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In the next section we introduce the type theory and notation required. For simplicity, a version of type theory without the Logical Framework is used, letting us obtain weaker versions of the results (since the language of this type theory is not rich enough to express the stronger results). In Section 3 we give the results for epimorphisms. In Section 4 we first construct sums (or binary co-products) of setoids, using only the minimal universe, and then prove both positive and negative results for disjoint sums. Then, in Section 5, we show how to adapt all the proofs to a presentation of type theory with the Logical Framework. Finally, in Section 6, we note how assuming disjoint sums suffices to reconstruct a minimal universe.

2. Type theory, setoids and notation

We will work in a version of Martin-Löf type theory with the type formers Π (for function spaces and the universal quantifier), Σ (for dependent products and the existential quantifier), $\vee$ (for the disjunction), and N_0, N_1 and N_2 for the empty, one- and two-point sets, respectively. For notational convenience, we will use $\forall$ and $\exists$ rather than Π and Σ where this is motivated by the context, and, similarly, we will write $\perp$ for the empty set. Also, we will use $*$ to denote the unique element of N_1, and tt and ff to denote the two elements of N_2 (so we see N_2 as the set of Boolean truth values). We will use $\mathsf{inl}()$ and $\mathsf{inr}()$ to denote the two constructors for disjunctions. These types all come with the usual elimination constants, systematically written $\vee$-elim, and so on.

To this, we add a minimal universe decoding function L, with domain N_2, such that $L(\mathsf{tt}) = \mathsf{N}_1$ and $L(\mathsf{ff}) = \mathsf{N}_0$.

From the given type formers, we also construct $\rightarrow$ (non-dependent function spaces and implication) and $\times$ (Cartesian products and conjunction, which is also written as &), as Π and Σ with constant families. Function application is written as $f(a)$ (rather than with an explicit application operator). The pairing construction will be written $\langle\cdot,\cdot\rangle$, both for the Cartesian products and for the Σ-types, and we also derive projection functions π_1, π_2 for the Cartesian product.

These will suffice to carry out all constructions in this paper. For the negative results, we may also, without loss of generality, assume that the type theory contains the natural numbers N, finite sets N_i of all sizes, W-types and the identity types $\mathsf{Id}_{\_}(\cdot,\cdot)$.

Thus, the type theory used is essentially that of Martin-Löf (1984), but with intensional identity types, and with the type universes either replaced simply by the minimal universe L or removed.

We define, for completeness, and to fix notation, a setoid $\mathscr{A}$ to consist of the following terms-in-context:

- a type A, the *carrier* of $\mathscr{A}$;
- a term $x =_{\mathscr{A}} y$, the *equality* of $\mathscr{A}$, such that $x, y : A \vdash x =_{\mathscr{A}} y$ type;
- a term $\mathsf{refl}^x_{\mathscr{A}}$ such that $x : A \vdash \mathsf{refl}^x_{\mathscr{A}} : x =_{\mathscr{A}} x$;
- a term $\mathsf{sym}^{x,y}_{\mathscr{A}}(p)$ such that $x, y : A, p : x =_{\mathscr{A}} y \vdash \mathsf{sym}^{x,y}_{\mathscr{A}}(p) : y =_{\mathscr{A}} x$;
- a term $\mathsf{trans}^{x,y,z}_{\mathscr{A}}(p,q)$ such that

$$x, y, z : A, p : x =_{\mathscr{A}} y, q : y =_{\mathscr{A}} z \vdash \mathsf{trans}^{x,y,z}_{\mathscr{A}}(p,q) : x =_{\mathscr{A}} z.$$

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In other words, a setoid consists of a type together with a binary relation, and proofs that this relation is an equivalence relation. For simplicity, the equivalence relation and proofs are all written as operators (the relation infix), and not as terms subjected to substitutions; the index on the setoid equality may also be left out when it is clear from the context.

Also, a map $\mathscr{A} \to \mathscr{B}$ of setoids consists of a function $f : A \to B$ (that is, an element of the non-dependent function space), together with a proof

$$\mathsf{ext}_f : (\forall x, y : A)(x =_{\mathscr{A}} y \to f(x) =_{\mathscr{B}} f(y))$$

that it respects the equality. Two such maps are equal if they are extensionally equal, resulting in a setoid $[\mathscr{A} \to \mathscr{B}]$ of setoid maps.

It is easy to verify that setoids and their maps (all taken in the empty context) form a category **Setoid**. When we consider the category of setoids for a particular model $\mathscr{M}$ of type theory, we will write it as $\mathbf{Setoid}_{\mathscr{M}}$.

3. Epimorphisms and surjections

Recall that a morphism is an *epimorphism* if it is *right cancellable*. In other words, the setoid map $f : \mathscr{A} \to \mathscr{B}$ is an epimorphism if for every setoid $\mathscr{C}$ and all parallel maps $g, h : \mathscr{B} \to \mathscr{C}$, the equality $g =_{[\mathscr{B} \to \mathscr{C}]} h$ follows from the equality of the composites $g \circ f =_{[\mathscr{A} \to \mathscr{C}]} h \circ f$.

A morphism $f : \mathscr{A} \to \mathscr{B}$ is *surjective* if $(\forall b : B)(\exists a : A)(f(a) =_{\mathscr{B}} b)$. (This is sometimes known as being *onto*).

Proposition 1. Every epimorphism of setoids is surjective.

This is of course well known, but all of the proofs I am aware of use a much stronger type theory. For example, the proof in Mines *et al.* (1988) is impredicative (it makes use of power sets), while other proofs (for example, in Coquand *et al.* (2005)) make use of larger universes.

Proof. Suppose $f : \mathscr{A} \to \mathscr{B}$ is a map of setoids. Define a setoid $\widehat{\mathscr{B}}$ having:

— as carrier the type $\mathsf{N}_2 \times B$;
— equality $x =_{\widehat{\mathscr{B}}} y$ given by

$$(L(\pi_1(x)) \to (\exists a : A)(f(a) =_{\mathscr{B}} \pi_2(x))) \leftrightarrow (L(\pi_1(y)) \to (\exists a : A)(f(a) =_{\mathscr{B}} \pi_2(y)));$$

— easy proofs of reflexivity, symmetry and transitivity.

We also define two maps $g, h : \mathscr{B} \to \widehat{\mathscr{B}}$ whose underlying functions are given by $g(b) = \langle \mathsf{ff}, b \rangle$ and $h(b) = \langle \mathsf{tt}, b \rangle$. The extensionality of these maps is easy to prove (even without using the *ex falso* rule).

We can also show that $g \circ f =_{[\mathscr{A} \to \widehat{\mathscr{B}}]} h \circ f$, since the existential statements are clearly true (and again, the *ex falso* rule is not essential).

Now, if the map f is an epimorphism, it follows that $g =_{[\mathscr{B} \to \widehat{\mathscr{B}}]} h$, so for any $b : B$, we have $g(b) =_{\widehat{\mathscr{B}}} h(b)$. Unfolding the definitions, this says that

$$(\mathsf{N}_0 \to (\exists a : A)(f(a) =_{\mathscr{B}} b)) \leftrightarrow (\mathsf{N}_1 \to (\exists a : A)(f(a) =_{\mathscr{B}} b)),$$

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and since the left-hand side follows immediately by *ex falso*, and N_1 is inhabited, it follows that $(\exists a\!:\!A)(f(a) =_{\mathscr{B}} b)$, so the map f is surjective. □

Proposition 2. There is a model of type theory without universes in which Proposition 1 does not hold.

Proof. Note first that this is not a non-derivability result since the statement considered requires quantification over all setoids, and hence over all small types, so it is not actually expressible in the language considered.

Consider the model $\mathscr{M}_{\mathrm{S}}$ of type theory constructed in Smith (1988). Note that the setoids are essentially trivialised in this model – since the model is 'proof irrelevant', the equality is either always inhabited, or always empty (this is, essentially, the content of Smith (1988, Lemma 1)). By reflexivity, the latter can happen only if the carrier is empty – thus setoids are either empty or extensionally one-point setoids. In particular, this means that *any* two parallel setoid maps will be (extensionally) equal. Consequently, *all* maps of setoids are epimorphisms.

But if we consider the empty setoid, with carrier N_0 and equality given by N_0 (and thus, no equalities hold), and the one-point setoid with carrier N_1 and equality given by N_1 (and thus all elements are equal), there is, in fact, a map (for example, given as the constant map with value $*$) from the first to the second[†]. But this map is clearly not surjective – in fact, surjectivity for this map is the proposition $(\forall x\!:\!\mathsf{N}_1)(\exists y\!:\!\mathsf{N}_0)\mathsf{N}_0$, whose negation is provable.

We have thus constructed a map that is provably not surjective, but is nevertheless an epimorphism in $\mathbf{Setoid}_{\mathscr{M}_{\mathrm{S}}}$, which is the category of setoids drawn from Smith's model of type theory without universes. □

The other direction, showing that all surjective maps are epimorphisms, goes through without universes (bearing in mind that being an epimorphism is not a proposition – it quantifies over all setoids, and thus implicitly over all types).

4. Disjoint sums

We say, as in Carboni *et al.* (1993), that a category has *disjoint sums* if it has sums (binary co-products), the injections for the sums are monic, and for any two objects X and Y, the pullback

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & X+Y \end{array}$$

exists and is initial.

We will show that **Setoid** has disjoint sums, but that this is not provable without universes. But first we need to remind ourselves of the pullback construction: given setoid

† We can in fact show that these setoids are initial and terminal in **Setoid** without using universes.

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maps

$$\mathscr{A} \xrightarrow{\;f\;} \mathscr{X} \xleftarrow{\;g\;} \mathscr{B},$$

the pullback $\mathscr{A} \times_{\mathscr{X}} \mathscr{B}$ has:

— carrier $(\Sigma x{:}A \times B)(f(\pi_1(x)) =_{\mathscr{X}} g(\pi_2(x)))$;
— equality $\langle x, p\rangle =_{\mathscr{A}\times_{\mathscr{X}}\mathscr{B}} \langle y, q\rangle$ given by $\pi_1(x) =_{\mathscr{A}} \pi_1(y) \,\&\, \pi_2(x) =_{\mathscr{B}} \pi_2(y)$;
— reflexivity, symmetry and transitivity immediately from those of $\mathscr{A}$ and $\mathscr{B}$.

We can now define the projections, verify that the resulting square commutes and then prove that the universal property holds.

The first thing we need to prove is that the category **Setoid** has sums. There is a standard construction, which given setoids $\mathscr{A}$ and $\mathscr{B}$ takes the disjoint union of types $A \vee B$ as carrier, and then defines the equality relation by case distinction – saying, essentially, that two elements of the form $\mathsf{inl}(a)$ and $\mathsf{inl}(a')$ are equal if $a =_{\mathscr{A}} a'$, that two elements of the form $\mathsf{inr}(b)$ and $\mathsf{inr}(b')$ are equal if $b =_{\mathscr{B}} b'$, and that two elements of the forms $\mathsf{inl}(a)$ and $\mathsf{inr}(b)$ are never equal. This will then provide the required sum. Unfortunately, the case distinction essentially amounts to an application of a 'large elimination rule' (Smith 1989), and thus to a much stronger type theory. We will replace this by a slightly more intricate construction to prove the following proposition.

Proposition 3. The category **Setoid** has sums.

Proof. We begin with some auxiliary constructions. Suppose X and Y are types. Then so is $X \vee Y$. We will first construct a function $\alpha{:}X \vee Y \to \mathsf{N}_2$. Note that $z{:}X \vee Y \vdash \mathsf{N}_2$ type, and that $x{:}X \vdash \mathsf{tt}{:}\mathsf{N}_2$ and $y{:}Y \vdash \mathsf{ff}{:}\mathsf{N}_2$. From this we obtain

$$z{:}X \vee Y \vdash \vee\text{-}\mathsf{elim}([z{:}X \vee Y]\mathsf{N}_2, [x{:}X]\mathsf{tt}, [y{:}Y]\mathsf{ff}, z){:}\mathsf{N}_2,$$

and hence, by abstraction, the required α (here in the more recent notational style, with explicit binders for the variables).

We can now define dependent functions

$$\mathsf{exl}{:}(\Pi x{:}X \vee Y)(L(\alpha(x)) \to X)$$
$$\mathsf{exr}{:}(\Pi x{:}X \vee Y)(L(\neg\alpha(x)) \to Y)$$

(where $\neg$ denotes the Boolean negation). For exl, given $x{:}X \vee Y$, we need to produce a function $L(\alpha(x)) \to X$. Proceeding by $\vee$-elim, we certainly have $x{:}X \vee Y \vdash X$ type. Also $x{:}X, p{:}L(\alpha(\mathsf{inl}(x))) \vdash x{:}X$, so, abstracting, we get

$$x{:}X \vdash \lambda([p{:}L(\alpha(\mathsf{inl}(x)))]x){:}L(\alpha(\mathsf{inl}(x))) \to X.$$

On the other hand,

$$y{:}Y, p{:}L(\alpha(\mathsf{inr}(y))) \vdash \bot\text{-}\mathsf{elim}_X(p){:}X$$

(since $L(\alpha(\mathsf{inr}(y)))$ is the empty set), and we can again abstract to obtain an element of $L(\alpha(\mathsf{inr}(y))) \to X$. Applying the disjunction elimination and abstracting now gives the required function exl. The construction of exr is similar.

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We are now ready to define the sum. Suppose we are given setoids $\mathscr{A}$ and $\mathscr{B}$. We define† a new setoid $\mathscr{A}+\mathscr{B}$ having:

— carrier $A \vee B$;
— equality $x =_{\mathscr{A}+\mathscr{B}} y$ given by

$$(\Sigma p\colon L(\alpha(x)))(\Sigma q\colon L(\alpha(y)))(\mathsf{exl}(x,p) =_{\mathscr{A}} \mathsf{exl}(y,q)) \vee \\ (\Sigma p\colon L(\neg\alpha(x)))(\Sigma q\colon L(\neg\alpha(y)))(\mathsf{exr}(x,p) =_{\mathscr{B}} \mathsf{exr}(y,q));$$

— reflexivity – note that given $x\colon A \vee B$ we may again proceed by $\vee$-elim, so for given $a\colon A$, we see that $\mathsf{inl}(\langle *, \langle *, \mathsf{refl}^a_{\mathscr{A}}\rangle\rangle)$ proves the equality required, and, similarly, for $b\colon B$, the term $\mathsf{inr}(\langle *, \langle *, \mathsf{refl}^b_{\mathscr{B}}\rangle\rangle)$ will do;
— symmetry – this is an immediate consequence of the symmetry in $\mathscr{A}$ and $\mathscr{B}$;
— transitivity – given $x, y, z\colon A \vee B$, we just need to consider cases for y, and then for the two proofs (in all cases where we do not have an element of N_0 to hand, we may instead apply the transitivity of $\mathscr{A}$ or $\mathscr{B}$ to obtain the required result).

We must now define the two injections, and prove the universal property.

The injection $\iota_1\colon \mathscr{A} \to \mathscr{A}+\mathscr{B}$ is given by the map sending a to $\mathsf{inl}(a)$. Note that since

$$a, a'\colon A, p\colon a =_{\mathscr{A}} a' \vdash \mathsf{inl}(\langle *, \langle *, p\rangle\rangle)\colon \iota_1(a) =_{\mathscr{A}+\mathscr{B}} \iota_1(a'),$$

this is an extensional map. The second injection $\iota_2\colon \mathscr{B} \to \mathscr{A}+\mathscr{B}$ is defined similarly.

For the universal property, suppose we have setoid maps

$$\mathscr{A} \xrightarrow{\ f\ } \mathscr{C} \xleftarrow{\ g\ } \mathscr{B}.$$

We need to define a setoid map $\binom{f}{g}\colon \mathscr{A}+\mathscr{B} \to \mathscr{C}$. For the mapping, suppose we have $x\colon A \vee B$. Using $\vee$-elim (which is unproblematic since we are constructing a *non-dependent* function), we may perform case distinction on the form of x:

— For an x of the form $\mathsf{inl}(a)$, we have $f(a)\colon C$.
— For an x of the form $\mathsf{inr}(b)$, we have $g(b)\colon C$.

This yields a function $A \vee B \to C$.

We still need to show that the map $\binom{f}{g}$ is extensional. So we suppose $x, y\colon A \vee B$ and consider cases for the forms of both x and y:

— If x and y have the forms $\mathsf{inl}(a)$ and $\mathsf{inl}(a')$, we must show that

$$((\Sigma p\colon \mathsf{N}_1)(\Sigma q\colon \mathsf{N}_1)(a =_{\mathscr{A}} a') \vee (\Sigma p\colon \mathsf{N}_0)(\Sigma q\colon \mathsf{N}_0)(\mathsf{exr}(\mathsf{inl}(a), p) =_{\mathscr{B}} \mathsf{exr}(\mathsf{inl}(a'), q))) \\ \to f(a) =_{\mathscr{C}} f(a').$$

So suppose p is a proof of equality in the sum, and make another case distinction on the form of p. For p of the form $\mathsf{inl}(q)$, we obtain a proof of $a =_{\mathscr{A}} a'$ from q, and are done by the extensionality of f. If, instead, p has the form $\mathsf{inr}(q)$, we can obtain an element of N_0 from q, and, using *ex falso*, we are done.
— If x and y have the forms $\mathsf{inr}(b)$ and $\mathsf{inr}(b')$, we can use a similar proof using the extensionality of g.

† A similar definition, making additional use of the Id-type, was given in Maietti (2009, page 331).

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— Suppose x has the form $\mathsf{inl}(a)$, and y has the form $\mathsf{inr}(b)$. We must show that

$$((\Sigma p\colon \mathsf{N}_1)(\Sigma q\colon \mathsf{N}_0)(a =_{\mathscr{A}} \mathsf{exl}(\mathsf{inr}(b), q)) \vee (\Sigma p\colon \mathsf{N}_0)(\Sigma q\colon \mathsf{N}_1)(\mathsf{exr}(\mathsf{inl}(a), p) =_{\mathscr{B}} b)) \\ \rightarrow f(a) =_{\mathscr{C}} g(b).$$

Again supposing p to be a proof of equality in the sum, and making a case distinction on the form of p, we see that we can extract an element of N_0 in both cases, and thus apply *ex falso* to conclude each branch.

— The other mixed case is similar.

We still need to show two things:

(1) that the diagram

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\iota_1} & \mathscr{A}+\mathscr{B} & \xleftarrow{\iota_2} & \mathscr{B} \\ \| & & \Big\downarrow {\scriptstyle \binom{f}{g}} & & \| \\ \mathscr{A} & \xrightarrow{f} & \mathscr{C} & \xleftarrow{g} & \mathscr{B} \end{array}$$

commutes;

(2) that the map $\binom{f}{g}$ is the unique map making the diagram commute.

Item (1) follows by direct computation. So to prove (2), suppose $h\colon \mathscr{A}+\mathscr{B} \rightarrow \mathscr{C}$ is any other map making the diagram above commute. Take an arbitrary $x\colon A \vee B$. We must show $h(x) =_{\mathscr{C}} \binom{f}{g}(x)$. We proceed by case distinction on x. For an x of the form $\mathsf{inl}(a)$, we have

$$h(x) =_{\mathscr{C}} h(\mathsf{inl}(a)) =_{\mathscr{C}} h(\iota_1(a)) =_{\mathscr{C}} f(a) =_{\mathscr{C}} \tbinom{f}{g}(\mathsf{inl}(a)) =_{\mathscr{C}} \tbinom{f}{g}(x).$$

For an x of the form $\mathsf{inr}(b)$, we have

$$h(x) =_{\mathscr{C}} h(\mathsf{inr}(b)) =_{\mathscr{C}} h(\iota_2(b)) =_{\mathscr{C}} g(b) =_{\mathscr{C}} \tbinom{f}{g}(\mathsf{inr}(b)) =_{\mathscr{C}} \tbinom{f}{g}(x).$$

Thus, applying $\vee$-elim, we are done. □

We are now in position to prove the following proposition.

Proposition 4. The category **Setoid** has disjoint sums.

Proof. Proving that the two injections are monomorphisms is easy. Since all pullbacks exist, it is sufficient to show that the carrier of the pullback

$$\begin{array}{ccc} \mathscr{P} & \longrightarrow & \mathscr{B} \\ \Big\downarrow & & \Big\downarrow {\scriptstyle \iota_2} \\ \mathscr{A} & \xrightarrow{\iota_1} & \mathscr{A}+\mathscr{B} \end{array} \qquad (1)$$

is empty. But the carrier is in fact

$$(\Sigma x\colon A \times B)(\iota_1(\pi_1(x)) =_{\mathscr{A}+\mathscr{B}} \iota_2(\pi_2(x))) \\ \equiv (\Sigma x\colon A \times B)((\Sigma p\colon \mathsf{N}_1)(\Sigma q\colon \mathsf{N}_0)(\mathsf{exl}(\mathsf{inl}(\pi_1(x)), p) =_{\mathscr{A}} \mathsf{exl}(\mathsf{inr}(\pi_2(x)), q)) \vee \\ (\Sigma p\colon \mathsf{N}_0)(\Sigma q\colon \mathsf{N}_1)(\mathsf{exr}(\mathsf{inl}(\pi_1(x)), p) =_{\mathscr{B}} \mathsf{exr}(\mathsf{inr}(\pi_2(x)), q))),$$

and this is clearly empty since we may extract a proof of type

$$(\Sigma p\colon \mathsf{N}_1)(\Sigma q\colon \mathsf{N}_0)(\ldots) \vee (\Sigma p\colon \mathsf{N}_0)(\Sigma q\colon \mathsf{N}_1)(\ldots),$$

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and then, by doing a case distinction on the form of this proof, we can extract an element of N_0. □

Proposition 5. There is a model of type theory without universes in which Proposition 4 does not hold.

Proof. Note that, for the reasons mentioned in the proof of Proposition 2, this is not a non-derivability result.

Consider again the category **Setoid** in the model of type theory of Smith (1988). Consider the one-point setoid $\mathscr{I}$ having carrier N_1 and equality given by N_1, and suppose its sum $\mathscr{I} + \mathscr{I}$ with itself exists[†]. This sum has an inhabited carrier (since, for example, $\iota_1(*)$ is an element), and must hence be (extensionally) a one-point setoid. Let us now consider the pullback. It has as carrier the set

$$(\Sigma x : \mathsf{N}_1 \times \mathsf{N}_1)(\iota_1(\pi_1(x)) =_{\mathscr{I}+\mathscr{I}} \iota_2(\pi_2(x))),$$

and this is inhabited, since $x =_{\mathscr{I}+\mathscr{I}} y$ is inhabited for all x and y. Hence the pullback is *also* a one-point setoid, and, therefore, is not initial (since we know there is an empty setoid). □

For an alternative proof of Proposition 5, suppose again that the sum $\mathscr{I} + \mathscr{I}$ exists, and call its carrier J. Consider the function $f : \mathsf{N}_2 \to J$ defined by recursion with $f(\mathsf{tt}) \equiv \iota_1(*)$ and $f(\mathsf{ff}) \equiv \iota_2(*)$. Suppose $\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt}, \mathsf{ff})$. Clearly, $f(\mathsf{tt}) =_{\mathscr{I}+\mathscr{I}} f(\mathsf{tt})$, so, using the elimination rule for the identity type, we get some $p : f(\mathsf{tt}) =_{\mathscr{I}+\mathscr{I}} f(\mathsf{ff})$. But then $\langle\langle *, *\rangle, p\rangle$ is an element of the empty pullback. So disjoint sums imply that $\neg\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt}, \mathsf{ff})$, which we know (from Smith (1988)) is not provable without universes and the result then follows immediately.

5. Setoids in the Logical Framework

Modern presentations of type theory (Nordström *et al.* 2000; Nordström *et al.* 1990) are formulated in the Logical Framework. In these presentations, there are only three type formers: the dependent function space, the special type Set (consisting of codes for inductively defined types) and its 'decoding function' El. The usual type formers turn into set formers, but otherwise remain unchanged. The type Set thus looks somewhat like a universe, and all work is done inside Set. However, Set is not a universe, and the proof irrelevant model of Smith (1988) can be extended to cover the Logical Framework, as done in Fridlender (2002).

Fridlender's model is constructed by giving an interpretation of the Logical Framework version of type theory into a non-trivial (that is, having at least two elements) extensional model for the untyped lambda calculus, with domain D. Given a valuation ρ, that is, an assignment of an element of D to each variable symbol, a (pre)type A is interpreted as a subset $[\![A]\!]_\rho \subseteq D$, a (pre)object a as an element $[\![a]\!]_\rho \in D$, and so on for the other

[†] Since the category of setoids is 'collapsed' in this model, a little thought should be enough to convince yourself that this sum does exist, as do all sums – their carrier being given by the disjunction, with equality relation given by N_1. But since this is not a sum in every model, there can be no internal proof that this really is a sum.

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syntactic categories. Of particular interest is Set, which is interpreted in every valuation as $\{\top, \bot\}$ (where $\top = \lambda x.\lambda y.x$ and $\bot = \lambda x.\lambda y.y$), and El, which is interpreted as the function sending an element $d \in D$ to the set $\{e \in D \mid d = \top\}$ (thus, in particular, $\mathsf{El}(\top) = D$ and $\mathsf{El}(\bot) = \emptyset$).

After making the necessary changes to the definition of a setoid, the constructions of Propositions 1, 3 and 4 work essentially unchanged (just recall that the universe decoding function L is now Set-valued). However, one change is worthy of mention: the equality of a setoid $\mathscr{A}$ can now be given as a typed term $=_{\mathscr{A}} : (A, A)\mathsf{Set}$ rather than as a term-in-context. This will allow us to put setoids in the context, which is key to formulating stronger results.

For the negative results (Propositions 2 and 5), the key observation was that the setoids in Smith's model are trivialised, in the sense that every setoid is either empty or (extensionally) a one-point setoid. This was a consequence of the proof irrelevance. In Fridlender's model, this is still true since here, too, families of sets over a given set are constant (see Fridlender (2002, Section 3.5), particularly Corollary 3.19). Having made this observation, the proofs go through exactly as before.

This can then be summed up by the following proposition.

Proposition 6. The category of setoids, in the Logical Framework version of type theory with the minimal universe L has disjoint sums, and all its epimorphisms are surjections. There is also a model $\mathscr{M}$ of type theory without universes, but in the Logical Framework, such that the associated category $\mathbf{Setoid}_{\mathscr{M}}$ of setoids has non-surjective epimorphisms and lacks disjoint sums.

So far we have only shown the weaker results, namely that there are models of type theory for which certain statements fail. Working with type theory in the Logical Framework means, however, that we have a much richer language to work with. In fact, this language is rich enough to express the statements considered, which enables us to achieve the following non-derivability results we have been aiming for.

Proposition 7. The statement that every epimorphism of setoids is a surjection, expressed (in shortened notation) as

$$\mathscr{A}, \mathscr{B} : \mathsf{setoid}, f : \mathscr{A} \to \mathscr{B}, e : \mathsf{epi}_f \vdash (\forall b : B)(\exists a : A)(f(a) = b) \ \mathsf{true},$$

is not derivable in the Logical Framework version of type theory without universes.

(For greater readability, we have written $\mathscr{A} : \mathsf{setoid}$ and $f : \mathscr{A} \to \mathscr{B}$ as shorthand for the typed terms from the definitions of setoids and setoid maps, respectively, and epi_f for the type

$$(\mathscr{C} : \mathsf{setoid})(g, h : \mathscr{B} \to \mathscr{C})(g \circ f = h \circ f \to g = h).$$

Note the essential use of the Logical Framework in this type.)

Proof. Suppose the judgement were derivable. Then we could also derive the judgement

$$e : \mathsf{epi}_f \vdash (\forall b : \mathsf{N}_1)(\exists a : \mathsf{N}_0)(f(a) = b) \ \mathsf{true},$$

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where f is the unique-up-to-extensional-equality map from the empty setoid to the standard one-point setoid, as considered in the proof of Proposition 2. In fact, this judgement is obtained from the previous one by carrying out a substitution. Since it is provable that the map f is *not* surjective, this would also imply the derivability of the judgement

$$e : \mathsf{epi}_f \vdash \mathsf{N}_0 \ \mathsf{true}.$$

We now consider the model $\mathscr{M}_{\mathrm{F}}$ of type theory constructed in Fridlender (2002), which will also serve as a reference for all notation. Taking ρ_0 to be an arbitrary valuation, we compute the interpretation in $\mathscr{M}_{\mathrm{F}}$ of the type epi_f. The type epi_f written out in full is

$$\begin{aligned}
&(C : \mathsf{Set})(=_C : (C, C)\mathsf{Set})(\forall \mathsf{refl}_C : (\forall x : C)(x =_C x))\\
&(\forall \mathsf{sym}_C : (\forall x, y : C)(\forall p : x =_C y)(y =_C x))\\
&(\forall \mathsf{trans}_C : (\forall x, y, z : C)(\forall p : x =_C y)(\forall q : y =_C z)(x =_C z))\\
&(\forall g : (\forall x : \mathsf{N}_1)C)(\forall \mathsf{ext}_g : (\forall x, y : \mathsf{N}_1)(\forall p : x =_1 y)(g(x) =_C g(y)))\\
&(\forall h : (\forall x : \mathsf{N}_1)C)(\forall \mathsf{ext}_h : (\forall x, y : \mathsf{N}_1)(\forall p : x =_1 y)(h(x) =_C h(y)))\\
&(\forall \mathsf{eq} : (\forall x : \mathsf{N}_0)(g(f(x)) =_C h(f(x))))(\forall x : \mathsf{N}_1)(g(x) =_C h(x)),
\end{aligned}$$

where we have made use of the shorthands introduced in Fridlender (2002, Section 4.2, page 788). For obvious reasons, much of the syntax will be elided in the following calculations. So we have

$$\begin{aligned}
\llbracket\mathsf{epi}_f\rrbracket_{\rho_0} &= \{d \in \mathscr{D} \mid (\forall e \in \mathsf{Set})(d \cdot e \in \llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(e)\}\\
&= \{d \in \mathscr{D} \mid \ d \cdot \top \in \llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(\top) \ \&\\
&\qquad\qquad d \cdot \bot \in \llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(\bot)\}.
\end{aligned}$$

Now note that finding an element d of this set is equivalent to finding elements

$$t \in \llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(\top)$$

and

$$f \in \llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(\bot)$$

since we may take $d = \lambda x.x \cdot t \cdot f$ to get $d \cdot \top = t$ and $d \cdot \bot = f$.

Before starting the search for terms t and f, we will briefly consider the interpretation of universally quantified statements. A term $(\forall x : C)D$ denotes a set. Taking an arbitrary valuation ρ, we have

$$\llbracket(\forall x : C)D\rrbracket_\rho = \llbracket\forall(C, [x]D)\rrbracket_\rho = \llbracket\forall\rrbracket_\rho \cdot \llbracket C\rrbracket_\rho \cdot \llbracket [x]D\rrbracket_\rho.$$

Following Fridlender (2002, pages 787–789), we find that $\llbracket\forall\rrbracket_\rho = \lambda x.\lambda y.x \cdot (y \cdot *) \cdot \top$, for $*$ an arbitrary closed expression. Thus we compute

$$\llbracket(\forall x : C)D\rrbracket_\rho = \llbracket C\rrbracket_\rho \cdot \llbracket D\rrbracket_{(\rho, x = *)} \cdot \top.$$

To find our t, we compute the relevant conditions: t must belong to the set

$$\begin{aligned}
&\llbracket [C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \to [=_C]\mathsf{El}(\cdots))\rrbracket_{\rho_0}(\top)\\
&\quad = \{d \in \mathscr{D} \mid (\forall e \in \mathscr{D})((\forall f \in \mathscr{D})(\forall g \in \mathscr{D})(e \cdot f \cdot g \in \mathsf{Set}) \to d \cdot e \in \llbracket\mathsf{El}(\cdots)\rrbracket_{\rho_1})\},
\end{aligned}$$

where ρ_1 is the valuation $(\rho_0, C = \top, =_C = e)$ (the calculation is long, but straightforward). The interesting case of the constraint is, of course, when we consider e such that $(\forall f \in \mathscr{D})(\forall g \in \mathscr{D})(e \cdot f \cdot g \in \mathsf{Set})$. Then, applying Fridlender (2002, Theorem 3.17), we have two cases, namely $(\forall f \in \mathscr{D})(\forall g \in \mathscr{D})(e \cdot f \cdot g = \top)$ and $(\forall f \in \mathscr{D})(\forall g \in \mathscr{D})(e \cdot f \cdot g = \bot)$. We handle these cases separately, and compute $[\![\mathsf{El}(\cdots)]\!]_{\rho_1} = \mathsf{El}([\![\cdots]\!]_{\rho_1})$. Long calculations making repeated use of both our characterisation of the universal quantifier and the respective assumptions on e yield $[\![\cdots]\!]_{\rho_1} = \top$ in both cases, so we get that the condition on t is that it must lie in the set

$$\{d \in \mathscr{D} \mid (\forall e \in \mathscr{D})((\forall f \in \mathscr{D})(\forall g \in \mathscr{D})(e \cdot f \cdot g \in \mathsf{Set}) \rightarrow d \cdot e \in \mathscr{D})\} = \mathscr{D}$$

(since $\mathsf{El}(\top) = \mathscr{D}$). So, in fact, there is no restriction on t, and any element of the model will do.

To find f, we again compute the relevant conditions: f must belong to the set

$$[\![[C]((\mathsf{El}(C), \mathsf{El}(C))\mathsf{Set} \rightarrow [=_C]\mathsf{El}(\cdots))]\!]_{\rho_0}(\bot) = \{d \in \mathscr{D} \mid (\forall e \in \mathscr{D})(d \cdot e \in \mathsf{El}([\![\cdots]\!]_{\rho_2}))\},$$

where ρ_2 is the valuation $(\rho_0, C = \bot, =_C = e)$. A lengthy computation now tells us that $[\![\cdots]\!]_{\rho_2} = \top$, and hence that there are no restrictions on f either.

Thus, we have a valuation ρ sending every variable to, say $\lambda x.x \cdot \top \cdot \top$, and this valuation respects the typing of the context.

We now apply Fridlender (2002, Theorem 3.15) and get an element of $[\![\mathsf{N}_0]\!]_\rho = \emptyset$, which gives a contradiction. Hence the original judgement is not derivable. $\square$

There is also a similar non-derivability result for sums. This result is a bit more complicated, the reason being that we have used the universe not only to prove the disjointness of sums, but also to define the sums themselves, so we will only show that the disjointness is not derivable without universes.

Proposition 8. The Logical Framework version of type theory without universes does not prove that there are disjoint sums of setoids.

Proof. Let us assume that there are sums, or in other words that there are terms making the judgements

$$\begin{aligned}
\mathscr{A}, \mathscr{B} : \mathsf{setoid} &\vdash \mathscr{A} + \mathscr{B} : \mathsf{setoid} \\
\mathscr{A}, \mathscr{B} : \mathsf{setoid} &\vdash \iota_1 : \mathscr{A} \rightarrow \mathscr{A} + \mathscr{B} \\
\mathscr{A}, \mathscr{B} : \mathsf{setoid} &\vdash \iota_2 : \mathscr{B} \rightarrow \mathscr{A} + \mathscr{B} \\
\mathscr{A}, \mathscr{B}, \mathscr{C} : \mathsf{setoid}, f : \mathscr{A} \rightarrow \mathscr{C}, g : \mathscr{B} \rightarrow \mathscr{C} &\vdash \binom{f}{g} : \mathscr{A} + \mathscr{B} \rightarrow \mathscr{C}
\end{aligned}$$

derivable (note that these four judgements expand to give eleven judgements in all).

Now we can compute the required pullback $\mathscr{P}$ (see diagram (1)), and disjointness of sums then corresponds to the derivability of the judgement $\mathscr{A}, \mathscr{B} : \mathsf{setoid} \vdash P \rightarrow \mathsf{N}_0$ true. In particular, considering the sum of the one-point setoid $\mathscr{I}$ with itself, and the resulting pullback $\mathscr{P}_1$, we obtain by substitution the derivability of the judgement $\vdash P_1 \rightarrow \mathsf{N}_0$ true, where

$$P_1 = (\Sigma x : \mathsf{N}_1 \times \mathsf{N}_1)(\iota_1(\pi_1(x)) =_{\mathscr{I} + \mathscr{I}} \iota_2(\pi_2(x))).$$

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We can next find a term $j = \mathsf{N}_2\text{-}\mathsf{elim}([x]J, \iota_1(*), \iota_2(*), x)$ such that $x{:}\mathsf{N}_2 \vdash j{:}J$ (writing J for the carrier of $\mathscr{I}+\mathscr{I}$), namely by noting that $\iota_1(*)$ and $\iota_2(*)$ are both elements of J and then applying the elimination rule for N_2. Now note that we can use identity elimination to derive the judgement

$$p{:}\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt},\mathsf{ff}) \vdash \mathsf{Id}_J(j(\mathsf{tt}), j(\mathsf{ff}))\ \mathsf{true},$$

that is

$$p{:}\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt},\mathsf{ff}) \vdash \mathsf{Id}_J(\iota_1(*), \iota_2(*))\ \mathsf{true},$$

and hence, by another identity elimination, also obtain a term q and a derivation of the judgement

$$p{:}\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt},\mathsf{ff}) \vdash q : \iota_1(*) =_{\mathscr{I}+\mathscr{I}} \iota_2(*).$$

But then $\langle\langle *, *\rangle, q\rangle : P_1$, and since we derived the judgement $\vdash P_1 \to \mathsf{N}_0$ true, we obtain a derivation of $p{:}\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt},\mathsf{ff}) \vdash \mathsf{N}_0$ true. Therefore, we have $\vdash \neg\mathsf{Id}_{\mathsf{N}_2}(\mathsf{tt},\mathsf{ff})$ true. But this was shown to be non-derivable in Fridlender (2002). □

6. Recovering the universe

So far we have shown that the availability of a universe is necessary and sufficient for proving that the category **Setoid** has disjoint sums. But a slightly stronger result is actually true, namely that if the category **Setoid** has disjoint sums, we can construct a small universe.

Proposition 9. If the sum $\mathscr{I}+\mathscr{I}$ of the standard one-point setoid with itself is disjoint, there is a small universe L' with domain N_2 such that $L'(\mathsf{ff})$ is empty and $L'(\mathsf{tt})$ is inhabited. In particular, if all sums are disjoint, we can construct such a small universe L'.

Proof. Let J be the carrier of $\mathscr{I}+\mathscr{I}$. We construct a non-dependent function $f{:}\mathsf{N}_2 \to J$ by recursion. Since $\vdash \iota_1(*){:}J$ and $\vdash \iota_2(*){:}J$, we have

$$x{:}\mathsf{N}_2 \vdash \mathsf{N}_2\text{-}\mathsf{elim}([x]J, \iota_1(*), \iota_2(*), x) : J,$$

and hence, abstracting, we obtain f.

Now define $L'(x)$ as $f(\mathsf{tt}) =_{\mathscr{I}+\mathscr{I}} f(x)$. We have $x{:}\mathsf{N}_2 \vdash L'(x)$ type. Furthermore,

$$L'(\mathsf{tt}) \equiv (f(\mathsf{tt}) =_{\mathscr{I}+\mathscr{I}} f(\mathsf{tt})),$$

so $\mathsf{refl}^{f(\mathsf{tt})}_{\mathscr{I}+\mathscr{I}} : L'(\mathsf{tt})$, that is, $L'(\mathsf{tt})$ is inhabited. Also,

$$L'(\mathsf{ff}) \equiv (f(\mathsf{tt}) =_{\mathscr{I}+\mathscr{I}} f(\mathsf{ff})) \equiv (\iota_1(*) =_{\mathscr{I}+\mathscr{I}} \iota_2(*)).$$

Thus, if $p{:}L'(\mathsf{ff})$, then

$$\langle\langle *, *\rangle, p\rangle : (\Sigma x{:}\mathsf{N}_1 \times \mathsf{N}_1)(\iota_1(\pi_1(x)) =_{\mathscr{I}+\mathscr{I}} \iota_2(\pi_2(x))),$$

which is the carrier for the pullback. But the pullback is empty (since we assumed the sum to be disjoint), so $L'(\mathsf{ff})$ is also empty. □

The universe we recover from disjoint sums is not quite the universe L we started with. The difference is that $L(\mathsf{tt}) \equiv \mathsf{N}_1$, but all we know about $L'(\mathsf{tt})$ is that it is inhabited.

However, this difference is very minor, and all the proofs and constructions presented here work just as well with the weaker assumption. In fact, this lets us combine earlier results, to prove

Proposition 10. The sum of two inhabited setoids is disjoint if and only if *all* sums of setoids are disjoint.

Proof. One direction is of course trivial. For the other direction, first note that the proof of Proposition 9 does not make use of any properties of N_1 beyond it being inhabited. Therefore, an easy adaptation of the proof shows that if a sum of two inhabited setoids is disjoint, we may construct a small universe L'. Now, inspection shows that the proofs of Propositions 3 and 4 make no essential use of the definitional equality $L(\mathsf{tt}) \equiv \mathsf{N}_1$, but only of the fact that $L(\mathsf{tt})$ is inhabited. So with only minor modifications to the construction and proof, we get the required result. □

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