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A new triad resonance between co-propagating surface and interfacial waves

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In a two-layer density-stratified fluid it is known, due to Ball (*J. Fluid Mech.*, vol. 19, 1964, p. 465), that two oppositely travelling surface waves may form a triad resonance with an interfacial wave. Ball claims 'there are no other interactions' between two surface waves and one interfacial wave. Contrary to this, here we present a new class of triad resonance that occurs between two co-propagating surface waves and one interfacial wave. While in Ball's resonance the interfacial wave has a wavelength of about half of two surface waves, in the new resonance presented here the interfacial wave has a much higher wavelength compared to those of surface waves. This, together with the unidirectionality of the participant triplet, makes the realization of the new resonance more likely in real ocean scenarios. We further show, via theoretical analysis and direct simulation, that, unique to this new class of resonance, the triad inevitably undergoes a cascade of (near-) resonance interaction that spreads the energy of initial waves to a number of lower and higher harmonics. The significance of the resonance studied here is, particularly, more emphasized in the littoral zones, where the spectrum refracts toward a unidirectional wave train.

Key words: nonlinear instability, stratified flows, waves/free-surface flows

1. Introduction

In the context of water waves, resonance transfers a significant amount of energy from a group of waves to another. Resonance is different from weak nonlinear interactions in that resonant wave amplitude may grow as large as the amplitude of initial waves. It is now established that resonance interactions between surface and/or internal waves play an important role in the evolution of the ocean spectrum (see e.g. Watson, West & Cohen 1976; Dysthe & Das 1981; Hammack & Henderson 1993; Craig, Guyenne & Sulem 2010).

Leading-order resonance may occur at second order (in the perturbation expansion of nonlinear governing equations) between a triplet of waves, and is called a triad resonance. In a homogeneous fluid it is known that, except in the limit of very shallow water or if surface tension intervenes, a triad resonance does not occur (Dyachenko & Zakharov 1994) and leading-order resonance occurs at third order between four waves, i.e. the quartet resonance (Phillips 1960; Longuet-Higgins 1962).

In a two-layer density-stratified fluid, however, triad resonance may occur between surface and interfacial waves. Specifically, two classes of triad resonance have been

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discovered and investigated so far. Class I triad resonance forms between two counter-propagating surface waves and one interfacial wave (Ball 1964; Joyce 1974). For typical ocean parameters, the two surface waves have almost the same wavelengths, hence forming a standing wave pattern on the surface, and the wavelength of the interfacial wave is about half that of surface waves. Class II triad resonance occurs between two counter-propagating interfacial waves and one surface wave (Wen 1995; Hill & Foda 1996; Jamali, Seymour & Lawrence 2003). In class II, the two interfacial waves must have almost the same wavelengths, forming a standing wave pattern on the interface, while the surface wave is much longer and has a frequency about double that of interfacial waves.

Here we present a new class of triad resonance (class III) between surface and interfacial waves that is significantly different in characteristics from classical class I and II. Specifically, in class III resonance the triplet of waves – two surface and one interfacial – travel in the same directions, and the interfacial wave has typically a much longer wavelength compared to the two surface waves. We will show that, unique to class III, the triad of waves inevitably leads to a cascade of multiple simultaneous (near-) resonance interactions that spreads the energy of initial waves into waves with lower and higher frequencies. To address the problem of amplitude growth of original triplet and new resonant harmonics, a multiple-scale analysis based on conservation of total energy is formulated here, and its performance is validated and discussed by comparing with direct simulation results from a high-order spectral scheme. Illustrative examples show the potential for a significant energy transfer and spreading in finite time.

Contrary to classes I and II, class III resonance only requires co-propagating waves; also, the participant interfacial wave has to be much longer than the surface waves. Both of these requisites are more likely to be realized in the real ocean and hence to influence the evolution of ocean spectra. The latter requisite also offers a new potential mechanism for the (still disputable) generation mechanism of long interfacial waves (Garrett & Munk 1979; Farmer & Armi 1999). Class III triad resonance may also accelerate the damping of high-frequency surface waves travelling over muddy seafloors (e.g. Sheremet & Stone 2003; Alam, Liu & Yue 2011) by transferring their energy to long interfacial waves that easily get damped by the bottom action. Importance of class III triad resonance is more emphasized in littoral zones and over continental shelves, where the spectrum refracts to a unidirectional wave train. We finally note that class III triad resonances may also form between two interfacial waves and one surface wave, but for very strong density ratios, atypical of real oceans, and therefore is not pursued here.

2. Governing equations

Consider free propagating waves on the surface and the interface of a two-layer density-stratified fluid. Depths and densities of the upper and lower layers are, respectively, given by h_u, ρ_u and h_ℓ, ρ_ℓ. We define a Cartesian coordinate system with the x-axis on the mean free surface and the z-axis positive upward. In each layer, we assume that the fluid is homogeneous, incompressible, immiscible and inviscid, and the motion is irrotational and described by the velocity potential $\phi_u(x, z, t)$ or $\phi_\ell(x, z, t)$. The equations governing the motion of a two-layer fluid are (ignoring surface tension):

$$\nabla^2 \phi_u = 0, \quad -h_u + \eta_\ell < z < \eta_u, \tag{2.1a}$$

$$\nabla^2 \phi_\ell = 0, \quad -h_u - h_\ell < z < -h_u + \eta_\ell, \tag{2.1b}$$

$$\eta_{u,t} + \eta_{u,x}\phi_{u,x} - \phi_{u,z} = 0, \quad z = \eta_u, \tag{2.1c}$$

$$\phi_{u,t} + \tfrac{1}{2}(\phi_{u,x}^2 + \phi_{u,z}^2) + g\eta_u = 0, \quad z = \eta_u, \tag{2.1d}$$

$$\eta_{\ell,t} + \eta_{\ell,x}\phi_{u,x} - \phi_{u,z} = 0, \quad z = -h_u + \eta_\ell, \tag{2.1e}$$

$$\eta_{\ell,t} + \eta_{\ell,x}\phi_{\ell,x} - \phi_{\ell,z} = 0, \quad z = -h_u + \eta_\ell, \tag{2.1f}$$

$$\left.\begin{aligned} \rho_u[\phi_{u,t} + \tfrac{1}{2}(\phi_{u,x}^2 + \phi_{u,z}^2) + g\eta_\ell] - \rho_\ell[\phi_{\ell,t} + \tfrac{1}{2}(\phi_{\ell,x}^2 + \phi_{\ell,z}^2) + g\eta_\ell] = 0,\\ z = -h_u + \eta_\ell, \end{aligned}\right\} \tag{2.1g}$$

$$\phi_{\ell,z} = 0, \quad z = -h_u - h_\ell, \tag{2.1h}$$

where $\eta_u = \eta_u(x, t)$, $\eta_\ell = \eta_\ell(x, t)$, are respectively the free surface and interfacial wave elevations, and g is the gravitational acceleration. The linearized system of equations (2.1*a*)–(2.1*h*) admits propagating wave solution with its frequency ω and wavenumber k satisfying the dispersion relation (Lamb 1932, art. 223, p. 387):

$$\begin{aligned} \mathscr{D}(k, \omega) \equiv \omega^4(\mathscr{R} + \coth kh_u \coth kh_\ell) - \omega^2 gk(\coth kh_u + \coth kh_\ell)\\ + g^2k^2(1 - \mathscr{R}) = 0. \end{aligned} \tag{2.2}$$

where $\mathscr{R} = \rho_u/\rho_\ell$ is the density ratio. For a given ω, (2.2) possesses two pairs of real roots for the wavenumber k (Ball 1964) where the wave associated with the higher/lower wavenumber is called an interfacial/surface mode wave. For a weak stratification, a surface/interfacial mode wave has a higher amplitude on the surface/interface. Throughout this paper we use subscripts $_{s,i}$ (capital or lowercase) to refer to surface and interfacial mode wavenumbers and frequencies, respectively.

3. Resonance condition

Two free propagating waves with wavenumbers and frequencies (k_p, ω_p) and (k_q, ω_q) are in a triad resonance with a third wave $(k_r = k_p \pm k_q, \omega_r = \omega_p \pm \omega_q)$ if $\mathscr{D}(k_r, \omega_r) = 0$, i.e. if (k_r, ω_r) is also a free propagating wave. Under the resonance energy may transfer (significantly) from original free waves to the resonant wave, and the amplitude of the resonant wave may become of the same order as original waves. In a homogeneous fluid it is known that triad resonance cannot occur (Dyachenko *et al.* 1994), except in the limit of shallow water, for which waves are non-dispersive, or if the effect of surface tension is taken into account (McGoldrick 1965). In a two-layer density stratified fluid, Ball (1964) showed that two oppositely travelling surface waves may resonate an interfacial wave (class I). Later Wen (1995) showed that two oppositely travelling interfacial waves form a triad resonance with a surface wave if three waves satisfy the resonant condition (class II).

Figure 1 graphically demonstrates these two classes of triad resonance. Solid curves and dashed curves are, respectively, surface and interfacial solutions of the dispersion relation (2.2). For precision we further indexed each curve with letters S, I to indicate surface and interfacial waves, and subscripts $_{r,l}$ to indicate, respectively, right-going and left-going waves. Any point on the $S_{l,r}$ or $I_{l,r}$ branches represents a free propagating surface or interfacial wave, respectively. Referring to figure 1(*a*), let us consider a free surface wave $OA = (k_{S1}, \omega_{S1})$ and draw a curve parallel to I_l from the point A: we named it I_l'. The curve I_l' will intersect S_l at the point B, and clearly $OA + AB = OB$. Note that $OB = (k_{S2}, \omega_{S2})$ is a left-going surface wave and $AB = (k_I, \omega_I)$ is a left-going interfacial wave. Therefore the relation $OA + AB = OB$ leads to $k_{S1} + k_I = k_{S2}$ and $\omega_{S1} + \omega_I = \omega_{S2}$, which are the resonance conditions between triplet of waves of (k_{S1}, k_{S2}, k_I).

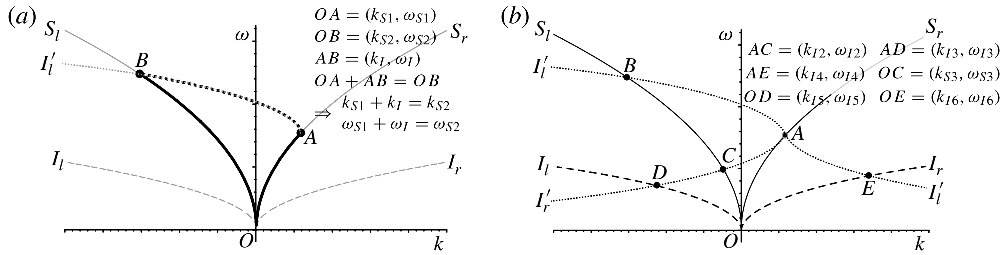

FIGURE 1. Geometric construction for class I and II triad resonance in a two-layer density-stratified fluid. Curves (S_r, S_ℓ,——) are surface wave solutions and (I_r, I_ℓ,- - -) are interfacial wave solutions of the dispersion relation (2.2), and (I_r', I_ℓ', · · ·) are plots of interfacial wave solutions originating at A. Resonant waves are where $I_{r,l}'$ (dotted) intersects with $S_{r,l}$, $I_{r,l}$ (solid and dashed) and are denoted by filled black circles. (*a*) A sample case of class I triad resonance where two surface waves (OA, OB) are in resonance with an interfacial wave AB. (*b*) All class I ($OA+AB=OB$, $OA+AC=OC$) and class II ($OA+AD=OD$, $OA+AE=OE$) triad resonances.

All possible triad resonance scenarios (a total of four) of class I and II are shown in figure 1(*b*) and an argument similar to figure 1(*a*) applies to each case. Triangles OAD and OAE are indicators of triad resonances between, respectively, (k_{S1}, k_{I3}, k_{I5}) and (k_{S1}, k_{I4}, k_{I6}), each of which is composed of two counter-propagating interfacial waves and one surface wave (Wen 1995); triangles OAB and OAC are indicators of triad resonances between, respectively, (k_{S1}, k_{S2}, k_I) and (k_{S1}, k_{S3}, k_{I2}), each of which is composed of two counter-propagating surface waves and one interfacial wave (Ball 1964).

Ball (1964) claims that the triad resonances between two surface waves and one interfacial wave 'always involve two external (i.e. surface) waves moving in the opposite direction to one another'. Contrary to this claim, here we present a new class of resonance (class III) that occurs between two short surface waves and one long interfacial wave where all three waves are co-propagating. The resonance condition for class III is presented schematically in figure 2, where the convention for curves is the same as in figure 1. Referring to figure 2, we see that the relations $OA+AF=OF$ and $OA+AG=OG$ are, respectively, triad resonance conditions between waves $(k_{s1}, k_{s2+}, k_{i1})$ and $(k_{s1}, k_{s2-}, k_{i2})$, i.e.

$$\begin{cases} k_{s1} + k_{i1} = k_{s2+}, \\ \omega_{s1} + \omega_{i1} = \omega_{s2+}, \end{cases} \qquad \begin{cases} k_{s1} - k_{i2} = k_{s2-}, \\ \omega_{s1} - \omega_{i2} = \omega_{s2-}. \end{cases} \tag{3.1}$$

Relations (3.1) are class III triad resonance conditions between co-propagating surface and interfacial waves. It is to be noted that for intersection points F, G to exist it is necessary that $\theta_{i_0} > \theta_{s_k}$, where $\theta_{i_0} = \mathrm{d}\omega_i/\mathrm{d}k|_{k=0} = C_{g,i_0}$ and $\theta_{s_k} = \mathrm{d}\omega_s/\mathrm{d}k|_{k=k_{s_1}} = C_{g,s_k}$. Therefore the necessary condition for the class III triad resonance is

$$C_{g,i_0} > C_{g,s_k}. \tag{3.2}$$

The necessary condition (3.2) indicates that class III triad resonance obtains only beyond a certain threshold. This perhaps explains why this class was overlooked by Ball (1964).

To answer the question of whether or not this necessary condition is satisfied, in a general case, the intersection points F, G have to be sought numerically. Figure 3(*a*) plots the resonance condition for class I and III for an ocean of $\mathscr{R} = 0.95$

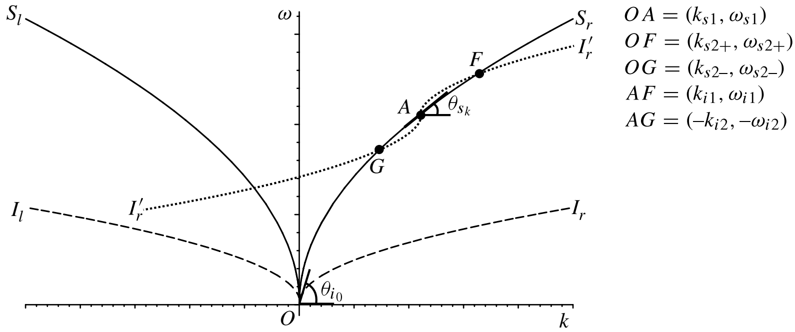

FIGURE 2. Schematic representation of the new class III triad resonance between two co-propagating surface waves and one interfacial wave. For legend see caption of figure 1.

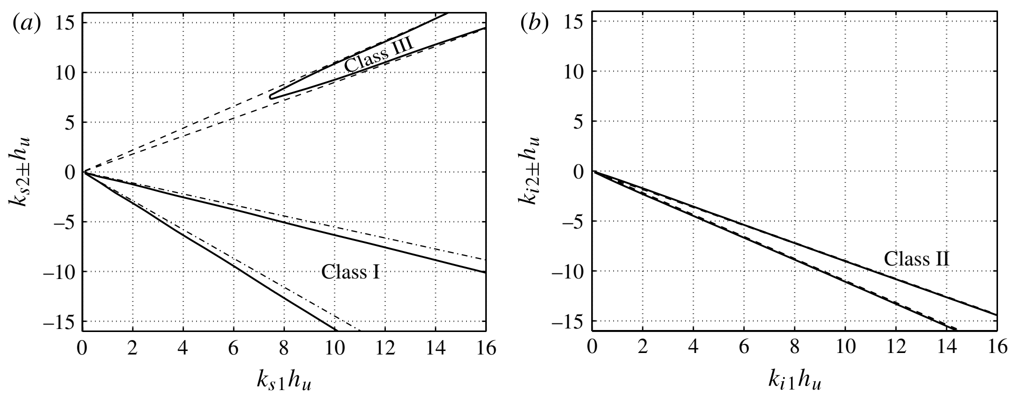

FIGURE 3. Resonance condition for class I, II and III triad resonance in a two-layer density-stratified fluid with $R = 0.95$, $\tilde{h} = 0.33$. (*a*) Any chosen point on a solid curve represents a triad resonance between two surface waves k_{s1}, k_{s2} and one interfacial wave $k_i = |k_{s2} - k_{s1}| \times \mathrm{sign}(k_{s2} + k_{s1})$. Curves are resonance conditions found from finite-depth dispersion relations ((3.1), ——), asymptotic conditions with deep-layer assumptions for class III ((4.4), - - -) and class I ((4.5), – · –). (*b*) Any point on a solid curve represents a triad resonance between two interfacial waves k_{i1}, k_{i2} and one surface wave $k_s = k_{i1} + k_{i2}$. Curves are resonance conditions found from finite-depth dispersion relations (——), asymptotic conditions with deep-layer assumptions for the class II ((4.5), – · –).

(typically used in experimental investigations of two-layer models; see e.g. Joyce 1974) and $\tilde{h} = 0.33$, where $\tilde{h} = h_u/(h_h + h_\ell)$. In figure 3(*a*) any point on the solid curve represents a triad resonance between its corresponding values of k_{s1}, k_{s2} and $k_i = |k_{s2} - k_{s1}| \times \mathrm{sign}(k_{s2} + k_{s1})$, where the last term in the expression corrects the sign and obtains by investigating figure 2 (we assume all frequencies are positive, therefore waves with negative wavenumbers travel in the opposite direction to waves with positive wavenumbers). Note that class III appears beyond a certain wavenumber ($k_{s1}h_u \approx 7.5$), which explains how it may have been overlooked in the past. Figure 2(*b*) plots the resonance condition between two interfacial wave components, k_{i1}, k_{i2}, and a surface wave of $k_s = k_{i1} + k_{i2}$. In contrast to figure 3(*a*) (class I), in this case counter-propagating waves k_{i1}, k_{i2} have close wavelengths and the resonant surface wave k_s is much longer than $k_{i1,2}$. We finally note that class III triad resonance can also occur between two co-propagating interfacial waves and one surface wave, but for very strong density ratios (in the limit of deep layers it can be shown that $R > 1/3$), which is non-physical for typical oceans.

4. Limiting cases

If we define dimensionless frequency $\sigma = \omega\sqrt{H/g}$, where $H = h_u + h_\ell$, and upper/lower/total shallowness by $\mu_u = kh_u$, $\mu_\ell = kh_\ell$ and $\mu_H = kH$, then the dispersion relation (2.2) in dimensionless variables can be rewritten in the form

$$\sigma^4(\mathscr{R} + \coth\mu_u \coth\mu_\ell) - \sigma^2\mu_H(\coth\mu_u + \coth\mu_\ell) + \mu_H^2(1 - \mathscr{R}) = 0. \tag{4.1}$$

Limiting cases of a two-layer density-stratified fluid are when the stratification is weak i.e. $1 - R = \nu^2$ and $\nu \ll 1$, and/or when upper/lower layers are either deep or shallow, i.e. $\mu_{u,\ell,H} \lessgtr 1$. If $\nu \ll 1$, and for general μ_u, μ_ℓ, μ_H, surface (σ_s) and interfacial (σ_i) wave solutions of (4.1), correct to $O(\nu)$, are

$$\sigma_s^2 = \mu_H \tanh\mu_H, \quad \sigma_i^2 = \frac{\mu_H \nu^2}{\coth\mu_u + \coth\mu_\ell}, \tag{4.2}$$

and the necessary condition for the class III resonance, (3.2), is obtained in a closed form,

$$\frac{1}{2}\sqrt{\frac{\mu_H \tanh\mu_H}{\mu_u\mu_\ell}}\left(1 + \frac{2\mu_H}{\sinh 2\mu_H}\right) < \nu. \tag{4.3}$$

From (4.3), it is seen that for a given total water depth H, resonance is more likely when $h_u = h_\ell$ (since the left-hand side of the equation is at a minimum). For very long waves, i.e. $\mu_H \ll 1$, the necessary condition is simplified to $\nu > 2$, which never satisfies. For short waves, i.e. $\mu_H \gg 1$, and if $\mu_u = \mu_\ell$, then the necessary condition is $\mu_H > 1/\nu^2$.

Referring to figure 2, class III triad resonance exists among k_{s1}, k_{s2} and k_{i1} if (3.1) is satisfied. If we consider a weak stratification and deep water assumption, i.e. $1/\nu, \mu_u, \mu_\ell \gg 1$ for all three waves participating in the resonance, then after some algebra closed-form expressions for the resonance condition are obtained (see dashed lines in figure 3*a*):

$$\frac{k_{s2}}{k_{s1}} = 1 \pm 2\nu^2. \tag{4.4}$$

Note that if the necessary condition (4.3) is not satisfied, there will be no resonance. A similar expression is obtained for class I and II triad resonances, respectively,

$$\frac{k_{s2}}{k_{s1}} = -1 \pm 2\nu, \quad \frac{k_{i2}}{k_{i1}} = -1 \pm 2\nu^2. \tag{4.5}$$

These asymptotic expressions are also shown in figure 3(*a*,*b*) with dot-dashed lines.

5. Amplitude growth rate

One interesting feature of class III resonance that significantly distinguishes it from class I and II is that all three interacting waves in a resonance triplet move in the same direction. This property, as we will discuss shortly, paves the way for the energy spread from initially few (a minimum of two) waves to several higher and lower

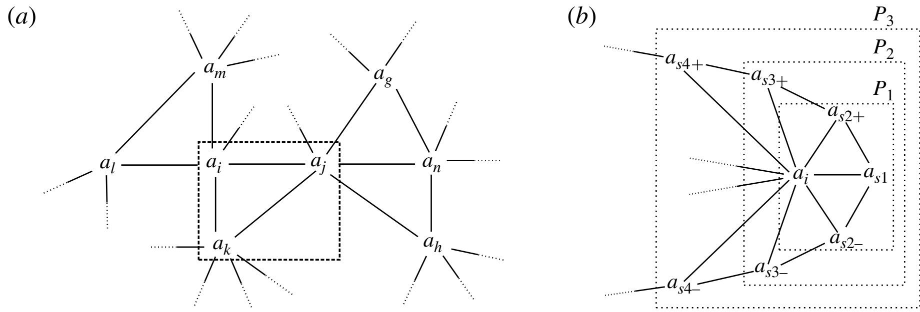

FIGURE 4. (*a*) Schematic representation of the interaction path of a group of waves that form a matrix of multiple triad resonances. (*b*) Schematic representation of a cascade of triad resonances in a typical class III resonance (see § 5). $P_{1,2,3}$ (i.e. triads enclosed in the dotted boxes) respectively indicate inclusion of primary, secondary and tertiary triad resonances.

harmonics, a phenomenon that does not occur in class I and II. Before discussing the concept of multiple resonance in class III, we note that in general, if the number of free surface/interfacial waves satisfies $n > 3$, then we may find that $m > 1$ triad resonances are occurring simultaneously (of course $m \leqslant \binom{n}{3} = n!/[3!(n-3)!]$). This can be schematically represented if we assign a triangle to each triplet of waves forming a triad resonance, as is shown in figure 4(*a*). For example, since vertices a_i, a_j and a_k form a triangle in figure 4(*a*), then we must have $k_i \pm k_j = k_k$ and $\omega_i \pm \omega_j = \omega_k$, with each wave satisfying the dispersion relation (2.2). Note that, for example, the wave a_i also belongs to another triad triangle (a_i, a_l, a_m), and wave a_j belongs also to the triplets of (a_j, a_g, a_n), (a_j, a_n, a_h), and more.

For class III triad resonance, referring to figure 3(*a*), it is seen that the resonance condition curve is nearly symmetric about the bisector of the upper half-plane (see (4.4)). Therefore in (3.1) we must have $k_{i1} \approx k_{i2}$ $(= k_{s2} - k_{s1} \approx 2\nu^2 k_{s1}$ in the limit of deep layers). This implies that for a given surface wave (k_{s1}) one interfacial wave $(k_i \approx 2\nu^2 k_{s1})$ can simultaneously resonate both k_{s2-} and k_{s2+} (see also figure 2). This is shown schematically in figure 4(*b*) (box P_1). Physically speaking, the simultaneous two-triad resonances transfer energy from k_{s1} to two nearby waves of $k_{s2\pm}$ (at leading-order nonlinearity). To make the matter more complicated, since waves $k_{s2\pm}$ are close to k_{s1}, the interfacial wave k_i can form a new set of (strictly speaking almost) triad resonance between $k_{s2\pm}, k_i$ and a new wave $k_{s3\pm} = k_{s2\pm} \pm k_i$ (see figure 4*b*, box P_2). The chain may easily continue to a number of steps until the detuning becomes sufficiently large that further (near-) resonance interaction is no longer of leading-order importance.

To obtain analytical solutions for the evolution of amplitude of initial as well as (all) resonant waves in a class III triad resonance, here we use a multiple-scale argument based on conservation of total energy. Consider a free progressive surface or an interfacial wave in the form

$$\eta = p\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} = a\cos(kx - \omega t + \psi), \tag{5.1}$$

where the real parameter $a = 2|p|$ is the physical amplitude of the wave and the phase satisfies $\tan\psi = \mathrm{Im}(p)/\mathrm{Re}(p)$. The total energy of this wave is independent of the constant phase ψ, and is given by

$$E = qa^2, \quad q = \tfrac{1}{2}\rho_\ell g[\mathscr{R} + \lambda^2(1-\mathscr{R})], \tag{5.2}$$

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where $\lambda = \cosh kh_u - gk/\omega^2 \sinh kh_u$. For an n-wave system the total energy is therefore $E_{total} = \sum_{i=1,n} q_i a_i^2$, where subscript i indexes each individual wave. Amplitudes of resonant waves, however, vary slowly in time: in other words they vary as a function of a slow time τ (usually defined in terms of surface steepness $\epsilon = ka \ll 1$, i.e. $\tau = \epsilon t$). From conservation of total energy we must have $\mathrm{d}E/\mathrm{d}\tau = 0$, therefore

$$\sum_{i=1,n} q_i a_i \frac{\mathrm{d}a_i}{\mathrm{d}t} = 0. \tag{5.3}$$

If $n = 3$, (5.3) admits a solution of the form (Alam, Liu & Yue 2010)

$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = \alpha_1 a_2 a_3, \quad \frac{\mathrm{d}a_2}{\mathrm{d}t} = \alpha_2 a_1 a_3, \quad \frac{\mathrm{d}a_3}{\mathrm{d}t} = \alpha_3 a_1 a_2, \tag{5.4}$$

with $\sum_{i=1,2,3} q_i \alpha_i = 0$. The coefficients α_i are readily found from regular perturbation (see the Appendix) and by matching the growth rate at $\tau = 0$. The set of ordinary differential equations (5.4) has a closed-form solution in terms of Jacobian elliptic functions, and shows interesting behaviour for special values (e.g. Alam *et al.* 2010, for the case of quartet Bragg resonance in homogeneous waters).

For $n > 3$, simultaneous interactions may occur between different groups of triads. For instance, for the case of figure 4(*a*), from regular perturbation expansion (see the Appendix) it turns out that, to leading order,

$$\frac{\mathrm{d}a_i}{\mathrm{d}\tau} = \alpha_{ijk} a_j a_k + \alpha_{iml} a_m a_l + \cdots, \tag{5.5}$$

where $\alpha_{ijk}, \alpha_{iml}, \ldots$ are obtained from the Appendix. Generalizing the idea to an n-wave system, we arrive at the solution to the energy conservation equation (5.3) in the form

$$\frac{\mathrm{d}a_p}{\mathrm{d}\tau} = \sum \alpha_{pqr} a_q a_r, \tag{5.6}$$

with the condition $\alpha_{pqr} q_p + \alpha_{rpq} q_r + \alpha_{qrp} q_q = 0$, where a_p, a_q and a_r are vertices of each individual triangle, i.e. forming a triad resonance (the order of indices of α is not important). As before, the coefficients α_{pqr} are found from regular perturbation (see the Appendix). The general solution to (5.6) is to be found via numerical quadrature. Note that triangles in figure 4 are not limited to remain in a two-dimensional plane.

6. Direct simulation and discussion of results

When more than just a few waves interact simultaneously, it is algebraically tedious – if not impossible – to track their interactions. This fact becomes more emphasized if various nonlinear interactions, such as resonances, near-resonances (detuned interactions), higher/lower harmonic generations, and effects of higher-order couplings/nonlinearities, are to be taken into account. To address the problem of many (typically $N = O(10^4)$) waves interacting and to consider an arbitrary order of nonlinearity (typically $M = O(10)$ in terms of perturbation expansions), we have recently extended a direct simulation scheme based on a high-order spectral method (HOS), originally derived to study nonlinear wave–wave (Dommermuth & Yue 1987) and wave–bottom (Liu & Yue 1998) interactions, to a two-layer density-stratified fluid with finite-depth upper and lower layers (Alam, Liu & Yue 2009, where extensive convergence tests and validations are also provided). In this section we use HOS to study long-time behaviour of class III triad resonance of surface and interfacial waves

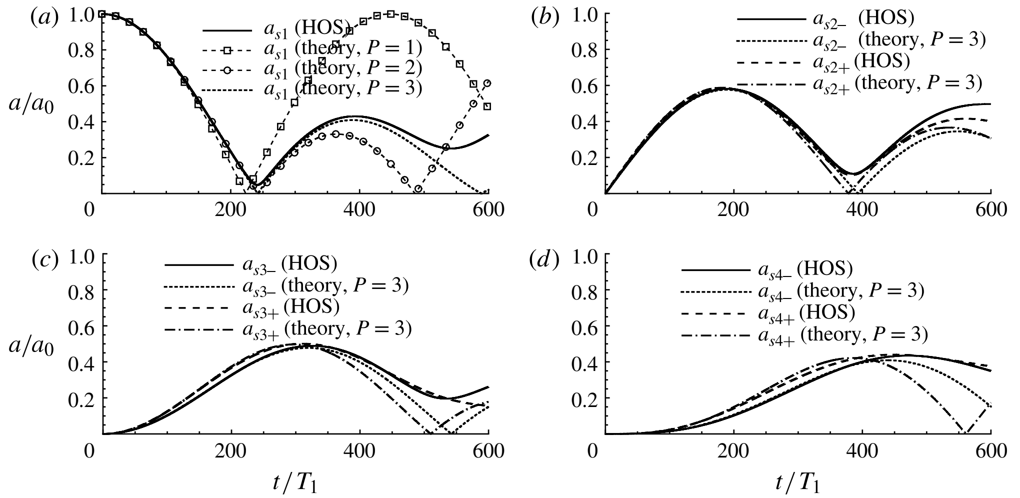

FIGURE 5. Evolution of amplitude of interacting waves as a function of time as predicted from numerical simulation and theory. (a) Evolution of a_{s1} from numerical computation (——) is compared with the theory of (5.6) for $P = 1$ (– □ –), $P = 2$ (– ○ –), and $P = 1$ (· · ·). (b–d) Plots of evolution of $a_{s2\pm}$, $a_{s3\pm}$ and $a_{s4\pm}$, respectively, as predicted by HOS (——, - - -) and theory of (5.6) (· · ·, – · –).

and also as a validation tool for the theoretical analysis of § 5. For the following direct simulations we use $N = 2048$, $M = 3$ and the time step $\omega_{s1}\delta t = 0.1$. All presented results converge for the chosen parameters.

We first consider a short surface wave $\mu_{s1} = 8$ ($\epsilon_{s1} = 0.008$) and a relatively long interfacial wave $\mu_i = 0.3$ ($\epsilon_i = 0.001$ on the interface). In a water of $\tilde{h} = 0.33$ and $\mathscr{R} = 0.95$, it can be shown that these two waves form two simultaneous triad resonances with free surface waves of $\mu_{s2-} = 7.7$ and $\mu_{s2+} = 8.3$. These resonant waves are themselves in a (near-) resonance interaction with k_i and, respectively, two new waves of $\mu_{s3-} = 7.4$ and $\mu_{s3+} = 8.6$, and the cascade of interaction continues to $\mu_{s4-} = 7.1$ and $\mu_{s4+} = 8.9$ and further on. Figure 5 compares results of evolution of amplitude of original surface wave (a_{s1}) and resonant surface waves ($a_{s2\pm,3\pm,4\pm}$) obtained from direct simulation and theoretical analysis (5.6). In figure 5(a), variation of a_{s1} is plotted as a function of time as predicted by HOS, and compared with quadrature of (5.6) for $P = 1, 2, 3$. While a very good agreement is obtained for initial-time evolution, the effect of secondary ($P = 2$) and tertiary ($P = 3$) interactions becomes of significant importance for later times. The analytical solution with only primary resonance, $P = 1$, predicts a perfect modulation of energy between a_{s1} and $a_{s2\pm}$, whereas including secondary and tertiary resonances clearly shows that energy flows to new waves ($a_{3\pm,4\pm}$) (see figure 5b–d). The variation in amplitude of resonant waves $a_{2\pm,3\pm,4\pm}$ is plotted in figure 5(b) to figure 5(d), compared with the analytical results of (5.6) with $P = 3$. Note that the amplitude of $a_{s2\pm}$ starts to increase at $t/T_1 = 0$ while the growth in amplitude of $a_{s3\pm}$ lags until $a_{s2\pm}$ develops, and similarly $a_{s4\pm}$ lags $a_{s3\pm}$. Direct simulation and analytical results are in very good agreement initially and, as expected, slowly diverge for longer times.

The resonance presented here also occurs if incident surface waves are slightly oblique to each other. The resonance condition now has to be written in the vector form $\boldsymbol{k}_{s1} + \boldsymbol{k}_i = \boldsymbol{k}_{s2}$ with $\omega_{s1} \pm \omega_i = \omega_{s2}$. If the surface wave $\boldsymbol{k}_2$ has an angle $0 < \theta_{12} \ll 1$

with respect to surface wave $\boldsymbol{k}_1$, then we always have $|\theta_{r1}| > |\theta_{12}|$, where θ_{r1} is the angle of resonant interfacial wave $\boldsymbol{k}_r$ with respect to $\boldsymbol{k}_1$. If θ_{12} further increases, the case of oppositely travelling surface waves of Ball (1964) is retrieved. A similar analysis (to that of Ball 1964) can be performed to show that, for a weak stratification, as θ_{12} increases from zero the strength of resonance decreases until $\theta_{12} = \pi/2$, where the coefficients of amplification are zero.

7. Conclusion

We have presented a new triad resonance (class III) between surface and interfacial waves in a two-layer density-stratified fluid. This resonance forms between a triplet of co-propagating two surface waves and one interfacial wave whose wavelength is much longer than those of surface waves. By investigating the resonance condition and the dispersion relation, we have shown that class III is inevitably followed by a cascade of triad (near-) resonances transferring the energy of originally few waves to a number of higher/lower harmonics. Using energy conservation and ideas from multiple scales, we have derived analytical sets of coupled ordinary differential equations governing the evolution of amplitudes of all interacting waves. These results were validated against direct simulations of a high-order spectral scheme and it is shown that effects of secondary and tertiary resonances, over long times, may significantly alter the evolution of the original waves.

Class III triad resonance between surface and interfacial waves can transfer energy from short surface waves to much longer interfacial waves, and hence may offer another potential mechanism for the generation of interfacial waves (e.g. Garrett & Munk 1979). If the seafloor is muddy, this phenomenon may further help the drainage of energy from the high-frequency part of the spectrum (e.g. Sheremet & Stone 2003). In addition, the formation of a cascade of triad (near-) resonances may affect the stability of narrow-band spectra in stratified waters.

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Appendix. Regular perturbation solution for the triad resonance

Consider three free waves, $k_1, k_2, k_r = k_1 + k_2$, forming a triad resonance triplet. If initial surface amplitudes of these waves are, respectively, $a_{1,2} \neq 0, a_r = 0$, then a_r (initially) grows in time according to

$$a_r = \gamma a_1 a_2 t + \text{non-growing terms}, \tag{A 1}$$

and the growth rate γ is

$$\gamma = -\omega_r/(2g^2 k_r) \times [C_4\omega_r^4 + C_3\omega_r^3 + C_2\omega_r^2 + C_1\omega_r + C_0]/[C_{-2}\omega_r^2 + C_{-0}], \tag{A 2}$$

where

$$C_0 = -2k_r^2 M_2 g^2 (1-R) su_r sl_r, \quad C_1 = -g^2 k_r (M_3 - cu_r M_1)(1-R) sl_r, \tag{A 3}$$

$$\left.\begin{aligned} C_2 &= gM_2 k_r sl_r cu_r (1+R) + gk_r(2M_2 su_r cl_r - M_5 sl_r), \\ C_3 &= -g(M_4 - M_3) cl_r - gM_1(cu_r cl_r + R su_r sl_r), \end{aligned}\right\} \tag{A 4}$$

$$\left.\begin{aligned} C_4 = -M_2(cl_r cu_r + R su_r sl_r), \quad C_{-0} &= -2gk_r(1-R) su_r sl_r,\\ C_{-2} &= cl_r su_r + sl_r cu_r, \end{aligned}\right\} \tag{A 5}$$

$$M_1 = 1/2(k_1+k_2)(k_2 a_1 A_2 + A_1 a_2 k_1), \tag{A 6}$$

$$M_2 = 1/2B_1k_1B_2k_2 - 1/2A_1k_1A_2k_2 - 1/2a_1B_2k_2\omega_2 - 1/2a_2B_1k_1\omega_1, \tag{A 7}$$

$$M_3 = -1/2(k_1+k_2)(k_1b_2B_1su_1 - k_1b_2A_1cu_1 + b_1B_2k_2su_2 - k_2A_2cu_2b_1), \tag{A 8}$$

$$M_4 = 1/2(k_1+k_2)(b_2D_1cl_1k_1 + D_2b_1k_2cl_2), \tag{A 9}$$

$$\begin{aligned} M_5 = {} & -1/2b_1D_2sl_2k_2\omega_2 + 1/2\mathscr{R}A_1su_1k_1B_2cu_2k_2 - 1/2\mathscr{R}k_1B_1su_1k_2A_2cu_2\\ & + 1/2\mathscr{R}k_1B_1su_1k_2B_2su_2 - 1/2b_2D_1sl_1k_1\omega_1 - 1/2\mathscr{R}b_1\omega_2A_2su_2k_2\\ & + 1/2\mathscr{R}k_1A_1cu_1k_2A_2cu_2 + 1/2D_1sl_1k_1D_2sl_2k_2 + 1/2\mathscr{R}b_1\omega_2B_2cu_2k_2\\ & - 1/2D_1cl_1k_1D_2cl_2k_2 + 1/2\mathscr{R}b_2\omega_1B_1cu_1k_1 - 1/2\mathscr{R}k_1A_1cu_1k_2B_2su_2\\ & - 1/2\mathscr{R}A_1su_1k_1A_2su_2k_2 + 1/2\mathscr{R}B_1cu_1k_1A_2su_2k_2 - 1/2\mathscr{R}B_1cu_1k_1B_2cu_2k_2\\ & - 1/2\mathscr{R}b_2\omega_1A_1su_1k_1, \end{aligned} \tag{A 10}$$

$$\left.\begin{aligned} cu_i = \cosh(k_ih_u), \quad su_i = \sinh(k_ih_u), \quad cl_i &= \cosh(k_ih_\ell),\\ sl_i = \sinh(k_ih_\ell), \quad i &= 1, 2, r \end{aligned}\right\} \tag{A 11}$$

$$\left.\begin{aligned} b_i = \left(\cosh k_ih_u - \frac{gk_i}{\omega_i^2}\sinh k_ih_u\right), \quad A_i = -\frac{g}{\omega_i}, \quad B_i &= -\frac{\omega_i}{k_i},\\ D_i = -\frac{b_i\omega_i}{k_i \sinh k_ih_\ell}, \quad i &= 1, 2. \end{aligned}\right\} \tag{A 12}$$

Note that if waves k_1, k_2 have initial angular phases ψ_1, ψ_2, the resonant wave has an angular phase of $\psi_r = \psi_1 + \psi_2$ and all three angular phases stay constant over time (see e.g. McGoldrick 1965).

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