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Adding Edges to Increase the Chromatic Number of a Graph

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If $n \geqslant k+1$ and G is a connected n-vertex graph, then one can add $\binom{k}{2}$ edges to G so that the resulting graph contains the complete graph K_{k+1}. This yields that for any connected graph G with at least $k+1$ vertices, one can add $\binom{k}{2}$ edges to G so that the resulting graph has chromatic number $> k$. A long time ago, Bollobás suggested that for every $k \geqslant 3$ there exists a k-chromatic graph G_k such that after adding to it any $\binom{k}{2} - 1$ edges, the chromatic number of the resulting graph is still k. In this note we prove this conjecture.

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1. Introduction

This note is another contribution to the old theme of sparse graphs with high chromatic number [4].

For a positive integer k and a connected graph G with at least $k+1$ vertices, let $f(G,k)$ denote the minimum number m of edges such that after adding m edges (and any number of vertices) to G we obtain a graph with chromatic number at least $k+1$. Since G is connected, we can add $\binom{k}{2}$ edges to some subtree of G with k edges so that the resulting graph contains the complete graph K_{k+1}. Thus $f(G,k) \leqslant \binom{k}{2}$ for every connected graph G with at least $k+1$ vertices. One may expect that if in addition G is k-chromatic,

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then $f(G,k) < \binom{k}{2}$. However, in the 1970s, Bollobás [1] suggested that for every $k \geqslant 3$ there exists a k-chromatic connected graph G_k with at least $k+1$ vertices such that $f(G_k,k) = \binom{k}{2}$. The goal of this note is to prove this conjecture.

In [3], the authors used uniform hypergraphs of large girth to modify the classic construction of Tutte (see [2]) to obtain a sequence of sparse graphs $G(k,g)$ with chromatic number k and girth g. We shall use these graphs to prove our result. The properties of these graphs we shall need are collected in the following lemma.

Lemma 1.1. *Let $k \geqslant 4$ and $g \geqslant 3$. Then $G = G(k,g)$ has chromatic number k and girth g; furthermore, its vertex set has partition $V_3 \cup V_4 \cup \cdots \cup V_k$ such that*

(a) *the sets $V_4, \ldots, V_k$ are independent, and $G[V_3]$ is the disjoint union of cycles;*
(b) *for all $3 \leqslant i < j \leqslant k$, each vertex $v \in V_i$ has exactly one neighbour in V_j; in particular, if $v \in V_3$ then $d(v) = k-1$.*

Our main result, stated next, will be proved in Section 2.

Theorem 1.2. *Let $k \geqslant 4, g \geqslant k^4$, $G = (V,E) = G(k,g)$ and let $H = (V',E')$ be a graph with $\chi(H) \geqslant k+1$. Then $|E(H) \setminus E(G(k,g))| \geqslant \binom{k}{2}$.*

2. Proof of Theorem 1.2

As every graph of chromatic number $k+1$ contains a subgraph of minimum degree k, it suffices to show that if H has minimum degree k then

$$|E(H) \setminus E(G(k,g))| \geqslant \binom{k}{2}. \tag{2.1}$$

Note that $|V'| \geqslant \delta(H) + 1 \geqslant k+1$. Assuming, as we may, that $V' \subset V$, set $G' = G[V']$. As remarked at the beginning, (2.1) holds if G' is acyclic. Assume that G' contains a cycle. Since every cycle of G (and so of G') contains at least $g \geqslant k^4$ vertices, $|V'| \geqslant k^4$.

For $i = 3,\ldots,k$, set $V'_i = V_i \cap V'$ and let W_i denote the set of vertices in V'_i incident with the edges in $E(H') - E(G)$. In particular $V'_3 = W_3$, since in G the vertices in V_3 have degree $k-1$. We will prove that

$$|W| \geqslant k^2 \text{ for some } i \text{ with } 3 \leqslant i \leqslant k. \tag{2.2}$$

This implies $|E'| \geqslant k^2/2$, and so the theorem.

Suppose (2.2) is false, *i.e.*, $|W_i| < k^2$ for every i; thus our task is to arrive at a contradiction. Since

$$|V'| = \left|\bigcup_{i=3}^{k} V'_i\right| \geqslant k^4,$$

there is an i with $|V'_i| > k^3$ and so $|V'_i \setminus W_i| > k^3 - k^2$. Choose the minimum such i. Then $i > 3$ because otherwise we would have $k^2 > |W_3| = |V'_3| > k^3$.

Since $|V'_i \setminus W_i| > k^3 - k^2$, and each vertex in $V'_i \setminus W_i$ has at least $k-(k-i) = i$ neighbours in $V'_3 \cup \cdots \cup V'_{i-1}$, *and* no vertex in $V'_3 \cup \cdots \cup V'_{i-1}$ has more than one neighbour

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in $V_i \supset V_i' \setminus W_i$, the set $V_i' \setminus W_i$ has at least $i|V_i' \setminus W_i|$ neighbours in $V_3' \cup \cdots \cup V_{i-1}'$. In particular,

$$|V_3' \cup \cdots \cup V_{i-1}'| \geqslant i|V_i' \setminus W_i| > i(k^3 - k^2),$$

and so there is a $j \in \{3, \ldots, i-1\}$ such that

$$|V_j'| \geqslant \frac{ik^2(k-1)}{i-3} > k^3,$$

contradicting the minimality of i, and completing the proof of (2.2) and the theorem. □

Note that the proof yields more than is claimed in the theorem. We did not make use of $\chi(H) \geqslant k$; we used only that H has a subgraph of minimum degree k. Briefly, *if* $H \supset G(k,g)$ *has minimum degree at least* k *then* $|E(H) \setminus E(G(k,g))| \geqslant \binom{k}{2}$, *provided* $k \geqslant 4$ *and* $g \geqslant k^4$.

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References

[1] Bollobás, B. personal communication.
[2] Descartes, B. (1954) Solution to advanced problem No 4526. *Amer. Math. Monthly* **61** 532.
[3] Kostochka, A. and Nešetřil, J. (1999) Properties of Descartes' construction of triangle-free graphs with high chromatic number. *Combin. Probab. Comput.* **8** 467–472.
[4] Nešetřil, J. (2013) A combinatorial classic: Sparse graphs with high chromatic number. In *Erdős Centennial*, Springer, pp. 383–407.

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