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RESEARCH PAPER

Reexamination of the Serendipity Theorem from the stability viewpoint

Akira Momota[1]*, Tomoya Sakagami[2] and Akihisa Shibata[3]

[1]Ritsumeikan University, Kusatsu, Shiga, Japan, [2]Kumamoto Gakuen University, Kumamoto, Japan and [3]Kyoto University, Kyoto, Japan
*Corresponding author. E-mail: momoakir123@yahoo.co.jp

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Abstract

This paper reexamines the Serendipity Theorem of Samuelson (1975) from the stability viewpoint, and shows that, for the Cobb–Douglas preference and CES technology, the most-golden golden-rule lifetime state being stable depends on parameter values. In some situations, the Serendipity Theorem fails to hold despite the fact that steady-state welfare is maximized at the population growth rate, since the steady state is unstable. Through numerical simulations, a more general case of CES preference and CES technology is also examined, and we discuss the realistic relevance of our results. We present the policy implication of our result, that is, in some cases, the steady state with the highest utility is unstable, and thus a policy that aims to achieve the social optima by manipulating the population growth rate may lead to worse outcomes.

Key words: Overlapping generations model; population growth; Samuelson's Serendipity Theorem; stability

JEL classification: C62; E13; I18

1. Introduction

Using the framework of Diamond's (1965) overlapping generations model, Samuelson (1975) investigated the optimal population growth rate that maximizes steady-state social welfare. Concretely, he derived the golden-rule allocation by solving the steady-state social planning problem for a given population growth rate and selecting the population growth rate that maximizes social welfare from golden-rule allocations. Furthermore, he found that, when the population growth rate is set at the derived optimal rate, the competitive equilibrium achieves "the most-golden golden-rule (MGG) lifetime state" without any government intervention. He named this result the Serendipity Theorem; since then, several researchers have reexamined its generality, as we will see later in this section. As for policy implications, this theorem is usually interpreted to mean that to achieve a socially optimal allocation, the only thing that the government must do is to set the population growth rate at the optimal level. This paper reexamines the theorem from the stability viewpoint and highlights a potential pitfall concerning its policy implications.

There are two important studies on the policy implications of this theorem: Deardorff (1976) and Michel and Pestieau (1993). Analyzing the case of imperfect capital depreciation, Deardorff (1976) showed that, if both utility and production functions are of Cobb–Douglas type, the steady-state equilibrium derived by Samuelson gives the lowest welfare. Furthermore, Deardorff (1976) pointed out that, when the production function has no upper bound on the per capita output, we can improve steady-state welfare unboundedly by reducing the population growth rate and, thus, Samuelson's solution cannot be the global optimum.[1]

Assuming perfect capital depreciation, Michel and Pestieau (1993) further analyzed the case in which both production and utility functions are of CES type. They showed that (i) when the elasticity of substitution in production is below unity, the Serendipity Theorem mostly holds, and (ii) when the production function is Cobb–Douglas, in which the share of capital income is below 1/2 (i.e., $f(k_t) = Ak_t^{\alpha}$ with $\alpha < 1/2$) and the elasticity of substitution in consumption is below unity, the Serendipity Theorem holds.[2]

The Serendipity Theorem has been examined in detail by many researchers. de la Croix *et al.* (2012) extended the theorem to the case of risky lifetime and Felder (2016) examined the second-order conditions concerning the extended theorem. Pestieau and Ponthiere (2014) constructed a four-period overlapping generations model in which each agent bears children in the second and third periods, and showed the robustness of the Serendipity Theorem to the introduction of different ages of motherhood. Moreover, Pestieau and Ponthiere (2017) also proved that the Serendipity Theorem still holds in a four-period overlapping generations model in which each agent works in the second and third periods with age-dependent labor productivity and bears children only in the second period.[3,4]

However, it is well-recognized in the literature that an issue remains unsolved; a stability analysis of the Serendipity Theorem is necessary. For example, de la Croix *et al.* (2012, p. 901, footnote 3) stated: "(I)mposing the optimum fertility makes the competitive economy converge toward the most golden rule steady state, provided this one is unique and stable. Otherwise, the convergence would not occur, and the social optimum could not be decentralized by means of the fertility rate." Based on this acknowledgement, we reexamine the Serendipity Theorem from the stability viewpoint. Since this is the first study on the Serendipity Theorem from the stability viewpoint, we employ the most standard framework introduced by Michel and Pestieau (1993). In doing so, to clarify the basic mechanism of our results, we first examine two special cases: (i) CES preference and Cobb–Douglas technology, and (ii) Cobb–Douglas preference and CES technology, and we *analytically* examine the existence and stability conditions of steady-state competitive equilibria. Then, using numerical simulations, we examine a more general case of CES preference and CES technology, and evaluate the empirical relevance of our analysis.

The remainder of this paper is organized as follows. Section 2 derives the laissez-faire equilibrium and obtains a formula for steady-state welfare evaluation. Section 3 summarizes the Serendipity Theorem by Samuelson (1975) and Michel and Pestieau's (1993) results. In Section 4, we show that, for CES preference and Cobb–Douglas technology, a unique, globally stable laissez-faire steady state with positive capital always exists, and thus, the Serendipity Theorem actually holds. We also show that, for Cobb–Douglas preference and CES technology, whether the MGG lifetime state is stable depends on parameter values. As such, in some situations, the Serendipity Theorem fails to hold despite the fact that the steady-state welfare is maximized for the population growth rate, since the steady state is unstable. In Section 5, we use numerical simulations to consider a more general case of CES preference and CES technology, and discuss the realistic relevance and policy implications of our results. Section 6 concludes the paper.

2. Laissez-faire equilibrium allocation

We employ the standard neoclassical production function, $Y_t = F(K_t, L_t)$, where K_t, L_t, and Y_t represent the total capital stock, labor input, and output, respectively. Defining $y_t \equiv Y_t/L_t$ and $k_t \equiv K_t/L_t$, we obtain the following per capita production function:

$$y_t = f(k_t), \tag{1}$$

where $f'(k_t) > 0$ and $f''(k_t) < 0$ hold. Following Michel and Pestieau (1993), we assume that the capital fully depreciates during the production process.

We consider a two-period overlapping generations model, where agents live for two periods as young and old. Each young agent has one unit of labor, supplies it inelastically, and retires on becoming old. We refer to young agents in period t as generation t. The population size of generation t is denoted by N_t, and its gross growth rate is n ($N_{t+1} = nN_t$). We treat n as an exogenous parameter, set by the government. The utility function of generation t is represented by

$$U_t = U(c_t,\ d_{t+1}), \tag{2}$$

where c_t and d_{t+1} stand for consumption when young and old, respectively. We assume that U is homothetic and quasi-concave.

In a laissez-faire economy, firms' optimal conditions are given by

$$w_t = w(k_t) \equiv f(k_t) - k_t f'(k_t),\ \ r_t = r(k_t) \equiv f'(k_t), \tag{3}$$

where w_t and r_t are the wage and gross interest rates, respectively.

The budget constraints of generation t are given by

$$c_t + s_t = w_t, \tag{4}$$

$$d_{t+1} = r_{t+1} \cdot s_t\,. \tag{5}$$

The first-order condition of generation t is

$$U_c(c_t,\ d_{t+1}) = r_{t+1} U_d(c_t,\ d_{t+1}), \tag{6}$$

where U_c and U_d denote the partial derivatives with respect to c_t and d_{t+1}, respectively. As proved in de la Croix and Michel (2002, Proposition 1.12), when preferences are homothetic, the saving function is linear with respect to w. Thus, we can express the saving function as follows.

$$s_t = s(w_t, r_{t+1}) = \frac{1}{\theta(r_{t+1})} w_t, \tag{7}$$

where $\theta(r_{t+1})$ is the inverse of the marginal propensity to save.

There are three markets in this economy: goods, capital, and labor. The equilibrium conditions of the capital and labor markets are represented by $s_tN_t = K_{t+1}$ and $L_t = N_t$, respectively. Dividing both sides of the capital market equilibrium condition by L_{t+1} gives

$$s_t = nk_{t+1}. \tag{8}$$

Using (3), (7), and (8), we obtain the equilibrium dynamics of the model as follows.[5]

$$nk_{t+1}\theta(r(k_{t+1})) = w(k_t). \tag{9}$$

As a final remark, we derive a general formula, which is independent of specifications of preferences and technologies, for the steady-state welfare evaluation of a laissez-faire equilibrium. Let us evaluate (2) in a steady state. Steady state is defined as the situation in which $k_t = k_{t+1} = k$. Therefore, w_t, r_t, c_t, d_t, and s_t are time invariant, and we denote them by w, r, c, d, and s, respectively. Differentiating (2) with respect to n and applying the envelope theorem to the result, we obtain

$$\frac{\mathrm{d}U}{\mathrm{d}n} = U_c(c, d)(-kf''(k))\frac{1}{r}(r-n)\frac{\mathrm{d}k}{\mathrm{d}n}\ , \tag{10}$$

which indicates that the welfare effect of a change in the population growth rate in a laissez-faire economy depends on the signs of $r - n$ and $\mathrm{d}k/\mathrm{d}n$. A formal proof of (10) is presented in Appendix A. This formula is utilized later.

3. Serendipity theorem

Consider a social planning problem in which, given n, the government directly maximizes steady-state social welfare under a resource constraint. As the resource constraint in a steady state is represented by

$$c + \frac{d}{n} + nk = f(k), \tag{11}$$

the government's problem is to choose c, d, and k to maximize $U(c, d)$ subject to (11). Deardorff (1976) terms the solution to this problem the *golden age allocation*. Formally, the golden age allocation (c_g, d_g, k_g) is defined as the solution to the following equations:

$$f'(k_g) = n, \tag{12}$$

$$\frac{U_c(c_g,\ d_g)}{U_d(c_g,\ d_g)} = n, \tag{13}$$

$$c_g + \frac{d_g}{n} + nk_g = f(k_g). \tag{14}$$

Assuming that (12) has a unique and interior solution,[6] we can determine k_g as a function of n. Given k_g, (13) and (14) give a unique set of c_g and d_g. Therefore, we can express c_g, d_g and k_g as $c_g = c_g(n)$, $d_g = d_g(n)$, and $k_g = k_g(n)$, respectively.

The Samuelson problem is to choose n to maximize steady-state utility, $U(c_g(n), d_g(n))$, among feasible golden age allocations. He called the resulting situation *the MGG lifetime state.*

Let us denote the solution by n_G, which is given by

$$n_G \equiv \arg \max_n U\left(f(k_g) - nk_g - \frac{d_g}{n},\ d_g \right).$$

If the solution to this problem is interior, using (12) and (13), we obtain the following equation:

$$k = \frac{d}{n^2}\ .$$

Now, we define *the MGG allocation*. The MGG allocation is defined as the golden age allocation when the population growth rate is set at n_G. Formally, the MGG allocation, (c_G, d_G, k_G, n_G), is defined as the solution to the following equations:

$$f'(k_G) = n_G, \tag{15}$$

$$\frac{U_c(c_G,\ d_G)}{U_d(c_G,\ d_G)} = n_G, \tag{16}$$

$$c_G + \frac{d_G}{n_G} + n_G k_G = f(k_G), \tag{17}$$

$$k_G = \frac{d_G}{n_G^2}\ . \tag{18}$$

From (15), (16), and (17), it is easily understood that c_G, d_G, and k_G are solved as functions of n_G, as stated earlier.

Using n_G, we can state Samuelson's theorem as follows.

Samuelson's Serendipity Theorem

Under n_G, the competitive equilibrium without any government intervention automatically achieves the MGG allocation.

This theorem states that to attain the MGG allocation, it is sufficient for the government to set the population growth rate at the optimal rate. However, to prove the theorem, the existence of n_G as an interior solution and the stability of a competitive equilibrium under n_G must also be proved.

Concerning the existence of n_G as an interior solution, Michel and Pestieau (1993) derived the existence conditions for the case

$$U(c_t,\ d_{t+1}) = \begin{cases} \left(\frac{1}{1+\beta}c_t^{1-(1/\mu)} + \frac{\beta}{1+\beta}d_{t+1}^{1-(1/\mu)}\right)^{\mu/(\mu-1)} & \text{for } \mu > 0,\ \mu \neq 1 \\ c_t^{1/(1+\beta)} d_{t+1}^{\beta/(1+\beta)} & \text{for } \mu = 1 \end{cases}, \tag{19}$$

$$f(k_t) = \begin{cases} A[(1-\alpha) + \alpha k_t^{\sigma}]^{\frac{1}{\sigma}} & \text{for } \sigma < 1,\ \sigma \neq 0 \\ Ak_t^{\alpha} & \text{for } \sigma = 0 \end{cases}. \tag{20}$$

In (19), β is the relative weight between c and d, and $\mu > 0$ is the elasticity of substitution in consumption (the degree of intertemporal substitution). Similarly, in (20), A is the technological level, α a constant satisfying $0 < \alpha < 1$, and $\sigma < 1$ is a parameter related to the elasticity of substitution in production; specifically, $1/(1-\sigma)$ represents the elasticity of substitution in production.[7]

The sufficient conditions for the existence of an interior solution are summarized as follows.

Michel and Pestieau's* (1993) *result

If either of the following conditions is satisfied, there exists an interior solution for n_G:

(i) σ *is negative (if* $\mu > 1$*, an additional condition is required in that either* $\mu < 2 - 1/(1-\sigma)$ *or* A *is large);*

(ii) σ *is zero,* $0 < \mu < 1$ *(the production technology is of the Cobb–Douglas type:* $f(k_t) = Ak_t^{\alpha}$*), and* $\alpha < 1/2$.

The results of their analysis are summarized in Table 1 of Michel and Pestieau (1993), while our Table 1 reproduces their results in our notation.

4. Existence and stability of a laissez-faire equilibrium

The following analysis focuses on two special cases of Michel and Pestieau's (1993) work, that is, the case of $\sigma = 0$ (Cobb–Douglas technology) and that of $\mu = 1$ (Cobb–Douglas preference). We examine the stability of the steady states of competitive equilibrium and whether the MGG allocation is supported by a stable steady-state competitive equilibrium in subsections 4.1 and 4.2, respectively.

4.1. CES preference and Cobb–Douglas technology

Here, we analyze the case of $\sigma = 0$. Since the CES utility function is homothetic, we can utilize the difference equation (9) describing the equilibrium dynamics of the model:

$$nk_{t+1}(\beta^{-\mu} \cdot r(k_t)^{1-\mu} + 1) = w(k_t), \tag{21}$$

where we use $\theta(r(k_t)) \equiv \beta^{-\mu} \cdot r(k_t)^{1-\mu} + 1$ under the CES utility function.

From (21), regarding steady-state competitive equilibrium, we have the following proposition:

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Table 1. Michel and Pestieau (1993)

	$0 < \mu < 1$	$\mu = 1$	$\mu > 1$
$\sigma < 0$	An interior solution exists	An interior solution exists	An interior solution exists under Condition (B)
$\sigma = 0$	An interior solution exists under Condition (A)	No interior solution	No interior solution
$\sigma > 0$	No interior solution	No interior solution	No interior solution

Condition (A): $f(k) = Ak^{\alpha}$ and $\alpha < ½$.
Condition (B): A is large enough or $\mu < 2 - [1/(1 - \sigma)]$.

Proposition 1

(i) *The nontrivial steady-state capital stock,* $k^* > 0$, *exists and is unique.*
(ii) *The nontrivial steady state is globally stable.*

Proof:

(i) Because of the Cobb–Douglas technology, we have

$$\frac{w(k_t)}{r(k_t) \cdot k_t} = \frac{1 - \alpha}{\alpha} . \tag{22}$$

In steady state, (21) and (22) are reduced to $nk\theta(r(k)) = w(k)$ and $w(k) = (1 - \alpha)rk/\alpha$, respectively. Combining these two equations, we have

$$\frac{\theta(r)}{r} = \frac{1}{n}\frac{1 - \alpha}{\alpha} , \tag{23}$$

which determines the steady-state interest rate. Since $\theta(r)/r = \beta^{-\mu} \cdot r^{-\mu} + r^{-1}$ is strictly decreasing with respect to r, $\lim_{r \to 0} \theta(r)/r = +\infty$, and $\lim_{r \to +\infty} \theta(r)/r = 0$, (23) always has a unique solution of r, that is, the steady-state equilibrium interest rate. We denote this solution by r^*. The steady-state capital stock k^* is uniquely determined by (3), $r^* = f'(k^*)$ because $f'' < 0$.

(ii) Defining Φ as $\Phi(k_{t+1}) \equiv nk_{t+1}\theta(r(k_{t+1}))$, we can express equilibrium dynamics as $\Phi(k_{t+1}) = w(k_t)$. As Figure 1 shows, the nontrivial steady-state k^* is globally stable if $\Phi(k) < w(k)$ for $0 < k < k^*$, $\Phi(k) = w(k)$ for $k = k^*$, and $\Phi(k) > w(k)$ for $k^* < k$. Therefore, we show that $\Phi(k)$ and $w(k)$ satisfy these relations. Since k^* is the unique solution of $\Phi(k)/w(k) = 1$ in the region of $k > 0$, we only have to show that $\Phi(k)/w(k)$ is a strictly increasing function of k. From (22), we have

$$\frac{\Phi(k)}{w(k)} = \frac{\alpha}{1 - \alpha} n \frac{\theta(r)}{r} . \tag{24}$$

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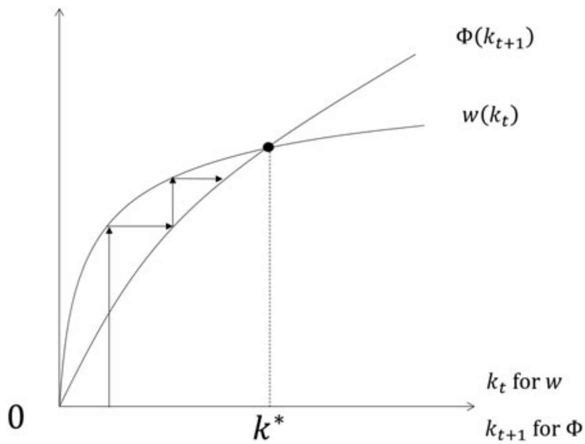

Figure 1. Stable steady-state equilibrium.

Since $\theta(r)/r$ is a strictly decreasing function of r and $r'(k) < 0$, $\Phi(k)/w(k)$ is strictly increasing with respect to k. This proves the global stability of the nontrivial steady state.

Combining Michel and Pestieau's (1993) results and Proposition 1, we have

Proposition 2 *Assume that $\sigma = 0$. Then, if $\alpha < 1/2$, the unique MGG lifetime state is globally stable and, thus, the Serendipity Theorem holds.*

4.2. Cobb–Douglas preference and CES technology

Subsequently, we analyze the case of Cobb–Douglas preference ($\mu = 1$) and CES technology.

4.2.1. Saving behavior and capital market equilibrium

We express the Cobb–Douglas-type utility function of generation t as

$$U(c_t,\ d_{t+1}) = c_t^{\frac{1}{1+\beta}} d_{t+1}^{\frac{\beta}{1+\beta}}.^{8} \tag{25}$$

Under (25), the saving function is derived as

$$s_t = \frac{\beta}{1+\beta} w_t.$$

Therefore, the capital market equilibrium condition, (8), is reduced to

$$k_{t+1} = \frac{\beta}{1+\beta} \frac{1}{n} w_t. \tag{26}$$

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4.2.2. CES technology

From the results of Michel and Pestieau (1993), reproduced in Table 1 above, a necessary condition for the Serendipity Theorem to hold is $\sigma < 0$. Therefore, it suffices to focus on the case of $\sigma < 0$ in analyzing the existence and stability of a steady-state equilibrium.

Under CES technology, the optimal conditions for a representative firm, (3), are

$$w_t = w(k_t) = A(1-\alpha)[(1-\alpha) + \alpha k_t^{\sigma}]^{1/\sigma - 1},$$

and

$$r_t = r(k_t) = A\alpha[(1-\alpha)k_t^{-\sigma} + \alpha]^{(1-\sigma)/\sigma}. \tag{27}$$

As we focus on the case of $\sigma < 0$, from (20) and (27) we have:

[1] $r(k) = f'(k) > 0$ and $r'(k) = f''(k) < 0$;
[2] $f(0) = 0, \quad \lim_{k \to 0} f'(k) = A\alpha^{\frac{1}{\sigma}}$, and $\lim_{k \to +\infty} f'(k) = 0$.

Similarly, it is easy to show the following properties of $w(k_t)$:

[3] $w'(k) > 0$;
[4] $w(0) = 0, \quad \lim_{k \to 0} w'(k) = 0, \quad \lim_{k \to +\infty} w'(k) = 0$;
[5] Define $k^+ = \left[-\frac{\alpha\sigma}{(1-\alpha)(1-\sigma)}\right]^{-\frac{1}{\sigma}}$. Then, for $k > 0$, $w''(k) \gtreqless 0 \Leftrightarrow k \lesseqgtr k^+$.

Properties [3], [4], and [5] imply that the $w(k)$ function is S-shaped.

4.2.3. Equilibrium dynamics and steady states

Combining (26) and (27), we obtain the equilibrium dynamics of the laissez-faire economy in terms of k:

$$k_{t+1} = \frac{\beta}{1+\beta}\frac{1}{n} w(k_t) \equiv h(k_t, \ n). \tag{28}$$

Since $[\beta/(1+\beta)](1/n)$ is constant for a given n, $h(k_t, n)$ is proportional to $w(k_t)$. Therefore, the shape of $h(k_t, n)$ is similar to that of the $w(k_t)$ curve. It should be noted that a rise in n rotates the $h(k_t, n)$ curve clockwise around the origin. Consequently, we define a critical value of the population growth rate, $\check{n}$, as shown in Figure 2, where $\check{n}$ is the highest population growth rate under which a steady-state equilibrium exists. From Figure 2, when $n = \check{n}$, the $h(k_t, n)$ curve is tangent to the 45-degree line from below. We denote the value of k at the tangent point by $\check{k}$. By definition, we have $\partial h(\check{k}, \check{n})/\partial k_t = 1$, and from this condition we can compute $\check{n}$ and $\check{k}$ as[9]

$$\check{n} = \frac{\beta}{1+\beta} A[\alpha(1-\sigma)]^{1/\sigma} \frac{-\sigma}{1-\sigma} > 0, \check{k} = \left[-\frac{1-\alpha}{\sigma\alpha}\right]^{1/\sigma}. \tag{29}$$

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Now, we have the following proposition:

Proposition 3 *Assume that* $\mu = 1$ and $\sigma < 0$. *Then, it follows that,*

(i) *if* $0 < n < \check{n}$, *there exist two nontrivial steady-state capital stocks* k^u *and* $k^s (> k^u)$, *and if* $n > \check{n}$, *no nontrivial steady state exists,*
(ii) *when* $0 < n < \check{n}$, *the lower steady-state equilibrium,* k^u, *is unstable, while the higher steady-state equilibrium,* k^s, *is stable.*

Proof:

(i) In Figure 2, the $h(k_t, n)$ curve is tangent to the 45-degree line when $n = \check{n}$ (Point E). Remember that $h(k_t, n)$ rotates counterclockwise around the origin as n decreases. Additionally, note that the slope of $h(k_t, n)$, $\lim_{k_t \to 0} \frac{\partial}{\partial k_t} h(k_t, n) = 0$, and $\lim_{k_t \to +\infty} \frac{\partial}{\partial k_t} h(k_t, n) = 0$ hold for any n because $\lim_{k \to 0} w'(k) = 0$, $\lim_{k \to +\infty} w'(k) = 0$. Thus, for $0 < n < \check{n}$, there are two intersections of the $h(k_t, n)$ curve and the 45-degree line. On the other hand, $h(k_t, n)$ rotates clockwise around the origin as n increases; thus, for $n > \check{n}$, there is no intersection.
(ii) At k^s and k^u in Figure 3, $k_{t+1} = h(k_t, n) = k_t$, and thus, k remains there. Suppose that $0 < k_t < k^u$ or $k^s < k_t$. Then, the $h(k_t, n)$ curve is located below the 45-degree line, that is, $k_{t+1} = h(k_t, n) < k_t$, and thus, the time path of k is decreasing in this area. For $k^u < k_t < k^s$, $k_{t+1} = h(k_t, n) > k_t$, and thus, the time path of k is increasing. From these observations, it follows that k^u is unstable and k^s is stable.

Changing n in Figure 2, we have the following corollary concerning comparative statics on the population growth rate.

Corollary *Assume that* $\mu = 1$ and $\sigma < 0$, *and further* $0 < n < \check{n}$. *Then, a marginal increase in* n *increases* k^u *and decreases* k^s.

Proof: Since an increase in n rotates the $h(k_t, n)$ curve clockwise around the origin, the result follows directly from Figure 4.

4.2.4. Steady-state welfare

Since there are two steady-state equilibria for $0 < n < \check{n}$, we must compare their welfare levels to derive the socially optimum allocation. The formula for steady-state welfare, (10), indicates that the welfare effect of a change in n depends on the signs of $r - n$ and $\mathrm{d}k/\mathrm{d}n$. From the Corollary, we know that $\mathrm{d}k^s/\mathrm{d}n < 0$ and $\mathrm{d}k^u/\mathrm{d}n > 0$, and thus, it follows from (10):

$$\mathrm{d}U^s/\mathrm{d}n \gtreqless 0 \Leftrightarrow r/n \lesseqgtr 1 \text{ and } \mathrm{d}U^u/\mathrm{d}n \lesseqgtr 0 \Leftrightarrow r/n \lesseqgtr 1, \tag{30}$$

where U^s and U^u represent steady-state utilities in the stable and unstable steady state, respectively. This result indicates that, to examine how a change in n affects steady-state utility, we should identify how a change in n affects the steady-state value of r/n.

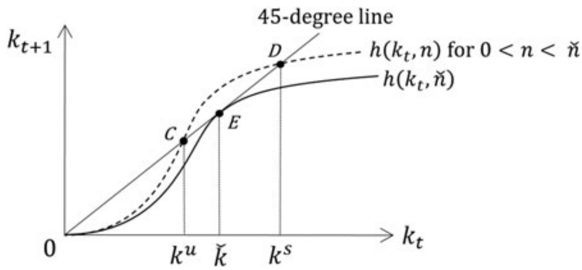

Figure 2. The graph of $h(k_t,\ \check{n})$.

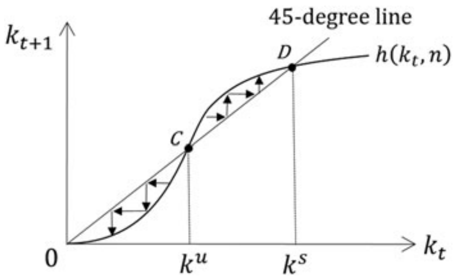

Figure 3. Two nontrivial steady states if $0 < n < \check{n}$.

The steady-state value of r is obtained as the solution to the following equation:

$$\frac{r}{n} = \varphi(r) \equiv \frac{1+\beta}{\beta}\frac{\alpha}{(r/A\alpha)^{\sigma/(1-\sigma)} - \alpha}. \tag{31}$$

The derivation of (31) is given in Appendix C. Concerning $\varphi(r)$, under the assumption $\sigma < 0$, we can observe the following properties:

Properties of $\varphi(r)$

(i) $\varphi(r) > 0\ \forall r \in (0,\ A\alpha^{1/\sigma})$;
(ii) $\varphi'(r) > 0,\ \lim_{r\to 0}\varphi'(r) = +\infty,\ \lim_{r\to 0}\varphi(r) = 0,\ \lim_{r\to A\alpha^{\frac{1}{\sigma}}}\varphi(r) = +\infty$;

(iii) *$\varphi(r)/r$ is minimized at* $\check{r} \equiv A\alpha[(1-\sigma)\alpha]^{(1-\sigma)/\sigma}$.

Properties (i) and (ii) except for the second one are straightforwardly confirmed. Regarding the second property of (ii), differentiating $\varphi(r)$ with respect to r gives

$$\varphi'(r) = \frac{\alpha[(1+\beta)/\beta][-\sigma/(1-\sigma)]}{[(r/A\alpha)^{1/2(1-\sigma)} - \alpha(r/A\alpha)^{(1-2\sigma)/2(1-\sigma)}]^2},$$

and thus, it follows that $\lim_{r\to 0}\varphi'(r) = +\infty$. Next, consider (iii). Because

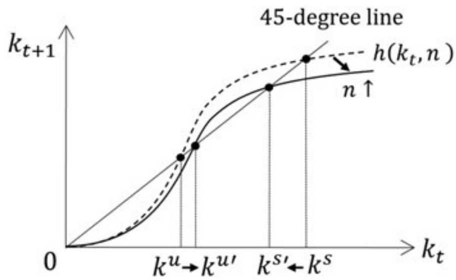

Figure 4. The effect of an increase in n.

$\varphi(r)/r = [(1+\beta)/\beta]\alpha \dfrac{1}{(A\alpha)^{-\sigma/(1-\sigma)} r^{1/(1-\sigma)} - \alpha r}$, $\varphi(r)/r$ is minimized at $\check{r} \equiv A\alpha[(1-\sigma)\alpha]^{(1-\sigma)/\sigma}$, at which the denominator is maximized.

Based on properties (i) to (iii), we can depict the $\varphi(r)$ curve as in Figure 5.[10] The steady-state value of r is determined by the abscissa of the intersections of two loci: the r/n locus (the straight line with $1/n$ slope) and the $\varphi(r)$ locus. As described in subsection 4.2.2, $r \to 0$ as $k \to +\infty$ under the assumption of $\sigma < 0$. Therefore, $\varphi(r)$ is not defined at $r = 0$. That is, the two loci do not intersect at the origin.

As shown in Proposition 3(i), there exist two nontrivial steady states for $0 < n < \check{n}$, implying that (31) has two different solutions. The dotted line in Figure 5 depicts the r/n locus for $n \in (0, \check{n})$. As is obvious from the figure, there are two intersection points for $n \in (0, \check{n})$ and the values on the vertical axis of the intersection points represent the steady-state values of r/n.

Due to the law of diminishing marginal productivity, r is negatively related to k, and the lower value of r/n, i.e., Point D in Figure 5, corresponds to a higher value of k, i.e., Point D in Figure 2, which is the stable steady state. Therefore, we denote the value of r/n in this stable steady state by r^s/n. Similarly, the higher value of r/n, Point C in Figure 5, corresponds to the lower value of k, i.e., Point C, in Figure 2. Thus, Point C in Figure 5 represents the unstable steady-state equilibrium and we denote the value of r/n in the unstable steady state by r^u/n.

When $n = \check{n}$, we have a unique, nontrivial steady state. It should be noted here that Point E in Figure 5 represents the same state as Point E in Figure 2. From (27) and (29), which define $\check{r}$ and $\check{k}$, we confirm this fact and obtain $\check{r}/\check{n} = -(1/\sigma)[(1+\beta)/\beta]$.

When $n > \check{n}$, there is no nontrivial steady state as shown in Proposition 3 (i). Thus, the r/n locus and $\varphi(r)$ have no intersection points.

Now, through Figure 5 we can prove the following proposition:

Proposition 4 *Assume that* $\mu = 1$ and $\sigma < 0$. *Then, it follows that*

(i) r^s/n *is an increasing function of* n and, r^u/n *is a decreasing function of* n.

(ii) $\lim_{n\to 0}\left(\dfrac{r^s}{n}\right) = 0$, and $\lim_{n\to 0}\left(\dfrac{r^u}{n}\right) = +\infty$.

(iii) $\lim_{n\to \check{n}}\left(\dfrac{r^s}{n}\right) = \lim_{n\to \check{n}}\left(\dfrac{r^u}{n}\right) = -\dfrac{1}{\sigma}\dfrac{1+\beta}{\beta}$.

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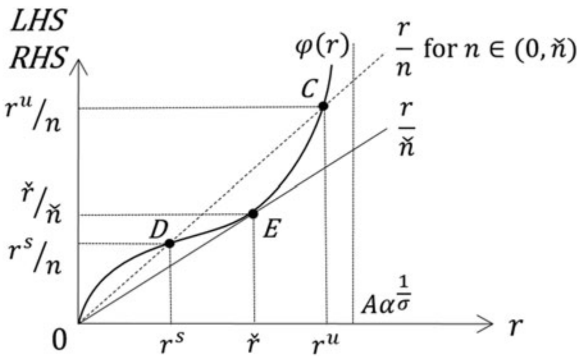

Figure 5. The graphs of the $\varphi(r)$ curve and the r/n line.

Proof:

(i) As n decreases, the slope of the r/n locus becomes steeper, and thus, the upper intersection of the two loci, r^u/n, moves upward while the lower intersection, r^s/n, moves downward. In other words, r^u/n is a decreasing function of n while r^s/n is an increasing function of n.

(ii) As n approaches zero, the slope of r/n locus diverges to $+\infty$. Noting that $\lim_{r\to 0} \varphi'(r) = +\infty$ and $\lim_{r\to 0} \varphi(r) = 0$, it is seen that Point D in Figure 5, $(r^s, r^s/n)$, approaches the origin. Thus, $\lim_{n\to 0} (r^s/n) = 0$ holds. On the other hand, because $\lim_{r\to A\alpha^{\frac{1}{\sigma}}} \varphi(r) = +\infty$, Point C $(r^u, r^u/n)$ approaches $(A\alpha^{1/\sigma}, +\infty)$. Thus, $\lim_{n\to 0} (r^u/n) = +\infty$ holds.

(iii) As n approaches $\check{n}$, both Points C and D converge to Point E. When $n = \check{n}$, there is only one steady state, and the stable and unstable steady states coincide: $r^s/n = r^u/n = \check{r}/\check{n} \,(= -(1/\sigma)[(1+\beta)/\beta])$.

From Proposition 4, we can depict the relationship between n and r/n as per Figure 6. Figure 6 is obtained by changing n in Figure 5 and plotting the locus of the intersections of the $\varphi(r)$ curve and the r/n line in the $(n, r/n)$ plane. The upward-sloping curve (solid curve) indicates the relationship between n and r^s/n (stable solution) while the downward-sloping curve (dashed curve) represents the relationship between n and r^u/n (unstable solution). Observe that these curves intersect at $(\check{n}, -(1/\sigma)[(1+\beta)/\beta])$. The Points C, D, and E in Figure 6 correspond to Points C, D, and E in Figure 5 (and thus, in Figure 2), respectively.

We next examine whether or not the Serendipity Theorem holds by checking the stability and welfare of steady states. The critical value of r/n is 1, as (30) shows. Using Figure 6 and taking into account the location of $r/n = 1$ (the $r/n = 1$ line is drawn as a horizontal line in Figure 6 because the horizontal axis is r/n), we can derive the following proposition:

Proposition 5 *Assume that $\sigma < 0$. The Serendipity Theorem holds if and only if* $-(1/\sigma)[(1+\beta)/\beta] > 1$.

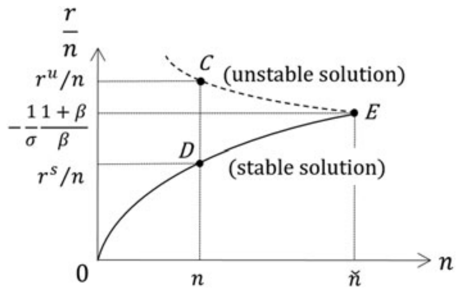

Figure 6. The relationship between r/n and n.

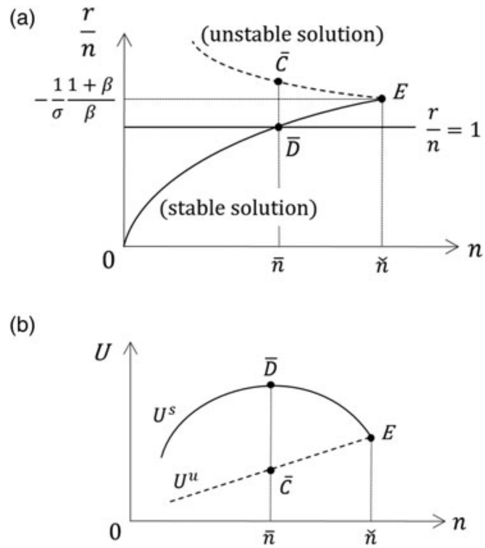

Figure 7. (a) The relationship between r/n and n in the case of $-(1/\sigma)[(1+\beta)/\beta] > 1$. (b) The relation between U and n in the case of $-(1/\sigma)[(1+\beta)/\beta] > 1$.

Proof: Let $\bar{n}$ denote the population growth rate where the sideways chevron-shaped graph in Figure 6 and the $r/n = 1$ line intersect. Assume that $-(1/\sigma)[(1+\beta)/\beta] > 1$. Then, the $r/n = 1$ line is located below Point E, and we obtain $\bar{n}$ as per Figure 7a. Points $\bar{C}$ and $\bar{D}$ represent the laissez-faire steady-state equilibria under $n = \bar{n}$, which are unstable and stable, respectively.

First, let us consider the upward-sloping curve passing Point $\bar{D}$ (the solid curve). As has been stated earlier, this curve corresponds to the "stable" steady state in each n. We see that, on the upward-sloping curve, $r/n < 1$ holds when $0 < n < \bar{n}$, and $r/n > 1$ holds when $\bar{n} < n < \check{n}$. Using (30), we confirm that $\mathrm{d}U^s/\mathrm{d}n > 0$ when $0 < n < \bar{n}$, and $dU^s/dn < 0$ when $\bar{n} < n < \check{n}$. Consequently, the relationship between n and U^s is drawn as per Figure 7b.

Next, we consider the downward-sloping curve passing through Point $\bar{C}$ (the dashed curve). This curve corresponds to the "unstable" steady state in each n. Note that $r/n > 1$ always holds on this curve. From (30), we obtain $\mathrm{d}U^u/\mathrm{d}n > 0$. Moreover, it is obvious that $U^s = U^u$ when $n = \check{n}$. Thus, we can depict U^u as per Figure 7b. Thus, Point $\bar{D}$ attains the highest utility among the laissez-faire steady state, and Michel and Pestieau (1993) showed

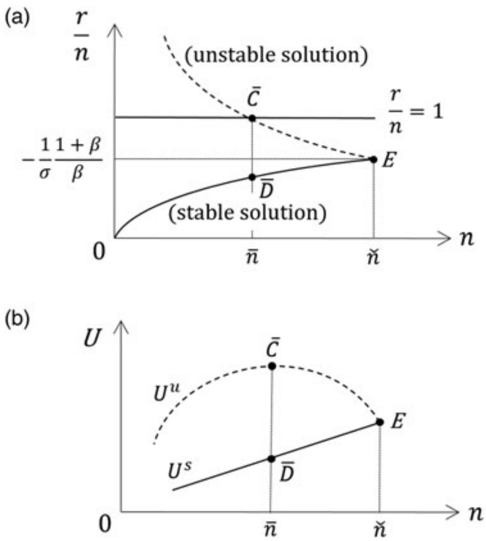

Figure 8. (a) The relationship between r/n and n in the case of $-(1/\sigma)[(1+\beta)/\beta] \le 1$. (b) The relation between U and n in the case of $-(1/\sigma)[(1+\beta)/\beta] \le 1$.

that Point $\bar{D}$ corresponds to the MGG allocation (i.e., $\bar{n} = n_G$). Note that Point $\bar{D}$ is stable, and as such, the Serendipity Theorem certainly holds in this case.

On the other hand, if $-(1/\sigma)[(1+\beta)/\beta] < 1$, the $r/n = 1$ line is located above Point E and we obtain $\bar{n}$ as per Figure 8a. Applying a similar argument to obtain Figure 7b from Figure 7a, we obtain Figure 8b from Figure 8a. We observe that Point $\bar{C}$ provides the highest steady-state utility; Michel and Pestieau (1993) showed that Point $\bar{C}$ corresponds to the MGG allocation (and again, $\bar{n} = n_G$). However, Point $\bar{C}$ is in the unstable region and the utility cannot be attained as a competitive equilibrium. The attainable maximum steady-state utility is provided by Point E, which is a corner solution, meaning that the Serendipity Theorem fails to hold.

5. Numerical analysis of a more general case: CES preference and CES technology

5.1. Basic setup

In subsection 4.2, to clarify the basic mechanism of our results, we restricted our attention to the case of CES technology and Cobb–Douglas preference. We showed that, if the elasticity of substitution in production is low, the MGG steady state is unstable, that is, the Serendipity Theorem fails to hold. In this section, to evaluate the realistic relevance of our analysis, we perform numerical simulations on a more general case – the case of CES preference and CES technology – by setting empirically plausible values of the key parameters.

As explained in Section 3, the MGG allocation is given by (15)–(18). Since we specify the utility and production functions to be of the CES types [see (19) and (20)], these four equations are summarized by the following two equations, the derivations of which are presented in Appendix D.

$$A\alpha[(1-\alpha)k_G^{-\sigma} + \alpha]^{(1-\sigma)/\sigma} = n_G, \tag{32}$$

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$$\alpha[\beta^{-\mu}n_G^{1-\mu}+1]=(1-\alpha)k_G^{-\sigma}. \tag{33}$$

From (32) and (33) we can solve for k_G and n_G. Since the elasticity of substitution in production, $1/(1-\sigma)$, is directly estimated in the empirical literature, it is convenient to introduce a new variable representing the elasticity of substitution in production, $\tau \equiv 1/(1-\sigma)$. Using τ, we can rewrite (32) and (33) as:

$$A\alpha = n_G[(1-\alpha)k_G^{(1-\tau)/\tau}+\alpha]^{1/(1-\tau)}, \tag{34}$$

$$\alpha[\beta^{-\mu}n_G^{1-\mu}+1]=(1-\alpha)k_G^{(1-\tau)/\tau}. \tag{35}$$

In our numerical simulations, we restrict our attention to the case of $0<\tau<1$ ($\Leftrightarrow \sigma<0$) and $0<\mu<1$, under which an MGG allocation exists, as proved by Michel and Pestieau (1993).[11]

The marginal propensity to save and the wage and gross interest rate under the double CES assumption are given by:

$$\theta(r_{t+1}) \equiv \beta^{-\mu}\cdot(r_{t+1})^{1-\mu}+1$$

$$\begin{aligned} w_t &= w(k_t) = A(1-\alpha)[(1-\alpha)+\alpha k_t^{-(1-\tau)/\tau}]^{-1/(1-\tau)}, \\ r_t &= r(k_t) = A\alpha[(1-\alpha)k_t^{(1-\tau)/\tau}+\alpha]^{-1/(1-\tau)}. \end{aligned} \tag{36}$$

We can express the equilibrium dynamics when the population growth rate is set at n_G as:

$$H(k_t,\ k_{t+1}) \equiv w(k_t)-n_G k_{t+1}(\beta^{-\mu}\cdot r(k_{t+1})^{1-\mu}+1)=0, \tag{37}$$

which is derived from (9) and (36). When $k_t = k_{t+1} = k_G$, $H(k_G, k_G) = 0$, and thus the MGG allocation constitutes a steady-state laissez-faire equilibrium under the population growth rate n_G. This is a restatement of the result of Samuelson (1975). It should be noted here that this does not mean that the MGG steady state is the unique steady state; there may be multiple steady states.

In order to check the local stability of the MGG steady state, $k_t = k_{t+1} = k_G$, we evaluate the following value:

$$\left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k_{t+1}=k_G} = -\frac{\partial H(k_G,\ k_G)}{\partial k_t}\Big/\frac{\partial H(k_G,\ k_G)}{\partial k_{t+1}}. \tag{38}$$

If the absolute value of $\mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G}$ is greater than unity, then the MGG steady state is locally unstable, and thus, the Serendipity Theorem fails to hold.

Regarding (38), we obtain the following lemma.

Lemma 1 *When* $\tau=\mu=0.5$, *it always holds that* $\mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G}=1$, *regardless of the values of* A, α, and β.

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The proof is given in Appendix E. Lemma 1 indicates that, when $\tau = \mu = 0.5$, a tangent bifurcation occurs at the MGG steady state. That is, the MGG steady state is represented as Point E in Figure 2, regardless of the values of A, α, and β.

5.2. Numerical simulations

5.2.1. Choice of parameter values

We briefly explain our choice of values of structural parameters, α, β, δ, A, τ, and μ. We begin with δ, α, and A.

Since our overlapping generations model consists of two periods, the length of one period could be reasonably considered to be about 30 years; thus, the depreciation rate would be large. Following de la Croix and Doepke (2003), who performed similar numerical simulations of a two-period overlapping generations model, we assume that the capital entirely depreciates after one period, that is, $\delta = 1$. Moreover,we specify $\alpha = 0.4$. Under this value, the steady-state labor income is computed as 0.6–0.7, consistent with most empirical estimates of the labor income share.[12] Following de la Croix and Doepke (2003), the benchmark value of the productivity parameter, A, is set at unity. In subsection 5.2.4, we examine the robustness of our numerical simulation results by changing the value of A.

We next observe the range of β. Several studies have constructed the value of β by following the business cycle literature, which often employs 0.99 as the per quarter discount factor. When we apply this discount rate to our economy, we set $\beta = 0.3$ because $(0.99)^{4\times 30} \approx 0.3$.[13] Moreover, we report the case of $\beta = 1$ in order to check how changes in β affect the stability of the MGG steady state.[14]

The elasticity of substitution in production, τ, and the degree of intertemporal substitution, μ, may be controversial because the estimated values of τ and μ have a wide range. Chirinko (2008) stated that evidence suggests that the elasticity of substitution in production, τ, ranges from 0.40 to 0.60, while Klump *et al.* (2008) obtained 0.7 as the elasticity of substitution in production, τ. Based on these results, this paper specifies that τ lies within the range [0.4, 0.7].[15]

Concerning intertemporal substitution μ, Table 1 of Havránek *et al.* (2015), which summarizes individual countries' estimated value of intertemporal substitution for 45 countries, shows a very wide range of estimated values of μ. Given that two-thirds of the mean elasticities lie in the range [0.1, 0.7], we consider this to be the range of μ.[16]

5.2.2. Stability of the MGG steady state

First, given $\beta = 0.3$, $A = 1$, and $\alpha = 0.3$ (we call this the benchmark case), we examine the relationship between the stability of the MGG steady state and the two key parameters, τ and μ. Figure 9 depicts the stable region, where $0 < \mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G} < 1$, and the unstable region, where $\mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G} > 1$, in the $\tau - \mu$ plane. The solid curve represents the locus that satisfies $\mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G} = 1$, that is, the border of the two regions. The key results are: the border is downward sloping, and the MGG steady state is unstable when τ and μ take small values.[17] Furthermore, the border passes through (0.5, 0.5), as Lemma 1 shows. Through a numerical examination, under our parameter settings, we confirm that the graph of the dynamical system, $H(k_t, k_{t+1}) = 0$, is expressed as an S-shaped curve, such as in Figure 3.

Next, let us examine how changes in the value of β affect the result. We change the value of β from 0.3 to 1 (the other parameters are fixed at the benchmark values), and consider how the border changes. Figure 10 depicts the result. The dashed and solid

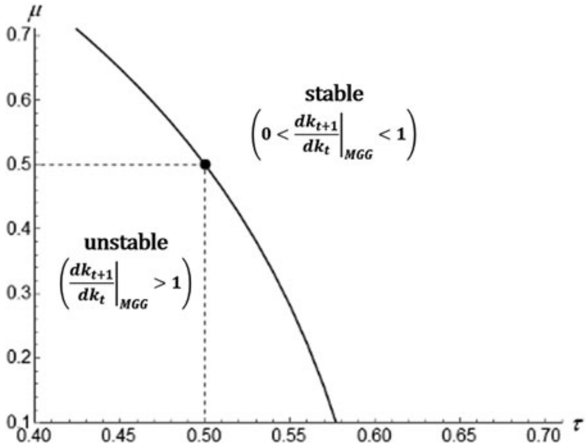

Figure 9. Benchmark case ($A = 1$, $\beta = 0.3$, $\alpha = 0.3$).

curves represent the border in the benchmark case (the same as Figure 9), and the high β case, respectively. Again, the border is downward sloping and passes through (0.5, 0.5). We observe that the unstable region expands for $0.4 < \tau < 0.5$, while it shrinks for $0.5 < \tau$. This suggests that, in a country where the elasticity of substitution in production is low, a higher propensity to save tends to make the MGG steady state unstable.

Finally, let us examine how changes in the value of A affect the unstable region. Here, we change the value of A from 1 to 5 (the other parameters are fixed at the benchmark values). The result is depicted in Figure 11.

The dashed and solid curves represent the border in the benchmark case (the same as Figure 9), and the high A case, respectively. Figure 11 shows that, the MGG steady state is unstable when τ and μ take small values, in the same way as Figures 9 and 10, and that, an increase in A makes the unstable region smaller for $0.4 < \tau < 0.5$, while it expands the unstable region for $0.5 < \tau$. This implies that, in an economy where the elasticity of substitution in production is low, an improvement in the TFP tends to stabilize the MGG steady state.

5.3. Policy implications: a potential pitfall

We confirmed that the MGG steady state is unstable when τ and μ take small values, and thus, the Serendipity Theorem can fail to hold under empirically plausible parameter values. We argue that this has significant policy implications.

Assume that the MGG steady state is unstable. In this case, the MGG steady state corresponds to Point C in Figure 3, and this scenario follows that in Figure 8b. Suppose that the population growth rate is $n_0(>n_G)$, as in Figure 12. If the government does not take into consideration the stability issue and believes that the Serendipity Theorem is applicable, it decreases the population growth rate to n_G by expecting the economy to move from Point H to Point F. However, Point H is an unstable steady state, and thus, it is natural to assume that the economy is initially located at Point G. Therefore, this policy will move the economy from Point G to Point I (refer to the dotted arrows in Figure 13). As a result, steady-state welfare deteriorates, contrary

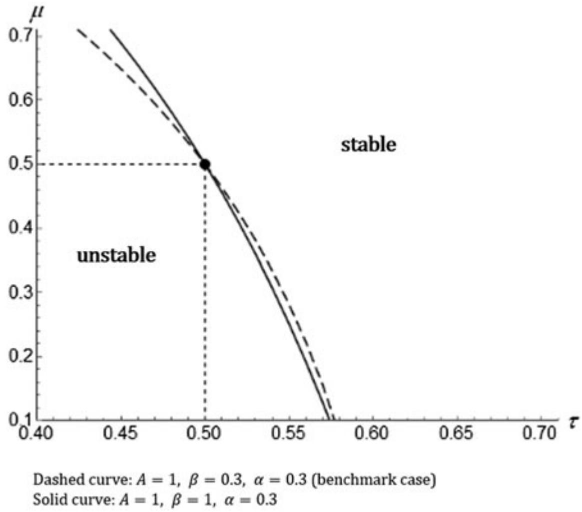

Figure 10. Effect of an increase in $\boldsymbol{\beta}$.

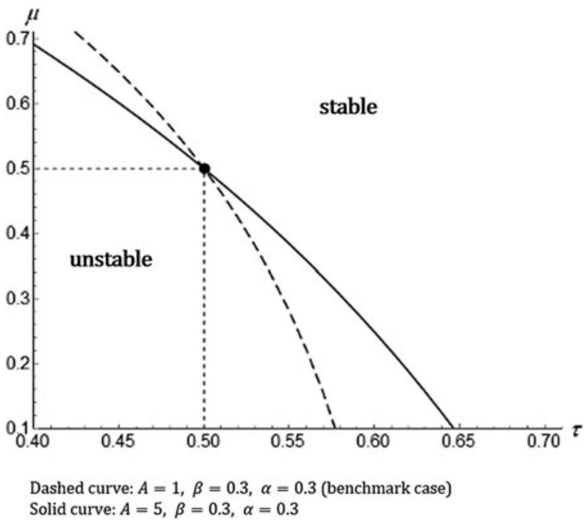

Figure 11. Effect of an increase in ***A***.

to the government's intention (see Figure 12). Moreover, if the economy is at Point H (the unstable steady state), changing the population growth rate from n_0 to n_G causes the economy to move from Point H to Point I (refer to the solid arrows in Figure 13). In this case, social welfare decreases to a greater degree than in the previous situation (see Figure 12).

Let us consider another example. Assume that in the situation in Figure 8b the population growth rate is given by $n_0(<n_G)$, as depicted in Figures 14 and 15, and that the

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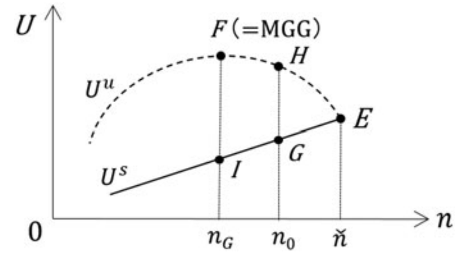

Figure 12. A potential pitfall (case of $-(1/\sigma)((1+\beta)/\beta) \leq 1$ and $n_0 > n_G$).

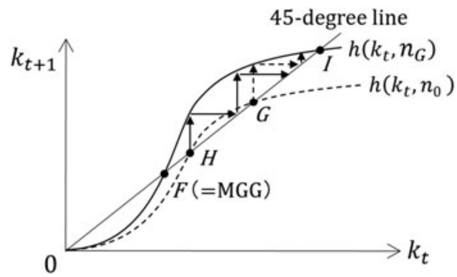

Figure 13. Wrong policy ($n_0 > n_G$).

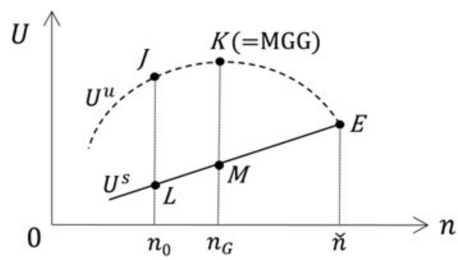

Figure 14. A potential pitfall (case of $-(1/\sigma)((1+\beta)/\beta) \leq 1$ and $n_0 < n_G$).

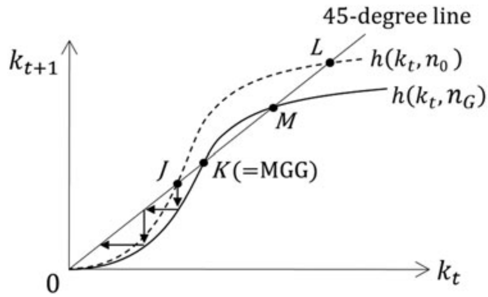

Figure 15. Wrong policy ($n_0 < n_G$).

economy is in the lower unstable steady state, Point J. If the government believes that it can apply the Serendipity Theorem, it will increase the population growth rate to n_G by expecting that the economy moves from Point J to K (see Figure 14). However, once n is increased, the economy moves toward the origin (refer to the solid arrows in Figure 15).

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As a result, the economy shrinks and welfare deteriorates, contrary to the government's intention.

These examples indicate the importance of the stability consideration in analyzing the Serendipity Theorem.

6. Concluding remarks

We reexamined the Serendipity Theorem of Samuelson (1975) from the stability viewpoint. Specifically, we investigated whether the MGG allocation can be supported by the *stable* steady-state competitive equilibrium. Moreover, we showed that, in the case of Cobb–Douglas preference and CES technology, the stability of the MGG lifetime state depends on parameter values and, in some situations, the Serendipity Theorem fails to hold despite steady-state welfare being maximized at the population growth rate as the steady state is unstable. Furthermore, using numerical simulations, we investigated a more general case, specifically that of CES preference and CES technology, and showed that the MGG lifetime state is unstable when the elasticities of substitution in consumption and production are small. These form the main contributions of this paper given that the previous studies on the Serendipity Theorem have not analyzed the stability issue in detail.

Notes

1 See also Samuelson (1976).

2 Since the Cobb–Douglas production function has no upper bound, the second result by Michel and Pestieau (1993) seems to contradict Deadorff's (1976) result. However, this discrepancy comes from the difference in the imposed assumption on the capital depreciation rate. When complete capital depreciation is assumed [as in Michel and Pestieau (1993)], Deadorff's method cannot be applied, and thus the case must be analyzed separately.

3 The limitations of the Serendipity Theorem were also discussed by several studies. Jaeger (1989) introduced an additional generation dependent on its parents and showed that even if an interior solution of the optimal population growth rate exists, the Serendipity Theorem does not hold in general. Kuhle (2007) and Jaeger and Kuhle (2009) showed that the Serendipity Theorem does not hold if there are government bonds.

4 Other related but slightly different lines of study include Abio (2003), Ponthière (2013), and Stelter (2016). The standard literature on the Serendipity Theorem assumes that households regard the population growth rate as exogenous and the government adopts average utility (Millian utilitarianism) as the social objective. Regarding the former, Abio (2003) adopted a version of the endogenous fertility model along the lines of Bental (1989) and Eckstein and Wolpin (1985), and investigated the existence of an interior solution of the socially optimal population growth rate. Concerning the latter, in a model with risky lifetime employing the ex post egalitarian criterion, which considers the welfare of the worst-off born agent, Ponthière (2013) derived the optimal population growth rate, while Stelter (2016) used a very pliable social welfare function covering Millian to Benthamite utilitarianism and examined the first best population growth rate.

5 Galor and Ryder (1989) and Konishi and Perera-Tallo (1997) analyzed sufficient conditions for the existence of a competitive, nontrivial steady-state equilibrium in a standard overlapping generations model with productive capital à la Diamond (1965).

6 The Inada condition ensures this situation. In the case of a CES production function, some parameter restrictions are required.

7 It should be also noted that our notations differ from Michel and Pestieau (1993) in that their τ corresponds to our $1/(1-\sigma)$ and their σ corresponds to our μ.

8 Instead, we can use the log-linear utility function: $\log c_t + \beta \log d_{t+1}$. This can be easily transformed into (25).

9 See Appendix B for the derivation of (29).
10 As easily understood, $\varphi(r)/r$ is the slope of the straight line connecting the origin and a point on the $\varphi(r)$ curve. Thus, in Figure 5, $\check{r}$ is determined by the point at which $\varphi(r)/r$ is minimized (Point E).
11 We can verify the existence of an MGG allocation as follows. Assuming $0 < \tau < 1$, the right-hand side of (34) increases with respect to n and k. As the left-hand side of (34) is constant, we can depict (34) as a downward sloping curve in the k-n plane. Moreover, it is easy to observe that $n \to A\alpha^{-\tau/(1-\tau)}$ as $k \to 0$, and $n \to 0$ as $k \to +\infty$. As $0 < \tau < 1$ and $0 < \mu < 1$, the left-hand side of (35) increases with respect to n and the right-hand side of (35) increases with respect to k. Thus, (35) is depicted as an upward sloping curve in the k-n plane, and $n \to 0$ as $k \to 0$ and $n \to +\infty$ as $k \to +\infty$. Thus, (34) and (35) have a unique intersection point, which determines a unique pair of k_G and n_G, and $0 < n_G < A\alpha^{-\tau/(1-\tau)}$.
12 The steady-state labor income share in the MGG allocation is computed as $(1-\alpha)/(1-\alpha+\alpha k_G^{\sigma})$. Small changes in the value of α have only a negligible effect on the steady-state labor income share. Moreover, α plays an insignificant role in our numerical simulations.
13 See de la Croix and Doepke (2003) and Momota and Horii (2013) for more details.
14 It is well-known that, in overlapping generations models, there are no theoretical restrictions on the size of β, and the estimated values of β are sometimes larger than unity; some examples include Hansen and Singleton (1983), Hotz *et al.* (1988), and Hurd (1989). Furthermore, İmrohoroğlu *et al.* (1995, 1998) and Kumru and Thanopoulos (2015) calibrate their model by assuming that the annual discount factors are greater than unity. Taking this into account, we examine the case of $\beta = 1$ for comparison.
15 Based on estimation of comparative statics results, Juselius (2008) concluded that the elasticity is smaller than 1.
16 Thimme (2017, p. 249) stated, "For models that assume that the representative agent consumes a single nondurable consumption good, it seems difficult to argue against values that are considerably lower and clearly below one."
17 Let us consider the relationship between Figure 9 and Proposition 5. According to Proposition 5, the MGG steady state is unstable when $\sigma < -13/3$ (or equivalently, $\tau < 3/16 \approx 0.18$) when $\beta = 0.3$. Remember that Proposition 5 focuses on the Cobb–Douglas preference case, $\mu = 1$. That is, the point $(\tau, \mu) = (3/16, 1)$ will be on the solid curve in Figure 9 if the values of τ and μ are considered empirically reasonable.

Supplementary material. The supplementary material for this article can be found at https://doi.org/10.1017/dem.2018.21.

Author ORCIDs. Akira Momota, 0000-0002-3363-7417.

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Appendix A

In this appendix, we derive the welfare evaluation formula (10). Differentiating $U(c, d)$ with respect to n gives

$$\frac{dU}{dn} = U_c(c, d)\frac{dc}{dn} + U_d(c, d)\frac{dd}{dn} . \tag{A.1}$$

Substituting (6) into (A.1), we have

$$\frac{dU}{dn} = U_c\left(\frac{dc}{dn} + \frac{1}{r}\frac{dd}{dn}\right). \tag{A.2}$$

From (4), (5), and the equilibrium condition of the economy, (9), we have

$$\frac{dc}{dn} = \frac{d(w-s)}{dn} = \frac{dw}{dn} - k - n\frac{dk}{dn}, \tag{A.3}$$

$$\frac{dd}{dn} = \frac{d(rnk)}{dn} = \frac{dr}{dn}nk + rn\frac{dk}{dn} + rk. \tag{A.4}$$

Thus, substituting (A.3) and (A.4) into (A.2) we obtain

$$\frac{dU}{dn} = U_c\left(\frac{dw}{dn} + \frac{nk}{r}\frac{dr}{dn}\right) = U_c\left(\frac{dw}{dk} + \frac{nk}{r}\frac{dr}{dk}\right)\frac{dk}{dn}. \tag{A.5}$$

Finally, substituting the derivatives of w and r in (3) with respect to k into (A.5), we have the following formula for welfare evaluation:

$$\frac{dU}{dn} = U_c(c, d)(-kf''(k))\frac{1}{r}(r-n))\frac{dk}{dn}.$$

Appendix B

This appendix derives (29) of the main text. As per Figure 3, when $n = \check{n}$, the $h(k_t, n)$ curve is tangent to the 45-degree line. We denote the value of k at the tangent point by $\check{k}$, that is,

$$h_k(\check{k}, \check{n}) = 1, \tag{B.1}$$

where h_k denotes the partial derivative of h with respect to k. As per Figure 3, at the tangent point, the following relation holds:

$$\frac{h(\check{k}, \check{n})}{\check{k}} = h_k(\check{k}, \check{n}).$$

Since $h(k_t,\ \check{n}) = \frac{\beta}{1+\beta}\frac{1}{\check{n}}w(k_t)$, this is equivalent to

$$\frac{w(\check{k})}{\check{k}} = w'(\check{k}),$$

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from which we can solve $\check{k}$ as follows:

$$\check{k} = \left[-\frac{1-\alpha}{\sigma\alpha}\right]^{1/\sigma}.$$

Substituting this into (B.1), we have

$$h_k(\check{k}, \check{n}) = \frac{1}{\check{n}}\frac{\beta}{1+\beta}A[\alpha(1-\sigma)]^{1/\sigma}\frac{-\sigma}{1-\sigma} = 1.$$

From this equation, we obtain

$$\check{n} = \frac{\beta}{1+\beta}A[\alpha(1-\sigma)]^{1/\sigma}\frac{-\sigma}{1-\sigma}.$$

Appendix C

In this appendix, we derive (31). First, we prove the following lemma.

Lemma 2

$$\frac{w(k)}{k} = A^{-\frac{\sigma}{1-\sigma}}\alpha^{-\frac{1}{1-\sigma}}r(k)^{\frac{1}{1-\sigma}} - r(k).$$

Proof: From (27), we have

$$\frac{w(k)}{k} = \frac{f(k)}{k} - r(k). \tag{C.1}$$

Since capital income is represented as

$$\frac{r(k)k}{f(k)} = \frac{\alpha}{(1-\alpha)k^{-\sigma}+\alpha},$$

we have

$$\frac{f(k)}{k} = \frac{r(k)}{\alpha}[(1-\alpha)k^{-\sigma}+\alpha]. \tag{C.2}$$

Using the equality between the interest rate and marginal productivity of capital, we can rewrite (C.2) as

$$\frac{f(k)}{k} = \frac{r(k)}{\alpha}\left(\frac{r(k)}{A\alpha}\right)^{\sigma/(1-\sigma)} = A^{-\sigma/(1-\sigma)}\alpha^{-1/(1-\sigma)}r(k)^{1/(1-\sigma)}. \tag{C.3}$$

Substituting (C.3) into (C.1), we have

$$\frac{w(k)}{k} = A^{-\sigma/(1-\sigma)}\alpha^{-1/(1-\sigma)}r(k)^{1/(1-\sigma)} - r(k).$$

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Let us next consider the steady state of (26), that is,

$$nk = \frac{\beta}{1+\beta} w(k).$$

If $k \neq 0$, this can be expressed as

$$n = \frac{\beta}{1+\beta} \frac{w(k)}{k}. \tag{C.4}$$

Using Lemma 2, we can rewrite (C.4) as follows:

$$n = \frac{\beta}{1+\beta} [A^{-\sigma/(1-\sigma)} \alpha^{-1/(1-\sigma)} r(k)^{1/(1-\sigma)} - r(k)]. \tag{C.5}$$

Since $\sigma < 0$, $\lim_{k \to +\infty} r(k) = 0$, and hence, $r \neq 0$ holds for all $0 < k < +\infty$. Dividing both sides of (C.5) by r and taking their reciprocals, we have

$$\frac{r}{n} = \frac{1+\beta}{\beta} \frac{\alpha}{(r/A\alpha)^{\sigma/(1-\sigma)} - \alpha}.$$

Therefore, we have derived (31) in the main text.

Appendix D

This appendix derives (32) and (33). Under the double CES assumption, (15) and (16) are given by

$$A\alpha[(1-\alpha) + \alpha k_G^{\sigma}]^{(1-\sigma)/\sigma} k_G^{\sigma-1} = n_G, \tag{D.1}$$

and

$$c_G = \beta^{-\mu} n_G^{-\mu} d_G. \tag{D.2}$$

(32) directly follows (D.1). From (17), (18), and (D.2), we have

$$A[(1-\alpha) + \alpha k_G^{\sigma}]^{1/\sigma} = n_G k_G [\beta^{-\mu} \cdot n_G^{1-\mu} + 2]. \tag{D.3}$$

Combining (D.1) and (D.3) gives (33).

Appendix E

Lemma 1 is proved here. We first solve k_G and n_G for $\tau = \mu = 0.5$ ($\sigma = -1$). In this case, (34) and (35) are given by

$$A\alpha = n_G[(1-\alpha)k_G + \alpha]^2, \text{ and} \tag{E.1}$$

$$\alpha[\beta^{-1/2} n_G^{1/2} + 1] = (1-\alpha)k_G. \tag{E.2}$$

Deleting k_G from (E.1) and (E.2), we obtain

$$\frac{A}{\alpha} = [\beta^{-1/2} n_G + 2n_G^{1/2}]^2 \,. \tag{E.3}$$

Noting that $\beta^{-1/2}n + 2n^{1/2} > 0$ and defining $z \equiv n^{1/2}$, (E.3) is transformed into the following quadratic equation of z:

$$\beta^{-1/2} z^2 + 2z = \left(\frac{A}{\alpha}\right)^{1/2}.$$

Since z must be positive, the solution to this equation is given by

$$z = n_G^{1/2} = \frac{\sqrt{1 + (A/\alpha\beta)^{1/2}} - 1}{\beta^{-1/2}} \,, \tag{E.4}$$

indicating that

$$n_G = \beta\left(\sqrt{1 + \left(\frac{A}{\alpha\beta}\right)^{1/2}} - 1\right)^2 . \tag{E.5}$$

Moreover, from (E.2) and (E.4), k_G is obtained as

$$k_G = \frac{\alpha}{1-\alpha}\sqrt{1 + \left(\frac{A}{\alpha\beta}\right)^{1/2}} \,. \tag{E.6}$$

We next derive the dynamics of the laissez-faire equilibrium for the case of $\tau = \mu = 0.5$ and $n = n_G$. From (36) and (37), we have

$$H(k_t,\ k_{t+1}) \equiv A(1-\alpha)[(1-\alpha) + \alpha k_t^{-1}]^{-2} - n_G k_{t+1}\left\{\left(\frac{A\alpha}{\beta}\right)^{1/2}[(1-\alpha)k_{t+1} + \alpha]^{-1} + 1\right\} = 0 \,. \tag{E.7}$$

Partially differentiating (E.7) with respect to k_t, we have

$$\begin{aligned}\frac{\partial H(k_G,\ k_G)}{\partial k_t} &= \frac{2A(1-\alpha)\alpha}{[(1-\alpha) + \alpha k_G^{-1}]^3 \cdot k_G^2} = \frac{2A(1-\alpha)\alpha}{[(1-\alpha) + \alpha k_G^{-1}]^2 \cdot k_G^2} \cdot \frac{1}{(1-\alpha) + \alpha k_G^{-1}} \\ &= \frac{2A(1-\alpha)\alpha}{[(1-\alpha)k_G + \alpha]^2} \cdot \frac{1}{(1-\alpha) + \alpha k_G^{-1}} \,.\end{aligned}$$

Substituting (E.1) into the first term of the right-hand side of the above equation yields

$$\frac{\partial H(k_G,\ k_G)}{\partial k_t} = 2(1-\alpha)n_G \cdot \frac{1}{(1-\alpha) + \alpha k_G^{-1}} = 2(1-\alpha)n_G \cdot \frac{k_G}{(1-\alpha)k_G + \alpha} \tag{E.8}$$

As (E.1) and (E.2) can be rewritten as

$$\left(\frac{A\alpha}{n_G}\right)^{1/2} = (1-\alpha)k_G + \alpha, \tag{E.9}$$

and

$$\left(\frac{n_G}{\beta}\right)^{1/2} = \frac{(1-\alpha)k_G - \alpha}{\alpha}, \tag{E.10}$$

respectively, we obtain the following equation by combining (E.9) and (E.10):

$$\left(\frac{A\alpha}{\beta}\right)^{1/2} = \frac{[(1-\alpha)k_G + \alpha] \cdot [(1-\alpha)k_G - \alpha]}{\alpha}. \tag{E.11}$$

Partially differentiating (E.7) with respect to k_{t+1}, we obtain

$$\begin{aligned}\frac{\partial H(k_G,\ k_G)}{\partial k_{t+1}} &= -\,n_G\left\{\left(\frac{A\alpha}{\beta}\right)^{1/2}\frac{1}{(1-\alpha)k_G+\alpha} + 1 - k_G\left(\frac{A\alpha}{\beta}\right)^{1/2}(1-\alpha)\frac{1}{((1-\alpha)k_G+\alpha)^2}\right\}\\ &= -\,n_G\left\{1 + \left(\frac{A\alpha}{\beta}\right)^{1/2}\frac{\alpha}{((1-\alpha)k_G+\alpha)^2}\right\}.\end{aligned}$$

Substituting (E.11) into the right-hand side of the above equation, we have

$$\frac{\partial H(k_G,\ k_G)}{\partial k_{t+1}} = -n_G\left(1 + \frac{(1-\alpha)k_G - \alpha}{(1-\alpha)k_G + \alpha}\right) = -n_G\frac{2(1-\alpha)k_G}{(1-\alpha)k_G + \alpha}. \tag{E.12}$$

Finally, substituting (E.8) and (E.12) into (38), we obtain

$$\left.\frac{\mathrm{d}k_{t+1}}{\mathrm{d}k_t}\right|_{k_t=k_{t+1}=k_G} = -\frac{\partial H(k_G,\ k_G)}{\partial k_t}\Big/\frac{\partial H(k_G,\ k_G)}{\partial k_{t+1}} = 1. \tag{E.13}$$

Thus, $\mathrm{d}k_{t+1}/\mathrm{d}k_t|_{k_t=k_{t+1}=k_G} = 1$, regardless of the values of A, α, and β.

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