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ON SKEWNESS AND DISPERSION AMONG CONVOLUTIONS OF INDEPENDENT GAMMA RANDOM VARIABLES

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Let $\{x_{(1)} \leq \cdots \leq x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$. A vector $\mathbf{x} \in \boldsymbol{R}^n$ majorizes another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$. A vector $\mathbf{x} \in \boldsymbol{R}^{+n}$ majorizes reciprocally another vector $\mathbf{y} \in \boldsymbol{R}^{+n}$ (written $\mathbf{x} \overset{rm}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j}(1/x_{(i)}) \geq \sum_{i=1}^{j}(1/y_{(i)})$ for $j = 1, \ldots, n$. Let $X_{\lambda_i,\alpha}$, $i = 1, \ldots, n$, be n independent random variables such that $X_{\lambda_i,\alpha}$ is a gamma random variable with shape parameter $\alpha \geq 1$ and scale parameter λ_i, $i = 1, \ldots, n$. We show that if $\boldsymbol{\lambda} \overset{rm}{\succeq} \boldsymbol{\lambda}^*$, then $\sum_{i=1}^{n} X_{\lambda_i,\alpha}$ is greater than $\sum_{i=1}^{n} X_{\lambda_i^*,\alpha}$ according to right spread order as well as mean residual life order. We also prove that if $(1/\lambda_1, \ldots, 1/\lambda_n) \overset{m}{\succeq} (1/\lambda_1^*, \ldots, 1/\lambda_n^*)$, then $\sum_{i=1}^{n} X_{\lambda_i,\alpha}$ is greater than $\sum_{i=1}^{n} X_{\lambda_i^*,\alpha}$ according to new better than used in expectation order as well as Lorenze order. These results mainly generalize the recent results of Kochar and Xu [7] and Zhao and Balakrishnan [14] from convolutions of independent exponential random variables to convolutions of independent gamma random variables with common shape parameters greater than or equal to 1.

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1. INTRODUCTION

Gamma random variable with scale parameter $\lambda > 0$, shape parameter $\alpha > 0$ and density function

$$f_{\lambda,\alpha}(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \qquad x > 0,$$

denoted by $X_{\lambda,\alpha}$, is one of the most used random variables in probability and statistics to model various stochastic phenomena; for instance, in life testing, the waiting time until death a unit is a random variable that is frequently modeled with a gamma distribution. On the other hand, convolution of independent random variables is of practical importance in various fields of probability and statistics. In particular, in reliability theory, the time to failure of an *standby* system consists of n components is a convolution of lifetimes of the components (cf. Barlow and Proschan [1]). As another example, let X_i denote the random value of the ith shock on a system, then if the convolutions of a number of X_is exceed the threshold of the system, then the system fails (cf. Marshall and Olkin [11]). Therefore, the study of the lifetime of a standby system or a cumulative damage threshold model is based on stochastic properties of convolutions of random variables. In this article we concentrate only on convolutions of independent gamma random variables differing in their scale parameters and occur frequently in probability and statistics, and we prove that a system with a lifetime equivalent to convolutions of independent gamma random variables differing in their scale parameters age faster and more dispersed in some sense if the vector of scale parameters satisfy certain restrictions.

The notions of dispersive order and right spread order have been introduced to compare the dispersion of two probability distributions.

X is said to be less dispersed than Y—denoted by $X \leq_{\text{disp}} Y$—if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha) \quad \text{for all } 0 < \alpha \leq \beta < 1.$$

A weaker order—called right spread order—has also been proposed to compare the variability of two distributions. X is said to be a less right spread than Y—denoted by $X \leq_{\text{RS}} Y$—if

$$\int_{F^{-1}(p)}^{\infty} \overline{F}(x)\,dx \leq \int_{G^{-1}(p)}^{\infty} \overline{G}(x)\,dx, \quad \text{for all } 0 \leq p \leq 1.$$

This order is equivalent to the *excess wealth* order in economics (cf. Kochar, Li, and Shaked [6]). In insurance, it is related to *stop-loss* order evaluated at a level of probability p (cf. Denuit, Dhaene, Goovaerts, and Kaas [2, pp. 149–182]).

It is known that

$$X \leq_{\text{disp}} Y \Longrightarrow X \leq_{\text{RS}} Y \Longrightarrow \text{Var}(X) \leq \text{Var}(Y).$$

The *convex transform order* [or more increasing failure rate (more IFR)], *star order* [or more increasing failure rate in average (more IFRA)], more new better than used in expectation order (more NBUE), and more decreasing mean residual life order (more DMRL) are some of the orders that have been proposed to compare the relative aging properties or skewness between two random variables (cf. Shaked and Shanhikumar [13] and Marshall and Olkin, [11]).

Let X and Y be two nonnegative random variables with distribution functions F and G, respectively, and denote their right continuous inverse functions by F^{-1} and G^{-1}, respectively. Then

(a) X is smaller than Y in the *convex transform order* if $G^{-1}F(x)$ is convex in x.

(b) X is smaller than Y in the *star transform order* if $G^{-1}F(x)$ is star-shaped $(X \le_* Y)$ in x, that is, $G^{-1}F(x)/x$ is increasing in x.

(c) X is smaller than Y in DMRL order if

$$\frac{(1/EY)\int_{G^{-1}(u)}^{\infty}\overline{G}(x)\,dx}{(1/EX)\int_{F^{-1}(u)}^{\infty}\overline{F}(x)\,dx} \quad \text{is increasing in } u \in [0,1].$$

(d) X is smaller than Y in the NBUE order—denoted by $X \le_{\text{NBUE}} Y$—if

$$\frac{1}{EX}\int_{F^{-1}(u)}^{\infty}\overline{F}(x)\,dx \le \frac{1}{EY}\int_{G^{-1}(u)}^{\infty}\overline{G}(x)\,dx \quad \text{for all } u \in [0,1].$$

It is well known that the above partial orderings imply that the random variables X ages faster than random variables Y. In particular, if we replace the function G with the exponential distribution, then the convex order implies that X is IFR, star order implies that X is IFRA, DMRL order implies that X is DMRL, and NBUE order implies that X is NBUE—that is, lifetimes with IFR, IFRA, DMRL and NBUE distributions age relatively faster than lifetimes with exponential distributions. For more details on this topic, the reader is referred to Balrow and Proschan [1], Marshall and Olkin [11], and Shaked and Shanthikumar [13].

When $E(X) = E(Y)$, it is seen that the RS order is equivalent to the NBUE. For more relations between the RS order and the above aging orderings, please refer to Ferandez-Ponce, Kochar and Muñoz-Perez [4] and Kochar et al. [6].

The notion of majorization is one of the tools that is useful for deriving various inequalities in statistics and probability.

Let $\{x_{(1)} \le \cdots \le x_{(n)}\}$ denote the increasing arrangement of the components of a vector $\mathbf{x} = (x_1, \ldots, x_n)$.

A vector $\mathbf{x}$ majorizes another vector $\mathbf{y}$ (written $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j} x_{(i)} \le \sum_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

A vector $\mathbf{x} \in \boldsymbol{R}^{+n}$ is p-larger than another vector $\mathbf{y} \in \boldsymbol{R}^{+n}$ (written $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if $\prod_{i=1}^{j} x_{(i)} \le \prod_{i=1}^{j} y_{(i)}$ for $j = 1, \ldots, n$.

A vector $\mathbf{x} \in \boldsymbol{R}^{+n}$ majorizes reciprocally another vector $\mathbf{y} \in \boldsymbol{R}^{+n}$ (written $\mathbf{x} \overset{rm}{\succeq} \mathbf{y}$) if $\sum_{i=1}^{j}(1/x_{(i)}) \ge \sum_{i=1}^{j}(1/y_{(i)})$ for $j = 1, \ldots, n$

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It is known that (cf. Kochar and Xu [7]) for any $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^{+n}$,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{rm}{\succeq} \mathbf{y}.$$

A vector $\mathbf{x} \in \boldsymbol{R}^{+n}$ weakly majorizes another vector $\mathbf{y} \in \boldsymbol{R}^{+n}$ (written $\mathbf{x} \succeq_w \mathbf{y}$) if $\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)}$, for $j = 1, \ldots, n$.

It is easy to see that for $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^{+n}$,

$$(1/x_1, \ldots, 1/x_n) \succeq_w (1/y_1, \ldots, 1/y_n) \Leftrightarrow \mathbf{x} \overset{rm}{\succeq} \mathbf{y}. \qquad \textbf{(1.1)}$$

Let $A \subset \boldsymbol{R}^{+n}$. A function $\phi : A \to \boldsymbol{R}$ is said to be Schur convex if it preserves the majorization ordering. We use the following theorem in the next section.

THEOREM 1.1: (Marshall and Olkin [10, p. 59]): *A real-valued function ϕ on the set $A \subset \boldsymbol{R}^n$ satisfies*

$$\boldsymbol{x} \succeq_w \boldsymbol{y} \text{ on } A \Rightarrow \phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$$

if and only if ϕ is Schur convex and increasing on A.

Marshall and Olkin [10] provided extensive and comprehensive details on the theory of majorization and its applications in statistics.

Let $X_{\lambda_i,\alpha}$, $i = 1, \ldots, n$, be n independent random variables such that $X_{\lambda_i,\alpha}$ are gamma random variables with shape parameter $\alpha \geq 1$ and scale parameter λ_i, $i = 1, \ldots, n$. Stochastic comparisons of convolutions of independent gamma random random variables with respect to the likelihood ratio order (cf. Shaked and Shanthikumar, [13, p. 42]) are studied in Korwar [6]. He proved that $\sum_{i=1}^{n} X_{\lambda_i,\alpha}$ is larger according to the likelihood ratio order if the vector $(\lambda_1, \ldots, \lambda_n)$ is more dispersed with respect to majorization. Khaledi and Kochar [5], further studied the above problem and proved that $\sum_{i=1}^{n} X_{\lambda_i,\alpha}$ is larger according to the hazard rate ordering (cf. Shaked and Shanthikumar [13, p. 16]) and dispersive ordering if the vector $(\lambda_1, \ldots, \lambda_n)$ is larger with respect to the p-larger order. Recently, Zhao and Balakrishnan [14] and Kochar and Xu [7], respectively, proved that if

$$(\lambda_1, \ldots, \lambda_n) \overset{rm}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Rightarrow \sum_{i=1}^{n} X_{\lambda_i,1} \geq_{\nabla} \sum_{i=1}^{n} X_{\lambda_i^*,1}, \qquad \textbf{(1.2)}$$

where ∇ order stands for RS order and mean residual life (mrl) order.

Kochar and Xu [7] also proved that

$$\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \overset{m}{\succeq} \left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_n^*}\right) \Rightarrow \sum_{i=1}^{n} X_{\lambda_i,1} \geq_{\triangle} \sum_{i=1}^{n} X_{\lambda_i^*,1}, \qquad \textbf{(1.3)}$$

where $\triangle$ order stands for NBUE and Lorenz order. For more details of the Lorenz order the reader is referred to Shaked and Shanthikumar [13, Sect. 3.A].

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In Section 2 we consider two independent random variables $X_{\lambda_1,\alpha}$ and $X_{\lambda_2,\alpha}$ and we show that for $\alpha \geq 1$, their convolutions are larger according to star ordering if the vector (λ_1, λ_2) as well as $(1/\lambda_1, 1/\lambda_2)$ are more dispersed with respect to majorization (Theorems 2.2). Then we generalize (1.2) and (1.3) from convolutions of independent exponential random variables to convolutions of independent gamma random variables $X_{\lambda_1,\alpha}, \ldots, X_{\lambda_n,\alpha}$ with $\alpha \geq 1$ (Theorem 2.5, Theorem 2.6, and Corollary 2.2).

2. MAIN RESULTS

We use the following results to prove the main results in this section.

THEOREM 2.1 (Lorch, [9]): *For each fixed $\beta > 0$, $\nu > -1$, and $\nu > -\beta/2$, the positive function $I_{\nu+\beta}(x)/I_\nu(x)$, $0 < x < \infty$, is increasing in x and* $\lim_{x\to\infty}(I_{\nu+\beta}(x)/I_\nu(x)) = 1$, *where $I_a(x)$ is a modified Bessel function with parameter a.*

LEMMA 2.1 (Saunders and Moran, [12]): *Let $\{F_a | a \in R\}$ be a class of distribution function, such that F_a is supported on some interval $(x_-^{(a)}, x_+^{(a)}) \subseteq (0, \infty)$ and has a density f_a that does not vanish on any subinterval of $(x_-^{(a)}, x_+^{(a)})$. Then*

$$X_{a^*} \leq_* X_a, \qquad a^* \leq a,$$

if and only if

$$\frac{F_a'(x)}{xf_a(x)} \quad \text{is decreasing in } x,$$

where F_a' is the derivative of F_a with respect to a.

To prove Theorem 2.4 we need to prove Theorem 2.2, which is of independent interest.

THEOREM 2.2: *Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_1^*}$, and $X_{\lambda_2^*}$ be independent gamma random variables with shape parameters $a \geq 1$ and scale parameters $\lambda_1, \lambda_2, \lambda_1^*$, and λ_2^*, respectively. Then for $\lambda_1 \neq \lambda_2 \neq \lambda_1^* \neq \lambda_2^*$,*

$$\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right) \overset{m}{\succeq} \left(\frac{1}{\lambda_1^*}, \frac{1}{\lambda_2^*}\right) \Longrightarrow X_{\lambda_1} + X_{\lambda_2} \geq_* X_{\lambda_1^*} + X_{\lambda_2^*}. \tag{2.1}$$

PROOF: Without loss of generality, assume that $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$. Let $1/\lambda_1 + 1/\lambda_2 = d$. Then the majorization assumption implies that (2.1) is equivalent to

$$0 < \frac{1}{\lambda} < \frac{1}{\lambda^*} < \frac{d}{2} \Longrightarrow X_\lambda + X_{\lambda/(\lambda d-1)} \geq_* X_{\lambda^*} + X_{\lambda^*/(\lambda^* d-1)}. \tag{2.2}$$

Let $f(y, a, \lambda)$ and $F(y, a, \lambda)$ denote the density function and the distribution function of $X_\lambda + X_{\lambda/(\lambda d-1)}$, respectively. From (3.1) of Korwar [8] and the above setting, $f(y, a, \lambda)$

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can be written as

$$f(y,a,\lambda) = \frac{\sqrt{\pi}\lambda^{a+1/2}y^{a-1/2}}{\Gamma(a)(\lambda d-2)^{a-1/2}\sqrt{\lambda d-1}}e^{-(\lambda+1/(d-1/\lambda))y/2}I_{a-1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right), \tag{2.3}$$

where

$$I_{a-1/2}(y) = \frac{2(y/2)^{a-1/2}}{\sqrt{\pi}\Gamma(a)}\int_0^1 (1-t^2)^{a-1}\cosh(ty)\,dt$$

is a modified Bessel function of the first kind. To prove the required result, we use Lemma 2.1. Using parts (i) and (ii) of Lemma A.1 in the Appendix and (2.3), we obtain that

$$\begin{aligned}
\frac{\partial F(y,a,\lambda)/\partial\lambda}{yf(y,a,\lambda)} &= \frac{a(d-2/\lambda)}{\lambda^2}\frac{f'(y,a+1,\lambda)}{yf(y,a,\lambda)} \\
&= \frac{a(d-2/\lambda)}{\lambda^2}\left[\frac{\lambda^2}{2a(\lambda d-1)} - \frac{\lambda^3 d}{2a(\lambda d-2)(\lambda d-1)}\right. \\
&\quad \left.\times \frac{I_{a+1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}{I_{a-1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}\right] \\
&= \frac{1}{2(\lambda d-1)}\left[(d-2/\lambda) - d\frac{I_{a+1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}{I_{a-1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}\right] \\
&= \frac{-1}{\lambda(\lambda d-1)} + \frac{d}{2(\lambda d-1)}\left[1 - \frac{I_{a+1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}{I_{a-1/2}([(\lambda d-2)/2(d-1/\lambda)]y)}\right].
\end{aligned}$$

Since $1/\lambda \in (0, d/2)$ implies that $(\lambda d-1) > 0$, the required result follows from Theorem 2.1. ■

THEOREM 2.3: *Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_1^*}$, and $X_{\lambda_2^*}$ be independent gamma random variables with shape parameters $a \geq 1$ and scale parameters $\lambda_1, \lambda_2, \lambda_1^*$ and λ_2^*, respectively. Then, for $\lambda_1 \neq \lambda_2 \neq \lambda_1^* \neq \lambda_2^*$,*

$$(\lambda_1,\lambda_2) \overset{m}{\succeq} (\lambda_1^*,\lambda_2^*) \Longrightarrow X_{\lambda_1}+X_{\lambda_2} \geq_* X_{\lambda_1^*}+X_{\lambda_2^*}. \tag{2.4}$$

PROOF: Without loss of generality, let $\lambda_1 > \lambda_2$ and $\lambda_1^* > \lambda_2^*$. Let $\lambda_1+\lambda_2 = d$. Then, by Lemma 2.1, (2.4) is equivalent to

$$h(\lambda) = \frac{\partial G(y,a,\lambda)/\partial\lambda}{yg(y,a,\lambda)} \qquad \text{is decreasing in } y \text{ for } \lambda \in (d/2, d), \tag{2.5}$$

where $g(y,a,\lambda)$ and $G(y,a,\lambda)$ are the density function and distribution functions of $X_\lambda + X_{d-\lambda}$, respectively.

Korwar [8] showed that for the above setting, the density function $g(y, a, \lambda)$ can be written as

$$g(y, a, \lambda) = \frac{\sqrt{\pi}(\lambda(d-\lambda))^a}{\Gamma(a)(y/(2\lambda - c))^{a-1/2} e^{-dy/2} I_{a-1/2}((\lambda - d/2)y)}. \qquad \textbf{(2.6)}$$

He also showed that

$$\frac{\partial}{\partial\lambda} G(y, a, \lambda) = \frac{d - 2\lambda}{2a\lambda^2(d-\lambda)^2}\{dag(y, a+1, \lambda) + \lambda(d-\lambda)yg(y, a, \lambda)\}.$$

Now, using these observations in (2.5) and the fact that $\lambda > d/2$, we obtain that

$$h(\lambda) = \frac{(d - 2\lambda)}{(2a\lambda(d-\lambda)) - d/(2a\lambda(d-\lambda))} \frac{I_{a+1/2}((\lambda - d/2)y)}{I_{a-1/2}((\lambda - d/2)y)}$$

is decreasing in y by Theorem 2.1. This proves the required result. ■

Remark 2.1: We conjecture that the results of Theorems 2.2 and 2.3 hold for $n > 2$ and other cases of λ_is and λ_i^*s

Now, we are ready to compare convolutions of independent gamma random variables with respect to right spread ordering.

THEOREM 2.4: *Let $X_{\lambda_1}, X_{\lambda_2}, X_{\lambda_1^*}$ and $X_{\lambda_2^*}$ be independent gamma random variables with common shape parameters $a \geq 1$ and scale parameters $\lambda_1, \lambda_2, \lambda_1^*$, and λ_2^*, respectively. Then*

$$(\lambda_1, \lambda_2) \overset{rm}{\succeq} (\lambda_1^*, \lambda_2^*) \Rightarrow X_{\lambda_1} + X_{\lambda_2} \geq_{\text{RS}} X_{\lambda_1^*} + X_{\lambda_2^*}.$$

PROOF: Without loss of generality we assume that $\lambda_1 \geq \lambda_2$ and $\lambda_1^* \geq \lambda_2^*$.

Case (a) $\lambda_1 > \lambda_2$, $\lambda_1^* > \lambda_2^*$ and $\lambda_i \neq \lambda_j^*$, $i = 1, 2$, $j = 1, 2$. Using (1.1) and Theorem 1.1, the required result follows if we show that

(i) $(1/\lambda_1, 1/\lambda_2) \overset{m}{\succeq} (1/\lambda_1^*, 1/\lambda_2^*) \Rightarrow X_{\lambda_1} + X_{\lambda_2} \geq_{RS} X_{\lambda_1^*} + X_{\lambda_2^*}$ and

(ii) $X_{\lambda_1} + X_{\lambda_2}$ is increasing in $(1/\lambda_1, 1/\lambda_2)$ with respect to RS order.

Since star order implies NBUE order, and NBUE order is equivalent to the right spread order with the same means, part (i) for this case follows from Theorem 2.2.

For $\lambda_1 \leq \lambda'$, $X_{\lambda_1} \geq_{\text{disp}} X_{\lambda'}$ which, in turn, implies $X_{\lambda_1} \geq_{\text{RS}} X_{\lambda'}$. Since X_{λ_2} has log-concave density, it follows from Theorem 3.C.7 of Shaked and Shanthikumar [13] that

$$X_{\lambda_1} + X_{\lambda_2} \geq_{\text{RS}} X_{\lambda'} + X_{\lambda_2},$$

which proves part (ii).

Case (b) $\lambda_1 > \lambda_2$, $\lambda_1 = \lambda_1^*$ and $\lambda_2 \neq \lambda_2^*$. For this case *rm* ordering implies that $1/\lambda_2 > 1/\lambda_2^*$, from which we have $X_{\lambda_2} \geq_{\mathrm{RS}} X_{\lambda_2^*}$. Now, again the required result follows from Theorem 3.C.7 of Shaked and Shanthikumar [13].

Case (c) $\lambda_1 > \lambda_2$, $\lambda_1 = \lambda_2^*$ and $\lambda_2 \neq \lambda_1^*$. The proof follows from similar kind of arguments used to prove case (b).

Case (d) $\lambda_1 > \lambda_2$ and $\lambda_1^* = \lambda_2^*$. If $\lambda_1 = \lambda_1^*$, then *rm* ordering implies that $1/\lambda_2 > 1/\lambda_2^*$. The required result follows from Theorem 3.C.7 of Shaked and Shanthikumar [13], since $X_{\lambda_2} \geq_{\mathrm{RS}} X_{\lambda_2^*}$ and $X_{\lambda_1} =_{\mathrm{st}} X_{\lambda_1^*}$.

If $\lambda_1 \neq \lambda_1^*$, from rm ordering, we have that $1/\lambda_2 \geq 1/\lambda_2^* = 1/\lambda_1*$ and $1/\lambda_1 + 1/\lambda_2 \geq 1/\lambda_1^* + 1/\lambda_2^* = 2/\lambda_1^*$, that is $\lambda_1^* \geq \lambda_H$, where λ_H is the harmonic mean of λ_1 and λ_2. If $\lambda_1^* = \lambda_H$, then it is easy to see that for integer $m \geq 1$,

$$(\lambda_1, \lambda_2) \overset{rm}{\succeq} (\lambda_H, \lambda_H + 1/m).$$

Using case (a), we obtain

$$X_{\lambda_1} + X_{\lambda_2} \geq_{\mathrm{RS}} X_{\lambda_H} + X_{\lambda_{H+1/m}}.$$

Now, it follows by limiting arguments that

$$X_{\lambda_1} + X_{\lambda_2} \geq_{\mathrm{RS}} Y_1 + Y_2,$$

where Y_1, Y_2 is a random sample of size 2 from gamma distribution with scale parameter λ_H and shape parameter $a \geq 1$.

If $\lambda_1^* > \lambda_H$, then $\lambda_2^* > \lambda_H$. These observations imply that

$$Y_1 + Y_2 \geq_{\mathrm{RS}} X_{\lambda_1^*} + X_{\lambda_2^*}.$$

Now, the required result follows from this and the case when $\lambda_1^* = \lambda_H$. ■

The following theorem extends the result of Theorem 2.4 from $n = 2$ to the case when $n > 2$.

THEOREM 2.5: *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then*

$$(\lambda_1, \ldots, \lambda_n) \overset{rm}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{RS} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

PROOF: A gamma random variable with shape parameter $a \geq 1$ has a log-concave density function, and convolution of independent random variables with log-concave densities has log-concave density (cf. Dharmadhiakri and Joag-Dev [3, p. 17]). Then the required result follows using the same kind of arguments used by Zhao and Balakrishnan [14] to prove their Theorem 4.1. ■

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Remark 2.2: Theorem 2.5 generalized Corollary 3.9 of Kochar and Xu [7] from convolutions of independent Erlang distributions to convolutions of gamma distributions with common shape parameters $a \geq 1$.

The following result is a generalization of Theorem 4.1 of Zhao and Balakrishnan [14] and Corollary 3.8 in Kochar and Xu [7] from convolutions of independent exponential distributions to convolutions of gamma distributions with common shape parameters $a \geq 1$.

THEOREM 2.6: *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then*

$$(\lambda_1, \ldots, \lambda_n) \overset{rm}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{\mathrm{mrl}} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

PROOF: Using the fact that convolutions of independent gamma random variables with common shape parameters $a \geq 1$ is DMRL and Theorem 2.5, the required result follows from Theorem 3.C.5 in Shaked and Shanthikumar [13]. ■

COROLLARY 2.1: *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$ and let $Y_1, \ldots, Y_n$ be a random sample from a gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_H, where λ_H is harmonic mean of λ_is. Then*

$$(\lambda_1, \ldots, \lambda_n) \overset{rm}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{\mathrm{mrl}} \sum_{i=1}^{n} Y_i.$$

This corollary provides a computable lower bound on the mrl function of convolutions of gamma random variables, which is sharper than those that can be obtained from Theorem 3.4 of Korwar [8] in terms of arithmetic mean and from Corollary 2.2 of Khaledi and Kochar [5] in terms of the geometric mean of λ_is. To justify these observation, in Figures 1 and 2 we plot the mean residual life functions of convolutions of two independent gamma random variables with bound given in terms of the arithmetic mean, geometric mean, and harmonic mean of λ_is. In Figure 1, we plot the mean residual functions for $\lambda_1 = 3.6$ and $\lambda_2 = 0.4$.

We also plot the mean residual life functions of convolutions of independent gamma random variables for different sets of λ_is:

$$(2, 6) \overset{rm}{\succeq} (5.2, 2.4) \overset{rm}{\succeq} (3, 6) \overset{rm}{\succeq} (4, 4),$$

which shows how *rm* ordering between λ_is will affect the mean residual life function of convolutions of gamma random variables.

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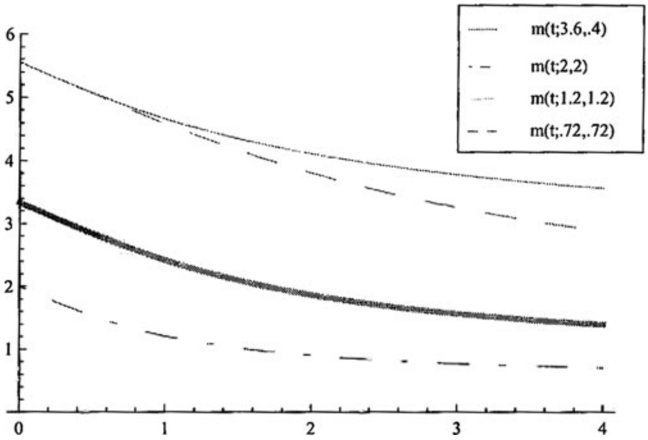

FIGURE 1. Mean residual function of convolutions of gamma random variables.

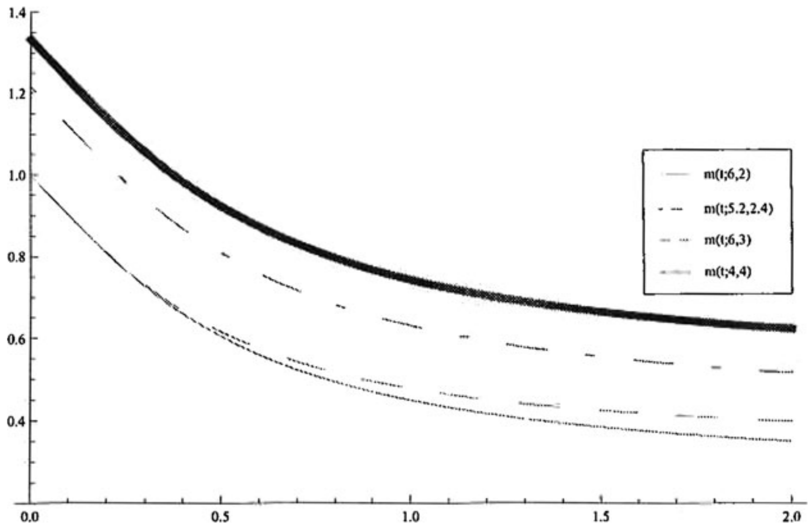

FIGURE 2. Mean residual function of convolutions of gamma random variables.

The following results also generalize Corollaries 3.4, 3.5, 3.6, and 3.7 of Kochar and Xu [7] from convolutions of exponential distributions to convolutions of gamma distributions with common shape parameters $a \geq 1$. The proofs is similar to those used in Kochar and Xu [7] to prove the corresponding results for convolutions of exponential distributions and hence are omitted.

COROLLARY 2.2: *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has a gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i,*

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for $i = 1, \ldots, n$. Then

$$\left(\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}\right) \overset{m}{\succeq} \left(\frac{1}{\lambda_1^*}, \ldots, \frac{1}{\lambda_n^*}\right) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i^*} \geq_{\triangle} \sum_{i=1}^{n} X_{\lambda_i},$$

where $\triangle$ order stands for NBUE and Lorenz order.

COROLLARY 2.3: *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has a gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$, and let $Y_1, \ldots, Y_n$ be independent gamma random variables with shape parameters $a \geq 1$ and the same scale parameter λ. Then*

(i) $\sum_{i=1}^n X_{\lambda_i} \geq_{\text{NBUE}} \sum_{i=1}^n Y_i$ *and*
(ii) $\sum_{i=1}^n X_{\lambda_i} \geq_{\text{RS}} \sum_{i=1}^n Y_i \Leftrightarrow E\left(\sum_{i=1}^n X_{\lambda_i}\right) \geq E\left(\sum_{i=1}^n Y_i\right)$.

Remark 2.3: For interesting applications of these results in reliability theory, economics, and actuarial science, we refer the reader to Kochar and Xu [7] and Zhao and Balakrishnan [14].

Remark 2.4: We recently became aware of the manuscript by Kochar and Xu [15] entitled "The tail behavior of convolutions of Gamma random variables". This paper includes a number of results that are similar to ours. While the two papers were written independently without knowledge of the existence of the other work, we wish to acknowledge the overlap in the two papers. We note that the motivation and methods of proof of the results that are common of these papers differ.

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APPENDIX

LEMMA A.1: *Let $f(y,a,\lambda)$ and $F(y,a,\lambda)$ denote the density function and the distribution function of $X_\lambda + X_{\lambda/(\lambda d-1)}$, respectively. Then*

(i)

$$\frac{\partial}{\partial\lambda}F(y,a,\lambda) = \frac{a(d-2/\lambda)}{\lambda^2}f'(y,a+1,\lambda),$$

where $f'(y,a,\lambda) = \partial f(y,a,\lambda)/\partial y$ and

(ii)

$$f'(y,a+1,\lambda) = -\frac{\lambda^2 d}{2(\lambda d-1)}f(y,a+1,\lambda) + \frac{\lambda^2 y}{2a(\lambda d-1)}f(y,a,\lambda).$$

PROOF: Using recurrence formula

$$I'_v(z) = I_{v+1}(z) + (v/z)I_v(z),$$

we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\lambda}f(x,a,\lambda)\\
&= \left\{\frac{(a+1/2)}{\lambda}f(x,a,\lambda) - \frac{d(a-1/2)}{(\lambda d-2)}f(x,a,\lambda) - \frac{d}{2(\lambda d-1)}f(x,a,\lambda)\right.\\
&\quad \left. - \frac{x}{2}\left(1-\frac{1}{(\lambda d-1)^2}\right)f(x,a,\lambda)\right\} + \frac{\sqrt{\pi}\lambda^{a+1/2}x^{a-1/2}}{\Gamma(a)(\lambda d-2)^{a-1/2}\sqrt{\lambda d-1}}e^{-(\lambda+1/(d-1/\lambda))x/2}\\
&\left\{\frac{x[(\lambda d-1)^2+1]}{2(\lambda d-1)^2}\right\}\left\{I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}x\right) + \frac{(a-1/2)}{[(\lambda d-2)/2(d-1/\lambda)]x}I_{a-1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}x\right)\right\}\\
&= \left\{\frac{(a+1/2)}{\lambda}f(x,a,\lambda) - \frac{d(a-1/2)}{(\lambda d-2)}f(x,a,\lambda) - \frac{d}{2(\lambda d-1)}f(x,a,\lambda)\right.\\
&\quad \left. - \frac{x}{2}\left(1-\frac{1}{(\lambda d-1)^2}\right)f(x,a,\lambda)\right\} + \frac{a[(\lambda d-1)^2+1]}{2(\lambda d-1)^2}\frac{(\lambda d-2)}{\lambda}f(x,a+1,\lambda)\\
&\quad + \frac{(a-1/2)[(\lambda d-1)^2+1]}{2(\lambda d-1)^2[(\lambda d-2)/2(d-1/\lambda)]}f(x,a,\lambda)
\end{aligned}$$

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$$
\begin{aligned}
&= f(x,a,\lambda)\left\{\frac{(a+1/2)}{\lambda} - \frac{d(a-1/2)}{(\lambda d-2)} - \frac{d}{2(\lambda d-1)} - \frac{x}{2}\left(1-\frac{1}{(\lambda d-1)^2}\right)\right.\\
&\quad\left. + \frac{(a-1/2)[(\lambda d-1)^2+1]}{(\lambda d-1)(\lambda^2 d-2\lambda)}\right\} + \frac{a[(\lambda d-1)^2+1]}{2(\lambda d-1)^2}\frac{(\lambda d-2)}{\lambda}f(x,a+1,\lambda)\\
&= f(x,a,\lambda)\left\{\frac{(a+1/2)}{\lambda} - \left[\frac{2d(a-1/2)(\lambda d-1)+d(\lambda d-2)}{2(\lambda d-1)(\lambda d-2)}\right] - \frac{x}{2}\left(1-\frac{1}{(\lambda d-1)^2}\right)\right.\\
&\quad\left. + \frac{(a-1/2)[(\lambda d-1)^2+1]}{\lambda(\lambda d-1)(\lambda d-2)}\right\} + \frac{a[(\lambda d-1)^2+1]}{2(\lambda d-1)^2}\frac{(\lambda d-2)}{\lambda}f(x,a+1,\lambda)\\
&= f(x,a,\lambda)\left\{\frac{a(\lambda d-2)}{\lambda(\lambda d-1)} + \frac{x}{2}\left(\frac{1}{(\lambda d-1)^2}-1\right)\right\} + f(x,a+1,\lambda)\frac{a[(\lambda d-1)^2+1]}{2(\lambda d-1)^2}\frac{(\lambda d-2)}{\lambda}\\
&= \frac{a(\lambda d-2)}{\lambda(\lambda d-1)}f(x,a,\lambda) + \frac{x}{2}\left(\frac{\lambda d(2-\lambda d)}{(\lambda d-1)^2}\right)f(x,a,\lambda)\\
&\quad + \frac{a(\lambda d-2)}{2\lambda}f(x,a+1,\lambda) + \frac{a(\lambda d-2)}{2\lambda(\lambda d-1)^2}f(x,a+1,\lambda)\\
&= a(d-2/\lambda)\left[\frac{f(x,a,\lambda)}{(\lambda d-1)} + \frac{f(x,a+1,\lambda)}{2}\right] + \frac{(d-2/\lambda)}{2(d-1/\lambda)^2}\left[\frac{a}{\lambda^2}f(x,a+1,\lambda) - xdf(x,a,\lambda)\right];
\end{aligned}
$$

that is,

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}f(x,a,\lambda) &= a(d-2/\lambda)\left[\frac{f(x,a,\lambda)}{(\lambda d-1)} + \frac{f(x,a+1,\lambda)}{2}\right]\\
&\quad + \frac{(d-2/\lambda)}{2(d-1/\lambda)^2}\left[\frac{a}{\lambda^2}f(x,a+1,\lambda) - xdf(x,a,\lambda)\right]. \qquad \textbf{(A.1)}
\end{aligned}
$$

Let $L(g)$ denote the Laplace transform of arbitrary function g; then it is easy to see that

$$
\begin{cases}
L(f(x,a,\lambda)) = \dfrac{\lambda^a}{(s+\lambda)^a}\dfrac{(1/(d-1/\lambda)^a}{(s+1/(d-1/\lambda))^a} = \left[\dfrac{\lambda^2}{(s+\lambda)(s(\lambda d-1)+\lambda)}\right]^a\\
L(f(x,a+1,\lambda)) = \left[\dfrac{\lambda^2}{(s+\lambda)(s(\lambda d-1)+\lambda)}\right]^{a+1}\\
L(xf(x,a,\lambda)) = -\dfrac{d}{ds}L(f(x,a,\lambda)) = \dfrac{a\lambda^{2a}[(\lambda d-1)(2s+\lambda)+\lambda]}{[(s+\lambda)(s(\lambda d-1)+\lambda)]^{a+1}}\\
\qquad = L(f(x,a+1,\lambda))\left\{\dfrac{2as(\lambda d-1)}{\lambda^2}+ad\right\}.
\end{cases}
$$

Taking Laplace transforms of both sides of (A.1) and using the above relations, we get

$$
\begin{aligned}
&L\left(\frac{\partial}{\partial\lambda}f(x,a,\lambda)\right)\\
&\quad = \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{2}L(f(x,a+1,\lambda)) + \frac{(d-2/\lambda)}{2(d-1/\lambda)^2}\\
&\qquad \times\left[\frac{a}{\lambda^2}L(f(x,a+1,\lambda)) - dL(f(x,a+1,\lambda))\left\{\frac{2as(\lambda d-1)}{\lambda^2}+ad\right\}\right]
\end{aligned}
$$

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$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{2}L(f(x,a+1,\lambda))$$
$$+ \frac{a(d-2/\lambda)}{2(\lambda d-1)^2}L(f(x,a+1,\lambda))\{1-\lambda^2 d^2 - 2sd(\lambda d-1)\}$$
$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{2(\lambda d-1)^2}L(f(x,a+1,\lambda))$$
$$\times \{(\lambda d-1)^2 + 1 - \lambda^2 d^2 - 2sd(\lambda d-1)\}$$
$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{2(\lambda d-1)^2}L(f(x,a+1,\lambda))$$
$$\times \{(\lambda d-1)[(\lambda d-1)-(\lambda d+1)-2sd]\}$$
$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{2(\lambda d-1)^2}L(f(x,a+1,\lambda))$$
$$\times \{(\lambda d-1)(-2-2sd)\}$$
$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a,\lambda)) + \frac{a(d-2/\lambda)}{(\lambda d-1)}L(f(x,a+1,\lambda))\{(-1-sd)\}$$
$$= \frac{a(d-2/\lambda)}{(\lambda d-1)}\{L(f(x,a,\lambda)) - (1+sd)L(f(x,a+1,\lambda))\}.$$

Now, using

$$L(f(x,a,\lambda)) = \frac{(s+\lambda)(s(\lambda d-1)+\lambda)}{\lambda^2}L(f(x,a+1,\lambda))$$

in the last equality, we obtain

$$\{L(f(x,a,\lambda)) - (1+sd)L(f(x,a+1,\lambda))\} = \frac{s^2(\lambda d-1)}{\lambda^2}L(f(x,a+1,\lambda));$$

that is,

$$L\left(\frac{\partial}{\partial\lambda}f(x,a,\lambda)\right) = \frac{s^2 a(d-2/\lambda)}{\lambda^2}L(f(x,a+1,\lambda))$$

and

$$L\left(\frac{\partial}{\partial\lambda}F(x,a,\lambda)\right) = L\left(\int_0^y \frac{\partial}{\partial\lambda}f(x,a,\lambda)\,dx\right)$$
$$= \frac{1}{s}L\left(\frac{\partial}{\partial\lambda}f(x,a,\lambda)\right)$$
$$= \frac{sa(d-2/\lambda)}{\lambda^2}L(f(x,a+1,\lambda))$$
$$= \frac{a(d-2/\lambda)}{\lambda^2}L(f'(y,a+1,\lambda)). \quad \textbf{(A.2)}$$

The last equality follows from

$$L(f'(t)) = sL(f(t)) - f(0).$$

This proves part (i).

We compute the $f'(y, a+1, \lambda)$ as follows:

$$f(y, a+1, \lambda) = \frac{\sqrt{\pi}\lambda^{a+3/2}y^{a+1/2}}{\Gamma(a+1)(\lambda d-2)^{a+1/2}\sqrt{\lambda d-1}} e^{-(\lambda+1/(d-1/\lambda))(y/2)} I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right).$$

Then

$$\begin{aligned} f'(y, a+1, \lambda) &= \frac{(a+1/2)}{y} f(y, a+1, \lambda) - \frac{1}{2}\left(\lambda + \frac{1}{d-1/\lambda}\right) f(y, a+1, \lambda) \\ &+ \frac{\sqrt{\pi}\lambda^{a+3/2}y^{a+1/2}}{\Gamma(a+1)(\lambda d-2)^{a+1/2}\sqrt{\lambda d-1}} e^{-(\lambda+1/(d-1/\lambda))(y/2)} \left\{\frac{(\lambda d-2)}{2(d-1/\lambda)}\right\} \\ &\times \left\{ I_{a+3/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right) + \frac{(a+1/2)}{[(\lambda d-2)/2(d-1/\lambda)]y} I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right)\right\}. \end{aligned}$$

Using the recurrence formula

$$I_{v-1}(x) = I_{v+1}(x) + \frac{2v}{x} I_v(x)$$

and letting $v = a + 1/2$, we can replace

$$\left\{ I_{a+3/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right) + \frac{(a+1/2)}{[(\lambda d-2)/2(d-1/\lambda)]y} I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right)\right\}$$

with

$$\left\{ I_{a-1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right) - \frac{(a+1/2)}{[(\lambda d-2)/2(d-1/\lambda)]y} I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right)\right\}.$$

Hence,

$$\begin{aligned} f'(y, a+1, \lambda) &= \frac{(a+1/2)}{y} f(y, a+1, \lambda) - \frac{1}{2}\left(\lambda + \frac{1}{d-1/\lambda}\right) f(y, a+1, \lambda) \\ &+ \frac{\sqrt{\pi}\lambda^{a+3/2}y^{a+1/2}}{\Gamma(a+1)(\lambda d-2)^{a+1/2}\sqrt{\lambda d-1}} e^{-(\lambda+(1/(d-1/\lambda))/(y/2)} \frac{(\lambda d-2)}{2(d-1/\lambda)} \\ &\times \left\{ I_{a-1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right) - \frac{(a+1/2)}{[(\lambda d-2)/2(d-1/\lambda)]y} I_{a+1/2}\left(\frac{\lambda d-2}{2(d-1/\lambda)}y\right)\right\}. \end{aligned}$$

After simplifications, we get

$$\begin{aligned} f'(y, a+1, \lambda) &= \frac{(a+1/2)}{y} f(y, a+1, \lambda) - \frac{1}{2}\left(\lambda + \frac{1}{d-1/\lambda}\right) f(y, a+1, \lambda) \\ &+ \frac{(\lambda d-2)}{2(d-1/\lambda)} \frac{\lambda y}{a(\lambda d-2)} f(y, a, \lambda) - \frac{(a+1/2)}{y} f(y, a+1, \lambda). \end{aligned}$$

We can write

$$f'(y, a+1, \lambda) = -\frac{\lambda^2 d}{2(\lambda d-1)} f(y, a+1, \lambda) + \frac{\lambda^2 y}{2a(\lambda d-1)} f(y, a, \lambda).$$

This completes the required result. ■