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Mode limitation and mode completion in collisionless plasmas

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The relativistically correct solution of the dispersion relation of linear plasma waves in an isotropic unmagnetized equilibrium electron plasma leads to two new effects unknown from the nonrelativistic dispersion theory. First, the number of damped subluminal modes is limited to a few (mode-limitation effect); secondly, for relativistic plasma temperatures the few individual modes complement each other in the sense that the dispersion relations $\omega_R = \omega_R(k)$ continuously match each other (mode-completion effect). The second effect does not occur at nonrelativistic temperatures.

1. Introduction

More than 50 years ago, Landau (1946) discovered the collective effect of collisionless damping of linear plasma waves in kinetic unmagnetized plasmas. Using the linearized Poisson equation for the electric field fluctuations and the linearized nonrelativistic Vlasov equation for the particle's phase-space density fluctuation, he calculated finite damping rates for the principal branch whose real part describes the usual longitudinal Langmuir waves. A more thorough analysis (Jackson 1960; Fried and Gould 1961) of this system of plasma equations indicated the coexistence of an infinite number of heavily damped modes. While the existence of the finite damping of the principal Langmuir branch has been demonstrated in a series of beautiful experiments (Malmberg and Wharton 1965), the theoretically predicted heavily damped higher modes have never been observed experimentally, nor has an experiment to detect them been proposed. Here we demonstrate that the prediction of the coexistence of an infinite number of heavily damped modes results from the use of the nonrelativistic form of the Vlasov equation in earlier work, and that this result is altered significantly if the relativistically correct Vlasov equation is used.

The relativistic kinetic theory of linear plasma wave modes in isotropic unmagnetized plasmas of arbitrary composition and arbitrary energy distribution function has recently been formulated (Schlickeiser and Kneller 1997). It allows the study of the time evolution of initially small fluctuations perturbing the assumed initial equilibrium state. Specializing to the textbook example of longitudinal fluctuations

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in a pure-electron equilibrium plasma with the normalized distribution

$$F(E) = \frac{\mu}{4\pi(m_e c)^3 K_2(\mu)} e^{-\mu E}, \tag{1}$$

where $\mu = m_e c^2/k_B T_e$ characterizes the plasma temperature and $K_2(z)$ denotes the modified Bessel function of second order, the dispersion relation for subluminal longitudinal waves takes the form (see Appendix A)

$$\Lambda^-(z) = 1 + \frac{\mu}{\kappa^2}\left[1 + \frac{z}{2}\frac{e^{-\mu}}{K_2(\mu)}\int_0^\infty du\, e^{-u}\left(1+\frac{u}{\mu}\right)^2 \right.$$
$$\times \ln \frac{z - \left[\left(\frac{2u}{\mu}+\frac{u^2}{\mu^2}\right)\Big/\left(1+\frac{u}{\mu}\right)^2\right]^{1/2}}{z + \left[\left(\frac{2u}{\mu}+\frac{u^2}{\mu^2}\right)\Big/\left(1+\frac{u}{\mu}\right)^2\right]^{1/2}}$$
$$\left. + i\pi z \frac{E_c^2 e^{-\mu E_c}}{K_2(\mu)}\left(1+\frac{2}{\mu E_c}+\frac{2}{\mu^2 E_c^2}\right)\right]. \tag{2}$$

$\kappa = kc/\omega_{p,e}$ denotes the normalized wavenumber, $z = 1/N = \omega/kc$ is the inverse index of refraction and $E_c = (1-z^2)^{-1/2}$.

We show in the following that the relativistic dispersion relation (2) implies two remarkable new effects: the effects of plasma mode limitation and plasma mode completion. Neither effect occurs in traditional non-relativistic plasma dispersion theory.

2. Analysis of the dispersion relation

The dispersion relation (2) is most conveniently solved by introducing the complex variable

$$x \equiv q + is = z\left[\frac{\mu}{2(1-z^2)}\right]^{1/2} \tag{3}$$

and separating the dispersion relation (2) into real and imaginary parts

$$\Re\Lambda^-(x) = 0, \tag{4a}$$
$$\Im\Lambda^-(x) = 0. \tag{4b}$$

In the nonrelativistic dispersion theory (Roos 1969) Picard's theorem implies for given wavenumber values $k \neq 0$ the existence of an infinite number of simple subluminal roots of the dispersion relation which are located in the fourth quadrant in the complex x plane ($q \geqslant 0,\ s < 0$), and represent damped subluminal modes.

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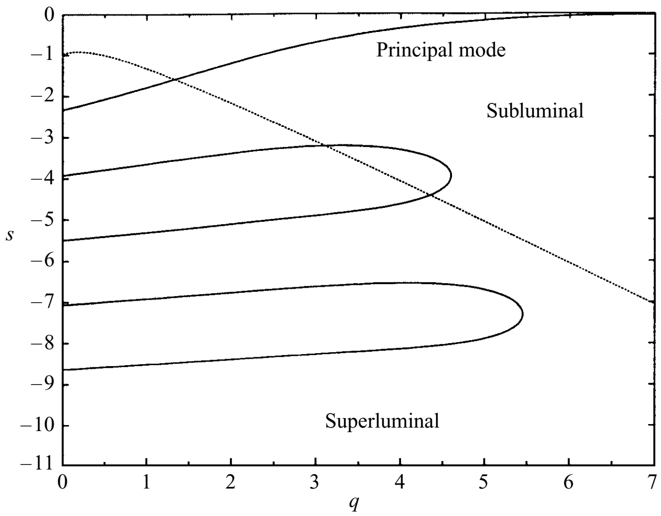

Figure 1. Locus in the complex $x\,(= q + is)$ plane of solutions of the subluminal dispersion relation in an electron equilibrium plasma. The curves are calculated numerically from the imaginary part (4*b*) of the dispersion relation (2) for an electron temperature value $\mu = m_e c^2/k_B T_e = 2$. The line $s_M(q)$ separates the complex x plane into regions with subluminal and superluminal solutions as indicated.

In general, the solution of (4*b*) defines the loci of points in the x plane, shown in Fig. 1, fixing the relation between q_l and s_l $(l = 0, 1, 2, \ldots)$ for any value of k^2. At any of these loci one can then compute the wavenumber values from (4*a*):

$$\Re[\Lambda^-(q_l + is_l)] = 0. \tag{5}$$

Only if $k^2 \geqslant 0$ do we have a bona fide solution of the dispersion relation, yielding with (3) in the form

$$\omega^2 = \frac{2x^2}{\mu + 2x^2}\,k^2c^2 \tag{6}$$

the variations of the real, $\omega_{R,l} = \omega_{R,l}(k)$, and imaginary, $\Gamma_l = \Gamma_l(k)$, parts for the different plasma modes after specifying the plasma temperature value μ.

3. Mode limitation

The dispersion relation (2) holds for subluminal waves with

$$\Re z \leqslant 1, \tag{7}$$

i.e. waves with phase speed less than the speed of light. The condition (7) does not arise in nonrelativistic kinetic plasma theories which formally correspond to the limit of an infinitely large speed of light $c \to \infty$ (Schlickeiser and Kneller 1997), so that (7) is automatically fulfilled since all waves are subluminal in this case.

Introducing also $z = r + iw$, the limiting condition for subluminal waves $(r = 1)$ according to (3) yields the two relations

$$\frac{w^2(3 + w^2)}{(1 + w^2)^2} = \frac{\mu}{2}\,\frac{s^2 - q^2}{(s^2 + q^2)^2}, \tag{8a}$$

$$\frac{w}{(1 + w^2)^2} = \frac{\mu q s}{2(s^2 + q^2)^2}. \tag{8b}$$

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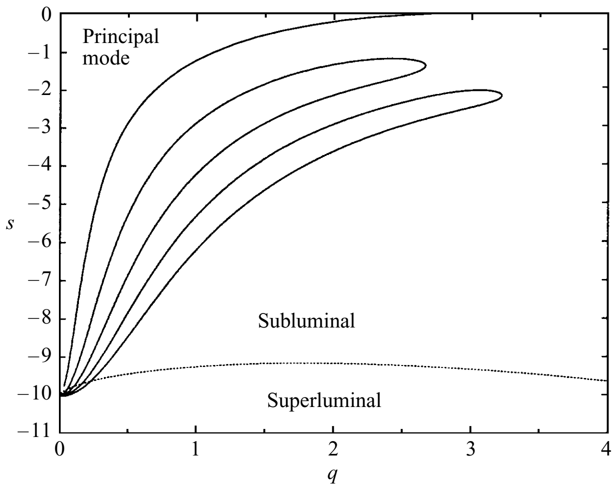

Figure 2. Locus in the complex $x\,(= q + is)$ plane of solutions of the subluminal dispersion relation in an electron equilibrium plasma for a nonrelativistic electron temperature value $\mu = m_e c^2/k_B T_e = 200$. Only the first three modes are calculated.

Since the left-hand side of (8a) is nonnegative for all values of w, we deduce for any value of μ that the limiting line ($r = 1$) is located in regions with

$$s^2 \geqslant q^2 \tag{9}$$

in the complex x plane.

Taking the ratio of (8a) and (8b) leads to the cubic equation

$$w^3 + 3w + \frac{q^2 - s^2}{qs} = 0, \tag{10}$$

which has one real solution in the fourth quadrant:

$$w = [(1 + Q^2)^{1/2} - Q]^{1/3} - [(1 + Q^2)^{1/2} + Q]^{1/3} \leqslant 0, \tag{11}$$

with

$$Q = \frac{q^2 - s^2}{2sq} \geqslant 0 \tag{12}$$

since $q \geqslant 0,\ s < 0$ and from (9). Inserting (11) into (8a) or (8b) fixes the limiting relation for subluminal solutions in terms of the function $s_M = s_M(q) < 0$ in the complex x plane.

Evidently only plasma mode solutions with

$$s_M(q) \leqslant s \leqslant 0 \tag{13}$$

are acceptable subluminal roots, limiting the number of modes to a finite value. This is illustrated in Fig. 1 for a temperature value $\mu = 2$ and in Fig. 2 for a temperature value $\mu = 200$. While at $\mu = 2$ only parts of the first two modes fulfil the condition (13), mode limitation at the nonrelativistic temperature $\mu = 200$ sets in at considerably higher modes.

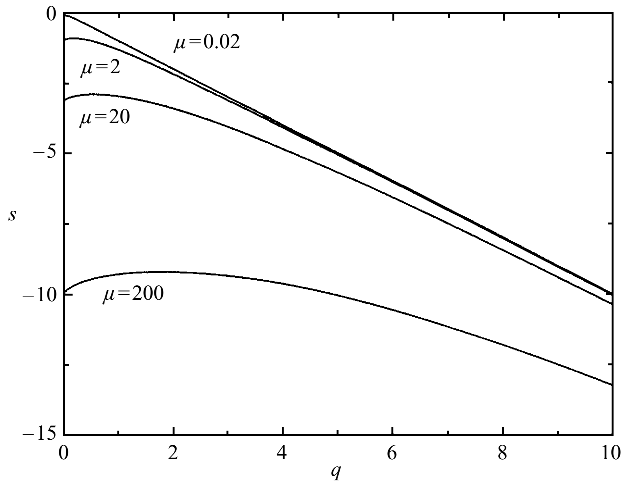

Figure 3. Limiting line $s_M(q)$ for four electron temperature values μ in the complex $x = (q+is)$ plane. The analytic approximation (14) is in good agreement with the numerically calculated curves.

In Appendix B we demonstrate that

$$s_M(q) \approx \begin{cases} -\dfrac{q_o}{\sqrt{3}} + \dfrac{3\sqrt{3}}{2}\dfrac{q^2}{q_0} & \text{for } 0 \leqslant q \leqslant q_1 = 0.17q_0, \\ -(q_0 q)^{1/2} & \text{for } q_1 \leqslant q \leqslant q_0, \\ -q - \dfrac{q_0^2}{8q}, & \text{for } q \geqslant q_0, \end{cases} \tag{14}$$

with

$$q_0 = \left(\tfrac{3}{2}\mu\right)^{1/2} = 1.225\mu^{1/2}. \tag{15}$$

The approximation (14) is in accord with the exact curves of $s_M(q)$ shown in Fig. 3 for different values of μ.

4. Mode completion

If we solve (5) for the two modes of Fig. 1 ($\mu = 2$) that are consistent with subluminal phase speeds, for the wavenumber values, and calculate the dispersion relation according to (6), we obtain the variation of the real parts of the frequency $\omega_R = \omega_R(k)$ and the damping rates $\Gamma = \Gamma(k)$ shown in Fig. 4. One notices that the frequency–wavenumber relations continuously complement each other, while the damping rates exhibit a sharp discontinuity at the joining wavenumber $k_j \approx 1.2\,\omega_{pe}/c$. At all wavenumber values the damping rates are much smaller than the real parts of the dispersion relation, so that the whole branch represents a weakly damped solution. The disjoint classification in the nonrelativistic theory of the low-frequency part as the principal weakly damped mode, and the high-frequency part as one of the heavily damped higher modes, must be an artefact resulting from using the nonrelativistic Vlasov equation.

The same phenomenon of mode completion occurs at the more relativistic plasma temperature $\mu = 0.02$ shown in Fig. 5. Again the two dispersion relations join

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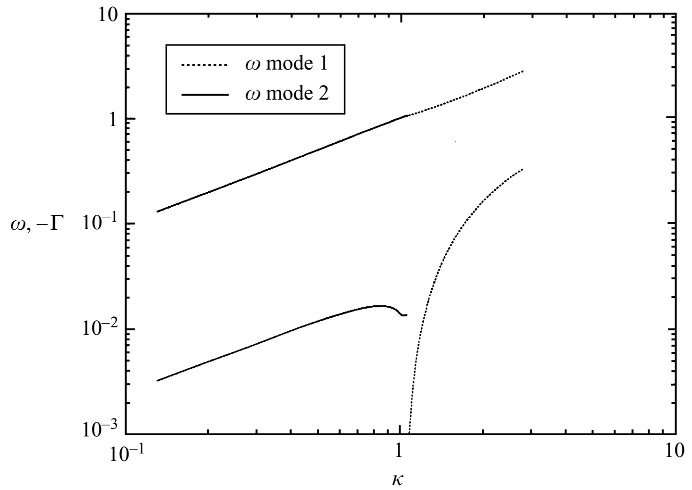

Figure 4. Mode completion in a mildly relativistic electron plasma with $\mu = 2$. The upper curve shows the variation of the real part of the frequency $\omega_R(\kappa)$ with wavenumber for the two acceptable modes (see also Fig. 1), whereas the lower curve shows the corresponding damping rates. Note the continuity of $\omega_R(\kappa)$ at the matching wavenumber.

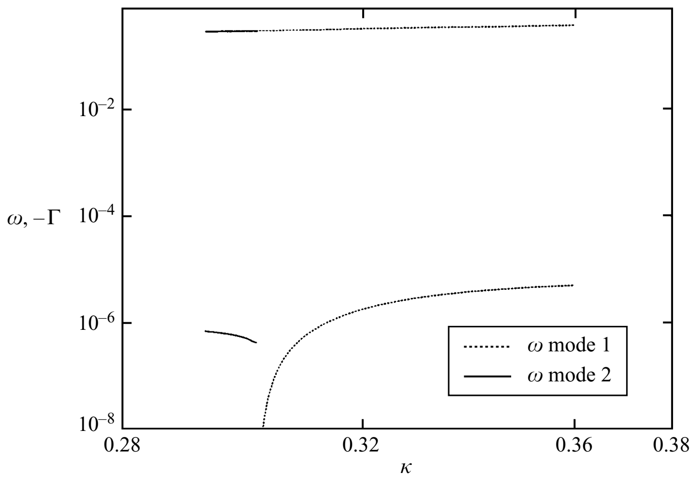

Figure 5. Mode completion in the ultrarelativistic electron plasma $\mu = 0.02$. The upper curve shows the variation of the real part of the frequency $\omega_R(\kappa)$ with wavenumber for the two acceptable modes, whereas the lower curve shows the corresponding damping rates. Note the continuity of $\omega_R(\kappa)$ at the matching wavenumber.

smoothly and complement each other, and are weakly damped over the whole wavenumber range.

Mode completion is a relativistic effect that does not occur at very nonrelativistic plasma temperatures (see Figs 2, 5 and 6 in Schlickeiser and Kneller 1997), where the individual modes do not join together into a single dispersion relation.

5. Summary and conclusions

We have discovered two new effects resulting from the relativistic solution of the dispersion relation for linear plasma waves in an isotropic unmagnetized equilib-

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rium plasma. The requirement of subluminal phase speeds of the waves leads to the limitation of the number of damped subluminal modes to a few as compared with the infinite number of modes found in nonrelativistic dispersion theories. Moreover, at relativistic plasma temperatures the dispersion relations for these modes continuously complement each other leading basically to a single weakly damped dispersion relation $\omega_R = \omega_R(k)$ over a broader wavenumber range than covered by the nonrelativistic principal mode. It will be of great interest to verify these new effects experimentally.

Appendix A. The longitudinal dispersion relation (2)

To determine the dielectric properties of a plasma, we determine the currents and charges induced by an electric field. In an unmagnetized plasma we start (see e.g. Bekefi 1966) from the linearized form of the Vlasov equation

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} = -q_a \mathbf{E} \cdot \frac{\partial F_a}{\partial \mathbf{p}}, \tag{A 1}$$

where $f_1(\mathbf{x}, \mathbf{p}, t)$ is the small perturbation of $F_a(\mathbf{p})$ caused by the RF field $\mathbf{E}$. $p = \gamma m_a v$ with $\gamma = [1 - (v/c)^2]^{-1/2}$ relates momentum and velocity of plasma particles of type a with charge q_a. All DC electric and magnetic fields are assumed to be zero.

The simplest way to solve (A 1) is to substitute the Fourier–Laplace transforms

$$f_1(\mathbf{x}, \mathbf{p}, t) = \bar{f}_1(\mathbf{k}, \mathbf{p}, \omega) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t) \tag{A 2a}$$

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_1(\mathbf{k}, \omega) \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t) \tag{A 2b}$$

into (A 1), yielding

$$\bar{f}_1 = \frac{-iq_a\, \mathbf{E}_1 \cdot \partial F_a/\partial \mathbf{p}}{\omega - \mathbf{k} \cdot \mathbf{v}}. \tag{A 3}$$

As usual, a positive imaginary part of the complex frequency ($\Gamma = \Im(\omega) > 0$) in (A 2) is assumed.

We also apply the substitutions (A 2) to the linearized Poisson equation

$$\nabla \cdot \mathbf{E} = 4\pi \sum_a q_a n_a \int d^3p\, f_1(\mathbf{x}, \mathbf{p}, t),$$

yielding, after inserting (A 3),

$$\mathbf{k} \cdot \mathbf{E}_1 + 4\pi \sum_a q_a^2 n_a \int d^3p\, \frac{\mathbf{E}_1 \cdot \partial F_a/\partial \mathbf{p}}{\omega - \mathbf{k} \cdot \mathbf{v}} = 0. \tag{A 4}$$

Writing (A 4) in the form (Bekefi 1966)

$$\Lambda(k, \omega)\, \mathbf{k} \cdot \mathbf{E}_1(\mathbf{k}, \omega) = 0, \tag{A 5}$$

we obtain for the longitudinal dispersion relation

$$\Lambda^+(k, \omega) = 1 + \sum_a \frac{\omega_{p,a}^2 m_a}{k^2} \int d^3p\, \frac{\mathbf{k} \cdot \partial F_a/\partial \mathbf{p}}{\omega - \mathbf{k} \cdot \mathbf{v}} = 0, \tag{A 6}$$

where $\omega_{p,a} = (4\pi q_a^2 n_a/m_a)^{1/2}$ denotes the plasma frequency of species a. The superscript + indicates that (A 6) holds in the positive ($\Gamma > 0$) complex-frequency plane. Introducing the momentum variables $y = p_\parallel/m_a c$ and $E = \gamma =$

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$[1+(p_\parallel^2+p_\perp^2)/m_a^2c^2]^{1/2}$ in terms of the cylindrical momentum variables $(p_\parallel, p_\perp, \phi)$, with $p_\parallel$ parallel to $\mathbf{E}_1$, reduces (A 6) to

$$\Lambda^+ = 1 + 2\pi \sum_a \frac{\omega_{p,a}^2 m_a}{k} \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp \frac{p_\perp \partial F_a/\partial p_\parallel}{\omega - kp_\parallel/m_a\gamma} = 0. \tag{A 7}$$

For isotropic plasmas $F_a(p_\parallel, p_\perp) = F_a(p)$ with $p = (p_\parallel^2 + p_\perp^2)^{1/2}$, (A 7) simplifies to

$$\Lambda^+ = 1 - \frac{2\pi}{k^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^\infty dE\, E\, \frac{\partial F_a}{\partial E} \int_{-(E^2-1)^{1/2}}^{(E^2-1)^{1/2}} dy \frac{y}{y - Ez}, \tag{A 8}$$

with the inverse index of refraction $z \equiv \omega/kc = 1/N$.

For superluminal waves ($\Re z > 1$) the value of the y integral in (A 8) is identical at positive and negative values of the imaginary frequency part Γ since the pole Ez lies outside the integration interval $[-(E^2-1)^{1/2}, (E^2-1)^{1/2}]$. However, for subluminal waves ($\Re z \leqslant 1$) we have to properly analytically continue the dispersion relation Λ from positive to negative values of Γ, provided the pole Ez lies inside the integration interval $[-(E^2-1)^{1/2}, (E^2-1)^{1/2}]$, which is equivalent to $E > E_c = (1-z^2)^{-1/2}$. The two integrals

$$\lim_{\Im z \to 0^+} \int_{-(E^2-1)^{1/2}}^{(E^2-1)^{1/2}} dy \frac{y}{y - Ez} = \lim_{\Im z \to 0^-} \int_{-(E^2-1)^{1/2}}^{(E^2-1)^{1/2}} dy \frac{y}{y - Ez} + 2\pi i E z \tag{A 9}$$

then differ by $2\pi i$ times the residue at the pole Ez, since the real frequency plane is approached from above and below, respectively. With

$$\int_{-(E^2-1)^{1/2}}^{(E^2-1)^{1/2}} dy \frac{y}{y - Ez} = 2(E^2-1)^{1/2} + Ez \ln \frac{Ez - (E^2-1)^{1/2}}{Ez + (E^2-1)^{1/2}},$$

the correct analytical continuation in the negative complex-frequency plane for subluminal waves is therefore

$$\begin{aligned}\Lambda^- = 1 &- \frac{4\pi}{k^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^\infty dE\, E(E^2-1)^{1/2} \frac{\partial F_a}{\partial E} \\ &- \frac{2\pi z}{k^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_1^\infty dE\, E^2 \frac{\partial F_a}{\partial E} \ln \frac{Ez - (E^2-1)^{1/2}}{Ez + (E^2-1)^{1/2}} \\ &- i\frac{4\pi^2 z}{k^2c^2} \sum_a \omega_{p,a}^2 (m_a c)^3 \int_{E_c}^\infty dE\, E^2 \frac{\partial F_a}{\partial E}.\end{aligned} \tag{A 10}$$

Adopting the electron equilibrium distribution (1), the first and third E integrals in (A 10) can be solved, and with the substitution $u = \mu(E-1)$, (A 10) is identical to (2) of the main text.

Appendix B. The function $s_M(q)$

The condition $Q = 1$ is equivalent to

$$s = s_1 \equiv -(1+\sqrt{2})q. \tag{B 1}$$

The line (B1) divides the complex (q, s) plane into the two regions: (a) $s_1 \leqslant s \leqslant -q$, where Q has values ($0 \leqslant Q \leqslant 1$) smaller than unity, and (b) $s \leqslant s_1$, where $Q \geqslant 1$ (see Fig. 3).

In region (a) we use the asymptotic behaviour of (11) and (8b) for small Q and w respectively, yielding $w(Q \ll 1) \approx -\frac{2}{3}Q \ll 1$ so that $|w| \ll 1$, implying for (8b), after rearrangement,

$$\frac{(s^2 - q^2)(s^2 + q^2)^2}{s^2 q^2} \approx \frac{3\mu}{2}. \tag{B 2}$$

Equation (9) suggests the ansatz

$$s^2 = Aq^2, \qquad A \geqslant 1, \tag{B 3}$$

so that (B 2) becomes

$$\frac{(A-1)(A+1)^2}{A} \approx \frac{3\mu}{2q^2} = \left(\frac{q_0}{q}\right)^2 \tag{B 4}$$

with $q_0 = (\frac{3}{2}\mu)^{1/2}$. For values of $q \geqslant q_0$ we approximate the left-hand side of (B 4) by $4(A-1)$, implying

$$A(q \geqslant q_0) \approx 1 + \left(\frac{q_0}{2q}\right)^2 ,$$

and, with (B 3),

$$s_M(q \geqslant q_0) \approx -q\left[1 + \left(\frac{q_0}{2q}\right)^2\right]^{1/2} \approx -q - \frac{q_0^2}{8q}. \tag{B 5}$$

For values of $q \leqslant q_0$ we approximate the left-hand side of (B 4) by A^2, implying

$$A(q \leqslant q_0) \approx \frac{q_0}{q}$$

and

$$s_M(q \leqslant q_0) \approx -(q_0 q)^{1/2}. \tag{B 6}$$

Since we are still in region (a) we have to require that $s_M(q) \geqslant s_1$ according to (B 1) which is fulfilled for (B 5) for all values of q, and for (B 6) only for values of $q \geqslant q_1$, where

$$q_1 = \frac{q_0}{3 + 2\sqrt{2}} = 0.17 q_0. \tag{B 7}$$

We therefore rewrite (B 6) as

$$s_M(q_1 \leqslant q \leqslant q_0) \approx -(q_0 q)^{1/2}. \tag{B 8}$$

For values of $q \leqslant q_1$ we are in region (b), where we use the asymptotic behaviour of (11) and (8b) for large Q and w respectively, i.e. $w(Q \gg 1) \approx -(2Q)^{1/3}$, yielding for (8$b$), after rearrangement,

$$\frac{(s^2 + q^2)^2}{s^2 - q^2} \approx \frac{\mu}{2}. \tag{B 9}$$

For $q \to 0$, (B 9) gives $s_M(q \to 0) = (\frac{1}{2}\mu)^{1/2} = q_0/\sqrt{3}$. The ansatz (B 3) reduces (B 9) to

$$\frac{(A+1)^2}{A-1} \approx \frac{\mu}{2q^2} = \frac{1}{3}\left(\frac{q_0}{q}\right)^2. \tag{B 10}$$

Since $q \leqslant q_1 = q_0/(3 + 2\sqrt{2})^{1/2}$, the right-hand side of (B 10) is greater than (17 +

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$12\sqrt{2})/3 \gg 1$. Equation (B 10) is then solved for large values of $A \gg 1$ by

$$A + 3 \approx \frac{1}{3}\left(\frac{q_0}{q}\right)^2, \tag{B 11}$$

since the second possibility $A \approx 1 + \epsilon$ with $\epsilon \ll 1$ leads to contradictions with the requirement that $s \leqslant s_1$ according to (B 1). Using (B 11) in (B 3) gives

$$s_M(0 \leqslant q \leqslant q_1) \approx -\frac{q_0}{\sqrt{3}}\left(1 - \frac{9q^2}{q_0^2}\right)^{1/2} \approx -\frac{q_0}{\sqrt{3}} + \frac{3\sqrt{3}}{2}\frac{q^2}{q_0}. \tag{B 12}$$

Collecting terms, we derive

$$s_M(q) \simeq \begin{cases} -\dfrac{q_0}{\sqrt{3}} + \dfrac{3\sqrt{3}}{2}\dfrac{q^2}{q_0} & \text{for } 0 \leqslant q \leqslant q_1, \\ -(q_0 q)^{1/2} & \text{for } q_1 \leqslant q \leqslant q_0, \\ -q - \dfrac{q_0^2}{8q} & \text{for } q \geqslant q_0 = \left(\dfrac{3}{2}\mu\right)^{1/2}, \end{cases} \tag{B 13}$$

which is identical to (14).

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