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ON ZETA FUNCTIONS ASSOCIATED WITH POLYNOMIALS

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We give direct proofs of meromorphic continuality on the whole complex plane of certain zeta functions $Z_{P,Q}(s)$ and $Z(P/Q, s)$ associated with a pair of polynomials P, Q. We calculate $Z_{P,Q}(-q)$ (q a non-negative integer) and give explicit formulas for the residues of $Z(P/Q, s)$ at poles.

INTRODUCTION

Let $Q(x) = \prod_{j=1}^{k} (x + \alpha_j)$ be a non-constant polynomial with real coefficients and $\mathrm{Re}\,(\alpha_j) > -1$ $(j = 1, \dots, k)$. Let $P(x) = b_0 + \dots + b_m x^m$ $(b_m \neq 0)$ be a polynomial of degree m with complex coefficients. Consider the Dirichlet series

$$Z_{P,Q}(s) = \sum_{n \geq 1} \frac{P(n)}{Q(n)^s}, \quad \left(\mathrm{Re}\,(s) > \frac{m+1}{k}\right).$$

Define polynomials $P_i(x)$ by $P_0(x) = x$, and, if $i \geq 0$, $P_i(x) = \sum_{j=1}^{i} a(j,i)x^j$ with $a(j,i) = \sum_{l=1}^{j} (-1)^{j-l} \binom{i+1}{j-l} l^i$. Let $C_r(i)$ $(i, r = 0, 1, \dots)$ be rational numbers defined by

$$\left(\frac{t}{1 - e^{-t}}\right)^{i+1} P_i(e^{-t}) = \sum_{r=0}^{\infty} \frac{C_r(i)}{r!} t^r.$$

Also put

$$A_{j,p}^{(k)}(s) = \frac{\Gamma(s + p_1) \dots \Gamma(s + p_k)}{\Gamma(s)^{k-1} \Gamma(ks + j)} \qquad (p_1 + \dots + p_k = j).$$

Our first result is the following

PROPOSITION 1.

(a) *$Z_{P,Q}(s)$ has a meromorphic continuation on the complex plane with at most simple poles at $s = (m + 1 - j)/k$ $(j = 0, 1, \dots)$, other than non-positive integers.*

(b) *For any non-negative integer q, we have*

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$$Z_{P,Q}(-q) = \frac{(-1)^q q!}{k} \sum_{l=0}^{m} b_l \sum_{r+j=kq+1+l} \frac{C_r(l)}{r!} \left[\sum_{p_1+\ldots+p_k=j} (-1)^j A_{j,p}^{(k)}(-q) \frac{\alpha_1^{p_1} \ldots \alpha_k^{p_k}}{p_1! \ldots p_k!} \right].$$

Particular cases treated before included: k arbitrary, $m = 0$ (see [**2**]) and $k = 2$, m arbitrary (see [**1**]).

Assuming $\deg P > \deg Q$, and $P(x) = \prod_{j=1}^{m} (x + \beta_j)$, $\operatorname{Re}(\beta_j) > -1$ $(j = 1, \ldots, m)$, we define

$$Z(P/Q, s) = \sum_{n \geq 1} \left(\frac{P(n)}{Q(n)} \right)^{-s}, \quad \left(\operatorname{Re}(s) > \frac{1}{m-k} \right).$$

Write

$$\frac{x^{m-k} Q(x) - P(x)}{P(x)} = \sum_{j=0}^{\infty} A_{-l_0-j}(P, Q) x^{-l_0-j},$$

and define the numbers $B_{-r}(P, Q, i)$ by the expression

$$\left(\sum_{j=0}^{\infty} A_{-l_0-j}(P, Q) x^{-l_0-j} \right)^i = \sum_{j=0}^{\infty} B_{-il_0-j}(P, Q, i) x^{-il_0-j} \quad (i = 0, 1, \ldots) \, .$$

We give a direct proof of meromorphic continuality of $Z(P/Q, s)$ on the complex plane, with an explicit formula for the residues. More involved proofs of this result can be found in (or deduced from) [**4, 3**].

Proposition 2.

(a) *$Z(P/Q, s)$ has a meromorphic continuation on the complex plane with at most simple poles at $s = (1-r)/(m-k)$ $(r = 0, 1, \ldots)$ other than zero.*

(b) *We have*

$$\operatorname{res}_{s=(1-r)/(m-k)} Z(P/Q, s) = \frac{1}{m-k} \sum_{il_0+j=r} \binom{\frac{1-r}{m-k}}{i} B_{-r}(P, Q, i), \qquad Z(P/Q, 0) = -\frac{1}{2}.$$

Proofs

Proof of Proposition 1: For $\operatorname{Re}(s) > 0$, we set

$$I_Q(s, t) = \frac{1}{\Gamma(s)^{k-1}} \int_E (u_1 \ldots u_k)^{s-1} e^{-t(\alpha_1 u_1 + \ldots + \alpha_k u_k)} du_1 \ldots du_{k-1},$$

where E is the standard simplex in $\mathbb{R}^{k-1}$ defined by $u_1, \ldots, u_{k-1} \geq 0$ and $u_k = 1 - u_1 - \ldots - u_{k-1} \geq 0$.

Lemma 1. *For $k \geq 2$ and $\mathrm{Re}(s) > (m+1)/k$, we have*

$$Z_{P,Q}(s) \cdot \Gamma(s) = \sum_{i=0}^{m} b_i \int_0^{\infty} \frac{t^{ks-1} P_i(e^{-t})}{(1-e^{-t})^{i+1}} I_Q(s,t)\, dt.$$

Proof: Modify the proof of Proposition 1 in [**2**]. ∎

Lemma 2. *$I_Q(s,t)$ has analytic continuation to an entire function of s.*

Proof: See [**2**, p.586]. ∎

Part (a) of Proposition 1 follows immediately from Lemmas 1 and 2. In the proof of part (b) we need the Taylor expansion:

$$I_Q(s,t) = \sum_{j=0}^{\infty} \Bigg[\sum_{p_1+\ldots+p_k=j} (-1)^j A_{j,p}^{(k)}(s) \frac{\alpha_1^{p_1} \ldots \alpha_k^{p_k}}{p_1! \ldots p_k!} \Bigg] t^j.$$

The assertion now follows from the calculation of $\mathrm{res}_{s=-q}\, Z_{P,Q}(s)\Gamma(s)$ in two ways. ∎

Proof of Proposition 2: We have

$$\begin{aligned}
\sum_{n \geq 1} \left(\frac{P(n)}{Q(n)} \right)^{-s} &= \sum_{n \geq 1} \left(1 + \frac{n^{m-k}Q(n) - P(n)}{P(n)} \right)^{-s} n^{-(m-k)s} \\
&= \sum_{i=0}^{\infty} \binom{s}{i} \sum_{n \geq 1} \left(\frac{n^{m-k}Q(n) - P(n)}{P(n)} \right)^{i} n^{-(m-k)s} \\
&= \sum_{i=0}^{\infty} \binom{s}{i} \sum_{j=0}^{\infty} B_{-il_0-j}(P,Q,i)\zeta\big((m-k)s + il_0 + j\big).
\end{aligned}$$

This gives meromorphic continuation of $Z(P/Q,s)$ on the complex plane, with explicit formulas for the residues. ∎

References

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