Proceedings of the Edinburgh Mathematical Society (2015) **58**, 333–354
DOI:10.1017/S0013091514000182

KOSZUL COMPLEXES AND POLE ORDER FILTRATIONS

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(Received 19 April 2012)

Abstract We study the interplay between the cohomology of the Koszul complex of the partial derivatives of a homogeneous polynomial f and the pole order filtration P on the cohomology of the open set $U = \mathbb{P}^n \setminus D$, with D the hypersurface defined by $f = 0$. The relation is expressed by some spectral sequences. These sequences may, on the one hand, in many cases be used to determine the filtration P for curves and surfaces and, on the other hand, to obtain information about the syzygies involving the partial derivatives of the polynomial f. The case of a nodal hypersurface D is treated in terms of the defects of linear systems of hypersurfaces of various degrees passing through the nodes of D. When D is a nodal surface in $\mathbb{P}^3$, we show that $F^2H^3(U) \neq P^2H^3(U)$ as soon as the degree of D is at least 4.

Keywords: projective hypersurfaces; singularities; Milnor algebra; syzygies; mixed Hodge structure; pole order filtration

2010 *Mathematics subject classification:* Primary 14C30; 13D40
Secondary 32S35; 13D02

1. Introduction

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded ring of polynomials in $x_0, \dots, x_n$ with complex coefficients and let us denote by S_r the vector space of homogeneous polynomials in S of degree r. For any polynomial $f \in S_N$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, \dots, f_n$ of f with respect to $x_0, \dots, x_n$. Following the notation of Eisenbud [**19**], for $n = 2$ we use x, y, z instead of x_0, x_1, x_2 and f_x, f_y, f_z instead of f_0, f_1, f_2.

We define the corresponding graded *Milnor algebra* (or *Jacobian algebra*) by

$$M(f) = S/J_f. \tag{1.1}$$

The study of such Milnor algebras is related to the singularities of the corresponding projective hypersurface $D\colon f = 0$ (see [**6**]) as well as to the mixed Hodge theory of the hypersurface D and of its complement $U = \mathbb{P}^n \setminus D$ (see the foundational article by

Griffiths [**23**] and also [**9**, **15**, **18**] and references therein). For mixed Hodge theory we refer the reader to [**25**].

In fact, such a Milnor algebra can be seen, up to a twist in grading, as the first (or the last) homology (or cohomology) of the Koszul complex of the partial derivatives $f_0, \dots, f_n$ in S (see [**6**] or [**11**, Chapter 6]). As such, it is related to certain natural E_1-spectral sequences associated with the pole order filtration and converging to the cohomology of the complement U introduced in [**9**] and discussed in detail in [**11**, Chapter 6].

In § 2 we recall and improve the construction of these spectral sequences and show that they degenerate at the E_2-terms when all the singularities of D are weighted homogeneous and $\dim D = 1$; in the curve case we use the more classical notation C instead of D (see Theorem 2.4 (iii)). In the curve case this result gives a positive answer to an old conjecture by the first author (see the claim just before Remark (3.11) in [**9**]). Such a degeneracy at the E_2-terms is also shown to occur for nodal surfaces (see Theorem 5.1 (i)).

In § 3 we assume that $n = 2$ and use this approach to determine the pole order filtration P^* on the cohomology group $H^2(U)$ for a number of cases (see Examples 3.2, 3.3 and 3.4, the latter being a new example where $F^2 \neq P^2$ on $H^2(U)$). In Example 3.1 we also describe these spectral sequences completely for the case in which C is a nodal curve.

In § 4 we discuss the syzygies of nodal hypersurfaces. For instance, we show that for a nodal curve there are no non-trivial relations

$$R_m\colon af_x + bf_y + cf_z = 0 \tag{1.2}$$

with a, b, c homogeneous of degree $m < N - 2$, and we describe completely the relations of degree $m = N - 2$ in terms of the irreducible factors f_j of f (see Theorem 4.1). Note that f_j has a different meaning for $n = 2$ and for $n > 2$. In [**17**] the vanishing part in Theorem 4.1 was extended to nodal hypersurfaces of arbitrary dimension using a different approach.

Definition 1.1. For a hypersurface $D\colon f = 0$ with isolated singularities we introduce three integers as follows.

(i) The *coincidence threshold* $\mathrm{ct}(D)$ is defined as

$$\mathrm{ct}(D) = \max\{q\colon \dim M(f)_k = \dim M(f_s)_k \text{ for all } k \leqslant q\},$$

with f_s a homogeneous polynomial in S of degree N such that $D_s\colon f_s = 0$, is a smooth hypersurface in $\mathbb{P}^n$.

(ii) The *stability threshold* $\mathrm{st}(D)$ is defined as

$$\mathrm{st}(D) = \min\{q\colon \dim M(f)_k = \tau(D) \text{ for all } k \geqslant q\},$$

where $\tau(D)$ is the total Tjurina number of D, i.e. the sum of all the Tjurina numbers of the singularities of D.

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(iii) The *minimal degree of a non-trivial syzygy* $\operatorname{mdr}(D)$ is defined as

$$\operatorname{mdr}(D) = \min\{q \colon H^n(K^*(f))_{q+n} \neq 0\},$$

where $K^*(f)$ is the Koszul complex of $f_0, \ldots, f_n$, and the grading is defined in the next section.

If a relation as in (1.2) is of minimal degree among the relations modulo the trivial relations (4.5), then one has $\operatorname{mdr}(D) = m$, i.e. our notion is the natural one. Moreover, it follows from (2.17) that one has

$$\operatorname{ct}(D) = \operatorname{mdr}(D) + N - 2. \tag{1.3}$$

By definition, it follows that for any such hypersurface D that is not smooth, we have $N - 2 \leqslant \operatorname{ct}(D) \leqslant (n+1)(N-2)$ and, using [**6**], we get $\operatorname{st}(D) \leqslant (n+1)(N-2)+1$. With this handy notation, we can state the following result, a consequence of the vanishings in Theorem 4.1 obtained via Hodge theory, using (2.17).

Theorem 1.2. *Let $C \colon f = 0$ be a nodal curve of degree N in $\mathbb{P}^2$. One then has $\operatorname{ct}(C) \geqslant 2N - 4$.*

Recall that the Hilbert–Poincaré series of a graded S-module E of finite type is defined by

$$\operatorname{HP}(E)(t) = \sum_{k \geqslant 0} (\dim E_k) t^k \tag{1.4}$$

and that we have

$$\operatorname{HP}(M(f_s)) = \frac{(1-t^{N-1})^{n+1}}{(1-t)^{n+1}}. \tag{1.5}$$

In particular, if we set $T = T(n, N) = (n+1)(N-2)$, it follows that $M(f_s)_j = 0$ for $j > T$ and $\dim M(f_s)_j = \dim M(f_s)_{T-j}$ for $0 \leqslant j \leqslant T$.

Theorem 1.2 determines the dimensions of $M(f)_q$ for all $q < 2N - 3$ in the case of a nodal curve C. The next dimension for such a curve is given by

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1,r} g_j = g + r - 1, \tag{1.6}$$

where $n(C) = \tau(C)$ is the total number of nodes of C and the g_j are the genera of the normalizations of the irreducible components C_j of C, whose number is r. The genus of the smooth curve $C_s \colon f_s = 0$ is given by

$$g = \frac{(N-1)(N-2)}{2} \tag{1.7}$$

(see (3.2) and (3.3)). For more general curves we have, as a consequence of Theorem 2.4, the following relation between the Milnor algebra $M(f)$ and the geometry of U.

Corollary 1.3. *Let $C\colon f=0$ be a curve in $\mathbb{P}^2$ of degree N having only isolated weighted homogeneous singularities. Then*

$$\dim M(f)_{2N-3} + \dim P^2H^2(U) = 2g + r - 1 = \dim H^2(U) + \tau(C),$$

where g is given by (1.7), r is the number of irreducible components of C and $\tau(C)$ is the total Tjurina number of C.

For a highly singular curve C, we can have much lower values for $\mathrm{ct}(C)$ than those given by Theorem 1.2, namely, $\mathrm{ct}(C) = N-2$ or $\mathrm{ct}(C) = N-1$ (see Example 4.2).

On the other hand, it follows from (1.6) that for a nodal curve C one has $\mathrm{ct}(C) = 2N-4$ if and only if C is not irreducible, i.e. if $r > 1$. One of the main results in [**16**], restated as the first equality in (1.6), implies that for a *rational nodal curve* (i.e. one for which $g_j = 0$ for $j = 1, \dots, r$) one has $\mathrm{st}(C) \leqslant 2N-3$. This yields the following corollary.

Corollary 1.4. *For a rational nodal curve C, the Hilbert–Poincaré series* $\mathrm{HP}(M(f))$ *is completely determined by the degree N and the number of nodes $n(C)$. In particular,* $\mathrm{st}(C) = 2N-3$ *unless C is a generic line arrangement, in which case* $\mathrm{st}(C) = 2N-4$ *for $N > 3$ and* $\mathrm{st}(C) = 1$ *for $N = 3$.*

For the case of hyperplane arrangements, an interesting approach to the study of the Jacobian ideal J_f is given in the recent paper [**8**].

At the other extreme, there are nodal curves with $\mathrm{ct}(C) = 3N-6$, as implied by the description given in Example 4.3 (i) of the Hilbert–Poincaré series $\mathrm{HP}(M(f))$ for any hypersurface having exactly one node.

To state the next result, we must recall some notation. For a finite set of points $\mathcal{N} \subset \mathbb{P}^n$ we define as

$$\operatorname{def} S_m(\mathcal{N}) = |\mathcal{N}| - \operatorname{codim}\{h \in S_m \mid h(a) = 0 \text{ for any } a \in \mathcal{N}\}$$

the *defect (or superabundance) of the linear system of polynomials in S_m vanishing at the points in $\mathcal{N}$* (see [**11**, p. 207]). In [**21**], this positive integer is called the *failure of $\mathcal{N}$ to impose independent conditions on homogeneous polynomials of degree m*. In § 4 we prove the following theorem.

Theorem 1.5. *Let $D\colon f=0$ be a degree N nodal hypersurface in $\mathbb{P}^n$ and let $\mathcal{N}$ denote the set of its nodes. Then*

$$\dim H^n(K^*(f))_{nN-n-1-k} = \operatorname{def} S_k(\mathcal{N})$$

for $0 \leqslant k \leqslant nN-2n-1$ and

$$\dim H^n(K^*(f))_j = \tau(D) = |\mathcal{N}|$$

for $j \geqslant n(N-1)$. In other words,

$$\dim M(f)_{T-k} = \dim M(f_s)_k + \operatorname{def} S_k(\mathcal{N})$$

for $0 \leqslant k \leqslant nN-2n-1$, where $T = T(n,N) = (n+1)(N-2)$. In particular, $\dim M(f)_T = \tau(D)$, i.e. $\mathrm{st}(D) \leqslant T$.

Note that this theorem determines the dimensions $\dim M(f)_j$ in terms of the defects of linear systems for any $j \geqslant N-1$, i.e. for all j, since the dimensions $\dim M(f)_j = \dim S_j$ for $j < N-1$ are well known. The last equality, namely $\dim M(f)_T = \tau(D)$, decreases by one the upper bound for $\operatorname{st}(D)$ obtained in [**6**, Corollary 9] in the case of nodal hypersurfaces. A similar result for hypersurfaces D having arbitrary isolated singularities is obtained in [**13**].

Illustrations of how to apply Theorem 1.5 are given in Example 4.3. Using Theorems 1.5 and 4.1 and Corollary 1.4, we get the following information on the *position of the nodes of a nodal curve*.

Corollary 1.6. *Let $C\colon f = 0$ be a degree N nodal curve in $\mathbb{P}^2$ and let $\mathcal{N}$ denote the set of its nodes. Then* $\operatorname{def} S_k(\mathcal{N}) = 0$ *for* $k > N-3$ *and* $\operatorname{def} S_{N-3}(\mathcal{N}) = r-1$, *where r is the number of irreducible components of C.*

Moreover, if the curve C is, in addition, rational, then all the defects $\operatorname{def} S_k(\mathcal{N})$ *are completely determined by the degree N and the number of nodes $n(C)$.*

In fact, a recent result by Kloosterman (see [**24**, Proposition 3.6]) implies that the first part of Corollary 1.6 holds for any curve C with the property that any singular point of C that is not a node is a unibranch singularity (see Remark 4.4 for more details on this).

In the final section we use Theorem 1.5 to determine the pole order filtration P^* on the cohomology groups $H^*(U)$ and the corresponding spectral sequences when D is a nodal surface. In particular, we obtain the following theorem.

Theorem 1.7. *Let $S\colon f = 0$ be a nodal surface in $\mathbb{P}^3$ of degree N and let $\mathcal{N}$ denote the set of its nodes. Then, if $U = \mathbb{P}^3 \setminus S$ and S_s is a smooth surface of degree N in $\mathbb{P}^3$, the following equalities hold:*

$$\dim \operatorname{Gr}^2_P(H^3(U)) = h^{1,1}(S_s) - 1 - \operatorname{def} S_{N-4}(\mathcal{N})$$

and

$$\dim \operatorname{Gr}^2_F(H^3(U)) = h^{1,1}(S_s) - 1 - |\mathcal{N}|.$$

In particular, $P^2H^3(U) = F^2H^3(U)$ if and only if the nodal surface S is smooth or $N < 4$.

This result complements the results in [**18**] (where arbitrary dimensions are considered, but only in the case of degrees $N = 3$ and $N = 4$) for the case of nodal surfaces, and answers the question posed therein as to whether the inequality $P^2H^3(U) \neq F^2H^3(U)$ holds for any surface with $|\mathcal{N}| = 1$ and $N \geqslant 4$.

Numerical experiments with the CoCoA package* and the Singular package† have played a key role in the completion of this work.

* CoCoA: a system for doing computations in commutative algebra. Available at http://cocoa.dima.unige.it.

† Singular 3-1-3: a computer algebra system for polynomial computations. Available at www.singular.uni-kl.de. Developed by W. Decker, G.-M. Greuel, G. Pfister and H. Schönemann.

2. Pole order filtrations, spectral sequences and Koszul complexes

Let X be a smooth complex quasi-projective variety and let $D \subset X$ be a reduced divisor. We denote by $i\colon D \to X$ and $j\colon U \to X$ the corresponding inclusions, where $U = X \setminus D$. Let Ω_X^* (respectively, Ω_U^*) denote the de Rham sheaf complex of regular differential forms on X (respectively, U). Grothendieck's theorem then states that

$$\mathbb{H}^*(U, \Omega_U^*) = H^*(U), \tag{2.1}$$

where $\mathbb{C}$-coefficients are used for the cohomology groups unless indicated otherwise. Moreover, as explained in [**7**], the isomorphism $j_*\Omega_U^* = Rj_*\Omega_U^*$ (a consequence of the fact that j is an affine morphism) implies the natural identification

$$\mathbb{H}^*(X, j_*\Omega_U^*) = H^*(U). \tag{2.2}$$

The sheaf complex $j_*\Omega_U^*$ has a natural decreasing filtration, called the *pole order filtration*, given by

$$P^s j_*\Omega_U^p = 0$$

if $p < s$ and given by

$$P^s j_*\Omega_U^p = \Omega_X^p((p-s+1)D) \tag{2.3}$$

if $p \geqslant s$ (see [**7**]). In other words, a rational differential form ω is in $P^s j_*\Omega_U^p$ if it has a pole of order at most $p-s+1$ along the divisor D (with special attention needed for the case of $p = s-1$). A word of warning: the corresponding filtration is denoted by F in [**9**] and is slightly different. However, the proof of the main results from [**9**] or [**11**] quoted below apply word for word to the present setup.

Using the filtration (2.3), we define the pole order filtration on the cohomology of U by setting

$$P^s H^*(U) = \operatorname{im}(\mathbb{H}^*(X, P^s j_*\Omega_U^*) \to \mathbb{H}^*(X, j_*\Omega_U^*) = H^*(U)). \tag{2.4}$$

The main result from [**7**] is the following theorem (see also [**27**] for another proof and conditions for equality).

Theorem 2.1. *Assume that the smooth variety X is proper and let F denote the Hodge filtration on the cohomology of U. Then $F^s H^*(U) \subset P^s H^*(U)$ for any s.*

From now on consider the case $X = \mathbb{P}^n$ and recall that Bott's vanishing theorem gives us

$$H^k(X, \Omega_X^p(sD)) = 0 \tag{2.5}$$

for any $k > 0$, $s > 0$ (see [**1**]). The polar filtration, even if it is an infinite filtration, gives rise to a spectral sequence

$$E_1^{p,q}(U) = \mathbb{H}^{p+q}(X, \operatorname{Gr}_P^p(j_*\Omega_U^*)) \tag{2.6}$$

whose limit term is exactly

$$E_\infty^{p,q}(U) = \operatorname{Gr}_P^p(H^{p+q}(U)). \tag{2.7}$$

Now, using the standard spectral sequence

$$E_1^{p,q} = H^p(X, \mathrm{Gr}_P^s(j_*\Omega_U^q)) \Rightarrow \mathbb{H}^{p+q}(X, \mathrm{Gr}_P^s(j_*\Omega_U^*)) \tag{2.8}$$

and the vanishings implied by (2.5), we obtain a description of the E_1-term of our spectral sequence without involving hypercohomology groups, namely,

$$E_1^{p,q}(U) = H^{p+q}(H^0(X, \mathrm{Gr}_P^p(j_*\Omega_U^*))). \tag{2.9}$$

This expression for $E_1^{p,q}(U)$ can be interpreted as follows. Let $A^*(U) = H^0(X, j_*\Omega_U^*)$ be the de Rham complex of regular forms defined on the affine open set U. It follows from Grothendieck's theorem (Theorem 2.1), that one has

$$H^m(A^*(U)) = H^m(U) \tag{2.10}$$

for any integer m. On the other hand, we have a very explicit description of these rational differential forms defined on U. Let $f = 0$ be a reduced equation for the divisor D and let N be the degree of the homogeneous polynomial f. Denote by $\Omega^p = H^0(\mathbb{C}^{n+1}, \Omega^p_{\mathbb{C}^{n+1}})$ the global (polynomial) differential p-forms on $\mathbb{C}^{n+1}$, regarded as a graded S-module in the usual way (i.e. $\deg(h\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_q}) = p + q$ if $h \in S_p$). Then a differential p-form $\omega \in A^p(U)$, for $p \geqslant 0$, is given by

$$\omega = \frac{\Delta(\gamma)}{f^s} \tag{2.11}$$

for some integer $s > 0$, $\gamma \in \Omega^{p+1}_{sN}$ and $\Delta\colon \Omega^{p+1} \to \Omega^p$ being the S-linear map given by the contraction with the Euler field (see [**11**, Chapter 6] for details). When ω is not a constant function on U (the case covered by $s = 1$ and $\gamma = a \cdot \mathrm{d}f$ for $a \in \mathbb{C}$), the minimal s in this formula is by definition the order of ω along the divisor D. We can define a polar filtration on the complex $A^*(U)$ by setting $P^sA^p(U) = 0$ if $p < s$ and

$$P^sA^p(U) = \left\{ \omega = \frac{\Delta(\gamma)}{f^{p-s+1}} \,\middle|\, \gamma \in \Omega^{p+1}_{(p-s+1)N} \right\} \tag{2.12}$$

if $p \geqslant s$. This decreasing filtration induces a spectral sequence

$$E_1^{p,q}(A) = H^{p+q}(\mathrm{Gr}_P^p(A^*(U))). \tag{2.13}$$

Using Bott's vanishing theorem (2.5) and (2.9), we see that this new spectral sequence coincides with the spectral sequence $E_1^{p,q}(U)$. In particular, they both induce the same filtration on their common limit, which is $H^*(U)$.

Note that $A^0(U)$ (respectively, $E_1^{0,0}(A) = H^0(\mathrm{Gr}_P^0(A^*(U)))$) contains the constant functions on U. Let us denote by $\tilde{A}^*(U)$ (respectively, $E_1^{p,q}(\tilde{A})$) the complex (respectively, the spectral sequence) obtained from the above complex $A^*(U)$ (respectively, spectral sequence $E_1^{p,q}(A)$) by replacing $A^0(U)$ (respectively, $E_1^{0,0}(A)$) by $A^0(U)/\mathbb{C}$ (respectively, $E_1^{0,0}(\tilde{A}) = E_1^{0,0}(A)/\mathbb{C}$). It is clear that the cohomology of the complex $\tilde{A}^*(U)$ (respectively, the limit of the spectral sequence $E_r^{p,q}(\tilde{A})$) is $\tilde{H}^*(U)$, the reduced cohomology of U.

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It turns out that the E_1-term of the spectral sequence $E_r^{p,q}(\tilde{A})$ can be described in terms of the *Koszul complex* of the partial derivatives f_j of f with respect to the variable x_j for $j = 0, \dots, n$ (see [**9**], [**11**, Chapter 6] and [**14**, Remark 2.10]). This Kozsul complex can be represented by the complex of graded S-modules

$$K^*(f)\colon 0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^{n+1} \to 0, \tag{2.14}$$

where the differentials are given by the wedge product with the differential $\mathrm{d}f$, and hence these differentials are homogeneous of degree N. This complex has a natural subcomplex

$$K'^*(f)\colon 0 \to \Omega'^0 \to \Omega'^1 \to \cdots \to \Omega'^{n+1} \to 0, \tag{2.15}$$

where $\Omega'^p = \bigoplus_{k\geqslant 0} \Omega^p_{kN}$.

Consider the associated double complex $(B, \mathrm{d}', \mathrm{d}'')$, with $B^{s,t} = \Omega^{s+t+1}_{(t+1)N}$ for $t \geqslant 0$ and $-1 \leqslant s+t \leqslant n$ and $B^{s,t} = 0$ otherwise, and differentials $\mathrm{d}' = \mathrm{d}$, the exterior derivative of a form, and $\mathrm{d}''(\omega) = -\mathrm{d}f \wedge \omega$. Note that $\mathrm{d}'\,\mathrm{d}'' + \mathrm{d}''\,\mathrm{d}' = 0$ and let $(B^*, D_f = \mathrm{d}' + \mathrm{d}'')$ be the associated total complex of this double complex. In fact, the complex B^* is the same as the reduced version of the subcomplex K'^*, but with a new differential.

As for any total complex, B^* comes with two natural decreasing filtrations, one such being

$$F^p B^k = \bigoplus_{s\geqslant p-1} B^{s,k-s}.$$

The contraction operator Δ defines a morphism of filtered complexes $\delta\colon B^* \to \tilde{A}^*(U)$ by setting

$$\delta(\omega) = \frac{\Delta(\omega)}{f^{t+1}} \quad \text{for } \omega \in B^{s,t}. \tag{2.16}$$

With this notation, we have the following result (see [**9**], [**11**, Chapter 6] and [**14**, Remark 2.10]).

Proposition 2.2. *Let $E_r^{p,q}(f)$ be the E_1-spectral sequence associated with the filtration F on (B^*, D_f). The following statements then hold.*

(i) *The morphism δ induces an isomorphism of E_1-spectral sequences*

$$E_r^{p,q}(f) \to E_r^{p,q}(\tilde{A}).$$

(ii) *There is a natural identification*

$$E_1^{p,q}(f) = H^{p+q+1}(K^*(f))_{(q+1)N}.$$

Remark 2.3.

(i) In the case $X = \mathbb{P}^n$, it is known that $F^1H^k(U) = H^k(U)$ for any integer $k > 0$ (see [**9**, Theorem 2.2] (there is an equals sign missing in the statement, but the proof of the equality is clearly done) or the proof of Corollary 1.32 [**11**, pp. 185–186]).

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(ii) One has $P^{k+1}H^k(U) = 0$ for any integer $k > 0$. To see this, just use the fact that the hypercohomology of a sheaf complex $\mathcal{F}^*$ with $\mathcal{F}^j = 0$ for $j < p$ satisfies $\mathbb{H}^j(\mathcal{F}^*) = 0$ for $j < p$. In particular, $P^2H^1(U) = 0$, i.e. we always have $\mathrm{Gr}_P^1(H^1(U)) = H^1(U)$ and $\mathrm{Gr}_P^j(H^1(U)) = 0$ for $j \neq 1$.

Assume now that the hypersurface D has only isolated singularities. The non-zero terms in the E_1-term of the spectral sequence $E_r^{p,q}(f)$ are sitting on two lines, given by $L\colon p + q = n$ and $L'\colon p + q = n - 1$. Indeed, one has to use the fact that in this case $H^m(K^*(f)) = 0$ for $m < n$ (see [**22, 26**]). For a term $E_1^{p,q}(f)$ situated on the line L, we have

$$E_1^{p,q}(f) = H^{n+1}(K^*(f))_{(q+1)N} = M(f)_{(q+1)N-n-1}.$$

We now describe the terms on the line L'. In order to do this, let $f_s \in S_N$ denote a polynomial of degree N defining a smooth hypersurface in $\mathbb{P}^n$. It is easy to show that

$$t^N \operatorname{HP}(H^n(K^*(f)))(t) = \operatorname{HP}(H^{n+1}(K^*(f)))(t) - \operatorname{HP}(H^{n+1}(K^*(f_s)))(t), \tag{2.17}$$

using the fact that Euler characteristics do not change when replacing a (finite-type) complex by its cohomology. Note also that

$$\operatorname{HP}(H^{n+1}(K^*(f_s))) = t^{n+1} \operatorname{HP}(M(f_s)) = t^{n+1} \frac{(1 - t^{N-1})^{n+1}}{(1 - t)^{n+1}} \tag{2.18}$$

is completely determined by the degree N. It follows that the term

$$E_1^{p,q}(f) = H^n(K^*(f))_{(q+1)N}$$

situated on the line L' has dimension

$$\dim H^n(K^*(f))_{(q+1)N} = \dim M(f)_{(q+2)N-n-1} - \dim M(f_s)_{(q+2)N-n-1}. \tag{2.19}$$

We now want to relate the spectral sequence $E_r^{p,q}(A)$ to some simpler, locally computable spectral sequences in the case when D has only isolated singularities, say at the points $a_1, \dots, a_m$. Consider the morphism of restriction

$$\rho\colon\ \mathrm{Gr}_P^p(j_*\Omega_U^*) \to i_{1*}\,\mathrm{Gr}_P^p((j_*\Omega_U^*)/\Omega_X^*) \mid \Sigma$$

obtained by factoring out the regular forms, taking the restriction from X to the singular locus Σ of D, and then extending via i_{1*}, where $i_1\colon \Sigma \to X$ is the inclusion. For $p < 0$ this morphism is easily seen to be a quasi-isomorphism, i.e. it induces isomorphisms at stalk level. For $p = 0$, the kernel K_ρ of ρ is the sheaf $\Omega_X^0 = \mathcal{O}_X$ (placed in degree 0). We know that, for the case $X = \mathbb{P}^n$,

$$\mathbb{H}^q(X, \mathcal{O}_X) = H^q(X, \mathcal{O}_X) = 0$$

for $q > 0$. It follows that the morphisms

$$\rho^k\colon \mathbb{H}^k(X, \mathrm{Gr}_P^0(j_*\Omega_U^*)) \to \mathbb{H}^k(X, i_{1*}\,\mathrm{Gr}_P^0((j_*\Omega_U^*/\Omega_X^*) \mid \Sigma))$$

are isomorphisms for any $k \geqslant 1$.

https://doi.org/10.1017/S0013091514000182 Published online by Cambridge University Press

As explained in [**9**] (with the notable difference that in [**9**] there is no quotient taken, which leads to an infinite-dimensional E_1-term), the complex $((j_*\Omega_U^*)/\Omega_X^*) \mid \Sigma$ is the direct sum of the complexes $\tilde{A}^*(D, a_j)$ for $j = 1, \dots, m$, where each $\tilde{A}^*(D, a_j)$ is the local analogue of the complex $\tilde{A}^*(U)$ above. These complexes come with a pole order filtration defined exactly as in the global case, and for each j there is an E_1-spectral sequence $E_r(D, a_j)$ with

$$E_1^{p,q}(D, a_j) = H^{p+q}(\mathrm{Gr}_P^p(\tilde{A}^*(D, a_j)))$$

and converging to $\tilde{H}^*(B_j \setminus D)$, where B_j is a small ball in X centred at a_j. It follows that ρ induces a morphism of E_1-spectral sequences

$$\rho^{p,q} \colon E_1^{p,q}(A) \to \bigoplus_{j=1,m} E_1^{p,q}(D, a_j)$$

with the property that $\rho^{p,q}$ is an isomorphism for any $p \leqslant 0$ and $p + q \geqslant 1$.

Moreover, when each singularity (D, a_j) is weighted homogeneous, it follows from the description of the local spectral sequence $E_1^{p,q}(D, a_j)$ (see [**9**, Example 3.6]) that all the differentials $d_1 \colon E_1^{n-1-t,t}(D, a_j) \to E_1^{n-t,t}(D, a_j)$ are isomorphisms for $t \geqslant n-1$. In this way we have proved the following improvement of Theorem (3.9) in [**9**]. (For the converse claim in (iii), see [**9**, Corollary 3.10].)

Theorem 2.4.

(i) *Let D be a hypersurface in $\mathbb{P}^n$ for $n \geqslant 2$, having only isolated singularities. Then the morphism of E_1-spectral sequences*

$$\rho^{p,q} \colon E_1^{p,q}(A) \to \bigoplus_{j=1,s} E_1^{p,q}(D, a_j)$$

is an isomorphism for any $p \leqslant 0$ and $p + q \geqslant 1$.

(ii) *If, in addition, the singularities of D are weighted homogeneous, then in the spectral sequence $E_1^{p,q}(A)$ the differential*

$$d_1 \colon E_1^{n-1-t,t}(A) \to E_1^{n-t,t}(A)$$

is injective for $t = n-1$ and is bijective for $t \geqslant n$.

(iii) *If D is a reduced curve in $\mathbb{P}^2$, then D has only isolated weighted homogeneous singularities if and only if the E_1-spectral sequences $E_r^{p,q}(U)$, $E_r^{p,q}(\tilde{A})$ and $E_r^{p,q}(f)$ degenerate at the E_2-term, i.e. $E_2 = E_\infty$ for any of these E_1-spectral sequences.*

This result, especially parts (ii) and (iii), is perhaps related to the results in [**3**] and [**4**].

3. Some examples of spectral sequences in the case of plane curves

Let $C \colon f = 0$ be a reduced curve in $\mathbb{P}^2$ of degree N. Let $C_j \colon f_j = 0$ for $j = 1, \dots, r$ be the irreducible components of C. The complement U has at most three non-zero cohomology groups. The first of them, $H^0(U)$, is one dimensional and of Hodge type $(0,0)$, so there is nothing of interest here. Moreover, $\tilde{H}^0(U) = 0$.

The second, $H^1(U)$, is $(r-1)$ dimensional and, for $r > 1$, is of Hodge type $(1,1)$ by Remark 2.3. It follows in this case that $P^1H^1(U) = F^1H^1(U) = H^1(U)$. Moreover, $H^1(U)$ has a basis given by

$$\omega_j = \frac{\mathrm{d}f_j}{N_j f_j} - \frac{\mathrm{d}f_r}{N_r f_r} \tag{3.1}$$

for $j = 1, \dots, r-1$, where $N_j = \deg(f_j)$ (see [**9**, Example 4.1]).

Example 3.1. We discuss first the case when $C\colon f = 0$ is a nodal curve in $\mathbb{P}^2$ of degree N. Using [**27**, Corollary 0.12] for $X = \mathbb{P}^2$, $i = 2$, it follows that $P^2H^2(U) = F^2H^2(U)$, since for a nodal curve $\alpha_f = 1$. We now look at the non-zero terms in the E_1-term of the spectral sequence $E_r^{p,q}(f)$. They are sitting on two lines given by $L\colon p + q = 2$ and $L'\colon p + q = 1$.

We look first at the terms on the line L. The term $E_1^{2,0}(f) = H^3(K^*)_N$ is isomorphic as a $\mathbb{C}$-vector space with $M(f)_{N-3}$, and hence has dimension g (as defined in (1.7)), which is determined by $N = \deg(f)$ alone. Hence, the corresponding limit term $E_\infty^{2,0}(U) = P^2H^2(U)$ has dimension at most g. On the other hand, $\dim F^2H^2(U) = g$ (see [**18**, Theorem 2.2] or a direct proof in [**16**, Proposition 4.1]). The above argument gives an alternative proof of the equality $F^2H^2(U) = P^2H^2(U)$ in this case.

The term $E_1^{1,1}(f) = H^3(K^*)_{2N}$ is isomorphic to $M(f)_{2N-3}$. To compute its dimension note that, by [**18**, Theorem 2.2], we have $\dim(I/J_f)_{2N-3} = \mathrm{Gr}_F^1(H^2(U))$, where I is the ideal in S of polynomials vanishing at all the singular points of C. It was shown in [**16**, Proposition 4.1] that

$$\dim(I/J_f)_{2N-3} = \sum_{j=1,r} g_j,$$

where g_j is the genus of the normalization of the curve C_j, for $j = 1, \dots, r$. On the other hand, we showed in [**16**, Lemma 4.2] that $\dim(S/I)_{2N-3} = n(C)$, the total number of nodes of C. It follows that

$$\dim M(f)_{2N-3} = n(C) + \sum_{j=1,r} g_j. \tag{3.2}$$

Moreover, the dimension of the corresponding limit term $E_\infty^{1,1}(U) = \mathrm{Gr}_F^1(H^2(U)) = \mathrm{Gr}_P^1(H^2(U))$ is $\sum_{j=1,r} g_j$, as noted above.

The term $E_1^{2-q,q}(f) = H^3(K^*)_{(q+1)N}$ for $q \geqslant 2$ is isomorphic to $M(f)_{(q+1)N-3}$, which has dimension $n(C)$. Furthermore, the corresponding limit terms $E_\infty^{2-q,q}(U) = \mathrm{Gr}_F^{2-q}(H^2(U))$ clearly vanish for $q \geqslant 2$.

We look now at the terms on the line L'. It follows from (3.2), (1.7) and the duality $\dim M(f_s)_{2N-3} = \dim M(f_s)_{N-3}$ that the term $E_1^{1,0}(f) = H^2(K^*)_N$ has dimension $n(C) + \sum_{j=1,r} g_j - g$. If we compare with the proof of Proposition 4.1 in [**16**], we see that the total number of nodes $n(C)$ is given by $\sum_{j=1,r} n_j + \sum_{1\leqslant i<j\leqslant r} d_i d_j$, where n_j is

the number of nodes on the curve C_j and d_k is the degree of the curve C_k. Using both Remark 2.3 and formula (4.1) in the proof of Proposition 4.1 in [**16**], we conclude that

$$\dim E_1^{1,0}(f) = \dim E_\infty^{1,0}(f) = r - 1. \tag{3.3}$$

The dimension of the other terms $E_1^{1-q,q}(f) = H^2(K^*)_{(q+1)N}$ for $q \geqslant 1$ is equal to $n(C)$. Moreover, the corresponding limit terms $E_\infty^{1-q,q}(U) = \mathrm{Gr}_F^{1-q}(H^1(U))$ clearly vanish for $q \geqslant 1$.

It follows that the differential $d_1 \colon E_1^{1,0}(f) \to E_1^{2,0}(f)$ is the zero map (not to decrease the dimension of $E_2^{1,0}(f)$, which is the dimension of the limit), a fact not shared by curves with general weighted homogeneous singularities as seen in Examples 3.2 and 3.3. The other differentials $d_1 \colon E_1^{1-q,q}(f) \to E_1^{2-q,q}(f)$ for $q \geqslant 1$ are all injective (any non-zero kernel would kill some terms needed in the limit via some d_r with $r \geqslant 2$), as happens for any curve with weighted homogeneous singularities in view of Theorem 2.4.

Example 3.2. Consider the curve $C \colon x(x^2y + xy^2 + z^3) = 0$, which is the union of a smooth cubic $C \colon x^2y + xy^2 + z^3 = 0$ and an inflectional tangent $L \colon x = 0$. It is easy to see that for this case $\tilde{H}^0(U) = 0$, $H^1(U)$ is one dimensional and $H^2(U)$ is two dimensional, with classes of Hodge type $(2,1)$ and $(1,2)$. In particular, $F^2H^2(U)$ is one dimensional.

On the other hand, the spectral sequence $E_1(f)$ has the following non-zero terms: $E_1^{1,0}$, which is two dimensional; $E_1^{2,0}$, which is three dimensional; and all $E_1^{p,q}$ for $p+q = 1$ or $p+q = 2$ and $q > 0$, which are five dimensional, since $\tau(C) = 5$. The computation for the other dimensions are based on formula (2.18) and a computation, using CoCoA or Singular, of the Hilbert–Poincaré series

$$\mathrm{HP}(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 5t^5 + \cdots$$

with stabilization threshold $\mathrm{st}(C) = 5$. It follows that $d_1 \colon E_1^{1,0} \to E_1^{2,0}$ has a one-dimensional kernel $E_2^{1,0} = H^1(U)$, and a two-dimensional cokernel $E_2^{2,0} = P^2H^2(U)$. In particular, the inclusion $F^2 \subset P^2$ is strict on $H^2(U)$, as mentioned in [**9**, Remark 2.6].

Example 3.3. Now consider the irreducible curve $C \colon x^2y^2 + xz^3 + yz^3 = 0$, which has two cusps A_2 as singularities. It is then easy to see that $\tilde{H}^0(U) = 0 = H^1(U)$, and that $H^2(U)$ is two dimensional, with classes of Hodge type $(2,1)$ and $(1,2)$. In particular, $F^2H^2(U)$ is one dimensional.

On the other hand, the spectral sequence $E_1(f)$ has the following non-zero terms: $E_1^{1,0}$, which is one dimensional; $E_1^{2,0}$, which is three dimensional; and all $E_1^{p,q}$ for $p+q = 1$ or $p+q = 2$ and $q > 0$, which are four dimensional, since $\tau(C) = 4$. Indeed, the computation, using CoCoA or Singular, in this case yields

$$\mathrm{HP}(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + \cdots$$

with stabilization threshold $\mathrm{st}(C) = 5$. It follows that $d_1 \colon E_1^{1,0} \to E_1^{2,0}$ is injective and has a two-dimensional cokernel $E_2^{2,0} = P^2H^2(U)$. In particular, the inclusion $F^2 \subset P^2$ is strict on $H^2(U)$, as mentioned in [**14**, Remark 2.5].

Example 3.4. Now consider the irreducible curve $C\colon x^3z^4 + xy^5z + x^7 + y^7 = 0$, which has a *non-weighted homogeneous singularity* located at (0:0:1) with Milnor number $\mu = 12$ and Tjurina number $\tau = 11$. It is then easy to see that $\tilde{H}^0(U) = 0 = H^1(U)$, and $H^2(U)$ has dimension 18, with classes of Hodge type $(2,1)$ and $(1,2)$. In particular, $\dim F^2H^2(U) = 9$.

On the other hand, the spectral sequence $E_1(f)$ has the following non-zero terms: $E_1^{2,0}$, which is 15 dimensional; and all $E_1^{p,q}$ for $p+q=1$ or $p+q=2$ and $q>0$, which are 11 dimensional, since $\tau(C) = 11$, except $E_1^{1,1}$ which is again 15 dimensional. Indeed, the computation using CoCoA or Singular in this case yields

$$\begin{aligned}\mathrm{HP}(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 25t^6 + 27t^7 \\ + 27t^8 + 25t^9 + 21t^{10} + 15t^{11} + 12t^{12} + 11t^{13} + \cdots\end{aligned}$$

with stabilization threshold $\mathrm{st}(C) = 13$. It follows that $d_1\colon 0 = E_1^{1,0} \to E_1^{2,0}$ is the zero map, and hence $\dim E_2^{2,0} = 15$. The other differentials $d_1\colon E_1^{1-t,t} \to E_1^{2-t,t}$ for $t \geqslant 1$ have a one-dimensional kernel, which can be seen using Theorem 2.4 (i) and [**10**, Proposition (3.4), Example (3.5) (i) and Corollary (4.3)], where it is shown that in this case the differentials $d_2\colon E_2^{1-t,t} \to E_2^{3-t,t-1}$ are injective for $t > 0$ in the local setting. It follows that $E_3 = E_\infty$ has the following non-zero terms: $E_3^{2,0}$ of dimension 14, and $E_3^{1,1}$ of dimension 4. In particular, one has

$$\dim F^2H^2(U) = 9 < 14 = \dim P^2H^2(U).$$

4. The syzygies of nodal hypersurfaces

First we give a geometric interpretation of a syzygy R_m as in (1.2) in the case $n = 2$ using [**2**, § 2.1]. Let F_f be the Milnor fibre of f, which is the smooth affine surface in $\mathbb{C}^3$ given by the equation $f(x,y,z) = 1$. Then there is a monodromy isomorphism $h\colon F_f \to F_f$ given by multiplication by $\lambda = \exp(2\pi \mathrm{i}/N)$ and an induced monodromy operator $h^1\colon H^1(F_f) \to H^1(F_f)$. The eigenvalues of h^1 are exactly the Nth roots of unity, and for each $k = 0, 1, \dots, N-1$ there is a rank 1 local system L_k on U such that

$$H^*(F_f)_{\lambda^k} = H^*(U, L_k), \tag{4.1}$$

where on the left-hand side we have the corresponding eigenspace and on the right-hand side we have the twisted cohomology of U with coefficients in L_k (for details see [**12**, Proposition 6.4.6]).

Let $\mathcal{L}_k$ be the Deligne extension of L_k over the nodal curve C such that the eigenvalues of the residue of the connection are contained in the interval $[0,1)$. In our case, the line bundle $\mathcal{L}_k$ is precisely $\mathcal{O}_{\mathbb{P}^2}(-k)$ (see [**2**, Equation (2.1.2)]) and we have the following relation with the Hodge filtration on $H^*(F_f)$:

$$\mathrm{Gr}_F^p H^{p+q}(F_f)_{\lambda^k} = H^q(\mathbb{P}^2, \Omega^p_{\mathbb{P}^2}(\log C) \otimes \mathcal{L}_k). \tag{4.2}$$

In particular, we get

$$\mathrm{Gr}_F^1 H^1(F_f)_{\lambda^k} = H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log C) \otimes \mathcal{L}_k). \tag{4.3}$$

Now, the curve C being nodal, it follows that $H^1(F_f)_{\lambda^k} = 0$ for $k = 1, \ldots, N-1$ (see [**12**, Corollary 6.4.14] for a stronger result).

Assume now that we have a non-zero syzygy R_m as in (1.2) with $m < N-2$. Consider the non-zero 2-form $\omega \in \Omega^2_{m+2}$ given by $\omega = a\,\mathrm{d}y \wedge \mathrm{d}z - b\,\mathrm{d}x \wedge \mathrm{d}z + c\,\mathrm{d}x \wedge \mathrm{d}y$ and note that $\mathrm{d}f \wedge \omega = 0$. The 1-form

$$\alpha = \frac{\Delta(\omega)}{f} \tag{4.4}$$

is an element of $H^0(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}(\log C) \otimes \mathcal{L}_k)$, with $k = N - 2 - m > 0$. To see this, use the formula for $\mathrm{d}\alpha$ given in [**11**, Equation (1.10), p. 181]. Moreover, $\alpha \neq 0$, since the kernel of $\Delta\colon \Omega^2 \to \Omega^1$ is the free S-module spanned by $\sigma = \Delta(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$ and $\mathrm{d}f \wedge \sigma = Nf\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \neq 0$. But this is in contradiction to $H^1(F_f)_{\lambda^k} = 0$ in view of (4.3).

Next we describe all the syzygies R_m as in (1.2) with $n = 2$ and $m = N-2$. This is the same as describing $H^2(K^*(f))_N$, and we know from the previous section that $\dim H^2(K^*(f))_N = r - 1$ (see (3.3)), i.e. this is essentially to lift the basis ω_j in (3.1) to a basis of $H^2(K^*(f))_N$. Note that

$$\omega_j = \frac{\alpha_j}{N_j N_r f},$$

where $\alpha_j = N_r f_1 \cdots \hat{f}_j \cdots f_r\,\mathrm{d}f_j - N_j f_1 \cdots \hat{f}_r\,\mathrm{d}f_r$ for $j = 1, \ldots, r-1$, with $\hat{f}_j$ meaning that the factor f_j is missing. Define $\beta_j = -f_1 \cdots \hat{f}_j \cdots \hat{f}_r\,\mathrm{d}f_j \wedge \mathrm{d}f_r$ and note that

$$\Delta(\beta_j) = -f_1 \cdots \hat{f}_j \cdots \hat{f}_r \Delta(\mathrm{d}f_j \wedge \mathrm{d}f_r) = \alpha_j.$$

For $r = 2$, β_1 is a good lifting since $\mathrm{d}f \wedge \beta_1 = 0$ and we are done. However, for $r > 2$, β_j is not a good lifting, since in general one has

$$\mathrm{d}f \wedge \beta_j = -\sum_{k \neq j;\, k \neq r} f^2/(f_k f_j f_r)\,\mathrm{d}f_k \wedge \mathrm{d}f_j \wedge \mathrm{d}f_r = f g_j\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

for some $g_j \in S_{N-3}$ that is non-zero in general. (A formula for g_j is given in Theorem 4.1 using the Jacobian determinant $\mathrm{Jac}(f_k, f_j, f_r)$ of the three functions f_k, f_j, f_r with respect to x, y and z.)

To correct this problem, we look for a modification of the form

$$\gamma_j = \beta_j + h_j \sigma,$$

where $h_j \in S_{N-3}$ and $\sigma = \Delta(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$ as above. Now $\mathrm{d}f \wedge \gamma_j = (fg_j + Nfh_j)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = 0$ if we choose $h_j = -g_j/N$. The resulting γ_j for $j = 1, \ldots, r-1$ yield a basis of $H^2(K^*(f))_N$.

Hence, we have proved the following result.

Theorem 4.1. *Let $C\colon f = 0$ be a nodal curve of degree N in $\mathbb{P}^2$. Then, for any $q < N$, $H^2(K^*(f))_q = 0$, and $H^2(K^*(f))_N$ is $(r-1)$ dimensional with a basis given by*

$$\gamma_j = -f_1 \cdots \hat{f}_j \cdots \hat{f}_r\,\mathrm{d}f_j \wedge \mathrm{d}f_r + h_j \sigma$$

for $j = 1, \dots, r-1$. Here r is the number of irreducible components of C, $f_j = 0$ are reduced equations for these components, $\sigma = \Delta(\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z)$, $h_1 = 0$ if $r = 2$, and

$$h_j = \frac{\sum_{k \neq j;\, k \neq r} f/(f_k f_j f_r) \operatorname{Jac}(f_k, f_j, f_r)}{N}$$

if $r > 2$.

For an arbitrary curve C having r irreducible components $C_j \colon f_j = 0$, the above elements γ_j yield $r-1$ linearly independent elements in $H^2(K^*(f))_N$, which are killed by d_1. It may happen that $\dim H^2(K^*(f))_N > r-1$, as we have seen in Example 3.2.

The corresponding vanishing result in the general case of nodal hypersurfaces is considered in [**17**], but in this general case there is no description of an explicit basis of the lowest-degree (possibly non-zero) syzygies as in Theorem 4.1. For an alternative proof of the vanishing part (without using Hodge theory) in a more general curve setting, see [**20**].

Example 4.2. In this example we look at some curves having low-degree relations R_m, as in (1.2).

(i) It is clear that a curve $C \colon f = 0$ admits a relation of degree $m = 0$ if and only if, up to a linear coordinate change, we have that the equation f is independent of z. In this case,

$$\operatorname{HP}(M(f))(t) = \frac{(1-t^{N-1})^2}{(1-t)^3}.$$

Hence, $\operatorname{ct}(C) = N-2$ (this is the minimal possible value) and $\operatorname{st}(C) = 2N-4$.

(ii) The curve $C \colon x^p y^q + z^N = 0$ for $p + q = N$ admits an obvious relation of degree 1, namely,

$$qxf_x - pyf_y = 0.$$

In this case, $\operatorname{ct}(C) = N-1$.

(iii) The curve $C \colon z^p(x^q + y^q) + x^N + y^N = 0$ for $p + q = N$ admits an obvious relation of degree $2p$, namely,

$$z^{p-1}x(qz^p + Ny^p)f_x + z^{p-1}y(qz^p + Nx^p)f_y - \frac{1}{p}(qz^p + Ny^p)(qz^p + Nx^p)f_z = 0.$$

It is easy to see that this relation is not a consequence of the trivial relations T_{ij} in (4.5). On the other hand, a computation in the case $N = 7$, $p = 4$ shows that

$$\begin{aligned}\operatorname{HP}(M(f))(t) = 1 &+ 3t + 6t^2 + 10t^3 + 15t^4 + 21t^5 + 25t^6 + 27t^7 \\ &+ 27t^8 + 25t^9 + 21t^{10} + 16t^{11} + 12t^{12} + 9t^{13} + 8t^{14} + \cdots\end{aligned}$$

with stabilization threshold $\operatorname{st}(C) = 14$. It follows that $\operatorname{ct}(C) = 10$, which implies via (1.3) that $\operatorname{mdr}(C) = 5$, i.e. the above relation does not in general have minimal degree. However, this is the case for $p = N-2$, when the curve C has a node at (0:0:1) and the corresponding relation has degree $2N-4$.

This is a very special case of Theorem 1.5 stated in § 1, which we now prove.

Proof. Choose the coordinates on $\mathbb{P}^n$ such that $H_0\colon x_0 = 0$ is transverse to D, i.e. the intersection $H_0 \cap D$ is smooth. It follows as in [**6**] that the partial derivatives $f_1, \dots, f_n$ of f form a regular system in S; in particular, they vanish at a finite set of points on $\mathbb{P}^n$, say $p_1, \dots, p_r$. Some of these points, say p_j for $j = 1, \dots, q$, are the nodes on D, i.e. the points in the set $\mathcal{N}$. It follows that the divisors $D_j\colon f_j = 0$ for $j = 1, \dots, n$ intersect transversely at any point $p_j \in \mathcal{N}$. To see this, one may work in the affine chart $x_0 = 1$, where $x_1, \dots, x_n$ may be used as coordinates, and use the definition of nodes as the singularities where the hessian of a (local) equation is non-zero.

Assume that we have a non-zero element in $H^n(K^*(f))_{nN-n-1-k}$ for some $0 \leqslant k \leqslant s$, with $s = nN - 2n - 1$. This is the same as having a relation

$$R_m\colon a_0 f_0 + a_1 f_1 + \dots + a_n f_n = 0,$$

where the $a_j \in S$ are homogeneous of degree $m = s - k$ and R_m is not a consequence of the relations

$$T_{ij}\colon f_j f_i - f_i f_j = 0. \tag{4.5}$$

Since p_j is not a singularity for D for $j > q$, it follows that $f_0(p_j) \neq 0$ in this range. Hence, for $j > q$, the relation R_m implies that the germ of the function induced by a_0 at p_j (dividing by some homogeneous polynomial b_j of degree m such that $b_j(p_j) \neq 0$) belongs to the ideal in $\mathcal{O}_{p_j}$ spanned by the local equations of the divisors $D_1, \dots, D_n$.

We now apply the Cayley–Bacharach theorem as stated in [**21**, Theorem CB7].

Let Γ be the zero-dimensional subscheme of $\mathbb{P}^n$ defined by the partial derivatives $f_1, \dots, f_n$. Let Γ' and Γ'' be the subschemes of Γ, residual to one another in Γ, and such that the support of Γ' is the set $\mathcal{N}' = \{p_{q+1}, \dots, p_r\}$ and the support of Γ'' is the set $\mathcal{N}$. Intuitively, Γ' is the 'restriction' of the scheme Γ to $\mathcal{N}'$, and Γ'' is the 'restriction' of the scheme Γ to $\mathcal{N}$. In particular, the scheme Γ'' is reduced.

Note that the above discussion implies that the dimension of the family of hypersurfaces a_0 of degree $m = s - k$ containing Γ' (modulo those containing all of Γ) is exactly the dimension of $H^n(K^*(f))_{nN-n-1-k}$.

On the other hand, for s as above and $0 \leqslant k \leqslant s$, the Cayley–Bacharach theorem states that this dimension is equal to the defect $\operatorname{def} S_k(\mathcal{N})$, thus proving the first claim in Theorem 1.5.

Next we have

$$\begin{aligned}\dim H^n(K^*(f))_j &= \dim H^{n+1}(K^*(f))_{j+N} - \dim H^{n+1}(K^*(f_s))_{j+N} \\ &= \dim M(f)_{j+N-n-1} - \dim M(f_s)_{j+N-n-1}.\end{aligned}$$

Moreover, $j \geqslant n(N-1)$ is equivalent to $j + N - n - 1 > (n+1)(N-2)$, and hence $\dim M(f)_{j+N-n-1} = \tau(D) = |\mathcal{N}|$ and $\dim M(f_s)_{j+N-n-1} = 0$, thus proving the second claim in Theorem 1.5. □

Example 4.3. We use the notation from Theorem 1.5 and set $T = (n+1)(N-2)$.

(i) If $|\mathcal{N}| = 1$, then $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k \geqslant 0$, and therefore we have $\operatorname{ct}(D) = \operatorname{st}(D) = T$.

(ii) If $|\mathcal{N}| = 2$, then $\operatorname{def} S_0(\mathcal{N}) = 1$ and $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k \geqslant 1$. It follows that $\operatorname{ct}(D) + 1 = \operatorname{st}(D) = T$.

(iii) If $|\mathcal{N}| = 3$, and the three nodes are not collinear, then $\operatorname{def} S_0(\mathcal{N}) = 2$ and $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k \geqslant 1$. It follows that $\operatorname{ct}(D) + 1 = \operatorname{st}(D) = T$ unless $n = 2$, when $\operatorname{ct}(D) = \operatorname{st}(D) = T - 1$.

For three collinear points, $\operatorname{def} S_0(\mathcal{N}) = 2$, $\operatorname{def} S_1(\mathcal{N}) = 1$ and $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k \geqslant 2$. It follows that $\operatorname{ct}(D) + 2 = \operatorname{st}(D) = T$ and $\dim M(f)_{T-1} = n + 2$.

To have some explicit examples of these two distinct situations, consider the following two curves of degree $N = 4$:

$$C\colon f = (x^3 + y^3 + z^3)x = 0$$

and

$$C'\colon f' = x^2y^2 + y^2z^2 + x^2z^2 - 2xyz(x + y + z) - (2xy + 3yz + 4xz)^2 = 0.$$

The curve C then has three collinear nodes and the corresponding Hilbert–Poincaré series is

$$\operatorname{HP}(M(f))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 4t^5 + 3t^6 + \cdots$$

with $\operatorname{st}(C) = 6$. In fact, the coefficients of t^k for $0 \leqslant k \leqslant 2N - 4 = 4$ are determined by Theorem 1.2 and the remaining terms are determined by Theorem 1.5.

In the same way, one may obtain

$$\operatorname{HP}(M(f'))(t) = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + \cdots$$

with $\operatorname{st}(C') = 5$, using the fact that C' has three non-collinear nodes located at (1:0:0), (0:1:0) and (0:0:1).

(iv) Here is one example of a sextic curve with six nodes. Consider the curve

$$C\colon f = x^2(x + z)^2(x - z)^2 - y^2(y - z)^2(y^2 + 2z^2) = 0.$$

The curve C then has six nodes, three of them on the line $y = 0$ (namely, (0:0:1), (1:0:1) and (−1:0:1)) and the other three on the line $y - z = 0$ (namely, (0:1:1), (1:1:1) and (−1:1:1)). The corresponding Hilbert–Poincaré series is

$$\begin{aligned}\operatorname{HP}(M(f))(t) = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + 18t^5 + 19t^6 \\ + 18t^7 + 15t^8 + 10t^9 + 7t^{10} + 6t^{11} + \cdots\end{aligned}$$

with $\operatorname{st}(C) = 11$. This result follows exactly by the same argument as above, using, in addition, the equalities $\operatorname{def} S_0(\mathcal{N}) = 5$, $\operatorname{def} S_1(\mathcal{N}) = 3$, $\operatorname{def} S_2(\mathcal{N}) = 1$ and $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k \geqslant 3$.

Remark 4.4. Let $C\colon f = 0$ be a degree N curve in $\mathbb{P}^2$ such that any singular point of C that is not a node is a unibranch singularity, and let $\mathcal{N}$ denote the set of its nodes. Then $\operatorname{def} S_k(\mathcal{N}) = 0$ for $k > N - 3$ and $\operatorname{def} S_{N-3}(\mathcal{N}) = r - 1$, where r is the number of irreducible components of C.

This can be derived as follows. Let I' be the ideal of functions in S vanishing at the points in $\mathcal{N}$. It was then shown in [**24**, Proposition 3.6] that there is a minimal resolution

$$0 \to \bigoplus_{i=1,t} S(-b_i) \to \bigoplus_{j=1,t+1} S(-a_j) \to S \to S/I' \to 0$$

such that $0 < a_j < N$ for any j, $0 < b_i \leqslant N$ for all i and

$$|\{i\colon b_i = N\}| = r - 1.$$

In fact, [**24**, Proposition 3.6] is stated only for curves with nodes and ordinary cusps, but the only point in the proof where one uses the ordinary cusps is to derive the equality (10), which may also be obtained in our slightly more general setting from [**11**, Diagram 3.14, p. 201].

The above resolution implies that the Hilbert–Poincaré series of S/I' is given by the following equality:

$$\mathrm{HP}(S/I')(t) = \frac{1 - \sum_j t^{a_j} + \sum_i t^{b_i}}{(1-t)^3}.$$

Since $\mathcal{N}$ is a finite set of points, it follows that this series can be rewritten as

$$\mathrm{HP}(S/I')(t) = \frac{Q(t)}{1-t},$$

where $Q(t)$ is a polynomial in t of degree at most $N-2$, the coefficient c_{N-2} of t^{N-2} being exactly $r-1$. It follows that $\dim(S/I')_k = |\mathcal{N}|$ for $k \geqslant N-2$ and that

$$\dim(S/I')_{N-3} = |\mathcal{N}| - c_{N-2} = |\mathcal{N}| - r + 1,$$

which proves our claim since one has $\operatorname{def} S_k(\mathcal{N}) = |\mathcal{N}| - \dim(S/I')_k$ for any k.

Alternatively, one may complete the proof using the formula for the defect or superabundance $\operatorname{def} S_k(\mathcal{N})$ as the difference between the Hilbert polynomial and the Hilbert function given just before the statement of Lemma 3.4 in [**24**].

Note that the other main results of our paper *do not* extend to this more general setting. For instance, the curve C constructed in Example 4.2 (ii) for $p = 2$, $q = 3$, $N = 5$ has as singularities two unibranch singularities located at (1:0:0) and (0:1:0) and has a relation of degree 1, i.e. $H^2(K^*(f))_3 \neq 0$, and hence Theorem 4.1 and its consequence, Theorem 1.2, fail in this case.

Moreover, Example 3.3 shows that the spectral sequences considered in § 2 in the presence of even ordinary cusps may exhibit different behaviour than in the case of nodes. Indeed, the differential $d_1\colon E_1^{1,0} \to E_1^{2,0}$ is trivial for a nodal curve and it is non-trivial in Example 3.3.

The same example shows that Theorem 1.5 also fails in this more general setting, since $\dim H^2(K^*(f))_4 = 1$ and $\operatorname{def} S_1(\mathcal{N}) = |\mathcal{N}| - \dim(S/I')_1 = 0 - 0 = 0$ since $\mathcal{N} = \emptyset$.

The resolutions constructed in [**16**] for the Jacobian ideals of Chebyshev curves show that there are no similar results to [**24**, Proposition 3.6] for such Jacobian ideals, not even for nodal curves.

Remark 4.5. For a nodal 3-fold $D\colon f=0$ in $\mathbb{P}^4$ of degree N, the fact that D is factorial (i.e. the quotient $S/(f)$ is a unique factorization domain) can be expressed as a vanishing property of a certain defect, namely, $\operatorname{def} S_{2N-5}(\mathcal{N})=0$ (see Cheltsov's paper [**5**, Remark 1.2]). It follows that [**5**, Theorem 1.4] can be restated as saying that $\operatorname{def} S_{2N-5}(\mathcal{N})=0$ when $|\mathcal{N}|<(N-1)^2$, which in turn may be restated in view of Theorem 1.5 as saying that the corresponding space of syzygies R_{2N-4} is trivial in such a case. On the other hand, [**17**, Theorem 2.1 (ii)] implies that $R_m=0$ for $m<2N-4$ and for *any* nodal 3-fold D in $\mathbb{P}^4$ of degree N.

5. The spectral sequence in the case of a nodal surface

Let $S\colon f=0$ be a nodal surface in $\mathbb{P}^3$ of degree N. Then S is a $\mathbb{Q}$-homology manifold satisfying $b_0(S)=b_4(S)=1$, $b_1(S)=b_3(S)=0$ and the middle Betti number $b_2(S)$ is computable. For example, using the formula $b_2(S)=b_2(S_s)-n(S)$, where S_s is a smooth surface in $\mathbb{P}^3$ of degree N, the corresponding second Betti number is given by

$$b_2(S_s)=\frac{(N-1)^4-1}{N}+2$$

and $n(S)=\tau(S)$ is the number of nodes, i.e. the cardinal of the set of nodes $\mathcal{N}$ of S. It follows that the complement U has at most two non-zero cohomology groups. The first of them, $H^0(U)$, is one dimensional and of Hodge type $(0,0)$, and so does not interest us. The second one, $H^3(U)$, is dual to $H^3_c(U)(-3)$, and $H^3_c(U)$ is isomorphic to $\operatorname{coker}(H^2(\mathbb{P}^3)\to H^2(S))$, the morphism being induced by the inclusion $i\colon S\to\mathbb{P}^3$. It follows that the mixed Hodge structure on $H^3(U)$ is pure of weight 4 with

$$h^{4,0}(H^3(U))=h^{0,4}(H^3(U))=0,$$
$$h^{3,1}(H^3(U))=h^{1,3}(H^3(U))=h^{2,0}(S)=h^{2,0}(S_s)=p_g(S_s),$$

where the geometric genus of S_s is given by

$$p_g(S_s)=\binom{N-1}{3}$$

and

$$h^{2,2}(H^3(U))=h^{1,1}(S)-1=h^{1,1}(S_s)-n(S)-1.$$

In particular, we have $P^1H^3(U)=F^1H^3(U)=H^3(U)$, as in Remark 2.3.

We now look at the non-zero terms in the E_1-term of the spectral sequence $E_r^{p,q}(f)$. They are sitting on two lines, given by $L\colon p+q=3$ and $L'\colon p+q=2$.

We look first at the terms on the line L. The term $E_1^{3,0}(f)=H^4(K^*)_N$ is isomorphic as a $\mathbb{C}$-vector space to $M(f)_{N-4}$, and hence has dimension $p_g=p_g(S_s)$. The corresponding limit term $E_\infty^{3,0}(U)=P^3H^2(U)$ therefore has dimension at most p_g. On the other hand, the above formulae for $h^{p,q}(H^3(U))$ imply that $\dim F^3H^2(U)=p_g$. In this case we conclude by Theorem 2.1 that $F^3H^3(U)=P^3H^3(U)$.

The term $E_1^{2,1}(f) = H^4(K^*)_{2N}$ is isomorphic to $M(f)_{2N-4}$. Theorem 1.5 and Griffiths's results for the smooth case in [**23**] imply that

$$\dim M(f)_{2N-4} = M(f_s)_{2N-4} + \operatorname{def} S_{2N-4}(\mathcal{N}) = h^{1,1}(S_s) - 1 + \operatorname{def} S_{2N-4}(\mathcal{N}). \quad (5.1)$$

The term $E_1^{1,2}(f) = H^4(K^*)_{3N}$ is isomorphic to $M(f)_{3N-4}$, and hence

$$\dim M(f)_{3N-4} = M(f_s)_{3N-4} + \operatorname{def} S_{N-4}(\mathcal{N}) = p_g + \operatorname{def} S_{N-4}(\mathcal{N}). \quad (5.2)$$

The term $E_1^{3-q,q}(f) = H^4(K^*)_{(q+1)N}$ for $q \geqslant 3$ is isomorphic to $M(f)_{(q+1)N-4}$, which has dimension $n(S)$. Moreover, the corresponding limit terms $E_\infty^{3-q,q}(U) = \operatorname{Gr}_P^{3-q}(H^2(U))$ clearly vanish for $q \geqslant 3$.

We look now at the terms on the line L'. By Theorem 1.5, the term $E_1^{2,0}(f) = H^3(K^*)_N$ has dimension $\operatorname{def} S_{2N-4}(\mathcal{N})$. On the other hand, $E_\infty^{2,0}(f) = 0$, which implies, in view of the equality $E_\infty^{3,0}(f) = E_1^{3,0}(f)$ established above, that in fact $\operatorname{def} S_{2N-4}(\mathcal{N}) = 0$.

The dimension of the term $E_1^{1,1}(f) = H^3(K^*)_{2N}$ is equal to $\operatorname{def} S_{N-4}(\mathcal{N})$, again by Theorem 1.5. And, again, $E_\infty^{1,1}(f) = 0$ implies, in view of the equality $E_\infty^{3,0}(f) = E_1^{3,0}(f)$ established above, that the differential $d_1\colon E_1^{1,1}(f) \to E_1^{2,1}(f)$ is injective.

The dimension of the other terms $E_1^{2-q,q}(f) = H^3(K^*)_{(q+1)N}$ for $q \geqslant 2$ is equal to $n(C)$, and the corresponding differentials $d_1\colon E_1^{2-q,q}(f) \to E_1^{3-q,q}(f)$ are injective by Theorem 2.4 (ii).

In this way we have proved the following theorem.

Theorem 5.1. *Let $S\colon f = 0$ be a nodal surface in $\mathbb{P}^3$ of degree N and let $\mathcal{N}$ denote the set of its nodes. The following statements then hold.*

(i) *The E_1-spectral sequences $E_r^{p,q}(U)$, $E_r^{p,q}(\tilde{A})$ and $E_r^{p,q}(f)$ degenerate at the E_2-term, i.e. $E_2 = E_\infty$ for any of these E_1-spectral sequences.*

(ii) *The subspace $P^3H^3(U) = F^3H^3(U)$ of $H^3(U)$ has dimension*

$$p_g = \binom{N-1}{3}.$$

(iii) *We have that*

$$\dim \operatorname{Gr}_P^2(H^3(U)) = h^{1,1}(S_s) - 1 - \operatorname{def} S_{N-4}(\mathcal{N})$$

and

$$\dim \operatorname{Gr}_F^2(H^3(U)) = h^{1,1}(S_s) - 1 - n(S).$$

In particular, $P^2H^3(U) = F^2H^3(U)$ if and only if the nodal surface S is smooth or $N < 4$.

(iv) $\operatorname{def} S_{2N-4}(\mathcal{N}) = 0$.

https://doi.org/10.1017/S0013091514000182 Published online by Cambridge University Press

Remark 5.2.

(i) Let I be the homogeneous ideal in S of polynomials vanishing on the set of nodes $\mathcal{N}$. The above formulae imply that

$$\dim \mathrm{Gr}_F^2(H^3(U)) = \dim(I/J_f)_{2N-4},$$

which is a special case of [**18**, Theorem 2.2].

(ii) The ideal I defined in (i) also occurs in the following formula, which is again a consequence of Theorem 5.1:

$$\dim P^2H^3(U) - \dim F^2H^3(U) = \dim(S/I)_{N-4}.$$

When the number of nodes is large, this difference can also be very large. For instance, if S is a Chebyshev surface in $\mathbb{P}^3$ whose affine equation is

$$T_N(x) + T_N(y) + T_N(z) + 1 = 0,$$

where $T_N(t)$ is the degree N Chebyshev polynomial in $\mathbb{C}[t]$, then $I_{N-4} = 0$ (see [**17**, Proposition 3.1]). It follows in this case that

$$\dim P^2H^3(U) - \dim F^2H^3(U) = \dim S_{N-4} = \binom{N-1}{3}.$$

Acknowledgements. A.D. was supported by the Institut Universitaire de France.

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