ESTIMATION OF FUTURE DISCRETIONARY BENEFITS IN TRADITIONAL LIFE INSURANCE

By

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Abstract

In the context of life insurance with profit participation, the future discretionary benefits (*FDB*), which are a central item for Solvency II reporting, are generally calculated by computationally expensive Monte Carlo algorithms. We derive analytic formulas to estimate lower and upper bounds for the *FDB*. This yields an estimation interval for the *FDB*, and the average of lower and upper bound is a simple estimator. These formulae are designed for real world applications, and we compare the results to publicly available reporting data.

Keywords

Solvency II, future discretionary benefits, market consistent valuation.

1. Introduction

As of 1. 1. 2016, the European Union has implemented a new regulatory regime (Solvency II Directive, 2009; Commission, 2014, 2015) which requires insurance companies operating in Member States to assign market consistent values to their balance sheet items. This requirement concerns, both, assets and liabilities and is therefore a full balance sheet approach.

Market consistent valuation in the context of life insurance with profit participation has been a developing subject over the past two decades. Early works in this context include Sheldon and Smith (2004), O'Brien (2009), Delong (2011), Wuthrich (2016). These works all highlight the interdependencies that exist between the insurer's asset portfolio and the policyholder's expected payoff. Indeed, the defining feature of with profit contracts is that the

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TABLE 1

RISK MARGIN VERSUS BEST ESTIMATE. AGGREGATE OF SOLO COMPANIES IN EEA FROM 2016 TO 2020 FOR LIFE INSURANCE (EXCLUDING HEALTH, INDEX-LINKED AND UNIT-LINKED). IN MILLION EURO AND PERCENT OF BEST ESTIMATE. SOURCE: ITEMS R0670 AND R0680 IN 'BALANCE SHEET EEA' IN EIOPA STATISTICS.

	2016	2017	2018	2019	2020
Best estimate	4,759,241.59	4,893,285.81	4,812,203.51	5,331,803.61	5,459,642.07
Risk margin	79,574.17	77,532.75	76,709.35	83,692.55	89,736.47
	1.67%	1.58%	1.59%	1.57%	1.64%

policyholder's benefit is a sum of, firstly, a guaranteed part depending, in particular, on the guaranteed technical interest rate, and, secondly, a bonus benefit depending, in particular, on the performance of the company's asset portfolio.

To obtain a more realistic model, the performance of the asset portfolio should be measured as a return on balance sheet items, and, accordingly, there are recent works which incorporate a strongly simplified, or stylized, version of a balance sheet projection (Gerstner *et al*., 2008, 2009; Engsner *et al*., 2017; Bacinello *et al*., 2021; Falden and Nyegaard, 2021).

However, to have a more accurate representation of the company's financial income, which is to be shared with policyholders, the return on assets should be derived according to local generally accepted accounting principles (local GAAP). This necessity is explained in detail in Dorobantu *et al*. (2020), Dhaene *et al*. (2017). Moreover, financial revenue can be controlled to a certain extent by management actions by realizing unrealized gains of individual assets, that is the difference of market and local GAAP book value (Dorobantu *et al*., 2020). Together with further management actions (e.g., strategic asset allocation, reinvestment strategy, profit sharing and profit declaration), this setup leads, in practice, to computationally expensive Monte Carlo algorithms in order to obtain realistic calculations of market consistent values of liabilities. See Vedani *et al*. (2017), Dorobantu *et al*. (2020), Dhaene *et al*. (2017), Albrecher *et al*. (2018).

Under Solvency II, the market consistent value of liabilities is defined as a sum of a best estimate and a risk margin (Directive, 2009, Article 77). For the purpose of life insurance with profit participation, the risk margin is generally several orders of magnitudes smaller than the best estimate (EIOPA Statistics), as shown in Table 1.

The best estimate of liabilities is defined as the expected value of discounted and probability weighted future policyholder and cost cash flows stemming from contracts which are active at valuation time. Thus, new business is not considered, and the expected value is to be taken with respect to a risk neutral measure. This definition of Directive (2009, Article 77) is clarified in Commission (2014).

Regarding life insurance with profit participation, the best estimate, denoted henceforth by BE, can be split into a sum of two parts: $BE = GB + FDB$; here GB denotes the value of those cash flows which are guaranteed at valuation

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time while *FDB* is the value of future discretionary benefits. Both, *FDB* and the total value *BE*, have to be reported individually by insurance companies on a quarterly basis. The significance of this splitting is that the guaranteed benefits, *GB*, are calculated by methods which are close to classical actuarial computations. If one allows dynamic policyholder behavior such that, for example, surrender depends on economic scenarios, then a stochastic cash flow model is needed to calculate *GB*. Nevertheless, in comparison to the future discretionary benefits, the guaranteed benefits are more straightforward to calculate and validate. This is so because the *FDB* depends directly on the company's surplus and thus in particular on management actions and financial revenue, while *GB* depends on these quantities only indirectly via the part that is affected by dynamic policyholder behavior. If a company chooses not to model dynamic policyholder behavior, then *GB* can be calculated by a deterministic actuarial cash flow model.

Due to the above described complexity, there exist no closed formula solutions for *FDB* calculation in the context of life insurance with profit participation. An analytic formula for an estimate of a lower bound for *FDB* has been derived in Hochgerner and Gach (2019), and this has been applied to validate the reported *FDB* of one major German life insurance company for the reporting year 2017. The purpose of this paper is to improve and extend the approach of Hochgerner and Gach (2019) in the following ways:

(1) In Equation (3.2), we derive a representation of the *FDB* which holds in a generic valuation framework. This representation cannot directly be applied to calculate the *FDB* since it relies on quantities which are just as difficult to obtain as the *FDB* itself. However, due to its generality, it can be used to validate the results of a given stochastic cash flow model, and it is the basis for the subsequently described estimations.

(2) Starting from Equation (3.2), we derive in Section 5 estimations of lower and upper bounds, $\widehat{LB}$ and $\widehat{UB}$, for the *FDB*. These estimated bounds are given by the analytic formulas (5.17) and (5.18), and all the data needed to evaluate the latter are listed in Table 2.

(3) The constituents of the lower and upper bounds, $\widehat{LB}$ and $\widehat{UB}$, are derived in an economically meaningful way such that there is an interpretation for each term and the dependency on market conditions at valuation time is plausible. The latter point concerns specifically the dependence on the market view of interest rate volatility. This, apart from the new upper bound, is an improvement over the lower bound formula given in Hochgerner and Gach (2019).

(4) Given $\widehat{LB}$ and $\widehat{UB}$, we obtain an estimator, $\widehat{FDB} = (\widehat{LB} + \widehat{UB})/2$, for *FDB*. This estimator is useful if the estimation error $\pm\epsilon$ with $\epsilon = (\widehat{UB} - \widehat{LB})/2$ is sufficiently small. This can be measured against the market value, MV_0, of the company's portfolio at valuation time.

(5) In Section 7, we apply these formulas to publicly available data of a German life insurance company for the reporting years 2017, 2018 and

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2019, and compare our results with their numerically calculated value of *FDB*. We find that $\epsilon < 1.5\% \, MV_0$ for all three reporting periods, and, moreover, the difference $\delta = FDB - \widehat{FDB}$ satisfies $\delta < 1\% \, MV_0$ in all cases. Hence, in all three cases, the true value lies within the estimation interval and the estimation is remarkably accurate, as measured by δ. See Table 14.

(6) Finally, in Section 7 we also perform a sensitivity analysis with respect to the parameters that are used in calculations of $\widehat{LB}$ and $\widehat{UB}$ for the German life insurer, and find that the results are quite stable.

The main tools used to obtain the representation (3.2) are the no-leakage principle of Hochgerner and Gach (2019) (see Remark 2.4) and an evolution Equation (2.14) for the statutory reserves of shared profits. Thus, this derivation depends in a generic sense on the local GAAP framework but not on specific management rules.

The derivation of the estimates, $\widehat{LB}$ and $\widehat{UB}$, depends on a number of assumptions. These are all listed and discussed in Section 4. While the assumptions are tailored to the German and Austrian markets, their derivation is quite generic and is applicable whenever the company's revenue is given by well-defined local GAA Principles (e.g., as discussed for the French market in Dorobantu *et al*., 2020). The main observation in this context is that book values are expected to be more stable than market value movements. This conclusion is drawn from accounting principles for book values of assets (e.g., strict lower of cost or market price Dorobantu *et al*., 2020), on one hand, and the use of surplus funds as buffer accounts for statutory reserves (Gerstner *et al*., 2008), on the other hand.

The bounds (5.17) and (5.18) can be readily applied to real world data, as we show in Section 7. Immediate and practical applications therefore include the following:

(1) Internal validation: companies may use $\widehat{LB}$ and $\widehat{UB}$ to validate their *FDB* calculations and thus their valuation models.

(2) External validation by parent companies: holdings may wish to validate the valuation models in their subsidiaries.

(3) External validation by supervisors or auditors.

(4) Sensitivity analysis. The estimator $\widehat{FDB}$ depends on economic quantities such as the prevailing interest rate and volatilities. These are at the same time important drivers for the full Monte Carlo best estimate calculation. Hence, one obtains a closed estimation formula which allows for efficient sensitivity analysis without the necessity of stochastic computation.

Clearly, the validation of the best estimate will be most effective when the control is paired with a statistical analysis of the second-order assumptions leading to *GB* and a verification that the contract specific features, which give rise to the guaranteed benefit cash flows, are correctly implemented. We do not view our estimation formula as a substitute for *FDB* calculation as required by Solvency II regulation.

The derivation of the bounds (5.17) and (5.18) relies on a detailed investigation of cash flows generated by local accounting principles. The notation that is used in this context is quite heavy and therefore collected in the Appendix at the end of the paper.

2. Framework

We consider a life insurance company selling traditional life insurance products. 'Traditional' in this context means that the company's gross surplus is shared between policyholder, shareholder and tax office. Thus, these contracts have a profit sharing feature.

Further we assume that the life insurance company under consideration is subject to the Solvency II regulatory regime.

Let us fix a yearly time grid, $t = 0, 1, \ldots, T$, where $t = 0$ corresponds to valuation time and T may be large. For a time-dependent quantity, f_t, we denote the increments by $\Delta f_t = f_t - f_{t-1}$.

2.1. Book value return on assets

Let C_t denote the amount of cash held by the company at time t. Let further $\mathcal{A}_t$ denote the set of assets, other than the cash account, in the company's portfolio at time t. Each asset, $a \in \mathcal{A}_t$, has at each time step a book value, BV_t^a, and a market value, MV_t^a. The difference is the unrealized gains or losses, $UG_t^a = MV_t^a - BV_t^a$. The total portfolio values are correspondingly

$$BV_t = \sum_{a \in \mathcal{A}_t} BV_t^a + C_t, \quad MV_t = \sum_{a \in \mathcal{A}_t} MV_t^a + C_t, \quad UG_t = MV_t - BV_t.$$

In accordance with Solvency II regulation, we assume that all market values are determined in an arbitrage-free manner. Concretely, we assume an interest rate model specified on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$ such that $\mathbb{Q}$ is the risk neutral measure for the money market numeraire $B_t = \Pi_{j=0}^{t-1}(1 + F_j)$, where F_t is the simple one year forward rate valid between t and $t + 1$. For example, this setting is satisfied for the LIBOR market model with respect to the spot measure (Brigo and Mercurio, 2006). Further, all stochastic processes shall be adapted to $(\mathcal{F}_t)$, and all expected values, $E[\,\cdot\,]$, are with respect to $\mathbb{Q}$. The discount factor from s to $t < s$ is given by $D(t, s) = \Pi_{j=t}^{s-1}(1 + F_j)^{-1} = B_t B_s^{-1}$. Hence, the value of a zero coupon bond paying one unit of currency at s is given by $P(0, s) = E[B_s^{-1}]$, and more generally, we have $P(t, s) = E[D(t, s)]$ for $t < s$.

Let $a \in \mathcal{A}_{t-1}$ and let cf_s^a be the cash flow generated by a at $s \geq t$. For example, if a is a bond, stock equity position or real estate investment, then cf_t^a would correspond to a coupon or principle payment, dividend yield or rental

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revenue, respectively. The cash flow, cf_s^a, goes to the cash account, C_s, and increases it accordingly. Cash flows can be influenced by market movements (affecting, e.g., dividend payments) or by management decisions. Indeed, the management may decide at any time, $s < T_a$, prior to the asset's maturity to sell the asset and realize the market value MV_s^a as a cash flow. This step is called realization because it converts the unrealized gains UG_s^a into a book value return. If an asset, a, is sold at $s < T_a$ then its book value is terminated, $BV_s^a = 0$, and its market value is converted to a cash flow, $cf_s^a = (cf_s^a)' + MV_a^s$, where $(cf_s^a)'$ are those cash flows (e.g., coupon payments or dividend yield) that result from holding a over the period $[s-1, s]$.

The cash flow process, cf_t^a, is an $(\mathcal{F}_t)$-adapted process.

To avoid notational difficulties, we set $cf_s^a = 0 = MV_s^a$ for $s > T_a$. At $t-1$ the forward rate F_{t-1} is known, that is it is $(\mathcal{F}_{t-1})$-adapted. We have the relation

$$
\begin{aligned}
MV_{t-1}^a &= E\left[\sum_{s=t}^{T} D(t-1,s) cf_s^a \middle| \mathcal{F}_{t-1}\right] + E\Big[D(t-1,T) MV_T^a \Big| \mathcal{F}_{t-1}\Big] \\
&= \Big(1+F_{t-1}\Big)^{-1}\Big(E\Big[cf_t^a \Big| \mathcal{F}_{t-1}\Big] + E\Big[MV_t^a \Big| \mathcal{F}_{t-1}\Big]\Big)
\end{aligned} \tag{2.1}
$$

where the first line involves the final market value, MV_T^a, at the end of the projection which satisfies $MV_T^a = 0$ if $T_a \leq T$.

The book value return, ROA_t^a, at t of a single asset $a \in \mathcal{A}_{t-1}$ is

$$
ROA_t^a = cf_t^a + \Delta BV_t^a, \tag{2.2}
$$

and that of the portfolio is

$$
ROA_t = \sum_{a \in \mathcal{A}_{t-1}} ROA_t^a + F_{t-1} C_{t-1} \tag{2.3}
$$

since we have fixed yearly time steps and F_{t-1} is the corresponding forward rate. Combining (2.2) with (2.1) yields the implication for the portfolio that

$$
\begin{aligned}
ROA_t &= E[ROA_t | \mathcal{F}_{t-1}] + ROA_t - E[ROA_t | \mathcal{F}_{t-1}] \\
&= \sum_{a \in \mathcal{A}_{t-1}} \Big(E[cf_t^a | \mathcal{F}_{t-1}] + E[\Delta BV_t^a | \mathcal{F}_{t-1}]\Big) \\
&\quad + F_{t-1} C_{t-1} + ROA_t - E[ROA_t | \mathcal{F}_{t-1}] \\
&= \sum_{a \in \mathcal{A}_{t-1}} \Big((1+F_{t-1}) MV_{t-1}^a - E[MV_t^a | \mathcal{F}_{t-1}] + E[\Delta BV_t^a | \mathcal{F}_{t-1}]\Big) \\
&\quad + F_{t-1} C_{t-1} + ROA_t - E[ROA_t | \mathcal{F}_{t-1}] \\
&= F_{t-1} BV_{t-1} + F_{t-1} UG_{t-1} - E[\Delta UG_t | \mathcal{F}_{t-1}] + ROA_t - E[ROA_t | \mathcal{F}_{t-1}],
\end{aligned} \tag{2.4}
$$

where we have used that $\sum_{a\in\mathcal{A}_{t-1}} E[UG_t^a|\mathcal{F}_{t-1}] = E[UG_t|\mathcal{F}_{t-1}]$. The significance of this equation is that it splits ROA_t into three terms: the forward yield on the total book value, $F_{t-1}BV_{t-1}$; a term depending on the $\mathcal{F}_{t-1}$-prediction of return due to realizations of unrealized gains, $F_{t-1}UG_{t-1} - E[\Delta UG_t|\mathcal{F}_{t-1}]$; finally, the difference between return and $\mathcal{F}_{t-1}$-predicted return. This interpretation is illustrated in the following toy example.

Remark 2.1 (Example). Fix an arbitrary $1 < t < T$ and assume the company's portfolio at time $t-1$ consists of two identical bonds, denoted by a_1 and a_2, with maturity $t+1$, fixed coupon payment $KN/2$ and notional $N/2$, each, with $K \geq 0$ and $N > 0$. Suppose further that the company employs the strict lower of cost or market value such that $BV_s = \min(N, MV_s)$ for $s = t-1$ and $s = t$. The cash flows generated by a_i are coupon and notional payments or, if the management decides to sell a_i at $s < t+1$, an additional payment of the prevailing market value at s but no further (e.g., notional) payments after this time. We assume that the time $t-1$ predicted management rules are such that bonds are held to maturity, that is, $E[cf_t^{a_i}|\mathcal{F}_{t-1}] = KN/2$ and $E[cf_{t+1}^{a_i}|\mathcal{F}_{t-1}] = (K+1)N/2$ for $i = 1, 2$.

The market value of the portfolio at t is then $MV_t = (1+F_t)^{-1}(1+K)N$. At $t-1$, assume that $K \geq F_{t-1}$ and $E[(1+F_t)^{-1}|\mathcal{F}_{t-1}](1+K) \geq 1$ such that

$$MV_{t-1} = (1+F_{t-1})^{-1}E[(1+F_t)^{-1}|\mathcal{F}_{t-1}](1+K)N + (1+F_{t-1})^{-1}KN \geq N,$$

whence $BV_{t-1} = \min(N, MV_{t-1}) = N$. The formula for MV_{t-1} is, in fact, independent of management decisions at t which reflects the fact that future investment strategies cannot affect the currently given market consistent value of a portfolio. Now we describe the constituents of Equation (2.4) under two management decisions at t: decision (1) is defined as the management's default strategy of not taking any action at t, while decision (2) shall mean that management decides to sell one of the two bonds at t, namely a_1. For example, this decision might depend on the observation of F_t.

In both cases, we clearly have $F_{t-1}BV_{t-1} = F_{t-1}N$, independently of management actions or market movements in F_t.

Decision (1): using (2.1), the second term in (2.4) is now calculated as

$$\begin{aligned}F_{t-1}UG_{t-1} - \Delta E[UG_t|\mathcal{F}_{t-1}] &= F_{t-1}(MV_{t-1} - N) - E[MV_t|\mathcal{F}_{t-1}] + MV_{t-1} \\ &\quad + E[\min(MV_t, N)|\mathcal{F}_{t-1}] - N \\ &= (K - F_{t-1})N \\ &\quad - E[\max((F_t - K)/(1+F_t), 0)|\mathcal{F}_{t-1}]N.\end{aligned}$$

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The third term gives

$$\begin{aligned} ROA_t - E[ROA_t|\mathcal{F}_{t-1}] &= cf_t^{a_1} + cf_t^{a_2} + \min((1+F_t)^{-1}(1+K)N, N) - N - KN \\ &\quad - E[\min((1+F_t)^{-1}(1+K)N, N)|\mathcal{F}_{t-1}] + N \\ &= \min((1+F_t)^{-1}(1+K)N, N) \\ &\quad - E[\min((1+F_t)^{-1}(1+K)N, N)|\mathcal{F}_{t-1}] \end{aligned}$$

since $cf_t^{a_1} = KN/2$ with respect to strategy (1) and $cf_t^{a_2} = KN/2$ in all cases.

Decision (2): the second term remains unchanged since it corresponds to an $\mathcal{F}_{t-1}$ prediction. This term therefore captures the contribution of unrealized gains to book value return, ROA_t, that is due to the expected evolution of the portfolio. On the other hand, the third term becomes now, due to $cf_t^{a_1} = KN/2 + MV_t^{a_1}$ (coupon payments reward for having held the asset over the period $[t-1, t]$ and thus precede market placements) and $BV_t^{a_1} = 0$ (termination of asset), to

$$\begin{aligned} ROA_t - E[ROA_t|\mathcal{F}_{t-1}] &= MV_t^{a_1} + \min((1+F_t)^{-1}(1+K)N, N)/2 \\ &\quad - E[\min((1+F_t)^{-1}(1+K)N, N)|\mathcal{F}_{t-1}] \\ &= (1+F_t)^{-1}(1+K)N/2 \\ &\quad + \min((1+F_t)^{-1}(1+K)N, N)/2 \\ &\quad - E[\min((1+F_t)^{-1}(1+K)N, N)|\mathcal{F}_{t-1}]. \end{aligned}$$

Notice that $MV_t^{a_1} = (BV_t^{a_1})' + (UG_t^{a_1})'$ with $(BV_t^{a_1})' = \min((1+F_t)^{-1}(1+K)N, N)/2$ where $(\cdot)'$ denotes the asset's value before selling. Hence, this corresponds to a realization of unrealized gains, $(UG_t^{a_1})' = \max((K-F_t)/(1+F_t), 0)N/2$, as a cash flow. If $K \leq F_t$, it follows that $(UG_t^{a_1})' = 0$, and in this case the decision to sell does not have any effect on the return because of the lower of market value or cost principle. The third term thus represents the unpredicted return (due to market movements or management actions at t).

Thus, this remark illustrates how market fluctuations and management decisions at t influence the third term, $ROA_t - E[ROA_t|\mathcal{F}_{t-1}]$, in (2.4) while leaving the other two terms unaffected. The above formulae can be simplified by considering the special cases $F_t = 0$ or $K = 0$ with $F_t > 0$.

2.2. Contracts and model points

We now describe the contracts which exist in the company's liability portfolio at valuation time. All contracts are assumed to have a minimum guaranteed benefit and bonus benefit which depends on the company's profit and is shared between policyholder, shareholder and tax office. Each contract gives rise to either a single maturity or mortality benefit, or multiple annuity benefits, or a surrender benefit, cost payments associated with the contract and premium payments accepted by the company. We refer to Gerber (1997) for a detailed exposition of life insurance mathematics.

Each contract, c, has at each time step, t, a mathematical reserve, $\widetilde{V}_t^c$, defined by actuarial principles and first order assumptions. Further, at valuation time the contract may have already received bonus declarations. The sum of these declared bonuses up to and including time $t=0$ are denoted by $(\widetilde{DB}_0^{\leq 0})^c$. If the contract is still active at $t>0$, this account remains unchanged, $(\widetilde{DB}_t^{\leq 0})^c = (\widetilde{DB}_0^{\leq 0})^c$. The sum of bonus declarations after valuation time up to and including time $t>0$ is denoted by $\widetilde{DB}_t^c$, and we have $\widetilde{DB}_0^c = 0$. The contract total reserve is thus $\widetilde{LP}_t^c = \widetilde{V}_t^c + (\widetilde{DB}_t^{\leq 0})^c + \widetilde{DB}_t^c$.

If $m>0$ is the contract's maturity, the policyholder receives a guaranteed minimum benefit, gbf_m^c, plus declared bonuses, that is, the benefit cash flow at m equals $gbf_m^c + (\widetilde{DB}_{m-1}^{\leq 0})^c + \widetilde{DB}_{m-1}^c$. Profit that is generated at m is not shared with contracts maturing at the same time. Annuity payments are analogous with a corresponding fraction of bonuses being paid out. If the policyholder dies at $0<t<m$, the death benefit similarly consists of a sum of a guaranteed minimum benefit, gbf_t^c, and declared bonuses, $(\widetilde{DB}_{t-1}^{\leq 0})^c + \widetilde{DB}_{t-1}^c$. Notice that, contrary to $\widetilde{DB}_{t-1}^c$, the cash flow due to $(\widetilde{DB}_{t-1}^{\leq 0})^c$ is already guaranteed at valuation time since these bonuses have already been declared. If the policyholder decides to surrender the contract at $0<t<m$, the resulting surrender benefit is $(1-\chi_t^c)\widetilde{LP}_{t-1}^c$ where $0 \leq \chi_t^c \leq 1$ is the fraction which gives rise to the surrender gain made by the company.

For modeling purposes, it is advantageous to describe model points instead of contracts. A model point is defined as either a single contract, or a collection of identical contracts, such that survival probabilities are already taken into account. Thus, if x is the model point associated with a contract c and p_0^t is the survival probability (incorporating mortality and surrender) between 0 and t the reserves are related by $LP_t^x = p_0^t \widetilde{LP}_t^c$. However, p_0^t may depend on the underlying economic scenario via dynamic policyholder behavior and therefore we refrain from introducing these probabilities explicitly. Rather, we let V_t^x, $(DB_t^{\leq 0})^x$, and DB_t^x denote the mathematical reserve, declared bonuses up to and including valuation time, and declared bonuses after valuation time, respectively, such that survival probabilities are taken into account. If policyholder behavior is dynamic with respect to economic scenarios, then V_t^x and $(DB_t^{\leq 0})^x$ are also scenario-dependent. Future declared bonuses, DB_t^x, depend on economic scenarios by construction. Notice that $(DB_t^{\leq 0})^x \leq (DB_0^{\leq 0})^x$, in general. Analogously, the probability weighted minimum guaranteed (maturity and mortality) benefits generated by model point x at t are denoted by gbf_t^x. The probability weighted cash flows, at t, due to $(DB_{t-1}^{\leq 0})^x$ are called $(gbf_t^{\leq 0})^x$. Those due to DB_{t-1}^x are called ph_t^x.

The total benefit cash flows are

$$gbf_t = \sum_{x \in \mathcal{X}_t} gbf_t^x, \qquad gbf_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} \left(gbf_t^{\leq 0}\right)^x, \qquad ph_t = \sum_{x \in \mathcal{X}_t} ph_t^x$$

and the total reserves are given correspondingly by

$$V_t = \sum_{x \in \mathcal{X}_t} V_t^x, \qquad DB_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} \left(DB_t^{\leq 0}\right)^x, \qquad DB_t = \sum_{x \in \mathcal{X}_t} DB_t^x,$$

where $\mathcal{X}_t$ denotes the set of model points active at time t.

2.3. Gross surplus and profit sharing

Let L_t denote the book value of liabilities at time t, and we assume that $L_t = LP_t + SF_t$ is a sum of two items: firstly, the life assurance provision, $LP_t = V_t + DB_t^{\leq 0} + DB_t$; and secondly, the surplus fund, SF_t. The surplus fund at time t, SF_t, consists of those profits that have not yet been declared to policyholders. As opposed to LP_t, SF_t belongs to the collective of policyholders and cannot be attributed to individual contracts. This setup follows the same logic as Gerstner *et al*. (2008) where V_t, $DB_t^{\leq 0} + DB_t$, and SF_t are referred to as the actuarial reserve, allocated bonus, and free reserve (buffer account), respectively.

The difference between total book value of assets and liabilities is the free capital, $BV_t = FC_t + L_t$. Since return on free capital is not shared with policyholders and therefore does not contribute to the future discretionary benefits that we are interested in, we assume without loss of generality that $FC_t = 0$, so that the initial book value of assets is equal to the initial value of liabilities. Further, we assume that all shareholder gains that are produced by the company over the projection time are directly paid out to shareholder and not accumulated within the company so that $FC_t = 0$ (cf. Hochgerner and Gach, 2019, A. 2.2):

$$BV_t = L_t = LP_t + SF_t. \tag{2.5}$$

Indeed, for purposes of best estimate calculation the free capital is not relevant as this and the corresponding revenue is not shared with policyholders. Moreover, assets used to cover statutory reserves may not be attributed separately to L_t and FC_t (Bundesministerium der Finanzen (BMF); Verordnung). Hence, setting $FC_t = 0$ leads to the appropriate scaling of revenue that is to be shared with policyholders. Thus, the equality (2.5) can always be achieved by replacing BV_0, MV_0, UG_0 by $BV_0' = BV_0 - FC_0$, $MV_0' = \frac{BV_0'}{BV_0} MV_0$, $UG_0' = \frac{BV_0'}{BV_0} UG_0$, respectively, and assuming that future shareholder gains are paid out as cash flows which leave the model. On the other hand, if the goal is to build a comprehensive asset liability model to simulate the shareholder's point of

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view then the free capital is of paramount importance. However, in the simulation of a run-off liability book as under Solvency II such a point of view is difficult to realize since the relation between FC_t and L_t quickly becomes unrealistic (without introducing new business). Notice also that the equity position in the balance sheet model of Gerstner *et al*. (2008) is a hybrid of free capital, in the above sense of FC_t, and hidden reserves, UG_t. We do retain UG_t in the projection since this is indispensable for best estimate calculation.

The profit sharing mechanism depends on the company's gross surplus with respect to local accounting rules. The gross surplus can be described verbally as the sum of book value return, increase or decrease of statutory reserves, and all relevant cash flows (premiums, benefits, costs). The gross surplus, gs_t, at t is therefore defined as

$$gs_t := ROA_t - \Delta V_t - \Delta DB_t^{\leq 0} - DB_t^- + DB_{t-1} + pr_t - gbf_t - gbf_t^{\leq 0} - ph_t - co_t,$$

where ROA_t is the book value return (2.4), DB_t^- is the account of declared bonuses before bonus declaration at t, pr_t are premium payments and co_t are all cost cash flows. If χ_t is the appropriately averaged surrender fee factor at t, the discrepancy between $-DB_t^- + DB_{t-1}$ and ph_t can be expressed as $-DB_t^- + DB_{t-1} - ph_t = \chi_t DB_{t-1}$. The analogous expression holds with an appropriately chosen factor $\chi_t^{\leq 0}$, representing surrender fees: $-\Delta DB_t^{\leq 0} = \chi_t^{\leq 0} DB_{t-1}^{\leq 0}$. Let further ρ_t denote the averaged technical interest rate at $t-1$ such that $-\Delta V_t + pr_t - gbf_t - co_t = -\rho_t V_{t-1} - tg_t$ where tg_t is the technical gains due to mortality and cost margins and surrender fees stemming from the mathematical reserves. We define $\gamma_t := (tg_t + \chi_t^{\leq 0} DB_{t-1}^{\leq 0} + \chi_t DB_{t-1})/LP_{t-1}$ and express the gross surplus as

$$gs_t = ROA_t - \rho_t V_{t-1} + \gamma_t LP_{t-1}. \tag{2.6}$$

Remark 2.2. The advantage of expressing the gross surplus in this form is that the effects of book value return on assets, guaranteed technical interest rate and technical gains (i.e., mortality, cost and surrender margin) can be isolated.

If gs_t is positive, it is the surplus shared between policyholder, shareholder and tax office. If it is negative, it is covered by the shareholder. Indeed, profit sharing is defined by legislation (Bundesministerium der Finanzen (BMF); Verordnung; Dorobantu *et al*., 2020) and requires that gs_t is shared between shareholders, policyholders and tax office according to

$$gs_t = sh_t + ph_t^* + tax_t, \tag{2.7}$$

where $sh_t = gsh \cdot gs_t^+ - gs_t^-$, $ph_t^* = gph \cdot gs_t^+$ and $tax_t = gtax \cdot gs_t^+$, and where gsh, gph and $gtax$ are positive numbers such that $gsh + gph + gtax = 1$. Furthermore, c^+ and c^- denote the positive and negative parts of a number c, respectively.

Notice that sh_t and tax_t constitute cash flows since this is money that leaves the company (and the model), while ph_t^* is an accounting flow since this corresponds to a quantity that is transferred within the company to a different account but is not paid out as a benefit at time t.

A fundamental principle of traditional life insurance is that profit sharing is *not* equal to profit declaration (Bundesministerium der Finanzen (BMF); Verordnung). This means that $ph_t^* = gph \cdot gs_t^+$ is not necessarily declared (or credited) to its full extent to specific policyholder accounts. Rather, the profit sharing mechanism in traditional life insurance is such that a part, $\nu_t \cdot ph_t^*$ with $0 \le \nu_t \le 1$, of ph_t^* is declared to the policyholder accounts. Management may choose the value ν_t at each accounting step t. However, declaration may also be augmented by additional contributions, $\eta_t \cdot SF_{t-1}$ with $0 \le \eta_t \le 1$, from the previously existing surplus fund SF_{t-1}. Again, management may choose the value η_t at each accounting step t. Typically, surplus fund contributions will take place when ph_t^* is small compared to management goals.

The total bonus declaration to DB_t at time t is therefore of the form

$$\nu_t \cdot ph_t^* + \eta_t \cdot SF_{t-1} \tag{2.8}$$

with the factor ν_t and η_t determined according to management rules.

As above, let ph_t denote the amount of discretionary benefits paid out at time t. This cash flow depends on declarations to the declared benefits account, DB_k, which have occurred at times $0 < k < t$. Declarations at valuation time, $t = 0$, belong by definition to $DB_t^{\le 0}$, and the resulting cash flows are already guaranteed at $t = 0$, whence these do not contribute to ph_t. Therefore, we have

$$ph_1 = 0. \tag{2.9}$$

For $k > 0$, let $0 \le \eta_k \le 1$ denote the amount of declaration from SF_{k-1} to DB_k. The numbers η_k and ν_k are, in general, unknown at $t = 0$ and depend on management rules. Let further $0 \le \mu_k^t \le 1$ denote the fraction of bonus declarations at k, $\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^*$, that is either paid out as a future discretionary benefit (in case of contract maturity or mortality) or kept by the company as a surrender fee (in case of premature contract termination), sg_t^*, at time t. This fraction is also unknown and may depend, in general, on dynamic policyholder behavior. The defining relation is thus

$$ph_t + sg_t^* = \sum_{k=1}^{t-1} \mu_k^t \Big(\eta_k \cdot SF_{k-1} + \nu_k \cdot ph_k^* \Big), \tag{2.10}$$

where $t \ge 2$. Since the sum of discretionary benefits and surrender fees cannot exceed the amount of previous declarations, we must have

$$\sum_{t=k+1}^{T} \mu_k^t \le 1. \tag{2.11}$$

To sum up the above discussion, passing from $t-1$ to t, DB_{t-1} is:

- increased by declarations $\eta_t \cdot SF_{t-1}$ from the surplus fund where $0 \leq \eta_t \leq 1$ is chosen by the management,
- increased by direct policyholder declarations $\nu_t \cdot ph_t^*$ where $0 \leq \nu_t \leq 1$ is chosen by the management,
- decreased by cash flows, ph_t, to policyholders whose contracts terminate at t, and
- decreased by accounting flows $sg_t^* := \chi_t \cdot DB_{t-1}$ with $0 \leq \chi_t \leq 1$ due to surrender fees. The fraction $\chi_t \cdot DB_{t-1}$ is freed up, in the sense that it is not attributed to specific contracts anymore, and thus contributes to the annual gross surplus.

Therefore, we have the iterative relation

$$DB_t = DB_{t-1} + \eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* - ph_t - sg_t^* \tag{2.12}$$

with starting point $DB_0 = 0$.

Consider the surplus fund SF_{t-1} at $t-1$. Going one time step further, it is increased by allocating $(1-\nu_t) \cdot ph_t^*$ to the fund, which is the part of ph_t^* not declared to the policyholders' accounts, and decreased by declaring $\eta_t \cdot SF_{t-1}$ to policyholder accounts. We thus obtain

$$SF_t = SF_{t-1} + (1-\nu_t) \cdot ph_t^* - \eta_t \cdot SF_{t-1} \tag{2.13}$$

with known starting point SF_0. Together with (2.12) this yields

$$\Delta\,(DB_t + SF_t) = DB_t + SF_t - DB_{t-1} - SF_{t-1} = +ph_t^* - ph_t - sg_t^*. \tag{2.14}$$

Crucially, the model-dependent fractions ν_t and η_t do not appear in this equation. This evolution equation for $DB_t + SF_t$ is the starting point for the subsequent analysis. The sum $DB_t + SF_t$ is increased at each time step by the total shared profit, ph_t^*, as opposed to the declaration (2.8), and therefore represents the statutory reserves of previously shared profit, although the totality of this sum is not a (single) balance sheet item.

For further reference, we observe that

$$\begin{aligned} PH^* := E\Big[\sum B_t^{-1} ph_t^*\Big] &= gph \cdot E\Big[\sum B_t^{-1} gs_t^+\Big] \\ &= gph \cdot E\Big[\sum B_t^{-1} gs_t\Big] + gph \cdot E\Big[\sum B_t^{-1} gs_t^-\Big] \\ &= gph \cdot (VIF + PH^* + TAX) + gph \cdot COG. \end{aligned} \tag{2.15}$$

Here we have used the splitting $gs_t = sh_t + ph_t^* + tax_t$, where sh_t and tax_t are the shareholder and tax cash flows as defined above, to obtain the value of in-force business

$$VIF = E\left[\sum_{t=1}^{T} B_t^{-1} sh_t\right] = E\left[\sum_{t=1}^{T} B_t^{-1}\left(gsh \cdot gs_t^+ - gs_t^-\right)\right],$$

the cost of guarantees

$$COG = E\left[\sum_{t=1}^{T} B_t^{-1} gs_t^-\right], \tag{2.16}$$

and

$$TAX = E\left[\sum_{t=1}^{T} B_t^{-1} tax_t\right] = E\left[\sum_{t=1}^{T} B_t^{-1} gtax \cdot gs_t^+\right].$$

2.4. Future discretionary benefits

According to Solvency II (Directive, 2009; Commission, 2014), the best estimate is the expectation of all future cash flows which are related to existing business. These cash flows are benefits, $gbf_t + gbf_t^{\leq 0} + ph_t$, premium income, pr_t, and costs, co_t. The best estimate is thus defined as

$$BE := E\left[\sum_{t=1}^{T} B_t^{-1}\left(gbf_t + gbf_t^{\leq 0} + ph_t + co_t - pr_t\right)\right]. \tag{2.17}$$

The value of future discretionary benefits is given by

$$FDB := E\left[\sum_{t=1}^{T} B_t^{-1} ph_t\right] \tag{2.18}$$

and the value of the guaranteed benefits is by definition

$$GB := BE - FDB = E\left[\sum_{t=1}^{T} B_t^{-1}\left(gbf_t + gbf_t^{\leq 0} + co_t - pr_t\right)\right]. \tag{2.19}$$

If actuarial variables are independent from economical ones, then $gbf_t + gbf_t^{\leq 0}$, co_t and pr_t, and hence GB, may be calculated from a purely deterministic model. This independence would exclude the possibility of dynamic policyholder behavior (e. g., surrender depends dynamically on a comparison of declared bonuses and the prevailing yield curve). However, for our results to hold, we do not need to make this assumption.

Lemma 2.3.

$$PH^* = \frac{gph}{1-gph}\left(MV_0 - E\left[B_T^{-1} MV_T\right] - GB - FDB + COG\right) \tag{2.20}$$

Proof. We use the no-leakage principle (Hochgerner and Gach 2019, Prop. 2.2) which states that $MV_0 = BE + VIF + TAX + E[B_T^{-1} MV_T]$. With the decomposition $BE = GB + FDB$ and definition (2.15), this implies that $PH^* = gph(PH^* + MV_0 - GB - FDB - E[B_T^{-1} MV_T] + COG)$. □

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Remark 2.4. The no-leakage principle (Hochgerner and Gach 2019, Prop. 2.2) essentially states that in a risk neutral model, all cash flows have to be accounted for and all future expected gains or losses have to be reflected in the initial market value. This is a general statement and uses only no arbitrage theory and the generally accepted accounting principles which define the cash flows leading to the quantities BE, VIF and TAX. However, the precise formulation of the accounting principles is not relevant in this context since it suffices that cash flows are well-defined and that there can be no other cash flows except those to the policyholder (including costs), to the shareholder and to the tax office.

3. A REPRESENTATION OF *FDB*

Equation (2.14) may be rephrased as $B_t^{-1}\Delta(DB_t + SF_t) = B_t^{-1}ph_t^* - B_t^{-1}ph_t - B_t^{-1}sg_t^*$. By virtue of $\Delta(f_t g_t) = (\Delta f_t)g_{t-1} + f_t \Delta g_t$, we obtain a discrete integration by parts formula

$$
\begin{aligned}
\sum_{t=1}^{T}\left(B_t^{-1}ph_t^* - B_t^{-1}ph_t - B_t^{-1}sg_t^*\right) &= \sum_{t=1}^{T} B_t^{-1}\Delta(DB_t + SF_t) \qquad (3.1)\\
&= \sum_{t=1}^{T}\Delta(B_t^{-1}(DB_t + SF_t))\\
&\quad - \sum_{t=1}^{T}(DB_{t-1} + SF_{t-1})\Delta B_t^{-1}\\
&= B_T^{-1}(DB_T + SF_T) - SF_0 + \sum_{t=1}^{T}(DB_{t-1}\\
&\quad + SF_{t-1})F_{t-1}B_t^{-1}
\end{aligned}
$$

since $\Delta B_t^{-1} = -F_{t-1}B_t^{-1}$. Taking the expectation of this equation and using (2.20), we find

$$
\begin{aligned}
&\frac{gph}{1-gph}\left(MV_0 - E[B_T^{-1}MV_T] - GB + COG\right) - \frac{1}{1-gph}FDB\\
&= PH^* - FDB\\
&= E\left[B_T^{-1}(DB_T + SF_T)\right] - SF_0 + E\left[\sum_{t=1}^{T}(DB_{t-1} + SF_{t-1})F_{t-1}B_t^{-1}\right]\\
&\quad + E\left[\sum_{t=1}^{T}B_t^{-1}sg_t^*\right]
\end{aligned}
$$

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Because of $MV_0 = BV_0 + UG_0 = LP_0 + SF_0 + UG_0$, rearranging yields:

Theorem 3.1.

$$FDB = SF_0 + gph\Big(LP_0 + UG_0 - GB\Big) + gph \cdot COG - I - II - III, \tag{3.2}$$

where

$$I := E\Big[B_T^{-1}\Big(DB_T + SF_T + gph\,\Big(UG_T + V_T + DB_T^{\leq 0}\Big)\Big)\Big]$$

$$II := (1-gph)E\left[\sum_{t=2}^{T} B_t^{-1} sg_t^*\right]$$

$$III := (1-gph)E\left[\sum_{t=1}^{T} F_{t-1}B_t^{-1}\,(DB_{t-1} + SF_{t-1})\right].$$

Remark 3.2. The estimation formula (3.2) is derived without any model specific assumptions and relies therefore only on general accounting rules and the application of the no-leakage principle in (2.20). The 'integration by parts' (3.1) transfers the problem of calculating $E[\sum B_t^{-1}\Delta(DB_t + SF_t)]$ to one of evaluating or estimating the 'boundary term' $E[B_T^{-1}(DB_T + SF_T)] - SF_0$ as well as II and III. The idea is now that these approximations should be feasible since SF_0 is known, $E[B_T^{-1}(DB_T + SF_T)]$ is expected to be negligible at run-off time T, and estimation errors in II and III concern only the surrender gains from future declared bonuses and the return, due to F_{t-1}, on $DB_{t-1} + SF_{t-1}$, respectively.

Remark 3.3. The interpretation of the constituents of (3.2) is as follows:

- Term SF_0 is not multiplied by gph. This makes sense since the surplus fund, while not assigned to individual contracts, already belongs to the policyholder collective (compare Directive, 2009, Article91). There cannot be a transfer of funds from SF_t to the shareholder or tax office. On the other hand, the return on SF_t is shared between all parties, whence the corresponding deduction in term III.
- Term $gph \cdot (LP_0 - GB)$: According to the local GAA principle of prudentiality (e.g., Bundesgesetz, 2016, 148(1)), the life assurance provisions are determined with respect to safety margins and $gph \cdot (LP_0 - GB)$ represents the policyholder share of this margin.
- Term $gph \cdot UG_0$ represents the policyholder share in the unrealized gains existing in the portfolio at valuation time.
- When the safety margins considered in the calculation of LP_0 are not sufficient (e.g., due to a very low interest rate environment), then cost of guarantees arise and manifest as shareholder capital injections. When the environment is such that losses are expected for all future valuation

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dates $1 \leq t \leq T$, then the company's management could choose to inject just enough shareholder capital to cover these losses so that COG balances the right hand side of (3.2) to yield $FDB = SF_0 - I - II - III$. Balancing the right-hand side of (3.2) would mean to realize hidden reserves, UG_t, before injecting new capital, and while this might be a realistic assumption, Equation (3.2) holds independently of all such management rules. Moreover, in practice it is difficult to determine this minimal amount precisely such that the possibly counter-intuitive appearance of COG in (3.2) represents the policyholder share of excess capital injections in $gph(LP_0 + UG_0 - GB + COG)$.

- Term I is related to the policyholder share of assets that remain in the company after run-off of the liability portfolio;
- Term II is the tax and shareholder share (since $1 - gph = gsh + gtax$) in the gross surplus due to the fraction of declared future profits, DB_t, that is freed up because of surrender fees.
- Term III captures the tax and shareholder shares in interests on allocated profits as well as on the surplus fund.

4. Assumptions

4.1. Liability run-off assumptions

Assumption 4.1. *The projection horizon T corresponds to the run-off time of the liability portfolio such that $SF_T = LP_T = UG_T = 0$ (cf. Hochgerner and Gach, 2019, A. 3.13).*

Assumption 4.2. *The expected life assurance provisions $E[LP_t]$ decrease geometrically: there is a fixed $1 \leq h < T$ such that $E[LP_t] = l_t^h\, LP_0$ where $l_t^h := 2^{-t/h}$ for $t < T$ and $l_T^h := 0$.*

Since the portfolio is in run-off there is a time, h, where $E[LP_h] = LP_0/2$. Continuing from h onwards there has to be a time, $h + h'$, such that $E[LP_{h+h'}] = E[LP_h]/2$. Assuming that the company's business model has been stable over time, we have time homogeneity in the sense that $h' = h$ and run-off of the liability book is geometric. This would not be satisfied if the company under consideration has taken up business only very recently but for companies with a longer history we view this as a very good approximation.

Assumption 4.3. *In expectation, the total declared bonuses are a fixed fraction of the life assurance provisions: $E[DB_t^{\leq 0} + DB_t] = \sigma\, E[LP_t]$ for all $0 \leq t \leq T$ and a fixed $0 \leq \sigma \leq 1$. Moreover, $E[DB_t^{\leq 0}]$ does not vanish too quickly: $E[DB_t] \leq \sigma_t E[LP_t]$ where $\sigma_t := t\sigma/h$ for $t \leq h$ and $\sigma_t := \sigma$ for $t > h$.*

Assumption 4.4. *The relation $SF_0/LP_0 =: \vartheta$ remains constant in expectation: $E[SF_t] = \vartheta\, E[LP_t]$ for all $0 \leq t \leq T$. (Cf. Hochgerner and Gach, 2019, A. 3.10)*

Assumptions 4.3 and 4.4 are also statements about time homogeneity. Management rules concerning bonus declarations should remain reasonably constant in the long run such that σ and ϑ do not vary too strongly. The relevant point in this context is that these quantities should not vary arbitrarily but follow from target rates set by management rules. Assumption 4.4 is comparable to the assumption concerning the 'annual interest rate' in Gerstner *et al*. (2008, Section 4.2).

4.2. Surrender assumption

Assumption 4.5. *The surrender gains, $sg_t^* = \chi_t DB_{t-1}$, can be estimated on average with the same factor, γ_t, as the technical gains in (2.6): $E[sg_t^*] \le E[\gamma_t DB_{t-1}]$.*

The factor γ_t comprises mortality, cost and surrender margins as a fraction of the full life assurance provision, LP_{t-1}. It is therefore reasonable to expect that the same factor can be used as an upper bound on the surrender margin arising from declared bonuses, DB_{t-1}, alone.

4.3. Bonus benefit assumptions

Because of Equation (2.13), the bonus benefit declaration at t can be expressed as $\eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* = SF_{t-1} - SF_t + ph_t^*$. Management rules generally strive to keep profit declarations stable while, in accordance with Assumptions 4.4 and 4.2, SF_t is expected to decrease geometrically over time. In order for SF_t to decrease in expectation, the bonus benefit declarations must be strictly positive in expectation. To achieve this, a fraction of the profit share, ph_t^*, must also be declared to policyholders, at least in expectation. We turn this reasoning into an assumption along all scenarios.

Assumption 4.6. *There is a fixed $0 < \nu < 1$ such that the declarations satisfy $\eta_t \cdot SF_{t-1} + \nu_t \cdot ph_t^* \ge \nu \cdot ph_t^*$ for all $1 \le t \le T$.*

Assumption 4.7. *Assume that μ_k^{s+1} is determined by the geometric run-off Assumption 4.2: $\mu_k^{s+1} = \frac{l_s^h - l_{s+1}^h}{l_k^h}$.*

Notice that, for fixed k, this definition entails $\sum_{s=k}^{T-1} \mu_k^{s+1} = 1 - l_T^h / l_k^h = 1$; cf. (2.11). That is, run-off is complete at T.

4.4. Gross surplus assumptions

According to (2.6) and (2.4), the gross surplus is given by

$$gs_t = F_{t-1}(LP_{t-1} + SF_{t-1}) + F_{t-1} UG_{t-1} - \Delta UG_t | \mathcal{F}_{t-1} + ROA_t - ROA_t | \mathcal{F}_{t-1} \\ - (\rho_t - \gamma_t) V_{t-1},$$

where ρ_t and γ_t may, in general, also depend on the stochastic interest rate curve via dynamic surrender.

For the purpose of estimating terms III and COG in (3.2), we make the following simplifying assumptions. The principle idea behind these assumptions is that the main source of stochasticity in gs_t is the forward rate F_{t-1} whence all other quantities are replaced by their expected values. The simplified model of gs_t will be denoted by $\widehat{gs}_t$.

Assumption 4.8. *In $\widehat{gs}_t$, the technical interest rate ρ_t and the technical gains γ_t are deterministic functions of t.*

Assumption 4.9. *In $\widehat{gs}_t$, the return ROA_t is predictable, that is $\mathcal{F}_{t-1}$-measurable, and realizations of unrealized gains are determined by a fixed number $1 < d < T$:*

(1) $ROA_t - E[ROA_t|\mathcal{F}_{t-1}] = 0$;
(2) $F_{t-1}UG_{t-1} - E[\Delta UG_t|\mathcal{F}_{t-1}] = P(0,t)^{-1}(l^d_{t-1} - l^d_t)UG_0$ *where* $l^d_t := 2^{-t/d}$ *for* $t < T$ *and* $l^d_T := 0$;

The motivation for item (2) is as follows. The quantity $F_{t-1}UG_{t-1} - E[\Delta UG_t|\mathcal{F}_{t-1}] =: cf^{UG}_t$ may be viewed as a cash flow due to realizations of unrealized gains as assets approach their maturities (an example of this reasoning is contained in Remark 2.1). Indeed, if an asset a has maturity T_a, then we must have $UG^a_{T_a} = MV^a_{T_a} - BV^a_{T_a} = 0$, and thus, UG^a_t tends to 0 as t approaches T_a. As a proxy for the number d, we take the duration of the portfolio. Assuming that cash flows, cf^{UG}_t, due to realizations of unrealized gains are known at valuation time $t = 0$, we obtain $UG_0 = \sum_{t=1}^T P(0,t)cf^{UG}_t$. Setting $\sum_{t=1}^T P(0,t)(F_{t-1}UG_{t-1} - E[\Delta UG_t|\mathcal{F}_{t-1}]) = \sum_{t=1}^T P(0,t)cf^{UG}_t = UG_0 = \sum_{t=1}^T (l^d_{t-1} - l^d_t)UG_0$ and insisting on equality of the summands leads to the above assumption.

Assumption 4.10. *The coefficient of variation of book valued items is negligible in comparison to that of market movements. Concretely, the coefficients of variations of DB_t, LP_t and SF_t are assumed to be negligible in comparison to those of F_t and B_t^{-1}.*

This assumption reflects the general principle that book values are expected to be more stable than market values since not all market movements are reflected in book values but rather lead to unrealized gains or losses (Dorobantu *et al*., 2020).

Invoking the above Assumptions 4.2, 4.9, 4.10, 4.8, 4.3 and 4.4, we define

$$\widehat{gs}_t := F_{t-1}E[BV_{t-1}] + P(0,t)^{-1}(l^d_{t-1} - l^d_t)UG_0 - \rho_t V_{t-1} + \gamma_t LP_{t-1} \tag{4.1}$$
$$= \left(F_{t-1} + P(0,t)^{-1}\frac{l^d_{t-1} - l^d_t}{l^h_{t-1}}\frac{UG_0}{(1+\vartheta)LP_0} - \frac{(1-\sigma)\rho_t - \gamma_t}{1+\vartheta}\right)(1+\vartheta)l^h_{t-1}LP_0$$

to be used as a simplified model for gs_t.

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5. Analytical lower and upper bounds for future discretionary benefits

The model dependent quantities in (3.2) are I, II, III and COG. Calculating these explicitly is just as difficult as calculating the FDB. The purpose of this section is therefore to derive analytical bounds for these quantities, that is bounds which can be calculated without a numerical model.

5.1. Estimating *I*

In accordance with Assumption 4.1, we estimate I by

$$\widehat{I} = 0. \tag{5.1}$$

Compare also with the second statement in Hochgerner and Gach (2019, Prop. 2.2).

5.2. Estimating *II*

The expression $sg_t^* = \chi_t DB_t$ in Term II corresponds to the fraction of DB_t that is freed up each year due to policyholder surrender fees and thus contributes to the company's surplus as a component of the surrender gains. Assumptions 4.10, 4.5, 4.3 and 4.2 imply that term II can be estimated as $II \leq \widehat{II}$ with

$$\widehat{II} := (1 - gph) \sum_{t=2}^{T} \gamma_t \sigma_t P(0, t) l_{t-1}^h LP_0. \tag{5.2}$$

Remark 5.1. The product $(1 - gph)\gamma_t\sigma_t$ is expected to be very small since this represents the shareholder and tax share of the surrender gains from future declared bonuses, such that $\widehat{II}$ should be also very small in comparison to MV_0.

5.3. Bounding *III* from above

The essential idea is to use the recursive relation (2.14) to obtain an upper bound for $DB_t + SF_t$. Equation (2.14) implies

$$DB_t + SF_t = SF_0 + \sum_{s=1}^{t} ph_s^* - \sum_{s=2}^{t} (ph_s + sg_s^*) = SF_0 + gph \cdot gs_t^+ \\ + \sum_{s=1}^{t-1} \Big(gph \cdot gs_s^+ - ph_{s+1} - sg_{s+1}^* \Big). \tag{5.3}$$

Assumptions 4.6 and 4.7 yield

$$\begin{aligned}\sum_{s=1}^{t-1}\left(ph_{s+1}+sg^*_{s+1}\right) &= \sum_{s=1}^{t-1}\sum_{k=1}^{s}\mu_k^{s+1}\left(\eta_k\cdot SF_{k-1}+\nu_k\cdot ph^*_k\right)\\ &\geq \nu\sum_{s=1}^{t-1}\sum_{k=1}^{s}\mu_k^{s+1}ph^*_k=\nu\sum_{k=1}^{t-1}\sum_{s=k}^{t-1}\frac{l^h_s-l^h_{s+1}}{l^h_k}ph^*_k\\ &=\nu\sum_{k=1}^{t-1}\frac{l^h_k-l^h_t}{l^h_k}ph^*_k,\end{aligned}$$

whence (5.3) satisfies

$$DB_t+SF_t\leq SF_0+gph\cdot gs^+_t+gph\cdot\sum_{s=1}^{t-1}\left(1-\nu(1-l^h_{t-s})\right)gs^+_s. \tag{5.4}$$

Thus,

$$\begin{aligned}III &= (1-gph)\sum_{t=0}^{T-1}E\left[B^{-1}_{t+1}F_t(DB_t+SF_t)\right] \qquad (5.5)\\ &\leq(1-gph)(1-P(0,T))SF_0+(1-gph)gph\sum_{t=1}^{T-1}E\left[B^{-1}_{t+1}F_t\cdot gs^+_t\right]\\ &+(1-gph)gph\sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\left(1-\nu(1-l^h_{t-s})\right)E\left[B^{-1}_{t+1}F_t\cdot gs^+_s\right].\end{aligned}$$

The expression $E[B^{-1}_{t+1}F_t\cdot gs^+_s]$ gives the fair value of the risk free return on gs^+_s in the period from t to $t+1$.

5.4. Bounding *III* from below

We use again Equation (5.3) and notice that Equations (2.10) and (2.13) imply

$$\begin{aligned}\sum_{s=2}^{t}\left(ph_s+sg^*_s\right) &= \sum_{s=2}^{t}\sum_{k=1}^{s-1}\mu^s_k\left(SF_{k-1}-SF_k+ph^*_k\right)\\ &=\sum_{k=1}^{t-1}\sum_{s=k+1}^{t}\mu^s_k\left(SF_{k-1}-SF_k+ph^*_k\right)\\ &\leq SF_0-SF_{t-1}+\sum_{k=1}^{t-1}ph^*_k\end{aligned}$$

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since $\sum_{s=k+1}^{t} \mu_k^s \leq 1$ due to (2.11). Inserting this in (5.3) yields $DB_t + SF_t \geq ph_t^* + SF_{t-1}$ for all $t \geq 1$, and therefore

$$
\begin{aligned}
III &= (1 - gph)E\left[\sum_{t=0}^{T-1} F_t B_{t+1}^{-1}(DB_t + SF_t)\right] \\
&\geq (1 - gph)F_0(1 + F_0)^{-1}SF_0 \\
&\quad + (1 - gph)E\left[\sum_{t=1}^{T-1} F_t B_{t+1}^{-1}\left(gph \cdot gs_t^+ + SF_{t-1}\right)\right].
\end{aligned}
$$

This estimate does not depend on any of the assumptions in Section 4. Neglecting, in accordance with Assumption 4.10, the variation of SF_{t-1} in comparison to that of F_{t-1}, and using Assumptions 4.4 and 4.2, yields

$$
\begin{aligned}
III \geq (1 - gph)&\left(F_0(1 + F_0)^{-1}SF_0 + \vartheta \sum_{t=1}^{T-1}(P(0,t) - P(0,t+1))l_{t-1}^h LP_0\right) \\
&+ gph(1 - gph)E\Big[\sum_{t=1}^{T-1} F_t B_{t+1}^{-1} gs_t^+\Big].
\end{aligned}
\tag{5.6}
$$

5.5. Estimating the return on the deferred caplet

Let us rewrite $F_t B_{t+1}^{-1} gs_s^+ = (B_t^{-1} - B_{t+1}^{-1})gs_s^+ = (D(s,t) - D(s,t+1))B_s^{-1}gs_s^+$ and abbreviate the coefficients of variations as

$$
CV_{s,t}^1 := CV\Big[D(s,t) - D(s,t+1)\Big], \qquad CV_s^2 := CV\Big[B_s^{-1}gs_s^+\Big]. \tag{5.7}
$$

Since $-1 \leq Corr_s^t := Corr[D(s,t) - D(s,t+1), B_s^{-1}gs_s^+] \leq 1$, it follows that

$$
\begin{aligned}
E\Big[(D(s,t) - D(s,t+1))B_s^{-1}gs_s^+\Big] &= E\Big[(D(s,t) - D(s,t+1))\Big]E\Big[B_s^{-1}gs_s^+\Big] \\
&\quad + Corr_s^t \cdot CV_{s,t}^1 CV_s^2 E\Big[(D(s,t) - D(s,t+1))\Big]E\Big[B_s^{-1}gs_s^+\Big] \\
&\leq \Big(P(s,t) - P(s,t+1)\Big)E\Big[B_s^{-1}gs_s^+\Big]\Big(1 + CV_{s,t}^1 CV_s^2\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&E\Big[(D(s,t) - D(s,t+1))B_s^{-1}gs_s^+\Big] \\
&\geq \Big(P(s,t) - P(s,t+1)\Big)E\Big[B_s^{-1}gs_s^+\Big]\Big(1 - CV_{s,t}^1 CV_s^2\Big).
\end{aligned}
$$

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Hence, (5.6) and (5.5) lead to

$$
\begin{aligned}
&(1-gph)\left(F_0(1+F_0)^{-1}SF_0+\vartheta\sum_{t=1}^{T-1}(P(0,t)-P(0,t+1))l^h_{t-1}LP_0\right)\\
&\qquad+gph(1-gph)\sum_{t=1}^{T-1}\Big(1-CV^1_{0,t}\,CV^2_t\Big)\Big(1-P(t,t+1)\Big)E\Big[B_t^{-1}gs_t^+\Big]\\
&\le III \qquad (5.8)\\
&\le(1-gph)(1-P(0,T))SF_0\\
&\qquad+(1-gph)gph\sum_{t=1}^{T-1}\Big(1+CV^1_{0,t}\,CV^2_t\Big)\Big(1-P(t,t+1)\Big)E\Big[B_t^{-1}gs_t^+\Big]\\
&\qquad+(1-gph)gph\sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\Big(1-\nu(1-l^h_{t-s})\Big)\Big(1+CV^1_{s,t}\,CV^2_s\Big)\\
&\qquad\Big(P(s,t)-P(s,t+1)\Big)E\Big[B_s^{-1}gs_s^+\Big].
\end{aligned}
$$

The term $E[B_s^{-1}gs_s^+]$ is the value of the caplet with payoff gs_s^+ at time s.

5.6. Estimating the caplet

Now we replace gs_s by its simplified model $\widehat{gs}_s$ defined in (4.1). This implies that the (simplified) caplet can be expressed as

$$E\Big[B_s^{-1}\widehat{gs}_s^+\Big]=\mathcal{O}_s^+(1+\vartheta)l^h_{s-1}LP_0, \qquad (5.9)$$

where

$$\mathcal{O}_s^{\pm}:=E\left[B_s^{-1}\Big(F_{s-1}+P(0,s)^{-1}\frac{l^d_{s-1}-l^d_s}{l^h_{s-1}}\frac{UG_0}{(1+\vartheta)LP_0}-\frac{(1-\sigma)\rho_s-\gamma_s}{1+\vartheta}\Big)^{\pm}\right] \qquad (5.10)$$

is the value at 0 of the caplet (corresponding to +) or floorlet (corresponding to −) with maturity $s-1$ and payment $(F_{s-1}+P(0,s)\frac{l^d_{s-1}-l^d_s}{l^h_{s-1}}\frac{UG_0}{(1+\vartheta)LP_0}-\frac{(1-\sigma)\rho_s-\gamma_s}{1+\vartheta})^{\pm}$ occurring at settlement date s. In the normal model, this value is given by the Black formula (Black, 1976; Brigo and Mercurio, 2006)

$$\mathcal{O}_s^{\pm}=P(0,s)\cdot\Big(\pm(F^0_{s-1}-k_s)\Phi(\pm\kappa_s)+\mathrm{IV}_s\sqrt{s}\phi(\pm\kappa_s)\Big), \qquad (5.11)$$

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where Φ and ϕ are the normal cumulative distribution and density functions, respectively. Further,

$$\kappa_s := \frac{F^0_{s-1} - k_s}{\mathrm{IV}_s \sqrt{s}},$$

where $F^0_{s-1} = P(0, s-1) - P(0, s)$ is the forward rate prevailing at time 0, the strike is given by

$$k_s := -P(0,s)^{-1} \frac{l^d_{s-1} - l^d_s}{l^h_{s-1}} \frac{UG_0}{(1+\vartheta)LP_0} + \frac{(1-\sigma)\rho_s - \gamma_s}{1+\vartheta} \tag{5.12}$$

and IV_s is the caplet implied volatility known from market data. Using the normal model at this point is, of course, an additional model choice.

Therefore, estimate (5.8) may be reformulated as

$$\widehat{III}_{lb} \leq III \leq \widehat{III}_{ub} \tag{5.13}$$

with

$$\begin{aligned}\widehat{III}_{lb} := {} & (1-gph)\left(F_0(1+F_0)^{-1}SF_0 + \vartheta \sum_{t=1}^{T-1} (P(0,t) - P(0,t+1)) l^h_{t-1} LP_0\right) \\ & + gph(1-gph)\sum_{t=1}^{T-1}\Big(1 - CV^1_{0,t}\, CV^2_t\Big)\Big(1 - P(t,t+1)\Big)\mathcal{O}^+_t(1+\vartheta) l^h_{t-1} LP_0\end{aligned} \tag{5.14}$$

$$\begin{aligned}\widehat{III}_{ub} := {} & (1-gph)(1-P(0,T))SF_0 \\ & + (1-gph)gph \sum_{t=1}^{T-1}\Big(1 + CV^1_{0,t}\, CV^2_t\Big)\Big(1 - P(t,t+1)\Big)\mathcal{O}^+_t(1+\vartheta) l^h_{t-1} LP_0 \\ & + (1-gph)gph \sum_{t=2}^{T-1}\sum_{s=1}^{t-1}\Big(1 - \nu(1 - l^h_{t-s})\Big)\Big(1 + CV^1_{s,t}\, CV^2_s\Big)\Big(P(s,t) \\ & - P(s,t+1)\Big)\mathcal{O}^+_s(1+\vartheta) l^h_{s-1} LP_0.\end{aligned}$$

5.7. Approximating *COG*

The shareholder cost of guarantees is defined in (2.16). We use again the simplified model (4.1) to estimate *COG* by

$$\widehat{COG} := E\left[\sum_{t=1}^{T} B_t^{-1} \widehat{gs}^-_t\right] = \sum_{t=1}^{T} \mathcal{O}^-_t (1+\vartheta) l^h_{t-1} LP_0, \tag{5.15}$$

where $\mathcal{O}^-_t$ is the value of the floorlet given in (5.11).

TABLE 2

LIST OF DATA NEEDED TO CALCULATE $\widehat{LB}$ AND $\widehat{UB}$.

(1)	the balance sheet items SF_0, LP_0, UG_0, GB;
(2)	the gross policyholder participation factor gph;
(3)	the initial discount curve $P(0,t)$ and interest rate implied volatilities IV_t;
(4)	the coefficients of variation $CV^1_{s,t}$ and CV^2_s;
(5)	duration factor d in years;
(6)	liability half life h in years;
(7)	surplus fund fraction ϑ;
(8)	bonus account factor σ;
(9)	bonus declaration bound ν;
(10)	expected technical interest rate ρ_t;
(11)	expected technical gains rate γ_t;
(12)	the information concerning the application of Article 91 as in Remark 5.2;
(13)	the projection time T;

5.8. Estimating *FDB*

Under the assumptions of Section 4, the terms COG, I, II and III can be estimated by $\widehat{COG}$, $\widehat{I}=0$, $II \leq \widehat{II}$ and $\widehat{III}_{lb} \leq III \leq \widehat{III}_{ub}$ as defined by (5.15), (5.1), (5.2) and (5.14), respectively. These estimates yield a lower bound, $\widehat{LB}$, and an upper bound, $\widehat{UB}$, for FDB:

$$\widehat{LB} \leq FDB \leq \widehat{UB}, \tag{5.16}$$

where

$$\widehat{LB} := SF_0 + gph\Big(LP_0 + UG_0 - GB\Big) - \widehat{II} - \widehat{III}_{ub} \tag{5.17}$$

$$\widehat{UB} := SF_0 + gph\Big(LP_0 + UG_0 - GB\Big) + gph \cdot \widehat{COG} - \widehat{III}_{lb}. \tag{5.18}$$

If the difference $\widehat{UB} - \widehat{LB}$ is sufficiently small (e.g., in comparison to BV_0), then $\widehat{FDB} = (\widehat{LB} + \widehat{UB})/2$ may be used as an estimator for FDB.

These estimation formulas are analytic in the sense that they do not depend on a numerical model. For the reader's convenience, we provide a compact list of data which have to be known or estimated in order to calculate these bounds in Table 2.

Remark 5.2. The estimations $\widehat{LB}$ and $\widehat{UB}$ regard the FDB as calculated with a stochastic cash flow model. The stochastic cash flow model is the numerical model used to generate cash flows relevant for best estimate calculation as in Gerstner et al. (2008), Vedani et al. (2017). In those EU member states that have authorized Article 91(2) of Directive 2009/138/EC (Directive, 2009), the surplus fund (in fact, the part that is not used to compensate losses) is not considered as a liability and is therefore not part of the life assurance provision. Thus, if a company chooses to deduct the part, denoted by $SF_0^{\text{Art. 91}}$, of the

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TABLE 3

ALLIANZ LEBENSVERSICHERUNGS-AG: PUBLIC DATA FOR 2017–2019, VALUES ARE IN BILLION EUROS.

Quantity	2017	2018	2019
L_0	189.8	201.2	219.6
UG_0	41.4	32.8	54.0
SF_0	10.4	11.0	11.5
Solvency II value of SF_0	10.9	10.5	11.3
GB	154.1	158.8	195.2
FDB	48.6	46.2	47.4

surplus fund that is not used to absorb losses (in the risk neutral average over all scenarios), this is subtracted from the FDB to yield the future discretionary benefits as reported to the supervisor and in financial statements, $FDB^{\text{Art. 91}} = FDB - SF_0^{\text{Art. 91}}$. Hence, this information has to be known and when relevant the corresponding quantity has to be subtracted from the bounds $\widehat{LB}$ and $\widehat{UB}$. If this is the case we, choose to approximate $SF_0^{\text{Art. 91}}$ by SF_0 itself in order to remain model free and thus use $\widehat{LB}' = \widehat{LB} - SF_0$ and $\widehat{UB}' = \widehat{UB} - SF_0$.

6. PUBLIC DATA

6.1. Allianz Lebensversicherungs-AG: publicly reported values

The data in Table 3 are taken from publicly available reports for the accounting years 2017–2019. The relevant references are listed in Table 4.

The value of UG_0 is already scaled to L_0, which is in line with the general assumption (2.5). The reason behind this scaling is that according to Bundesministerium der Finanzen (BMF), 3 only the fraction of the capital gains, corresponding to the assets scaled to cover the average value of liabilities in the accounting year under consideration, contribute to the gross surplus.

As for L_0, we adjust the local GAAP value of life insurance with profit participation for necessary regrouping of business, as explained in Allianz Lebensversicherungs-AG (2017b, p. 52), Allianz Lebensversicherungs-AG (2018b, p. 46), Allianz Lebensversicherungs-AG (2019b, p. 46) for the different accounting years 2017–2019.

6.2. Estimating technical gains from market data

For the German life insurance market, technical gains can be determined from tables 130 and 141 in Bundesanstalt (2019). The relevant items are stated below:

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TABLE 4

ALLIANZ LEBENSVERSICHERUNGS-AG: REFERENCES FOR THE DATA LISTED IN TABLE 3.

Quantity	Source for 2017	Source for 2018	Source for 2019
L_0+[i]	Allianz Lebensversicherungs-AG (2017b, p. 46, 52)	Allianz Lebensversicherungs-AG (2018b, p. 42, 46)	Allianz Lebensversicherungs-AG (2019b, p. 42, 46)
UG_0+[ii]	Allianz Lebensversicherungs-AG (2017a, p. 46)	Allianz Lebensversicherungs-AG (2018a, p. 42)	Allianz Lebensversicherungs-AG (2019a, p. 46)
SF_0+[iii]	Allianz Lebensversicherungs-AG (2017a, p. 55)	Allianz Lebensversicherungs-AG (2018a, p. 51)	Allianz Lebensversicherungs-AG (2019a, p. 55)
Solvency II value of SF_0+[iv]	Allianz Lebensversicherungs-AG (2017b, p. 52)	Allianz Lebensversicherungs-AG (2018b, p. 46)	Allianz Lebensversicherungs-AG (2019b, p. 46)
GB[v]	Allianz Lebensversicherungs-AG (2017b, p. 46)	Allianz Lebensversicherungs-AG (2018b, p. 42)	Allianz Lebensversicherungs-AG (2019b, p. 42)
FDB[vi]	Allianz Lebensversicherungs-AG (2017b, p. 46)	Allianz Lebensversicherungs-AG (2018b, p. 42)	Allianz Lebensversicherungs-AG (2019b, p. 42)

[i] Versicherung mit berschussbeteiligung.
[ii] Stille Reserven der einzubeziehenden Kapitalanlagen.
[iii] Rckstellung fr Beitragsrckerstattung abzglich festgelegte, aber noch nicht zugeteilte Teile.
[iv] berschussfonds.
[v] Bester Schtzwert: Wert fr garantierte Leistungen.
[vi] Bester Schtzwert: zuknftige berschussbeteiligung.

TABLE 5

BAFIN: PUBLIC DATA FOR 2017–2019, VALUES ARE IN BILLION EUROS.

Quantity	Symbol	2017	2018	2019
Gross surplus net of direct policyholder declarations[i]	a	8.3	9.9	11.3
Direct policyholder declarations[ii]	b	2.3	2.1	2.1
Share of gross surplus allocated to the surplus fund[iii]	c	6.4	8.1	9.3
Interest margin[iv]	d	3.5	5.2	6.1
Gross technical provisions for direct business[v]	e	991.4	1011.1	1069.1
Gross technical provisions of those contracts where the investment risk is carried by the policyholder[vi]	f	109.1	101.7	124.8

[i] Überschuss.
[ii] Direktgutschrift.
[iii] Zuführung zur RfB.
[iv] Kapitalanlagenergebnis 1 b).
[v] Tabelle 130, Versicherungstechnische Rückstellungen brutto, selbst abgeschlossenes Geschäft.
[vi] Bilanzposten a) 6 a) brutto: Tabelle 130, Versicherungstechnische Rckstellungen, soweit das Anlagerisiko vom Versicherungsnehmer getragen wird.

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TABLE 6
VALUES OF γ FOR 2017–2019.

	2017	2018	2019
$\widehat{\gamma}_{LP}$	0.80%	0.74%	0.78%

TABLE 7
VALUES OF $\widehat{\rho}$ FOR 2017–2019.

	2017	2018	2019
$\widehat{\rho}$	2.63%	2.52%	2.38%

We estimate the technical gains relative to the life assurance provisions by $\widehat{\gamma}_{LP} := (a+b-d)/(e-f)$ and find

6.3. Estimating ρ

The average technical interest rate of the Allianz Lebensversicherungs-AG can be derived from the distribution of life assurance provision over the guaranteed interest rates, with the following results:

The underlying data can be found in Allianz Lebensversicherungs-AG (2017a, p. 34), Allianz Lebensversicherungs-AG (2018a, p. 33), and Allianz Lebensversicherungs-AG (2019a, p. 37). (To obtain the values stated in Table 7, we have taken the upper end points where technical interest rate intervals are provided.)

6.4. Calculating *gph*

The net policyholder shares, *nph* for the accounting years 2017–2019, can be obtained via $nph = (b+c)/a$ from the values collected in the table below (see Allianz Lebensversicherungs-AG, 2017a, p. 9, Allianz Lebensversicherungs-AG, 2018a, p. 9, and Allianz Lebensversicherungs-AG, 2019a, p. 8 for accounting years 2017–2019).

The gross policyholder share, *gph*, is calculated from *nph* according to the relation $gph = (1-\tau)nph/(1-\tau \cdot nph)$. Applying the German tax rate of $\tau = 29.9\%$ (Bundesministerium der Finanzen (BMF) 2019, p. 16) yields the following table:

For the estimation of Term *III*, we fix $gph = 75.5\%$ as the average of the values in Table 9.

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TABLE 8

VALUES ARE IN BILLION EUROS.

Quantity	Symbol	2017	2018	2019
Gross surplus net of direct policyholder declarations[i]	a	2.6	3.1	3.6
Share of gross surplus allocated to the surplus fund[ii]	b	2.0	2.3	2.9
Direct policyholder declarations [iii]	c	0.1	0.1	0.2

[i]Bruttoüberschuss.
[ii]Zuführung zur RfB.
[iii]Direktgutschrift.

TABLE 9

VALUES OF *gph* FOR 2017–2019.

Quantity	2017	2018	2019
nph	80.8%	77.9%	85.6%
gph	74.7%	71.2%	80.6%

6.5. Discount rates

The following are the publicly available EIOPA discount rates for 2017, 2018 and 2019.

7. ESTIMATION OF $\widehat{FDB}$ FROM PUBLIC DATA

We use the publicly available data collected in Section 6 to find the estimation interval for $\widehat{FDB}$ according to (5.16) and compare the result with numerically calculated FDB contained in the public data.[1] In order to calculate $\widehat{LB}$ and $\widehat{UB}$, we have to know or estimate the data listed in Table 2. This is done as follows:

(1) the balance sheet items SF_0, LP_0, UG_0, GB: These are given in Table 3 with $L_0 - SF_0 = BV_0 - SF_0 = LP_0$.

(2) the gross policyholder participation factor gph: We use the average value of $gph = 75.5\%$ given in Table 9. The average is employed since this factor subsequently remains constant over the full projection time and should not depend on special circumstances at valuation time.

(3) the initial discount curve $P(0,t)$ and interest rate implied volatilities IV_t: The discount curve is the relevant EIOPA curve as listed in Tables 10, 11 and 12; the implied volatilities are taken from Bloomberg (end of year 2019) for available maturities, linearly interpolated between available maturities and extrapolated by keeping the last available volatility constant. This leads to $IV_t = 10 + 50(t-1)/21$ for $1 \le t \le 21$ and $IV_t = 50$

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TABLE 10

EURO DISCOUNT RATES AS OF 31.12.2017. THE RATES ARE WITH VOLATILITY ADJUSTMENT. SOURCE: RISK-FREE.

t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$
1	1.003	11	0.902	21	0.740	31	0.534	41	0.362	51	0.241
2	1.004	12	0.885	22	0.720	32	0.514	42	0.348	52	0.232
3	1.001	13	0.868	23	0.700	33	0.495	43	0.334	53	0.222
4	0.996	14	0.850	24	0.679	34	0.477	44	0.321	54	0.214
5	0.988	15	0.834	25	0.658	35	0.459	45	0.308	55	0.205
6	0.977	16	0.819	26	0.637	36	0.441	46	0.296	56	0.197
7	0.965	17	0.804	27	0.616	37	0.424	47	0.284	57	0.189
8	0.951	18	0.790	28	0.595	38	0.408	48	0.273	58	0.181
9	0.936	19	0.774	29	0.574	39	0.392	49	0.262	59	0.174
10	0.920	20	0.758	30	0.554	40	0.377	50	0.252	60	0.167

TABLE 11

EURO DISCOUNT RATES AS OF 31.12.2018. THE RATES ARE WITH VOLATILITY ADJUSTMENT. SOURCE: RISK-FREE.

t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$
1	1.001	11	0.890	21	0.722	31	0.525	41	0.360	51	0.244
2	1.001	12	0.872	22	0.703	32	0.506	42	0.347	52	0.234
3	0.998	13	0.853	23	0.684	33	0.488	43	0.334	53	0.225
4	0.992	14	0.835	24	0.664	34	0.470	44	0.321	54	0.217
5	0.983	15	0.818	25	0.644	35	0.453	45	0.309	55	0.208
6	0.972	16	0.803	26	0.623	36	0.436	46	0.297	56	0.200
7	0.958	17	0.788	27	0.603	37	0.420	47	0.285	57	0.192
8	0.943	18	0.773	28	0.583	38	0.405	48	0.274	58	0.185
9	0.926	19	0.757	29	0.563	39	0.389	49	0.264	59	0.178
10	0.908	20	0.740	30	0.544	40	0.375	50	0.254	60	0.171

TABLE 12

EURO DISCOUNT RATES AS OF 31.12.2019. THE RATES ARE WITH VOLATILITY ADJUSTMENT. SOURCE: RISK-FREE.

t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$	t	$P_{0,t}$
1	1.004	11	0.975	21	0.878	31	0.665	41	0.466	51	0.320
2	1.006	12	0.967	22	0.861	32	0.643	42	0.449	52	0.308
3	1.008	13	0.957	23	0.842	33	0.621	43	0.432	53	0.296
4	1.009	14	0.947	24	0.821	34	0.600	44	0.416	54	0.285
5	1.008	15	0.937	25	0.800	35	0.579	45	0.401	55	0.275
6	1.006	16	0.929	26	0.778	36	0.559	46	0.386	56	0.264
7	1.001	17	0.922	27	0.755	37	0.539	47	0.372	57	0.254
8	0.996	18	0.914	28	0.733	38	0.520	48	0.358	58	0.245
9	0.990	19	0.904	29	0.710	39	0.501	49	0.345	59	0.236
10	0.982	20	0.893	30	0.687	40	0.483	50	0.332	60	0.227

for $t \geq 21$, expressed in basis points. Additional sensitivity analysis is performed.

(4) the coefficients of variation $CV^1_{s,t}$ and CV^2_s: Since the product of two such coefficients is expected to be small, and it is the product that enters the calculation of $\widehat{LB}$ and $\widehat{UB}$, these are estimated as $CV^1_{s,t} \cdot CV^2_s = 0$. While this parameter choice is certainly very practical, it is not very well founded from a theoretical perspective, and an estimation from historical data would be a more justifiable approach.

(5) duration factor d: This factor is known only to the company under consideration. We set $d = 8$ and perform sensitivity analysis on this assumption.

(6) liability half life h: This factor is known only to the company under consideration. We set $h = 10$ and perform sensitivity analysis on this assumption.

(7) surplus fund fraction ϑ: We take $\vartheta = SF_0/LP_0$, and perform sensitivity analysis.

(8) bonus account factor σ: We choose $\sigma = 20\%$, and perform sensitivity analysis.

(9) bonus declaration lower bound ν: We choose $\nu = 75\%$, and perform sensitivity analysis.

(10) expected technical interest rate ρ_t: We use the values constant $\hat{\rho} = \rho_t$ as listed in Table 7, and perform sensitivity analysis.

(11) expected technical gains rate γ_t: We use the constant values $\hat{\gamma} = \gamma_t$ as listed in Table 6, and perform sensitivity analysis.

(12) the information concerning the application of Article 91 as in Remark 5.2: the company in question does apply Article 91 as stated in Allianz Lebensversicherungs-AG (2017b) and seen in Table 3. Hence, we subtract SF_0 from the bounds to estimate FDB.

(13) the projection time T: we have chosen $T = 50$ years to reflect the long term nature of life insurance.

Remark 7.1. The coefficient of variation CV^2_s cannot be estimated from market data. To estimate it correctly, one would need a full numerical model to calculate the variation of gs^+_s. To avoid the need for a numerical best estimate model, one may also approximate CV^2_s by $CV[B^{-1}_s \widehat{gs}_s]$ using the simplified model (4.1). However, management rules are usually structured so as to reduce variations in bonus declarations. Hence, we expect that CV^2_s should be sufficiently small so that the product $CV^1_{s,t} \cdot CV^2_s$ is negligible.

The results corresponding to these assumptions are referred to as the base case and are shown in Tables 13 and 14 in absolute value (billion Euros) and in percent of the initial market value $MV_0 = LP_0 + SF_0 + UG_0$, respectively. The

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TABLE 13

BASE CASE, DISPLAYED NUMBERS ARE IN BILLION EUROS.

	LP_0	SF_0	UG_0	GB	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\widehat{II}$	$\widehat{COG}$
2017	179.40	10.40	41.40	154.10	48.60	46.78	43.82	49.73	2.96	−1.82	1.10	0.50
2018	190.20	11.00	32.80	158.80	46.20	45.16	42.24	48.08	2.92	−1.04	1.07	0.85
2019	208.10	11.50	54.00	195.20	47.40	47.44	44.05	50.84	3.40	0.04	1.39	1.50

TABLE 14

BASE CASE, DISPLAYED NUMBERS ARE IN PERCENT OF $MV_0 = LP_0 + SF_0 + UG_0$.

	LP_0	SF_0	UG_0	GB	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\widehat{II}$	$\widehat{COG}$
2017	77.60	4.50	17.91	66.65	21.02	20.23	18.95	21.51	1.28	−0.79	0.48	0.21
2018	81.28	4.70	14.02	67.86	19.74	19.30	18.05	20.55	1.25	−0.44	0.46	0.36
2019	76.06	4.20	19.74	71.35	17.32	17.34	16.10	18.58	1.24	0.02	0.51	0.55

value FDB is the numerically calculated number as reported by the company, see Table 3.

We use throughout the notation $\delta = \widehat{FDB} - FDB$ and $\epsilon = (UB - LB)/2$, either as absolute values or relative to MV_0, as indicated. The estimation interval is thus given by $\widehat{FDB} \pm \epsilon$, and the estimation is considered successful if $|\delta| < \epsilon$ such that the true value, FDB, lies within this interval. This holds for the base case as well as for all sensitivities.

Table 14 shows that the estimation error, δ, compared to the true value is in all three cases below 1% of the initial market value, MV_0. We view this as quite a remarkable result for an analytically calculated approximation. Further, it is shown that the influence of $\widehat{II}$ on $\widehat{LB}$ is quite small. The estimated cost of guarantee, $\widehat{COG}$, increases noticeably from 2018 to 2019. This is due to the significantly lower interest rate curve, as can be seen by comparing Tables 10, 11 and 12. Generally speaking, one may also remark that the ratio SF_0/LP_0 has a significant impact on cost of guarantees: if SF_0 is large compared to LP_0 the basis, $BV_0 = SF_0 + LP_0$ for the return on assets is comparatively large and this is advantageous from the company's point of view since the guaranteed interest rate acts only on $V_0 \leq LP_0$.

Table 15 shows the corresponding results for an implied volatility curve which has been reduced by 50%. The effect is most pronounced on the estimation of the upper bound, $\widehat{UB}$, since it leads to a reduced cost of guarantees. Further, we notice that the estimation interval $\widehat{FDB} \pm \epsilon$ shrinks quite strongly. This makes sense since the estimation of a stochastic quantity should improve as the underlying volatility is reduced.

The notation Δ^{rel} in Table 15, as well as below, is understood relative to the base case and in percent of FDB, that is $\Delta^{\mathrm{rel}} X = 100(X - X_0)/FDB$ where X and X_0 are in billion Euros, and X_0 is taken from Table 13. Thus, all the

TABLE 15

SENSITIVITY: VOLATILITY IS REDUCED BY 50%, $IV'_t = IV_t/2$. LHS: DISPLAYED NUMBERS ARE IN PERCENT OF $MV_0 = LP_0 + SF_0 + UG_0$; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.16	18.97	21.35	1.19	−0.86	−0.35	0.08	−0.74
2018	19.74	19.18	18.08	20.29	1.10	−0.56	−0.61	0.13	−1.32
2019	17.32	17.16	16.13	18.19	1.03	−0.16	−1.01	0.19	−2.26

TABLE 16

SENSITIVITY: VOLATILITY INCREASED BY 50%, $IV'_t = 1.5\,IV_t$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.47	18.90	22.05	1.57	−0.55	1.15	−0.25	2.55
2018	19.74	19.62	17.99	21.26	1.64	−0.12	1.62	−0.32	3.59
2019	17.32	17.75	16.02	19.48	1.73	0.43	2.38	−0.44	5.17

TABLE 17

SENSITIVITY: $\rho' = 0.75\,\rho$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF $MV_0 = LP_0 + SF_0 + UG_0$; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.02	18.62	21.43	1.41	−1.00	−1.01	−1.60	−0.39
2018	19.74	19.07	17.72	20.41	1.34	−0.68	−1.19	−1.67	−0.69
2019	17.32	17.12	15.84	18.41	1.29	−0.20	−1.24	−1.52	−0.99

sensitivities in Table 15 are at most of the order of 5% FDB and would be almost 0% when compared to MV_0. This observation holds also for all the other sensitivities considered subsequently.

Table 16 shows the effect of increasing volatility by 50% which leads to a noticeable increase in $\widehat{COG}$ and therefore of $\widehat{UB}$. Increasing the volatility thus also implies a larger estimation error $\pm\epsilon$. This effect is most pronounced for 2019 because of the very low interest rate environment.

Table 17 shows that the effect of reducing the (constant) technical interest rate by 25% is quite small compared to FDB.

Table 18 shows that the effect of increasing the (constant) technical interest rate by 25% is quite small compared to FDB. However, it can also be seen that this conclusion depends on the specific circumstances, as the conditions corresponding to 2018 lead to a slightly more pronounced effect.

Table 19 shows that the effect of reducing the (constant) technical gains rate by 50% is quite small compared to FDB.

Table 20 shows that the effect of increasing the (constant) technical gains rate by 50% is quite small compared to FDB.

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TABLE 18

SENSITIVITY: $\rho' = 1.25\,\rho$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF $MV_0 = LP_0 + SF_0 + UG_0$; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.57	19.27	21.86	1.30	−0.46	1.58	1.50	1.69
2018	19.74	19.82	18.33	21.30	1.49	0.07	2.62	1.41	3.81
2019	17.32	17.63	16.35	18.91	1.28	0.31	1.69	1.43	1.90

TABLE 19

SENSITIVITY: $\gamma' = 0.5\,\gamma$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.57	19.44	21.71	1.14	−0.45	1.63	2.28	0.97
2018	19.74	19.76	18.49	21.03	1.27	0.02	2.34	2.23	2.45
2019	17.32	17.69	16.56	18.83	1.13	0.37	2.05	2.66	1.41

TABLE 20

SENSITIVITY: $\gamma' = 1.5\,\gamma$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	19.95	18.46	21.45	1.49	−1.07	−1.34	−2.35	−0.31
2018	19.74	19.01	17.58	20.43	1.43	−0.74	−1.47	−2.38	−0.56
2019	17.32	17.03	15.63	18.44	1.40	−0.29	−1.77	−2.70	−0.84

TABLE 21

SENSITIVITY: $\theta' = 0.5\,\theta$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.30	18.98	21.61	1.32	−0.72	0.29	0.12	0.49
2018	19.74	19.37	18.08	20.66	1.29	−0.37	0.35	0.15	0.54
2019	17.32	17.37	16.12	18.63	1.25	0.05	0.19	0.08	0.25

Table 21 shows that the effect of reducing θ, estimated by $\theta = SF_0/LP_0$, by 50% is quite small compared to FDB.

Table 22 shows that the effect of increasing θ, estimated by $\theta = SF_0/LP_0$, by 50% is quite small compared to FDB.

Table 23 shows that the effect of reducing σ, estimated by the chosen value $\sigma = 20\%$, by 50% is quite small compared to FDB.

Table 24 shows that the effect of increasing σ, estimated by the chosen value $\sigma = 20\%$, by 50% is quite small compared to FDB.

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TABLE 22

SENSITIVITY: $\theta' = 1.5\,\theta$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF *FDB*.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.17	18.93	21.41	1.24	−0.85	−0.31	−0.12	−0.47
2018	19.74	19.23	18.02	20.44	1.21	−0.51	−0.35	−0.15	−0.52
2019	17.32	17.31	16.08	18.54	1.23	−0.01	−0.17	−0.08	−0.25

TABLE 23

SENSITIVITY: $\sigma' = 0.5\,\sigma$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF *FDB*.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.48	19.36	21.60	1.12	−0.54	1.15	1.91	0.43
2018	19.74	19.63	18.43	20.83	1.20	−0.12	1.67	1.90	1.43
2019	17.32	17.60	16.48	18.71	1.12	0.27	1.50	2.19	0.76

TABLE 24

SENSITIVITY: $\sigma' = 1.5\,\sigma$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF *FDB*.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.01	18.55	21.47	1.46	−1.01	−1.09	−1.93	−0.21
2018	19.74	19.06	17.66	20.46	1.40	−0.68	−1.21	−1.97	−0.45
2019	17.32	17.10	15.72	18.49	1.38	−0.22	−1.37	−2.22	−0.55

TABLE 25

SENSITIVITY: $\nu' = 0.75\,\nu$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF *FDB*.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\text{rel}}\,\widehat{FDB}$	$\Delta^{\text{rel}}\,\widehat{LB}$	$\Delta^{\text{rel}}\,\widehat{UB}$
2017	21.02	20.10	18.69	21.51	1.41	−0.92	−0.64	−1.26	0.00
2018	19.74	19.19	17.83	20.55	1.36	−0.55	−0.56	−1.13	0.00
2019	17.32	17.23	15.88	18.58	1.35	−0.09	−0.61	−1.27	0.00

Table 25 shows that the effect of decreasing ν, estimated by the chosen value $\nu = 75\%$, by 25% is quite small compared to *FDB*.

Table 26 shows that the effect of increasing ν, estimated by the chosen value $\nu = 75\%$, by 25% is quite small compared to *FDB*. The parameter ν does not enter the estimation formula (5.18) for the upper bound and hence $\Delta^{\text{rel}}\,\widehat{UB} = 0$ in Tables 25 and 26.

Table 27 shows that the effect of increasing $d = 8$ to $d + 2$, while leaving $h = 10$ unchanged, is quite small compared to *FDB*.

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TABLE 26

SENSITIVITY: $\nu' = 1.25\,\nu$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\mathrm{rel}}\,\widehat{FDB}$	$\Delta^{\mathrm{rel}}\,\widehat{LB}$	$\Delta^{\mathrm{rel}}\,\widehat{UB}$
2017	21.02	20.36	19.22	21.51	1.15	−0.66	0.62	1.26	0.00
2018	19.74	19.41	18.27	20.55	1.14	−0.33	0.56	1.13	0.00
2019	17.32	17.45	16.32	18.58	1.13	0.13	0.63	1.27	0.00

TABLE 27

SENSITIVITY: $d' = 10 = h$ LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\mathrm{rel}}\,\widehat{FDB}$	$\Delta^{\mathrm{rel}}\,\widehat{LB}$	$\Delta^{\mathrm{rel}}\,\widehat{UB}$
2017	21.02	20.26	18.98	21.55	1.28	−0.76	0.12	0.10	0.16
2018	19.74	19.38	18.06	20.71	1.32	−0.36	0.43	0.04	0.82
2019	17.32	17.32	16.12	18.53	1.21	−0.00	−0.11	0.08	−0.32

TABLE 28

SENSITIVITY: $h' = 12 = h + 2$; LHS: DISPLAYED NUMBERS ARE IN PERCENT OF MV_0; RHS: DIFFERENCE TO BASE CASE IN PERCENT OF FDB.

	FDB	$\widehat{FDB}$	$\widehat{LB}$	$\widehat{UB}$	ϵ	δ	$\Delta^{\mathrm{rel}}\,\widehat{FDB}$	$\Delta^{\mathrm{rel}}\,\widehat{LB}$	$\Delta^{\mathrm{rel}}\,\widehat{UB}$
2017	21.02	20.15	18.69	21.61	1.46	−0.87	−0.39	−1.23	0.47
2018	19.74	19.24	17.80	20.68	1.44	−0.51	−0.30	−1.28	0.67
2019	17.32	17.37	15.88	18.85	1.48	0.04	0.17	−1.24	1.56

Table 28 shows that the effect of increasing $h = 10$ to $h + 2$, while leaving $d = 8$ unchanged, is quite small compared to FDB.

8. CONCLUSIONS

The bounds (5.17) and (5.18) have been derived in a manner which is quite basic from the mathematical point of view but seems at the same time adequate for real world applications. We view the accuracy of $\widehat{FDB}$ as demonstrated by $\delta = (\widehat{FDB} - FDB)/MV_0 < 1\%$ and $\epsilon = (\widehat{UB} - \widehat{LB})/MV_0 < 1.5\%$ in all three cases in Table 14 as quite remarkable.

However, it would certainly also be interesting to further refine the model: for example one could relax the Assumption 4.9 and attempt to model ROA_t, or also the difference $ROA_t - E[ROA_t|\mathcal{F}_{t-1}]$ along the lines of Dorobantu *et al*. (2020). Moreover, the estimation of the return on the deferred caplet in Equation (5.8) relies on simply estimating the absolute value of a correlation factor by 1, and this estimate could possibly be improved. At the same time, the product, $CV_{s,t}^1 CV_s^2$, of the coefficients of variation as mentioned in Table 2 has been simply set to 0 in Section 7. The validity of this parameter choice remains to be analyzed.

CONFLICTS OF INTEREST

The authors declare no competing interests.

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NOTE

1 The calculations have been carried out in *R*, and the script files can be provided upon request.

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APPENDIX: LIST OF SYMBOLS

Symbol	Meaning	Definition	Reference
A			
$\mathcal{A}_t$	set of assets, excluding cash, at time t	–	p. 5
B			
B_t	bank account at time t	$B_t = \prod_{j=0}^{t-1}(1+F_j)$	p. 5
BE	best estimate	$BE = E[\sum_{t=1}^{T} B_t^{-1}(gbf_t + gbf_t^{\leq 0} + ph_t + co_t - pr_t)]$	p. 14
BV_t	book value of the asset portfolio at time t	$BV_t = \sum_{a\in\mathcal{A}_t} BV_t^a + C_t$	p. 5
BV_t^a	book value of asset a	–	p. 5
C			
C_t	amount of cash held by the company at time t		p. 5
cf_t^a	cash flow of asset a at time t	–	p. 5
χ_t	surrender fee factor at time t	–	p. 11
COG	cost of guarantees	$COG = E\left[\sum_{t=1}^{T} B_t^{-1} gs_t^{-}\right]$	p. 14
co_t	cost cash flows at time t	–	p. 11
$CV_{s,t}^1$	first coefficient of variation	$CV_{s,t}^1 = CV\left[D(s,t) - D(s,t+1)\right]$	p. 23
CV_s^2	second coefficient of variation	$CV_s^2 = CV\left[B_s^{-1} gs_s^{+}\right]$	p. 23
D			
d	duration	–	p. 19
$D(t,s)$	discount factor from s to $t<s$	$D(t,s) = \prod_{j=t}^{s-1}(1+F_j)^{-1}$	p. 5

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Symbol	Meaning	Definition	Reference
DB_t	declared bonuses after valuation time	$DB_t = \sum_{x \in \mathcal{X}_t} DB_t^x$	p. 9
DB_t^x	declared bonuses after valuation time of model point x at time t	–	p. 9
DB_t^-	account of declared bonuses before bonus declaration at time t	–	p. 11
$DB_t^{\leq 0}$	declared bonuses up to and including valuation time	$DB_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} (DB_t^{\leq 0})^x$	p. 9
$(DB_t^{\leq 0})^x$	declared bonuses up to and including valuation time of model point x at time t	–	p. 9
Δf_t	increment of f_t	$\Delta f_t = f_t - f_{t-1}$	p. 5
E			
η_t	fraction of declaration of SF_{t-1} to DB_t	–	p. 12
F			
F_t	simple one year forward rate between t and $t+1$	–	p. 5
FC_t	free capital at time t	$FC_t = BV_t - L_t$	p. 10
FDB	value of future discretionary benefits	$FDB = E\left[\sum_{t=1}^{T} B_t^{-1} ph_t\right]$	p. 14
G			
γ_t	fraction of technical gains	$\gamma_t = (tg_t + \chi_t^{\leq 0} DB_{t-1}^{\leq 0} + \chi_t DB_{t-1})/LP_{t-1}$	p. 11
GB	value of guaranteed benefits	$GB = BE - FDB$	p. 14
gbf_t	guaranteed benefits at time t	$gbf_t = \sum_{x \in \mathcal{X}_t} gbf_t^x$	p. 14
gbf_t^x	guaranteed benefits generated by model point x at time t	–	p. 10
$gbf_t^{\leq 0}$	cash flows due to $DB_{t-1}^{\leq 0}$	$gbf_t^{\leq 0} = \sum_{x \in \mathcal{X}_t} (gbf_t^{\leq 0})^x$	p. 9
$(gbf_t^{\leq 0})^x$	cash flows due to $(DB_{t-1}^{\leq 0})^x$	–	p. 9
gph	policyholder share in gross surplus	–	p. 12
gs_t	gross surplus at time t	$gs_t = ROA_t - \Delta V_t - \Delta DB_t^{\leq 0} - DB_t^- + DB_{t-1} + pr_t - gbf_t - gbf_t^{\leq 0} - ph_t - co_t$	p. 11
gsh	share holder share in gross surplus	–	p. 11
$gtax$	tax paid on gross surplus at time t	–	p. 11
H			
h	half life of assurance provisions	–	p. 17
I			
IV_s	caplet implied volatility	–	p. 24
K			
k_s	strike	–	p. 24
L			
L_t	book value of liabilities at time t	$L_t = LP_t + SF_t$	p. 10
LP_t	life assurance provision at time t	$LP_t = V_t + DB_t^{\leq 0} + DB_t$	p. 10

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Symbol	Meaning	Definition	Reference
M			
μ_k^t	fraction of bonus declarations from time k paid out (or kept as surrender fee) at t	–	p. 12
MV_t	market value of the portfolio at time t	$MV_t = \sum_{a\in\mathcal{A}_t} MV_t^a + C_t$	p. 5
MV_t^a	market value of asset a at time t	–	p. 5
N			
ν	bonus declaration bound	–	p. 18
ν_t	declaration fraction of ph_t^*	–	p. 12
O			
O_s^+	value of the caplet with maturity $s-1$	–	p. 24
O_s^-	value of the floorlet with maturity $s-1$	–	p. 24
P			
$P(t,s)$	value of a zero coupon bond, with nominal of 1 at s, at time t	$P(t,s) = E[D(t,s)]$	p. 5
PH^*	time value of the accounting flows ph_t^*	$PH^* = E\left[\sum_{t=1}^T B_t^{-1} ph_t^*\right]$	p. 14
ph_t	amount of discretionary benefits paid out at time t	$ph_t = \sum_{x\in\mathcal{X}_t} ph_t^x$	p. 10
ph_t^*	policyholder accounting flow at time t	$ph_t^* = gph \cdot gs_t^+$	p. 12
ph_t^x	cash flows due to DB_{t-1}^x	–	p. 9
pr_t	premium payments at time t	–	p. 11
R			
ρ_t	average technical interest rate at time $t-1$	–	p. 11
ROA_t	book value return at time t	$ROA_t = \sum_{a\in\mathcal{A}_{t-1}} ROA_t^a + F_{t-1}C_{t-1}$	p. 6
ROA_t^a	book value return of asset a at time t	$ROA_t^a = cf_t^a + \Delta BV_t^a$	p. 5
S			
SF_t	surplus fund at time t	–	p. 10
sg_t^*	surrender fee at time t	–	p. 12
sh_t	share holder cash flow at time t	$sh_t = gsh \cdot gs_t^+ - gs_t^-$	p. 12
T			
T	projection horizon	–	p. 17
TAX	time value of tax	$TAX = E\left[\sum_{t=1}^T B_t^{-1} tax_t\right]$	p. 14
tax_t	tax cash flow at time t	$tax_t = gtax \cdot gs_t^+$	p. 12
tg_t	technical gains at time t	–	p. 11
θ	surplus fund fraction	–	p. 18
U			
UG_t	unrealized gains at time t	$UG_t = MV_t - BV_t$	p. 5
UG_t^a	unrealized gains of asset a at time t	$UG_t^a = MV_t^a - BV_t^a$	p. 5

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Symbol	Meaning	Definition	Reference
V			
V_t	mathematical reserves at time t	$V_t = \sum_{x\in\mathcal{X}_t} V_t^x$	p. 10
V_t^x	mathematical reserve of model point x at time t	–	p. 9
VIF	value of in-force business	$VIF = E\left[\sum_{t=1}^{T} B_t^{-1} sh_t\right]$	p. 13
X			
$\mathcal{X}_t$	set of model points active at time t	–	p. 10

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