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Positive predicate structures for continuous data[†]

MARGARITA KOROVINA[‡] and OLEG KUDINOV[§]

[‡]*A.P. Ershov Institute of Informatics systems, SB RAS, NSU, Novosibirsk, Russia*
Email: rita.korovina@gmail.com
[§]*Sobolev Institute of Mathematics, SB RAS, NSU, Novosibirsk, Russia*
Email: kud@math.nsc.ru

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In this paper, we develop a general framework for continuous data representations using positive predicate structures. We first show that basic principles of Σ-definability which are used to investigate computability, i.e., existence of a universal Σ-predicate and an algorithmic characterization of Σ-definability hold on all predicate structures without equality. Then we introduce positive predicate structures and show connections between these structures and effectively enumerable topological spaces. These links allow us to study computability over continuous data using logical and topological tools.

1. Introduction

The main goal of the research presented in this paper is to provide a logical framework for studying computability over discrete and continuous data in a common language. In order to achieve this goal, we represent data as a structure which might not have effective equality and employ Σ-definability theory.

Our approach is based on representations of data (discrete or continuous) by a suitable structure $\mathcal{A} = \langle A, \sigma_0 \rangle = \langle A, \sigma_P \cup \{\neq\} \rangle$, where A contains more than one element, and σ_P is a finite set of basic predicates. *We assume that all predicates $Q_i \in \sigma_P$ and $\neq$ occur only positively in existential and Σ-formulas and do not assume that the language σ_P contains equality*. We call these structures as *predicate structures*.

Definability is a very successful framework for generalized computability theory (Moschovakis 1974), descriptive complexity (Ajtai 1989; Immerman 1999), set-theoretic specifications (Hoges 1993) and databases (Sazonov 2001). One of the most interesting and practically important types of definability is Σ-definability, which generalizes recursive enumerability over the natural numbers (Barwise 1975; Ershov 1996; Hoges 1993; Sazonov 2001). However, the most developed part of definability and Σ-definability theories deals with abstract structures with equality (i.e., the natural numbers, trees, automata, etc). In the context, e.g., of continuous data, equality cannot be effectively represented.

It turns out that Σ-definability without equality is rather different from Σ-definability with equality. It has been shown in Morozov and Korovina (2008) that there is no

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effective procedure which given a Σ-formula with equality defining an open set produces a Σ-formula without equality defining the same set. Therefore, it is important to investigate which properties of Σ-definability hold on structures with equality likewise on structures without equality.

Some of the important properties of Σ-definability with respect to computability, i.e., existence of a universal Σ-predicate and an algorithmic characterization of Σ-definability have been proven over structures with equality (Ershov 1996) and over the real numbers without equality (Korovina 2003). In Sections 2–5, we show that these properties hold over every predicate structure. In order to do this, we develop new tools and techniques to overcome difficulties arising from possible absence of equality and particular properties of the reals.

In Section 6, we investigate predicate structures with a computably enumerable existential positive theory called *positive predicate structures*. We discuss links between positive predicate structures and effectively enumerable topological spaces which contain effective ω-continuous domains and computable metric spaces as proper subclasses (Korovina and Kudinov 2008). We show that a positive predicate structure $\mathcal{A}$ can be considered as a topological space $\left(A, \tau_\Sigma^A\right)$ with a base of topology τ_Σ^A consisting of the subsets of A defined by existential formulas. In this topology, *Σ-definability coincides with effective openness*. Therefore, if an effectively enumerable topological space can be structured then we can use Σ-definability for characterization of effective openness and computability.

On several examples we illustrate how to choose an appropriate finite language in such way that the τ_Σ^A-topology coincides with the usual topology. In Section 7, we prove that any computable metric space can be structured.

2. Σ-definability over predicate structures

We start by introducing basic notations and definitions. In this paper, we are working with an arbitrary structure $\mathcal{A} = \langle A, \sigma_0 \rangle = \langle A, \sigma_P \cup \{\neq\} \rangle$, where A contains more than one element, σ_P is a finite set of basic predicates.

Example 2.1.

1. The natural numbers: $\mathbb{N} = \langle \mathbb{N}, Q_1, Q_2, < \rangle$, where Q_1 and Q_2 have the following meanings:

$$\mathbb{N} \models Q_1(x) \leftrightarrow x = 0 \text{ and } \mathbb{N} \models Q_2(x, y) \leftrightarrow x = y + 1.$$

2. The real numbers: $\mathbb{R} = \langle \mathbb{R}, \mathcal{M}_E^*, \mathcal{M}_H^*, \mathcal{P}_E^+, \mathcal{P}_H^+, < \rangle$, where $\mathcal{M}_E^*, \mathcal{M}_H^*$ are interpreted as the open epigraph and the open hypograph of multiplication respectively, and $\mathcal{P}_E^+, \mathcal{P}_H^+$ are interpreted as the open epigraph and the open hypograph of addition respectively, e.g.,

$$\mathbb{R} \models \mathcal{M}_H^*(x, y, z) \leftrightarrow x \cdot y < z \text{ and } \mathbb{R} \models \mathcal{M}_E^*(x, y, z) \leftrightarrow x \cdot y > z;$$
$$\mathbb{R} \models \mathcal{P}_H^+(x, y, z) \leftrightarrow x + y < z \text{ and } \mathbb{R} \models \mathcal{P}_E^+(x, y, z) \leftrightarrow x + y > z.$$

3. The complex numbers: $\mathbb{C} = \langle \mathbb{C}, , P_1, \ldots, P_{12} \rangle$, where the predicates $P_1, \ldots, P_{12}$ have the following meanings for every $x,\ y,\ z \in \mathbb{C}$.

The first group formalizes relations between Re and Im of two complex numbers.

$$\mathbb{C} \models P_1(x,y) \leftrightarrow Re(x) < Re(y) \text{ and } \mathbb{C} \models P_2(x,y) \leftrightarrow Im(x) < Im(y);$$
$$\mathbb{C} \models P_3(x,y) \leftrightarrow Re(x) < Im(y) \text{ and } \mathbb{C} \models P_4(x,y) \leftrightarrow Im(x) < Re(y).$$

The second group formalizes properties of operations.

$$\mathbb{C} \models P_5(x,y,z) \leftrightarrow Re(x) + Re(y) < Re(z);$$
$$\mathbb{C} \models P_6(x,y,z) \leftrightarrow Re(x) + Re(y) > Re(z);$$
$$\mathbb{C} \models P_7(x,y,z) \leftrightarrow Re(x) \cdot Re(y) < Re(z);$$
$$\mathbb{C} \models P_8(x,y,z) \leftrightarrow Re(x) \cdot Re(y) > Re(z);$$
$$\mathbb{C} \models P_9(x,y,z) \leftrightarrow Im(x) + Im(y) < Im(z);$$
$$\mathbb{C} \models P_{10}(x,y,z) \leftrightarrow Im(x) + Im(y) > Im(z);$$
$$\mathbb{C} \models P_{11}(x,y,z) \leftrightarrow Im(x) \cdot Im(y) < Im(z);$$
$$\mathbb{C} \models P_{12}(x,y,z) \leftrightarrow Im(x) \cdot Im(y) > Im(z).$$

4. The function space: $C[0,1] = \langle C[0,1], P_1, \ldots, P_{10} \rangle$ where the predicates $P_1, \ldots, P_{10}$ have the following meanings for every $f,\ g \in C[0,1]$:
the first group formalizes relations between infimum and supremum of two functions.

$$C[0,1] \models P_1(f,g) \leftrightarrow \sup(f) < \sup(g);$$
$$C[0,1] \models P_2(f,g) \leftrightarrow \sup(f) < \inf(g);$$
$$C[0,1] \models P_3(f,g) \leftrightarrow \sup(f) > \inf(g);$$
$$C[0,1] \models P_4(f,g) \leftrightarrow \inf(f) > \inf(g).$$

The second group formalizes properties of operations on $C[0,1]$.

$$C[0,1] \models P_5(f,g,h) \leftrightarrow f(x) + g(x) < h(x) \text{ for every } x \in [0,1];$$
$$C[0,1] \models P_6(f,g,h) \leftrightarrow f(x) \cdot g(x) < h(x) \text{ for every } x \in [0,1];$$
$$C[0,1] \models P_7(f,g,h) \leftrightarrow f(x) + g(x) > h(x) \text{ for every } x \in [0,1];$$
$$C[0,1] \models P_8(f,g,h) \leftrightarrow f(x) \cdot g(x) > h(x) \text{ for every } x \in [0,1].$$

The third group formalizes relations between functions f and the identity function $\lambda x.x$.

$$C[0,1] \models P_9(f) \leftrightarrow f > \lambda x.x;$$
$$C[0,1] \models P_{10}(f) \leftrightarrow f < \lambda x.x.$$

In order to do any kind of computation or to develop a computability theory, one has to work within a structure rich enough for information to be coded and stored. For this purpose we extend the structure A by the set of hereditarily finite sets $\mathrm{HF}(A)$.

The idea that the hereditarily finite sets over A form a natural domain for computation is quite classical and is developed in detail in Barwise (1975) and Ershov (1996) for the case when σ_0 contains equality.

We construct the set of hereditarily finite sets, $\mathrm{HF}(A)$, as follows:

1. $\mathrm{HF}_0(A) \rightleftharpoons A$,
2. $\mathrm{HF}_{n+1}(A) \rightleftharpoons \mathcal{P}_\omega(\mathrm{HF}_n(A)) \cup \mathrm{HF}_n(A)$, where $n \in \omega$ and for every set B, $\mathcal{P}_\omega(B)$ is the set of all finite subsets of B,
3. $\mathrm{HF}(A) \rightleftharpoons \bigcup_{n\in\omega} \mathrm{HF}_n(A)$.

We define $\mathbf{HF}(A)$ as the following model:

$$\mathbf{HF}(A) \rightleftharpoons \langle \mathrm{HF}(A), \sigma_0 \cup \{U, \in\}\rangle \rightleftharpoons \langle \mathrm{HF}(A), \sigma\rangle\,,$$

where the binary predicate symbol $\in$ has the set-theoretic interpretation. Also we add the predicate symbol U for urelements (elements from A).

The natural numbers $0, 1, \ldots$ are identified with the (finite) ordinals in $\mathrm{HF}(A)$ i.e., $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}, \ldots$, so in particular, $n + 1 = n \cup \{n\}$ and the set ω is a subset of $\mathrm{HF}(A)$. In this paper, we follow the standard agreement that for formula definitions a countable list of variables $\{x_i\}_{i\in\omega}$ is used and a, b, c, x, y, t, z, s denote variables from this list.

The *atomic formulas* include $U(x)$, $\neg U(x)$, $x \neq y$, $x \in s$, $x \notin s$ and also, for every $Q_i \in \sigma_P$ of the arity n_i, $Q_i(y_1, \ldots, y_{n_i})$ which has the following interpretation:

$$\mathbf{HF}(A) \models Q_i(a_1, \ldots, a_{n_i}) \text{ if and only if}$$
$$\mathcal{A} \models Q_i(a_1, \ldots, a_{n_i}) \text{ and, for every } 1 \leqslant j \leqslant n_i,\ a_j \in A.$$

The set of $\exists$-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee$ and existential quantifiers.

The set of Δ_0-*formulas* is the closure of the set of atomic formulas under $\wedge, \vee$, bounded quantifiers $(\exists x \in y)$ and $(\forall x \in y)$, where $(\exists x \in y)\ \Psi$ means the same as $\exists x(x \in y \wedge\ \Psi)$ and $(\forall x \in y)\ \Psi$ as $\forall x(x \in y \to \Psi)$ where y ranges over sets.

The set of Σ-*formulas* is the closure of the set of Δ_0-formulas under $\wedge,\vee$, $(\exists x \in y)$, $(\forall x \in y)$ and $\exists x$, where y ranges over sets.

Remark 2.1. We recall that all predicates $Q_i \in \sigma_P$ and $\neq$ occur only positively in $\exists$-formulas and Σ-formulas. Hence, when σ_P does not contain equality as a basic predicate, it is not necessary that equality on the urelements (elements from A) is representable by a $\exists$-formula or by a Σ-formula.

Let $\mathcal{A}$ be a predicate structure. We are interested in Σ-definability of sets on A^n which can be considered as generalization of recursive enumerability. The analogy of Σ-definable and recursively enumerable sets is based on the following fact. Consider the structure $\mathbf{HF} = \langle \mathrm{HF}(\emptyset), \in\rangle$ with the hereditarily finite sets over $\emptyset$ as its universe and membership as its only relation. In $\mathbf{HF}$, the Σ-definable subsets of ω are exactly the recursively enumerable sets (Barwise 1975).

The notion of Σ-definability has a natural meaning also in the structure $\mathbf{HF}(A)$.

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Definition 2.1.

1. A relation $B \subseteq \mathrm{HF}(A)^n$ is Σ-*definable*, if there exists a Σ-formula $\Phi(\bar{a})$ such that

$$\bar{b} \in B \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{b}).$$

2. A function $f : \mathrm{HF}(A)^n \to \mathrm{HF}(A)^m$ is Σ-*definable*, if there exists a Σ-formula $\Phi(\bar{c}, \bar{d})$ such that

$$f(\bar{a}) = \bar{b} \leftrightarrow \mathbf{HF}(A) \models \Phi(\bar{a}, \bar{b}).$$

In a similar way, we introduce the notion of Δ_0-definability. Let $S(\mathrm{HF}(A))$ denote the set of all sets in $\mathrm{HF}(A)$ and $S'(\mathrm{HF}(A))$ denote the set of all nonempty sets in $\mathrm{HF}(A)$.

Lemma 2.1.

1. The predicates $S(x) \rightleftharpoons$ 'x is a set', $\varnothing(x) \rightleftharpoons$ 'x is the empty set', $n \in \omega$ and $\neg\varnothing(x) \rightleftharpoons$ 'x is not the empty set' are Δ_0-definable.
2. The predicate $S'(x) \rightleftharpoons$ 'x is a nonempty set' is Δ_0-definable.
3. The following predicates are Δ_0-definable: $x = y$, $x = y \cap z$, $x = y \cup z$, $x = \langle y, z\rangle$, $x = y \setminus z$ where all variables x, y, z range over sets.
4. If a function $f : \omega^n \to \omega^m$ is computable then it is Σ-definable.
5. Let $Fun(g)$ mean that $g : S'(\mathrm{HF}(A)) \to S'(\mathrm{HF}(A))$ is a finite function and $g \in S'(\mathrm{HF}(A))$. Then the predicate $Fun(g)$ is Δ_0-definable.
6. If $\mathbf{HF}(A) \models Fun(g)$ then the domain of g, denoted by $dom(g)$, is Δ_0-definable.
7. The set $FF \rightleftharpoons \{\gamma : \omega \to S'(\mathbf{HF}(A)) | \gamma \text{ is a finite function}\}$ is Σ-definable.

Proof. Proofs of all properties are straightforward except (4) which can be found in Ershov (1996). □

For finite functions $Fun(\gamma)$ let us denote $\gamma(x) = y$ if $\langle x, y\rangle \in \gamma$.

3. Gandy's theorem and inductive definitions

Let us recall Gandy's Theorem for $\mathbf{HF}(A)$ which will be essentially used in all proofs of the main results. Let $\Phi(a_1, \ldots, a_n, P)$ be a Σ-formula, where P occurs positively in Φ and the arity of P is equal to n. We think of Φ as defining an *effective operator* $\Gamma : \mathcal{P}(\mathrm{HF}(A)^n) \to \mathcal{P}(\mathrm{HF}(A)^n)$ given by

$$\Gamma(Q) = \{\bar{a} | (\mathbf{HF}(A), Q) \models \Phi(\bar{a}, P)\}.$$

Since the predicate symbol P occurs only positively, we have that the corresponding operator Γ is monotone, i.e., for all sets B and C, from $B \subseteq C$ follows $\Gamma(B) \subseteq \Gamma(C)$, and continuous with respect to Scott topology on $\mathcal{P}(\mathrm{HF}(A)^n)$ (see, e.g., Ershov (1996)). By monotonicity, the operator Γ has a least (w.r.t. inclusion) fixed point which can be described as follows. We start from the empty set and apply operator Γ until we reach the fixed point:

$$\Gamma^0 = \varnothing, \quad \Gamma^{n+1} = \Gamma(\Gamma^n), \quad \Gamma^\gamma = \cup_{n<\gamma}\Gamma^n,$$

where γ is a limit ordinal.

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One can easily check that the sets Γ^n form an increasing chain of sets: $\Gamma^0 \subseteq \Gamma^1 \subseteq \ldots$. By set-theoretical reasons, there exists the least ordinal γ such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$. This Γ^γ is the least fixed point of the given operator Γ.

Theorem 3.1 (Gandy's theorem for HF(A)).
Let $\Gamma : \mathcal{P}(\mathrm{HF}(A)^n) \to \mathcal{P}(\mathrm{HF}(A)^n)$ be an effective operator. Then, the least fixed-point of Γ is Σ-definable and the least ordinal such that $\Gamma(\Gamma^\gamma) = \Gamma^\gamma$ is less or equal to ω.

Proof. See Korovina (2003). □

Definition 3.1. A relation $B \subset A^n$ is called Σ-inductive if it is the least fixed-point of an effective operator.

Corollary 3.1. Every Σ-inductive relation is Σ-definable.

Proof. See Korovina (2003). □

4. Universal Σ-predicate

In order to obtain a result on the existence of a universal Σ-predicate we first prove Σ-definability of the predicate $TR^\forall$ introduced below.

We use a countable list of variables $\{x_i\}_{i\in\omega}$ and fix a standard effective Gödel numbering of formulas of the language σ by finite ordinals which are elements of $\mathrm{HF}(\emptyset)$. Let $\ulcorner\Phi\urcorner$ denote the code of a formula Φ. It is worth noting that the type of an expression is effectively recognizable by its code. We also can obtain effectively from the codes of expressions the codes of their subexpressions and vice versa. Since equality is Δ_0-definable in $\mathrm{HF}(\emptyset)$, we can use the well-known characterization which states that all effective procedures over ordinals are Σ-definable. Thus, for example, the following predicates

$$Code_{elem_0}(n, j) \rightleftharpoons n = \ulcorner U(x_j)\urcorner,$$
$$Code_{elem_i}(n, j_1, \ldots, j_{n_i}) \rightleftharpoons n = \ulcorner Q_i(x_{j_1}, \ldots, x_{j_{n_i}})\urcorner,$$
$$Code_\wedge(n, i, j) \rightleftharpoons n = \ulcorner\Phi \wedge \Psi\urcorner \wedge i = \ulcorner\Phi\urcorner \wedge j = \ulcorner\Psi\urcorner$$

are Σ-definable. Hence, in Σ-formulas we can use such predicates.

Let $FV(\Phi)$ denote the set of variables with free occurrences in Φ and $FF = \{\gamma : \omega \to S'(\mathbf{HF}(A)) | \gamma \text{ is a finite function}\}$ as defined in Lemma 1.

Proposition 4.1. For every $\mathcal{A}$ of cardinality > 1 there exists a Σ-definable set

$$TR^\forall \subseteq \omega \times FF$$

with the following properties.

1. Let n be the Gödel number of a Σ-formula Φ and $f : FV(\Phi) \to \mathrm{HF}(A)$ be an assignment function such that $\mathbf{HF}(A) \models \Phi[f]$. We define the finite function $\gamma : \omega \to S'(\mathrm{HF}(A))$ as follows $\gamma(i) = \{f(x_i)\}$ for all $i : x_i \in dom(f)$. Then $\langle n, \gamma\rangle \in TR^\forall$.
2. If $\langle n, \gamma\rangle \in TR^\forall$ then n is the Gödel number of a Σ-formula Φ and $\gamma : \omega \to S'(\mathrm{HF}(A))$ is a finite function such that, for every assignment function $f : FV(\Phi) \to \mathrm{HF}(A)$ with the property $f(x_i) \in \gamma(i)$, $\mathbf{HF}(A) \models \Phi[f]$.

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Proof. The predicate $TR^{\forall}$ is the least fixed point of the operator defined by the following formula:

$$\Psi(n,\gamma,P) \rightleftharpoons \text{Gödel}(n) \wedge \text{Correct}(n,\gamma) \wedge (\Psi_{elem}(n,\gamma) \vee \Psi_{\wedge}(n,\gamma,P) \vee \\ \Psi_{\vee}(n,\gamma,P) \vee \Psi_{\exists\in}(n,\gamma,P) \vee \Psi_{\forall\in}(n,\gamma,P) \vee \Psi_{\exists}((n,\gamma,P))),$$

where n, γ are free variables and P is a new predicate symbol. The formula $\Psi(n,\gamma,P)$ represents the inductive definition of the predicate $TR^{\forall}$ where the immediate subformulas have the following meaning. The first two formulas recognize the properties of n and γ. The formula Gödel(n) represents that n is the Gödel number of a Σ-formula Φ; the formula Correct(n,γ) represents that γ is a finite function from ω to $S'(\text{HF}(A))$ such that $i \in dom(\gamma)$ if and only if $x_i \in FV(\Phi)$. The formula $\Psi_{elem}(n,\gamma)$ defines the basis of the inductive definition and captures the cases when n is the Gödel number of an atomic formula. The remaining formulas represent inductive steps for conjunctions, disjunctions, bounded quantifiers, and existential quantifiers. By Lemma 2.1, the formulas Gödel(n) and Correct(n,γ) are equivalent to Σ-formulas. We illustrate constructions of the rest of the formulas. The basis of the inductive definition is given by the following formula:

$$\Psi_{elem}(n,\gamma) \rightleftharpoons \Psi_U(n,\gamma) \vee \Psi_{\neg U}(n,\gamma) \vee \Psi_{\in}(n,\gamma) \vee \\ \Psi_{\notin}(n,\gamma) \vee \Psi_{\neq}(n,\gamma) \vee \bigvee_{Q_i \in \sigma_P} \Psi_{Q_i}(n,\gamma),$$

where the subformulas can be done in the following way:

$$\begin{aligned}
\Psi_U(n,\gamma) &\rightleftharpoons \exists i \left(n = \ulcorner U(x_i) \urcorner \wedge \forall z \in \gamma(i)\, U(z)\right);\\
\Psi_{\neg U}(n,\gamma) &\rightleftharpoons \exists i \left(n = \ulcorner \neg U(x_i) \urcorner \wedge \forall z \in \gamma(i) \neg U(z)\right);\\
\Psi_{\in}(n,\gamma) &\rightleftharpoons \exists i \exists j \exists a (n = \ulcorner x_i \in x_j \urcorner \wedge S'(a) \wedge \gamma(j) = \{a\} \wedge \gamma(i) \subseteq a);\\
\Psi_{\notin}(n,\gamma) &\rightleftharpoons \exists i \exists j (n = \ulcorner x_i \notin x_j \urcorner \wedge \exists a (S'(a) \wedge \gamma(j) = \{a\} \wedge \gamma(i) \cap a = \varnothing)\\
&\qquad \vee \forall z \in \gamma(j)\, U(z) \vee \forall z \in \gamma(j)\, \varnothing(z));\\
\Psi_{\neq}(n,\gamma) &\rightleftharpoons \exists i \exists j (n = \ulcorner x_i \neq x_j \urcorner \wedge \forall z \in \gamma(i) \forall k \in \gamma(j) (U(z) \wedge U(k) \wedge z \neq k));\\
\Psi_{Q_i}(n,\gamma) &\rightleftharpoons \exists j_1 \ldots \exists j_{n_i} (n = \ulcorner Q_i(x_{j_1},\ldots,x_{j_{n_i}}) \urcorner \forall z_j \in \gamma(j_1) \ldots\\
&\qquad \forall z_{j_{n_i}} \in \gamma(j_{n_1}) Q_i(z_{j_1},\ldots,z_{j_{n_i}}), \text{for every basic predicate } Q_i \in \sigma_P.
\end{aligned}$$

Now, we construct the formulas for the inductive steps. For conjunctions and disjunctions:

$$\begin{aligned}
\Psi_{\wedge}(n,\gamma,P) &\rightleftharpoons \exists m \exists k \left(n = \ulcorner \Phi \wedge \Psi \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge k = \ulcorner \Psi \urcorner \wedge P(m,\gamma) \wedge P(k,\gamma)\right);\\
\Psi_{\vee}(n,\gamma,P) &\rightleftharpoons \exists m \exists k \left(n = \ulcorner \Phi \vee \Psi \urcorner \wedge m = \ulcorner \Phi \urcorner \wedge k = \ulcorner \Psi \urcorner \wedge (P(m,\gamma) \vee P(k,\gamma))\right).
\end{aligned}$$

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For bounded quantifiers:

$$\begin{aligned}\Psi_{\exists\in}(n,\gamma,P) \rightleftharpoons\ & \exists i \exists j \exists a \exists v \exists \gamma^* \exists m \left(n = \lceil \exists x_i \in x_j\, \Phi \rceil \wedge m = \lceil \Phi \rceil \wedge j \in dom(\gamma) \wedge \right.\\ & S'(a) \wedge \gamma(j) = \{a\} \wedge \gamma \cup \{\langle i, v\rangle\} = \gamma^* \wedge i \notin dom(\gamma) \wedge \\ & \left. P(m,\gamma^*) \wedge v \subseteq a\right);\\ \Psi_{\forall\in}(n,\gamma,P) \rightleftharpoons\ & \exists i \exists j \exists a \exists m (n = \lceil \forall x_i \in x_j \Phi \rceil \wedge m = \lceil \Phi \rceil \wedge j \in dom(\gamma) \wedge (\forall z \in \gamma(j)\\ & U(z) \vee \forall z \in \gamma(j)\, \varnothing(z) \vee (S'(a) \wedge \gamma(j) = \{a\} \wedge i \notin dom(\gamma)\\ & \wedge \forall b \in a \exists \gamma^* (i \in dom(\gamma^*) \wedge \gamma \subseteq \gamma^* \wedge b \in \gamma^*(i) \wedge P(m,\gamma^*))))).\end{aligned}$$

The formula $\Psi_{\exists}(n,\gamma,P)$ can be given as follows:

$$\begin{aligned}\Psi_{\exists}(n,\gamma,P) \rightleftharpoons \exists i \exists m \exists v \exists w \left(n = \lceil \exists x_i\, \Phi \rceil \wedge m = \lceil \Phi \rceil \wedge i \notin dom(\gamma) \wedge \right.\\ \left. S'(v) \wedge w = \gamma \cup \{\langle i, v\rangle\} \wedge P(m,w)\right).\end{aligned}$$

From Gandy's theorem (c.f. Section 3), it follows that the least fixed point $TR^{\forall}$ of the effective operator defined by Ψ is Σ-definable. □

Theorem 4.1. For every $n \in \omega$ there exists a Σ-formula $Univ_{n+1}(m, x_0, \ldots, x_n)$ such that for any Σ-formula $\Phi(x_0, \ldots, x_n)$

$$\mathbf{HF}(A) \models \Phi(r_0, \ldots, r_n) \leftrightarrow Univ_{n+1}(\lceil \Phi \rceil, r_0, \ldots, r_n).$$

Proof. It is easy to see that the following formula defines a universal Σ-predicate for the Σ-formulas of arity $n+1$.

$$\begin{aligned}\mathrm{Univ}_{n+1}(m, x_0, \ldots, x_n) \rightleftharpoons \exists y_0 \ldots \exists y_n \exists \gamma (S'(y_0) \wedge \cdots \wedge S'(y_n) \wedge \\ \gamma = \{\langle 0, y_0\rangle, \ldots, \langle n, y_n\rangle\} \wedge TR^{\forall}(m,\gamma) \wedge \bigwedge_{0 \leqslant i \leqslant n} x_i \in y_i).\end{aligned}$$

□

5. Semantic characterization of Σ-definability

In this section, we prove that a relation over A is Σ-definable if and only if it is definable by a disjunction of a recursively enumerable set of existential formulas in the language σ_0.

Definition 5.1. A partial finite injective function from X to A is called an assignment.

Definition 5.2. Let a set of distinct variables $X = \{x_i | i \in \omega\}$ and an assignment $f : X \to A$ be given. For $z \in \mathrm{HF}(X)$, define $sp(z)$ and $[z]_f$ as follows:

1. if z is a variable then $sp(z) = \{z\}$ and $[z]_f = f(z)$; if $f(z)$ is undefined then $[z]_f$ is undefined;
2. if z is the set $\{z_1, \ldots, z_k\}$ then $sp(z) = \bigcup_{i \leqslant k} sp(z_i)$ and $[z]_f = \{[z_1]_f, \ldots, [z_k]_f\}$; if, for some z_i, $[z_i]_f$ is undefined then $[z]_f$ is undefined;
3. if $z = \varnothing$ then $sp(z) = \varnothing$ and $[z]_f = \varnothing$.

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Definition 5.3. We say that $z \in \mathrm{HF}(X)$ structurally represents $y \in \mathrm{HF}(A)$ if $[z]_f = y$ for an assignment $f : X \to A$.

In the proposition below we use the language $\sigma_0^* \rightleftharpoons \sigma_0 \cup \{\top, \bot\}$, where $\top$ represents a logical truth which can be defined by the formula $\exists x \exists y (x \neq y)$ and $\bot$ represents a logical false which can be defined by the formula $\exists x (x \neq x)$.

Proposition 5.1. Suppose φ is a Δ_0-formula with s free variables and $y_1, \dots, y_s$ are elements of $\mathrm{HF}(A)$. Let $z_1, \dots, z_s \in \mathrm{HF}(X)$ structurally represent $y_1, \dots, y_s$ with the same assignment $f : X \to A$. Then, we can effectively construct a quantifier-free formula ψ in the language σ_0^* such that $FV(\psi) \subseteq sp(\{z_1, \dots, z_s\})$ and

$$\mathcal{A} \models \psi[f] \leftrightarrow \mathbf{HF}(A) \models \varphi([z_1]_f, \dots, [z_s]_f).$$

The choice of ψ depends on the tuple $\bar{z} = (z_1, \dots, z_s)$ and φ, and does not depend on f.

Proof. In order to simplify the proof, without loss of generality, we assume that every formula has subformulas distinguishing free variables. Using induction on the structure of a Δ_0-formula φ, we show how to obtain a required formula ψ.
Atomic case.
1. If $\varphi(\bar{t}) \rightleftharpoons Q(\bar{t})$ for $Q \in \sigma_P$ and $z_1, \dots, z_n$ represent $y_1, \dots, y_n \in \mathrm{HF}(A)$ then $\psi \rightleftharpoons Q(\bar{z})$. If $\varphi(t_1, t_2) \rightleftharpoons t_1 \neq t_2$, and z_1, z_2 structurally represent $y_1, y_2 \in \mathrm{HF}(A)$ then $\psi \rightleftharpoons z_1 \neq z_2$. The subcase $\varphi(t) \rightleftharpoons U(t)$ and $\varphi(t) \rightleftharpoons \neg U(t)$ can be considered by analogy.
2. Suppose $\varphi(t_1, t_2) \rightleftharpoons t_1 \in t_2$ and z_1, z_2 structurally represent y_1, $y_2 \in \mathrm{HF}(A)$. If $z_1 \in z_2$ then $\psi \rightleftharpoons \top$ else $\psi \rightleftharpoons \bot$. The subcase $\varphi(y_1, y_2) \rightleftharpoons y_1 \notin y_2$ can be considered by analogy.
Disjunction and *Conjunction*. If $\varphi \rightleftharpoons \varphi_1 \tau \varphi_2$, where τ is $\vee$ or $\wedge$, and ψ_1, ψ_2 are already constructed for φ_1, φ_2 then $\psi \rightleftharpoons \psi_1 \tau \psi_2$.
Bounded quantifier cases.
Suppose $\varphi(\bar{t}) \rightleftharpoons (\exists v \in t_j) v(v, \bar{t})$ and z_j structurally represents $y_j \in \mathrm{HF}(A)$. If $z_j \in X$, then the formula φ is false, so $\psi \rightleftharpoons \bot$. Suppose $z_j = \{z_j^1, \dots, z_j^k\}$. By inductive assumption, for every Δ_0-formula $v(z_j^i, \bar{t})$, where $1 \leqslant i \leqslant k$, there exists a required ψ_i. Put $\psi \rightleftharpoons \bigvee_{1 \leqslant i \leqslant k} \psi_i$. For the subcase $\varphi(\bar{t}) \rightleftharpoons (\forall v \in t_j) v(v, \bar{t})$, we put $\psi \rightleftharpoons \bigwedge_{1 \leqslant i \leqslant k} \psi_i$. □

Theorem 5.1. A set $B \subseteq A^n$ is Σ-definable if and only if there exists an effective sequence of existential formulas $\{\varphi_s(\bar{x})\}_{s \in \omega}$ in the language σ_0 such that

$$(x_1, \dots, x_n) \in B \leftrightarrow \mathcal{A} \models \bigvee_{s \in \omega} \varphi_s(x_1, \dots, x_n).$$

Proof. $\to$) Without loss of generality suppose B is Σ-definable by the formula $\exists t \psi(t, \bar{x})$. For every $y \in \mathrm{HF}(A)$ there exist $z \in \mathrm{HF}(X)$ which structurally represents y and we can effectively enumerate $\mathrm{HF}(X)$. Using Proposition 5.1 we effectively construct the set of formulas $\psi_j(\bar{x}_j, \bar{x})$ such that

$$\mathbf{HF}(A) \models \exists t\, \psi(t, \bar{x}) \leftrightarrow \mathcal{A} \models \bigvee_{j \in \omega} \exists \bar{x}_j\, \psi_j(\bar{x}_j, \bar{x}).$$

$\leftarrow$) Let $B \subset A^n$ be definable by $\bigvee_{s \in \omega} \varphi_s(x_1, \dots, x_n)$. By Theorem 4.1, there exists a universal Σ-predicate $Univ_n(m, \bar{x})$ for Σ formulas with variables from $\{x_1, \dots, x_n\}$. Let the computable

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function $f : \omega \to \omega$ enumerate the Gödel numbers of the formulas φ_i, $i \in \omega$. It is easy to see that the following formula is required.

$$\Phi(\bar{x}) \rightleftharpoons \exists i\, Univ_n(f(i), \bar{x}).$$

□

It is worth noting that both of the directions of this characterization are important. The right-to-left direction reveals an algorithmic property of Σ-definability, i.e., gives us an effective procedure which generates existential formulas approximating Σ-relations. The converse direction provides tools for descriptions of the results of effective infinite approximating processes by finite formulas.

6. Positive predicate structures and effectively enumerable topological spaces

In this section, we discuss links between predicate structures and topological spaces. Let us consider a predicate structure $\mathcal{A}$ with the topology $\tau_\Sigma^{\mathcal{A}}$ formed by a base which is the set of subsets definable by existential formulas in the language σ_0. We assume that the numbering of the base is induced by the Gödel numbering of the ∃-formulas in the language σ_0. The following proposition shows that the topology $\tau_\Sigma^{\mathcal{A}}$ is natural with respect to Σ-definability.

Theorem 6.1. Every subset of A is effectively open in the topology $\tau_\Sigma^{\mathcal{A}}$ if and only if it is Σ-definable.

Proof. The claim follows from Theorem 5.1. □

Below, we illustrate how to pick an appropriate finite language in such way that $\tau_\Sigma^{\mathcal{A}}$ coincides with the usual topology. First we consider the structures from Example 2.1.

Proposition 6.1.

1. For the structure $\mathbb{N} = \langle \mathbb{N}, Q_1, Q_2, < \rangle$ the topology $\tau_\Sigma^{\mathbb{N}}$ coincides with the discrete topology.
2. For the structure $\mathbb{R} = \left\langle \mathbb{R}, \mathcal{M}_E^*, \mathcal{M}_H^*, \mathcal{P}_E^+, \mathcal{P}_H^+ < \right\rangle$ the topology $\tau_\Sigma^{\mathbb{R}}$ coincides with the real line topology.
3. For the structure $\mathbb{C} = \langle \mathbb{C}, , P_1, \dots, P_{12} \rangle$ the topology $\tau_\Sigma^{\mathbb{C}}$ coincides with the plane topology.
4. For the structure $C[0,1] = (C[0,1], P_1, \dots, P_{10})$, the topology $\tau_\Sigma^{C[0,1]}$ coincides with the topology $\tau_{\|\ \|}$ induced by the supremum norm.

Proof. The first three claims are straightforward. Let us prove the last statement. ⊆). It is easy to see that $\{\bar{x} | \mathbf{HF}(C[0,1]) \models P_i(\bar{x})\} \in \tau_{\|\ \|}$ for every $1 \leqslant i \leqslant 10$. Since $\left(C[0,1], d_{\|\ \|}\right)^m$ is a metric space, a projection of an open set is again open. So, $\{\bar{x} | \mathbf{HF}(C[0,1]) \models Q(\bar{x}),\ Q \text{ is a } \exists\text{-formula}\} \in \tau_{\|\ \|}$. By induction, $\tau_\Sigma^{C[0,1]} \subseteq \tau_{\|\ \|}$.
⊇). First, recall that a base of the topology $\tau_{\|\ \|}$ is the following:

$$\tau_{\|\ \|}^* = \{\{f \mid \|f - p_i\| < \epsilon\} | p \text{ is a polynomial with rational coefficients}, \epsilon \in \mathbb{Q}\}.$$

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Since the set $\{p | p$ is a polynomial with rational coefficients$\}$ is dense in $C[0,1]$, it is sufficient to show that $f > p$ and $f < p$ are $\exists$-definable. This claim follows from the following equivalences:

$$f > 0 \leftrightarrow f + f > f;$$
$$f > 1 \leftrightarrow \exists g\,(f \cdot g > g \wedge g > 0);$$
$$f < 0 \leftrightarrow f + f < f;$$
$$f < 1 \leftrightarrow \exists g\,(f < 0 \vee g > 0 \wedge f \cdot g < g);$$
$$f > x^2 \leftrightarrow \exists g\,(g > \lambda x.x \wedge f > g \cdot g);$$
$$f < x^2 \leftrightarrow \exists g\,(g < \lambda x.x \wedge f < g \cdot g);$$
$$f > \frac{x}{n} \leftrightarrow \exists g\,(g > \lambda x.x \wedge (f + \cdots + f) > g) \text{ for } n \in \omega;$$
$$f < \frac{x}{n} \leftrightarrow \exists g\,(g < \lambda x.x \wedge (f + \cdots + f) < g) \text{ for } n \in \omega.$$

So, the set $\{f \,|\, ||f - p|| < \epsilon, p$ is a polynomial with rational coefficients, $\epsilon \in \mathbb{Q}\}$ is $\exists$-definable for every considered p and ϵ. Therefore $\tau_\Sigma^{\mathcal{C}} \supseteq \tau_{||\ ||}$. □

Now, we recall the definition of effectively enumerable topological spaces which contain computable metric spaces and ω-continuous domains as proper subclasses (Korovina and Kudinov 2008). Let (X, τ, v) be a topological space, where X is a nonempty set, $\tau^* \subseteq 2^X$ is a base of the topology τ and $v : \omega \to \tau^*$ is a numbering.

Definition 6.1 (Korovina and Kudinov 2008). A topological space (X, τ, v) is *effectively enumerable* if the following conditions hold.

1. There exists a computable function $g : \omega \times \omega \times \omega \to \omega$ such that

$$v(i) \cap v(j) = \bigcup_{n \in \omega} v(g(i, j, n)).$$

2. The set $\{i | v(i) \neq \varnothing\}$ is computably enumerable.

Definition 6.2 (Korovina and Kudinov 2008). An effectively enumerable topological space (X, τ, v) is *strongly effectively enumerable* if there exists a computable function $h : \omega \times \omega \to \omega$ such that

$$X \setminus cl(v(i)) = \bigcup_{j \in \omega} v(h(i, j)).$$

It is worth noting that the computable metric space are strongly effectively enumerable (Korovina and Kudinov 2008).

Now we consider a predicate structure $\mathcal{A}$. Let us denote

$$Th_\exists^{pos}(\mathcal{A}) = \{\varphi | \varphi \text{ is a } \exists\text{-sentence such that } \mathcal{A} \models \varphi\}.$$

In fact, it is existential positive theory of $\mathcal{A}$.

The following proposition shows a connection between predicate structures and effectively enumerable topological spaces.

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Theorem 6.2 (Korovina and Kudinov 2008). For every predicate structure $\mathcal{A}$ the following properties hold.

1. The topological space $\left(A, \tau_{\Sigma}^{\mathcal{A}}\right)$ is effectively enumerable if and only if $Th_{\exists}^{pos}(A)$ is computable enumerable.
2. If $Th_{\exists}^{pos}(A)$ is decidable then $\left(A, \tau_{\Sigma}^{\mathcal{A}}\right)$ is strongly effectively enumerable.

Proof. The claim is straightforward from the definition of effectively enumerable topological space. □

The Theorem 6.2 reveals the great importance of the predicate structures with computably enumerable $Th_{\exists}^{pos}(A)$. So, we would like to distinguish these structures as a special class.

Definition 6.3. A predicate structure $\mathcal{A} = \langle A, \sigma_0 \rangle$ is called positive predicate structure if $Th_{\exists}^{pos}(\mathcal{A})$ is computably enumerable.

Remark 6.1. It is worth noting that the structures from Example 2.1 are positive predicate structures.

Corollary 6.1. A predicate structure $\mathcal{A}$ is positive predicate structure if and only if the corresponding topological space $(A, \tau_{\Sigma}^{\mathcal{A}})$ is effectively enumerable.

Proof. The claim follows from Theorem 6.2. □

Definition 6.4. We say that an effectively enumerable space (X, τ, ν) can be structured if there exists a finite predicate language σ such that for the predicate structure $\langle X, \sigma \rangle$ the following properties hold:

1. for every $P \in \sigma$ the interpretation of P on X is effectively open;
2. τ coincides with $\tau_{\Sigma}^{\mathcal{X}}$ and
3. the sets $\nu(n)$ are uniformly Σ-definable.

Definition 6.5. We say that an effectively enumerable space (X, τ, ν) can be semi-structured if there exists a finite predicate language σ such that for the predicate structure $\langle X \sqcup \mathbb{N}, \sigma \cup \{Q_1, Q_2, R, N, P\} \rangle$, the following properties hold:

1. Q_1 and Q_2 have the same interpretation as in Example 2.1, R is interpreted as X and N is interpreted as $\mathbb{N}$, the interpretation of P is an effectively open set;
2. for every $P \in \sigma$ the interpretation of P on X is effectively open;
3. τ coincides with the restriction of $\tau_{\Sigma}^{\mathcal{M}}$ on X and
4. the sets $\nu(n)$ are uniformly Σ-definable.

The spaces $\mathbb{R}$, $C[0, 1]$ and many others (see Example 2.1) can be structured. It is worth noting that any effectively enumerable space (X, τ, ν) can be semi-structured, we just put

$$P(x, n) \leftrightarrow R(x) \wedge N(n) \wedge x \in \nu(n).$$

On the other hand, the following example illustrates that there are non-structured effectively enumerable spaces.

Example 6.1 (Non-structured space). Let us consider $\mathbb{N}$ with the base topology, consisting of cofinite subsets. It is easy to see that this effectively enumerable space is non-structured.

Indeed, any open predicate P of arity k on it should contain the power $(\omega \setminus m)^k$ for some fixed $m \in \omega$. By monotonicity and induction, any $\exists$-definable relation should contain the corresponding power of $\omega \setminus m$, so, for a finite language there are only finite number of $\exists$-definable subsets of $\mathbb{N}$, contrary to an infinite topology base.

7. Positive predicate structures for computable metric spaces

In this section, we show that computable metric spaces can be structured. For the definition of computable metric space we refer to Moschovakis (1976) and Weihrauch (2000). Let $\mathcal{M} = (M, b, \mathbf{B}, d)$ be a computable metric space, where $\mathbf{B} = \{b_i | i \in \omega\} \subseteq M$ is countable and dense in M, $b : \omega \to \mathbf{B}$ is a numbering, and $d : M \times M \to \mathbb{R}$ is a distance function computable on $(\mathbf{B}, \nu)$.

We define the corresponding predicate structure

$$\mathcal{M} = \langle M, \sigma_P \rangle = \langle M, R_0, S, D_1, D_2, D_3 \rangle,$$

where the predicates have the following meanings:

$$D_1(x, y, u, v) \rightleftharpoons d(x, y) < d(u, v),$$
$$D_2(y, z, v) \rightleftharpoons d(y, z) - d(y, v) < 1,$$
$$D_3(y, z, t, w, s) \rightleftharpoons 2\,(d(y, z) - d(y, v)) < d(t, w) - d(t, s),$$
$$R_0(x, y, z, v) \rightleftharpoons 2d(x, b_0) < d(y, z) - d(y, z),$$
$$S(x, y, z, v, a, b, c, d) \rightleftharpoons \bigvee_{n \in \omega} (R_n(x, y, z, v) \wedge R_{n+1}(a, b, c, d)),$$

where,

$$R_n(x, y, z, v) \rightleftharpoons 2d(x, b_n) < d(y, z) - d(y, z) \wedge (\forall i < n)\, 2d(x, b_i) > d(y, z) - d(y, v).$$

Theorem 7.1. Let $\mathbb{M}^n$ be a computable metric space and $\mathcal{M}$ be the corresponding predicate structure defined above. Then, the topology $\tau_\Sigma^{\mathcal{M}}$ coincides with the topology τ_d induced by the metric in an effective way that means that the lists of effectively open sets coincide and one can compute corresponding indices from each other.

Proof. $\subseteq$). By definition, the predicates S, D_1, D_2, D_3, and R_0 define sets which are open in the product topology. Since $\mathbb{M}$ is a metric space, a projection of an open set is again open. So, by induction, every $\exists$-definable subset of M belongs to τ_d.
$\supseteq$). It is sufficient to show that the balls $B(b_r, a)$, where $b_r \in \mathbf{B}$ and $a \in \mathbb{Q}^+$, are uniformly Σ-definable.

First we show by induction on n, that R_n are uniformly Σ-definable.
$n = 0$. By the definition $R_0 \in \sigma_P$.
Inductive step: $n \to n + 1$. Since by definition $R_n \cap R_m = \varnothing$ for $n \neq m$, we can Σ-define R_{n+1} as follows:

$$R_{n+1}(a, b, c, d) \rightleftharpoons \exists x \exists y \exists z \exists v\, (R_n(x, y, z, v) \wedge S(x, y, z, v, a, b, c, d)).$$

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The next step is to define predicates A_s^m with the following properties: for every $m \in \omega$ and $s \in \omega$, the set $\{x|\mathbf{HF}(M) \models A_s^m(x)\}$ is a subset of the ball $B(b_s, \frac{1}{2^m})$, and for all $x \in M$ and $m \in \omega$, there exists $s \in \omega$ such that $\mathbf{HF}(M) \models A_s^m(x)$.

Put

$$\begin{aligned} A_s^1(x) &\rightleftharpoons \exists y \exists z \exists v (R_s(x,y,z,v) \wedge d(y,z) - d(y,v) < 1); \\ A_s^{m+1}(x) &\rightleftharpoons \exists y \exists z \exists v \exists s_1 \ldots \exists s_m \exists w_1 \ldots \exists w_m \exists t_1 \ldots \exists t_m (R_s(x,y,z,v)) \wedge \\ &\qquad 2(d(y,z) - d(y,v)) < d(t_1, w_1) - d(t_1, s_1) \wedge \\ &\qquad d(t_m, w_m) - d(t_m, s_m) < 1 \wedge \\ &\qquad \bigwedge_{1 \leqslant i < m-1} 2(d(t_i, w_i) - d(t_i, s_i)) < d(t_{i+1}, w_{i+1}) - d(t_{i+1}, s_{i+1}). \end{aligned}$$

It is easy to see that A_i^j is equivalent to a Σ-formula in the language σ_P. By definition, the first property holds. We prove the second one. Let $x \in M$ and $m \in \omega$. We find the first $s \in \omega$ such that $2d(x, b_s) < d(y_m, z_m) < \frac{1}{2^m}$, where y_m, z_m, v_m are inductively constructed as follows:

$$\begin{aligned} &0 < d(y_0, z_0) - d(y_0, v_0) < 1; \\ &0 < d(y_1, z_1) - d(y_1, v_1) < \frac{d(y_0, z_0) - d(y_0, v_0)}{2}; \\ &0 < d(y_{i+1}, z_{i+1}) - d(y_{i+1}, v_{i+1}) < \frac{d(y_i, z_i) - d(y_i, v_i)}{2}. \end{aligned}$$

In order to avoid the case $d(x, b_i) = d(y_m, z_m)$ for $i < s$, we choose $v \in \mathbf{B}$ such that for all $i < s$ we have $2d(x, b_i) > d(y_m, z_m) - d(y_m, v)$ and $2d(x, b_s) < d(y_m, z_m) - d(y_m, v)$. Then $\mathbf{HF}(M) \models R_s(x, y_m, z_m, v)$ and $d(y_m, z_m) < \frac{1}{2^m}$. So $x \in A_s^m$. Now, we are ready to prove that the balls $B(b_r, a)$ are Σ-definable. For this we show the following equivalence:

$$d(x, b_r) < a \leftrightarrow \mathbf{HF}(M) \models \exists s \exists m \left(d(b_r, b_s) < a - \frac{1}{2^m} \wedge x \in A_s^m \right).$$

$\leftarrow$). If $x \in A_s^m$ then as we have shown above $d(x, b_s) < \frac{1}{m}$. So $d(b_r, x) < d(b_r, b_s) + d(x, b_s) < a$.

$\rightarrow$). Since $a - d(x, b_r) > 0$, we can find $N \in \omega$ such that $a - d(x, b_r) > \frac{1}{2^N}$. We already proved that for x and N there exists s such that $x \in A_s^{N+1}$. So $d(x, b_s) < \frac{1}{2^{N+1}}$. Finally, $d(b_r, b_s) < d(b_r, x) + d(x, b_s) < a - \frac{1}{2^{N+1}}$.

Therefore, the equivalence has been shown which completes the proof of the theorem. □

Corollary 7.1. Every computable metric space can be structured.

Proof. The claim follows from Theorem 7.1 and Korovina and Kudinov (2008).

Corollary 7.2. If $\mathbb{M}$ is a computable metric space then $Th_{\exists}^{pos}(M)$ is computably enumerable. Therefore, if $\mathbb{M}$ is a computable metric space then the corresponding $\mathcal{M}$ is a positive predicate structure.

Proof. The claim follows from Theorem 7.1 and Korovina and Kudinov (2008). □

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Corollary 7.3. Every subset of a computable metric space is effectively open if and only if it is Σ-definable.

Proof. The claim follows from Theorems 5.1 and 7.1. □

Corollary 7.4. Every function over a computable metric space is computable if and only if it is effectively continuous in τ_Σ^M topology.

Proof. The claim follows from Theorem 7.1 and Moschovakis (1976) and Weihrauch (1993). □

Corollary 7.5. A total function $F : M \to \mathbb{R}$ is computable if and only if the epigraph and the hypograph are Σ-definable.

Proof. The claim follows from Theorem 7.1 and Korovina and Kudinov (2008). □

8. Conclusion

A finite language is preferable in many applications where effective representations of continuous data are required. The main challenge in this work is to keep the language finite yet powerful to express computability. The obtained results show that computable metric spaces admit structurizations.

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