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ON THE SIZE, SPECTRAL RADIUS, DISTANCE SPECTRAL RADIUS AND FRACTIONAL MATCHINGS IN GRAPHS

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Abstract

We first establish a lower bound on the size and spectral radius of a graph G to guarantee that G contains a fractional perfect matching. Then, we determine an upper bound on the distance spectral radius of a graph G to ensure that G has a fractional perfect matching. Furthermore, we construct some extremal graphs to show all the bounds are best possible.

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1. Introduction

We deal only with finite and undirected graphs without loops or multiple edges. For graph theoretic notation and terminology not defined here, we refer to [4, 13].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G is the number $n = |V(G)|$ of its vertices and its *size* is the number $m = |E(G)|$ of its edges. A graph G is called *trivial* if $|V(G)| = 1$. Let $V_1 \subseteq V(G)$ and $E_1 \subseteq E(G)$. Then, $G - V_1, G - E_1$ are the graphs formed from G by deleting the vertices in V_1 and their incident edges, or the edges in E_1, respectively. For convenience, denote $G - \{v\}$ and $G - \{uv\}$ by $G - v$ and $G - uv$, respectively. For a given subset $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. As usual, P_n and K_n denote the path and the complete graph on n vertices.

For a vertex $v \in V(G)$, let $N_G(v)$ be the set of all neighbours of v in G. Then, $d_G(v) = |N_G(v)|$ is the *degree* of v in G. A vertex v of G is called a *pendant vertex* if $d_G(v) = 1$. A *quasi-pendant vertex* is a vertex being adjacent to some pendant vertex. A graph is *r-regular* if each vertex has the same degree r. The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G, in which any two distinct vertices are adjacent if and only if they are nonadjacent in G. For two graphs G_1 and G_2, we define $G_1 \cup G_2$

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to be their *disjoint union*. The *join* $G_1 \vee G_2$ is obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with every vertex of G_2 by an edge.

Let $V(G) = \{v_1, \ldots, v_n\}$. The *adjacency matrix* $A(G) = (a_{ij})$ of G is an $n \times n$ matrix in which the entry a_{ij} is 1 or 0 according to whether v_i and v_j are adjacent or not. The eigenvalues of the adjacency matrix $A(G)$ are also called *eigenvalues* of G. Note that $A(G)$ is a real symmetric nonnegative matrix. Hence, its eigenvalues are real and can be arranged in nonincreasing order as $\lambda_1(G) \geqslant \cdots \geqslant \lambda_n(G)$. The spectral radius of G is equal to $\lambda_1(G)$, written as $\rho(G)$.

Let G be a connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The *distance* between v_i and v_j, denoted by d_{ij}, is the length of a shortest path from v_i to v_j. The *Wiener index* of G is defined as $W(G) = \sum_{i<j} d_{ij}$. The *distance matrix* of G, denoted by $D(G)$, is the $n \times n$ real symmetric matrix whose (i, j)-entry is d_{ij}. We can order the eigenvalues of $D(G)$ as $\lambda_1(D(G)) \geqslant \lambda_2(D(G)) \geqslant \cdots \geqslant \lambda_n(D(G))$. By the Perron–Frobenius theorem, $\lambda_1(D(G))$ is always positive (unless G is trivial) and $\lambda_1(D(G)) \geqslant |\lambda_i(D(G))|$ for $i = 2, 3, \ldots, n$. We call $\lambda_1(D(G))$ the *distance spectral radius* of G, written as $\mu(G)$.

A subset S of $V(G)$ (respectively $E(G)$) is called an *independent set* (respectively a *matching*) if any two members of S are not adjacent in G. A matching with the maximum size in G is called a *maximum matching*. The *matching number* $\alpha'(G)$ is the size of a maximum matching in G. We call an edge subset S a *perfect matching* if each vertex of G is incident with an edge in S.

Brouwer and Haemers [2] proved that if G is an r-regular graph without perfect matchings, then G has at least three proper induced subgraphs H_1, H_2 and H_3, which are contained in the family

$$\mathscr{H} = \{H : |V(H)| \text{ is odd}, r|V(H)| - r + 2 \leqslant 2|E(H)| \leqslant r|V(H)| - 1\},$$

and that $\lambda_3(G) \geqslant \min\{\rho(H_i) : i = 1, 2, 3\} > \min\{2|E(H)|/|V(H)| : H \in \mathscr{H}\}$. Quite recently, O [11] showed that there is a close relationship between the spectral radius and prefect matching not only for regular graphs but also for general graphs. He established sharp upper bounds on the number of edges and the spectral radius of a graph without a perfect matching.

There are several interesting results on the distance spectral radius of G and its matching number. Ilić [6] characterised n-vertex trees with a given matching number which minimise the distance spectral radius. Liu [7] characterised graphs with minimum distance spectral radius in connected graphs on n vertices with fixed matching number. Zhang [15] and Lu and Luo [8] characterised unicyclic graphs with a perfect matching and a given matching number which minimise the distance spectral radius.

A fractional matching of a graph G is a function f giving each edge a number in $[0, 1]$ so that $\sum_{e\in\Upsilon(v)} f(e) \leqslant 1$ for all $v \in V(G)$, where $\Upsilon(v)$ is the set of edges incident to v. The *fractional matching number* of G, written as $\alpha'_f(G)$, is the maximum of $\sum_{e\in E(G)} f(e)$ over all fractional matchings f. If $f(e) \in \{0, 1\}$ for every edge e, then f is just a matching, or more precisely, the indicator function of a matching. A *fractional*

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perfect matching is a fractional matching f with $\sum_{e\in E(G)} f(e) = n/2$. Scheinermann and Ullman [12] showed that a graph G has a fractional perfect matching f if and only if $\sum_{e\in\Upsilon(v)} f(e) = 1$ for every $v \in V(G)$.

Summing the inequality constraints for all vertices yields $\sum_{v\in V(G)}\sum_{e\in\Upsilon(v)} f(e) \leqslant n$, so we always have $\alpha'_f(G) \leqslant n/2$. Since every matching can be viewed as a fractional matching, $\alpha'_f(G) \geqslant \alpha'(G)$ for all graphs G, but equality need not hold. For example, $\alpha'_f(G) = n/2$ for an r-regular graph G by setting each edge weight to $1/r$, but not every r-regular graph has a perfect matching. In 2016, O [10] determined the connection between the spectral radius and fractional matching number among connected graphs with given minimum degree.

Motivated by [10–12], it is natural and interesting to give some sufficient conditions to ensure that a graph contains a fractional perfect matching. Here, we focus on sufficient conditions including a structure graph condition, adjacency spectral graph condition and distance spectral graph condition.

Our first main result gives a sufficient condition to ensure a graph G contains a fractional perfect matching according to the size of G.

THEOREM 1.1. *Let G be an n-vertex connected graph. Then, G contains a fractional perfect matching if*

$$|E(G)| > \begin{cases} \frac{1}{8}(n-1)(3n-1) & \text{if } n = 3,5,7,9,11; \\ \frac{3}{8}n(n-2) & \text{if } n = 4,6,8,10; \\ 2 + \binom{n-2}{2} & \text{if } n \geqslant 12. \end{cases}$$

Our second main result gives a sufficient condition to ensure a graph G contains a fractional perfect matching according to the adjacency spectral radius of G.

THEOREM 1.2. *Let G be an n-vertex connected graph. Assume the largest roots of $x^2 - \frac{1}{2}(n-3)x - \frac{1}{4}(n^2-1) = 0$ and $x^3 - (n-4)x^2 - (n-1)x + 2n - 8 = 0$ are $\xi_1(n)$ and $\xi_2(n)$, respectively. Then, G has a fractional perfect matching if*

$$\rho(G) > \begin{cases} \sqrt{3} & \text{if } n = 4; \\ \frac{1}{2}(1+\sqrt{33}) & \text{if } n = 6; \\ \xi_1(n) & \text{if } n = 3,5,7,9; \\ \xi_2(n) & \text{if } n = 8 \text{ or } n \geqslant 10. \end{cases}$$

Our last main result gives a sufficient condition to ensure a graph G contains a fractional perfect matching with respect to the distance spectral radius of G.

THEOREM 1.3. *Let G be an n-vertex connected graph. Assume the largest roots of $x^2 - \frac{1}{2}(3n-5)x + \frac{1}{4}(n^2-8n+7) = 0$, $x^2 - \frac{1}{2}(3n-4)x + \frac{1}{4}(n^2-8n+4) = 0$ and $x^3 - (n-2)x^2 - (7n-17)x - 4n + 10 = 0$ are $\zeta_1(n)$, $\zeta_2(n)$ and $\zeta_3(n)$, respectively. Then, G*

contains a fractional perfect matching if

$$\mu(G) < \begin{cases} \zeta_1(n) & \text{if } n = 3, 5, 7, 9, 11; \\ \zeta_2(n) & \text{if } n = 4, 6, 8; \\ \zeta_3(n) & \text{if } n = 10 \text{ or } n \geqslant 12. \end{cases}$$

The proof techniques in the paper for our main results follow the idea of O [11]. Our paper is organised as follows. In Section 2, we give some preliminary results. In Section 3, we give the proofs of Theorems 1.1, 1.2 and 1.3. In the last section, we give several extremal graphs to show that all the bounds are best possible.

2. Some preliminaries

In this section, we present some necessary preliminary results, which will be used to prove our main results. The first one follows directly from [1, Theorem 6.8].

LEMMA 2.1 [1]. *Let G be a connected graph and let H be a proper subgraph of G. Then, $\rho(G) > \rho(H)$.*

Let M be a real symmetric matrix whose rows and columns are indexed by $V = \{1, \ldots, n\}$. Assume that M can be written as

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1s} \\ \vdots & \ddots & \vdots \\ M_{s1} & \cdots & M_{ss} \end{pmatrix}$$

according to the partition $\pi : V = V_1 \cup \cdots \cup V_s$, where M_{ij} denotes the submatrix (block) of M formed by the rows in V_i and the columns in V_j. Let q_{ij} denote the average row sum of M_{ij}. The matrix $M_\pi = (q_{ij})$ is called the *quotient matrix* of M. If the row sum of each block M_{ij} is a constant, then the partition is *equitable*.

LEMMA 2.2 [3, 14]. *Let M be a real matrix with an equitable partition π and let M_π be the corresponding quotient matrix. Then, every eigenvalue of M_π is an eigenvalue of M. Furthermore, the largest eigenvalues of M and M_π are equal.*

LEMMA 2.3 [9]. *Let G be a connected graph with two nonadjacent vertices $u, v \in V(G)$. Then, $\mu(G + uv) < \mu(G)$.*

The next lemma can be easily derived from the Rayleigh quotient [5].

LEMMA 2.4. *Let G be a connected graph with order n. Then,*

$$\mu(G) = \max_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T D(G) \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geqslant \frac{\mathbf{1}^T D(G) \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = \frac{2W(G)}{n},$$

where $\mathbf{1} = (1, 1, \ldots, 1)^T$.

Let $I(G)$ be the set of isolated vertices of G and let $i(G) = |I(G)|$. The next lemma gives a necessary and sufficient condition for a graph to contain a fractional perfect matching.

LEMMA 2.5 [12]. *A graph G contains a fractional perfect matching if and only if $i(G - S) \leqslant |S|$ for every set $S \subseteq V(G)$.*

In 2021, O [11] gave two sufficient conditions to ensure that a graph G contains a perfect matching according to the size and adjacent spectral radius of G. These results simplify our proof.

LEMMA 2.6 [11]. *Let G be an n-vertex connected graph. Then, G contains a perfect matching if*

$$|E(G)| > \begin{cases} 9 & \text{if } n = 6; \\ 18 & \text{if } n = 8; \\ 2 + \binom{n-2}{2} & \text{if } n = 4 \text{ or } n \geqslant 10 \text{ is even.} \end{cases}$$

LEMMA 2.7 [11]. *Let G be an n-vertex connected graph. Assume the largest root of $x^3 - (n-4)x^2 - (n-1)x + 2n - 8 = 0$ is $\xi_2(n)$. Then, G has a perfect matching if*

$$\rho(G) > \begin{cases} \frac{1}{2}(1 + \sqrt{33}) & \text{if } n = 6; \\ \xi_2(n) & \text{if } n = 4 \text{ or } n \geqslant 8 \text{ is even.} \end{cases}$$

3. Proofs of our main results

In this section, we give the proofs of Theorems 1.1, 1.2 and 1.3.

PROOF OF THEOREM 1.1. A perfect matching is obviously a fractional perfect matching of a graph. By Lemma 2.6, it is easy to see that our result is true when n is even. Hence, in what follows, we consider the remaining case when n is odd.

Suppose to the contrary that G has no fractional perfect matching. Choose a connected graph G whose size is as large as possible. By Lemma 2.5, there exists a set $S \subseteq V(G)$ such that $i(G - S) \geqslant |S| + 1$. According to the choice of G, both the induced graph $G[S]$ and each connected component of $G - S$ are complete graphs. Furthermore, G is just the graph $G[S] \vee (G - S)$.

Note that there is at most one nontrivial connected component in $G - S$. Otherwise, we can add edges among all nontrivial connected components to get a larger nontrivial connected component, which is a contradiction to the choice of G. For convenience, let $i(G - S) = i$ and $|S| = s \geqslant 1$. We proceed by considering two possible cases.

Case 1. $G - S$ has only one nontrivial connected component, say G_1.

Let $|V(G_1)| = n_1 \geqslant 2$. If $i \geqslant s + 2$, then we construct a new graph H_1 obtained from G by joining each vertex of G_1 with one vertex in $I(G - S)$ by an edge. Clearly, $i(H_1 - S) = i - 1 \geqslant s + 1$. By Lemma 2.5, H_1 has no fractional perfect matching. Note that $|E(H_1)| = |E(G)| + n_1 > |E(G)|$, giving a contradiction with the choice of G.

Since $i \geqslant s + 1$, we must have $i = s + 1$. Note that $n = n_1 + 2s + 1 \geqslant 2s + 3 \geqslant 5$ and $|E(G)| = s(s + 1) + \binom{n-s-1}{2}$. Then, by a direct calculation,

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$$\binom{n-2}{2} + 2 - |E(G)| = \frac{(s-1)(2n-3s-8)}{2}$$
$$\geqslant \frac{(s-1)(4s+6-3s-8)}{2} = \frac{(s-1)(s-2)}{2} \geqslant 0.$$

Thus, $|E(G)| \leqslant \binom{n-2}{2} + 2$ for $n \geqslant 5$. This is a contradiction for $n \geqslant 13$.

For $n = 5, 7, 9, 11$, by a direct calculation,

$$\frac{1}{8}(n-1)(3n-1) - \binom{n-2}{2} - 2 = -\frac{1}{8}n^2 + 2n - \frac{39}{8} = -\frac{(n-3)(n-13)}{8} > 0.$$

Thus, $|E(G)| \leqslant \binom{n-2}{2} + 2 < \frac{1}{8}(n-1)(3n-1)$, which is a contradiction.

Case 2. $G - S$ has no nontrivial connected component.

If $i \geqslant s+3$, we can create a new graph H_2 by adding an edge in $I(G-S)$. Then, $i(H_2 - S) \geqslant s+1$ and $H_2 - S$ has exactly one nontrivial connected component. Together with $|E(G)| < |E(H_2)|$ and Case 1, we obtain a contradiction. Thus, it suffices to consider $i = s+1$ in this case (since $n = 2s+2$ is even if $i = s+2$).

Assume $i = s+1$. Then, $n = 2s+1 \geqslant 3$ and $|E(G)| = \binom{s}{2} + s(s+1) = \frac{1}{2}(3s^2 + s)$. Obviously, $\frac{1}{8}(n-1)(3n-1) = \frac{1}{2}(3s^2 + s)$. We get a contradiction when $n = 3, 5, 7, 9, 11$. Comparing $|E(G)|$ with $\binom{n-2}{2} + 2$ gives

$$\binom{2s-1}{2} + 2 - |E(G)| = \frac{(s-1)(s-6)}{2}.$$

Thus, $|E(G)| \leqslant \binom{n-2}{2} + 2$ for $s \geqslant 6$, which is a contradiction for $n \geqslant 13$.

This completes the proof. □

Next, based on the idea in the proof of Theorem 1.1, we prove Theorem 1.2 by comparing the spectral radius rather than the number of edges.

Proof of Theorem 1.2. By a similar discussion as the proof of Theorem 1.1, we only consider n odd (based on Lemma 2.7). Suppose to the contrary that G has no fractional perfect matching. Choose a connected graph G of order n such that its adjacency spectral radius is as large as possible.

Note that there does not exist a fractional perfect matching in G. Hence, by Lemma 2.5, there exists a set $S \subseteq V(G)$ satisfying $i(G-S) \geqslant |S| + 1$. Together with Lemma 2.1 and the choice of G, the induced graph $G[S]$ as well as each connected component of $G - S$ is a complete graph and G is the join of $G[S]$ and $G - S$, that is, $G = G[S] \vee (G - S)$.

For convenience, let $i(G-S) = i$ and $|S| = s$. One may see that there exists at most one nontrivial connected component in $G - S$. Otherwise, we can add edges among all nontrivial connected components to get a nontrivial connected component

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of larger size, which gives a contradiction (based on Lemma 2.1). Hence, we proceed by considering the following two possible cases.

Case 1. $G - S$ has just one nontrivial connected component, say G_1.

In this case, one has $|V(G_1)| = n_1 \geqslant 2$. If $i \geqslant s + 2$, we construct a new graph H_1 obtained from G by joining each vertex of G_1 with one vertex in $I(G - S)$ by an edge. Then, G is a proper subgraph of H_1. By Lemma 2.1, $\rho(G) < \rho(H_1)$, which is a contradiction.

Now, we assume $i = s + 1$. Then, $n = n_1 + 2s + 1 \geqslant 2s + 3 \geqslant 5$. According to the partition $V(G) = S \cup I(G - S) \cup V(G_1)$, the quotient matrix of $A(G)$ equals

$$B_1 = \begin{pmatrix} s-1 & s+1 & n-2s-1 \\ s & 0 & 0 \\ s & 0 & n-2s-2 \end{pmatrix}.$$

Then, the characteristic polynomial of B_1 is

$$\Phi_{B_1}(x) = x^3 - (n-s-3)x^2 - (s^2+n-2)x + s(s+1)(n-2s-2).$$

Since the partition $V(G) = S \cup I(G-S) \cup V(G_1)$ is equitable, by Lemma 2.2, the largest root, say θ_1, of $\Phi_{B_1}(x) = 0$ equals the spectral radius of G.

Let $f(x) = x^3 - (n-4)x^2 - (n-1)x + 2n - 8$ and let $\xi_2(n)$ be the largest root of $f(x) = 0$. Next, we aim to show $f(\theta_1) < 0$. Substituting θ_1 for x in $f(x) - \Phi_{B_1}(x)$,

$$\begin{aligned} f(\theta_1) - \Phi_{B_1}(\theta_1) &= (1-s)\theta_1^2 + (s^2-1)\theta_1 - ns^2 + 2s^3 - sn + 4s^2 + 2n + 2s - 8 \\ &= (1-s)\theta_1^2 + (s-1)(s+1)\theta_1 - (s-1)(sn - 2s^2 + 2n - 6s - 8) \\ &= (s-1)[-\theta_1(\theta_1 - s - 1) - (s+2)n + 2s^2 + 6s + 8] \\ &\leqslant (s-1)[-\theta_1(\theta_1 - s - 1) - (s+2)(2s+3) + 2s^2 + 6s + 8]. \end{aligned}$$

Note that K_{s+2} is a proper subgraph of G. Hence, by Lemma 2.1, $\theta_1 > s + 1$. Then,

$$f(\theta_1) - \Phi_{B_1}(\theta_1) < (s-1)[-(s+2)(2s+3) + 2s^2 + 6s + 8] = (s-1)(2-s) \leqslant 0.$$

Bear in mind $\Phi_{B_1}(\theta_1) = 0$. So we obtain $f(\theta_1) = f(\theta_1) - \Phi_{B_1}(\theta_1) < 0$, which gives $\rho(G) = \theta_1 < \xi_2(n)$ when $n \geqslant 5$. This is a contradiction for $n \geqslant 11$.

Now consider $n = 5, 7, 9$. Let $\bar{f}(x) = x^2 - \frac{1}{2}(n-3)x - \frac{1}{4}(n^2-1)$ and let $\xi_1(n)$ be the largest root of $\bar{f}(x) = 0$. Since $\rho(G) = \theta_1 < \xi_2(n)$ when $n \geqslant 5$, we need to compare $\xi_2(n)$ with $\xi_1(n)$. By a direct calculation, $\xi_2(5) \approx 2.3429 < 3 = \xi_1(5)$, $\xi_2(7) \approx 4.1055 < 4.6056 \approx \xi_1(7)$, $\xi_2(9) \approx 6.0492 < 6.2170 \approx \xi_1(9)$, which is a contradiction.

Case 2. $G - S$ has no nontrivial connected component.

If $i \geqslant s + 3$, we can construct a new graph H_2 by adding an edge in $I(G - S)$. Then, $i(H_2 - S) \geqslant s + 1$ and $H_2 - S$ has one nontrivial connected component. Together with Lemma 2.1 and Case 1, we have $\rho(G) < \rho(H_2) \leqslant \xi_2(n)$, which is a contradiction. Thus, it suffices to consider $i = s + 1$ in this case.

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Assume $i = s + 1$. Consider the partition $V(G) = S \cup I(G - S)$. The quotient matrix of $A(G)$ with respect to the partition $S \cup I(G - S)$ is equal to

$$B_2 = \begin{pmatrix} s-1 & s+1 \\ s & 0 \end{pmatrix}$$

and the characteristic polynomial of B_2 equals $\Phi_{B_2}(x) = x^2 - (s-1)x - s(s+1)$. It is easy to see that the partition $V(G) = S \cup I(G - S)$ is equitable. By Lemma 2.2, the largest root, say θ_2, of $\Phi_{B_2}(x) = 0$ equals the spectral radius of G.

Note that $n = 2s + 1$. Then, $\Phi_{B_2}(x) = \bar{f}(x)$ and $\theta_2 = \xi_1(n)$. This is a contradiction for $n = 3, 5, 7, 9$. Next, we consider $n \geqslant 11$ so that $s \geqslant 5$. Substituting θ_2 for x in $f(x) - x\Phi_{B_2}(x)$ gives

$$\begin{aligned} f(\theta_2) - \theta_2\Phi_{B_2}(\theta_2) &= (-n + s + 3)\theta_2^2 + (s^2 + s - n + 1)\theta_2 + 2n - 8 \\ &= (2 - s)\theta_2^2 + (s^2 - s)\theta_2 + 4s - 6 \\ &= \theta_2[(2 - s)\theta_2 + s^2 - s] + 4s - 6. \end{aligned}$$

Since $\theta_2 = \frac{1}{2}(s - 1 + \sqrt{5s^2 + 2s + 1}) > \frac{1}{2}(s - 1 + \sqrt{4s^2 + 4s + 1}) = \frac{3}{2}s$, we have

$$\begin{aligned} f(\theta_2) - \theta_2\Phi_{B_2}(\theta_2) &< \theta_2\Big[(2 - s)\frac{3s}{2} + s^2 - s\Big] + 4s - 6 \\ &= \theta_2\Big(-\frac{1}{2}s^2 + 2s\Big) + 4s - 6 \\ &< -\frac{3}{4}s^3 + 3s^2 + 4s - 6. \end{aligned}$$

Let $p_1(x) = -\frac{3}{4}x^3 + 3x^2 + 4x - 6$. Then, we have $p_1'(x) = -\frac{9}{4}x^2 + 6x + 4$ and $p_1'(\frac{1}{3}(4 \pm 4\sqrt{2})) = 0$. Therefore, $p_1(x)$ is monotonically decreasing for $x \geqslant 4$ and $p_1(s) \leqslant p_1(5) = -4.75 < 0$ when $s \geqslant 5$. Thus, $f(\theta_2) = f(\theta_2) - \theta_2\Phi_{B_2}(\theta_2) < p_1(s) < 0$ for $s \geqslant 5$ and $\rho(G) = \theta_2 < \xi_2(n)$, which is a contradiction.

Together, Cases 1 and 2 complete the proof. □

Finally, we give the proof of Theorem 1.3, again using the idea in the proof of Theorem 1.1.

PROOF OF THEOREM 1.3. Suppose to the contrary that G has no fractional perfect matching. Choose a connected graph G of order n such that its distance spectral radius is as small as possible.

Let

$$\begin{aligned} h(x) &= x^2 - \tfrac{1}{2}(3n - 5)x + \tfrac{1}{4}(n^2 - 8n + 7), \\ \bar{h}(x) &= x^2 - \tfrac{1}{2}(3n - 4)x + \tfrac{1}{4}(n^2 - 8n + 4), \\ \tilde{h}(x) &= x^3 - (n - 2)x^2 - (7n - 17)x - 4n + 10. \end{aligned}$$

Assume that the largest roots of $h(x) = 0$, $\bar{h}(x) = 0$ and $\tilde{h}(x) = 0$ are $\zeta_1(n)$, $\zeta_2(n)$ and $\zeta_3(n)$ (simply ζ_1, ζ_2 and ζ_3), respectively.

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By Lemma 2.5, there exists a set $S \subseteq V(G)$ satisfying $i(G - S) \geqslant |S| + 1$. Together with Lemma 2.3 and the choice of G, the induced graph $G[S]$ as well as each connected component of $G - S$ is a complete graph and G is the join of $G[S]$ and $G - S$, that is, $G = G[S] \vee (G - S)$.

For convenience, let $i(G - S) = i$ and $|S| = s$. There exists at most one nontrivial connected component in $G - S$. Otherwise, we can obtain a new graph G' by adding edges among all nontrivial connected components to get a larger nontrivial connected component. By Lemma 2.3, we have $\mu(G) > \mu(G')$, which gives a contradiction with our choice. So in what follows, we proceed by considering the two possible cases.

Case 1. There is just one nontrivial connected component, say G_1, in $G - S$.

In this case, $|V(G_1)| = n_1 \geqslant 2$. If $i \geqslant s + 2$, we construct a new graph H_1 obtained from G by joining each vertex of G_1 with one vertex in $I(G - S)$ by an edge. Clearly, $i(H_1 - S) = i - 1 \geqslant s + 1$. By Lemma 2.5, H_1 also has no fractional perfect matching. In view of Lemma 2.3, $\mu(G) > \mu(H_1)$, which is a contradiction to our choice.

Now, we assume $i = s + 1$. Then, $n = n_1 + 2s + 1$. We compare the distance spectral radius of G with that of F_n, where $F_n = K_1 \vee (K_{n-3} \cup 2K_1)$ and $\mu(F_n) = \zeta_3(n)$ (see Theorem 4.3 below). According to the partition $V(G) = S \cup V(G_1) \cup I(G - S)$, the quotient matrix of $D(G)$ equals

$$M_1 = \begin{pmatrix} s-1 & n-2s-1 & s+1 \\ s & n-2s-2 & 2s+2 \\ s & 2n-4s-2 & 2s \end{pmatrix}$$

and the characteristic polynomial of M_1 is

$$\Phi_{M_1}(x) = x^3 - (n+s-3)x^2 - (2sn - 5s^2 + 5n - 6s - 6)x + ns^2 - 2s^3 - sn + 2s^2 - 4n + 6s + 4.$$

Since the partition $V(G) = S \cup V(G_1) \cup I(G - S)$ is equitable, by Lemma 2.2, the largest root, say σ_1, of $\Phi_{M_1}(x) = 0$ equals the distance spectral radius of G. Let

$$g_1(x) = \Phi_{M_1}(x) - \tilde{h}(x) = (s-1)[-x^2 + (-2n + 5s + 11)x + ns - 2s^2 + 6].$$

If $s = 1$, then $\Phi_{M_1}(x) = \tilde{h}(x)$ and $\sigma_1 = \zeta_3$. We aim to show $g_1(\zeta_3) < 0$ when $s \geqslant 2$.

For $s \geqslant 2$, we get $n = n_1 + 2s + 1 \geqslant 2s + 3 \geqslant 7$. By Lemma 2.4,

$$\zeta_3 = \mu(F_n) \geqslant \frac{2W(F_n)}{n} = \frac{n^2 + 3n - 10}{n} = n + 3 - \frac{10}{n} > n + \frac{3}{2}.$$

Let $h_1(x) = -x^2 + (-2n + 5s + 11)x + ns - 2s^2 + 6$ be a real function in x, where $x \in [n + \frac{3}{2}, +\infty)$. By a direct calculation, $h_1'(x) = -2x - 2n + 5s + 11$ and $h_1''(x) = -2 < 0$. Hence, $h_1'(x)$ is a decreasing function. Consequently, $h_1'(x) \leqslant h_1'(n + \frac{3}{2}) = -4n + 8 + 5s$. Note that $n \geqslant 2s + 3$. Thus, $h_1'(x) \leqslant -3s - 4 < 0$. That is to say, $h_1(x)$ is a decreasing function on $x \in [n + \frac{3}{2}, +\infty)$, and so

$$h_1(\zeta_3) < h_1(n + \tfrac{3}{2}) = -3n^2 + (6s + 5)n - 2s^2 + \tfrac{15}{2}s + \tfrac{81}{4}.$$

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Let $h_2(x) = -3x^2 + (6s+5)x - 2s^2 + 15/2s + 81/4$ be a real function in x, where $x \in [2s+3, +\infty)$. Then, $h_2(x)$ is monotonically decreasing for $x \geqslant 2s+3$ and

$$h_2(n) \leqslant h_2(2s+3) = -2s^2 - \frac{1}{2}s + \frac{33}{4} = -2\Big(s - \frac{\sqrt{265}-1}{8}\Big)\Big(s + \frac{\sqrt{265}+1}{8}\Big) < 0$$

for $s \geqslant 2$. Therefore, $g_1(\zeta_3) < 0$ for $s \geqslant 2$. Then, $\mu(G) = \sigma_1 \geqslant \zeta_3$ for $n \geqslant 5$, giving a contradiction for $n = 10$ or $n \geqslant 12$.

For $n = 5, 7, 9, 11$, we only need to compare ζ_3 with ζ_1 by the proof above. Let $g_2(x) = \tilde{h}(x) - xh(x)$. Then,

$$g_2(x) = \frac{1}{2}(n-1)x^2 - \Big(\frac{n^2}{4} + 5n - \frac{61}{4}\Big)x - 4n + 10.$$

By a direct calculation, $\zeta_1 = \frac{1}{4}(3n - 5 + \sqrt{(5n-3)(n+1)})$. Thus,

$$\begin{aligned} g_2(\zeta_1) &= \frac{n-3}{8}[n\sqrt{(5n-3)(n+1)} + 2n^2 - 11\sqrt{(5n-3)(n+1)} - 32n + 26] \\ &= \frac{n-3}{8}[(n-11)\sqrt{(5n-3)(n+1)} + (2n^2 - 32n + 26)] \\ &= \frac{n-3}{8}[(n-11)\sqrt{(5n-3)(n+1)} + 2(n - 8 - \sqrt{51})(n - 8 + \sqrt{51})]. \end{aligned}$$

Since $5 \leqslant n \leqslant 11$, we get $g_2(\zeta_1) < 0$. Then, $\zeta_3 > \zeta_1$, and so $\mu(G) > \zeta_1$, which is a contradiction.

Similarly, for $n = 6, 8$, it suffices to compare ζ_3 with ζ_2. By a direct calculation, we have $\zeta_2(6) \approx 7.2749 < 7.5546 \approx \zeta_3(6)$ and $\zeta_2(8) \approx 9.8990 < 10.0839 \approx \zeta_3(8)$. Then, $\mu(G) > \zeta_2$, which is a contradiction.

Case 2. There does not exist any nontrivial connected component in $G - S$.

If $i \geqslant s+3$, we can construct a new graph H_2 by adding an edge in $I(G-S)$. Then, $i(H_2 - S) \geqslant s+1$ and $H_2 - S$ has one nontrivial connected component. Together with Lemma 2.3 and Case 1, we obtain a contradiction. Thus, it suffices to consider $i = s+1$ and $i = s+2$.

For $i = s+1$, the quotient matrix of $D(G)$ for the partition $V(G) = S \cup I(G-S)$ is

$$M_2 = \begin{pmatrix} s-1 & s+1 \\ s & 2s \end{pmatrix}$$

and the characteristic polynomial of M_2 equals $\Phi_{M_2}(x) = x^2 - (3s-1)x + s^2 - 3s$. Since the partition $V(G) = S \cup I(G-S)$ is equitable, by Lemma 2.2, the largest root, say σ_2, of $\Phi_{M_2}(x) = 0$ equals the distance spectral radius of G.

Note that $n = 2s+1$. Then, $\Phi_{M_2}(x) = h(x)$ and $\sigma_2 = \zeta_1$. We can get a contradiction when $n = 3, 5, 7, 9, 11$. For $s = 6$ $(n = 13)$, one has $\mu(G) = \sigma_2 \approx 15.8655 > 15.8393 \approx \zeta_3$. Next, we consider $s \geqslant 7$. Note that $n = 2s+1 \geqslant 15$. By Lemma 2.4, it follows that $\zeta_3 \geqslant n + 3 - 10/n > n + 2 = 2s + 3$. Let $\bar{g}(x) = x\Phi_{M_2}(x) - \tilde{h}(x)$. Then,

$$\bar{g}(x) = -sx^2 + (s^2 + 11s - 10)x + 8s - 6.$$

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It is easy to check that $\bar{g}(x)$ is a monotonically decreasing function for $x \geqslant 2s+3$, and so $\bar{g}(\zeta_3) \leqslant \bar{g}(2s+3) = -2s^3 + 13s^2 + 12s - 36$.

Let $h_3(x) = -2x^3 + 13x^2 + 12x - 36$, where $x \in [7, +\infty)$. Then, $h_3'(x) = -6x^2 + 26x + 12 = -6(x - \frac{1}{6}(13 + \sqrt{241}))(x - \frac{1}{6}(13 - \sqrt{241})) < 0$ and $\bar{g}(\zeta_3) \leqslant h_3(s) \leqslant h_3(7) = -1 < 0$ when $s \geqslant 7$. Thus, $\mu(G) > \zeta_3(n)$ for $n \geqslant 13$, which is a contradiction.

For $i = s+2$, the quotient matrix of $D(G)$ for the partition $V(G) = S \cup I(G-S)$ is

$$M_3 = \begin{pmatrix} s-1 & s+2 \\ s & 2s+2 \end{pmatrix}$$

and the characteristic polynomial of the matrix M_3 is $\Phi_{M_3}(x) = x^2 - (3s+1)x + s^2 - 2s - 2$. Since the partition $V(G) = S \cup I(G-S)$ is equitable, by Lemma 2.2, the largest root, say σ_3, of $\Phi_{M_3}(x)$ equals the distance spectral radius of G.

Recall that $n = 2s+2$. It is easy to see that $\Phi_{M_3}(x) = \bar{h}(x)$ and $\sigma_3 = \zeta_2$. Thus, we get a contradiction when $n = 4, 6, 8$. If $s = 4$, then $n = 10$ and $\mu(G) = \sigma_3 \approx 12.5208 > 12.4504 \approx \zeta_3(10)$. Next, we consider $s \geqslant 5$. Note that $n \geqslant 12$ and $\zeta_3 \geqslant n + 3 - 10/n > n + 2 = 2s + 4$ by Lemma 2.4. Let

$$\tilde{g}(x) = x\Phi_{M_3}(x) - \tilde{h}(x) = -(s+1)x^2 + (s^2 + 12s - 5)x + 8s - 2.$$

By a direct calculation, $\tilde{g}(x)$ is a monotonically decreasing function for $x \geqslant 2s+4$ and $\tilde{g}(\zeta_3) < \tilde{g}(2s+4) = -2s^3 + 8s^2 + 14s - 38$.

Let $h_4(x) = -2x^3 + 8x^2 + 14x - 38$ be a real function in x, where $x \in [5, +\infty)$. Then, $h_4'(x) = -6s^2 + 16s + 14 = -6(x - \frac{1}{3}(4 + \sqrt{37}))(x - \frac{1}{3}(4 - \sqrt{37})) < 0$ and $\tilde{g}(\zeta_3) < h_4(s) \leqslant h_4(5) = -18 < 0$. Thus, $\mu(G) > \zeta_3(n)$ for $n \geqslant 12$ and $n = 10$, which is a contradiction.

Together, Case 1 and Case 2 complete the proof. □

4. Extremal graphs

In this section, we construct several graphs to show that the bounds established in Theorems 1.1, 1.2 and 1.3 are sharp.

THEOREM 4.1. *Let n be a positive integer.*

(i) *For* $n = 3, 5, 7, 9, 11$, *we have* $|E(\overline{K_{(n+1)/2}} \vee K_{(n-1)/2})| = \frac{1}{8}(n-1)(3n-1)$ *and* $\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}$ *has no fractional perfect matching.*

(ii) *For* $n = 4, 6, 8, 10$, *we have* $|E(\overline{K_{(n+2)/2}} \vee K_{(n-2)/2})| = \frac{3}{8}n(n-2)$ *and* $\overline{K_{(n+2)/2}} \vee K_{(n-2)/2}$ *has no fractional perfect matching.*

(iii) *For* $n \geqslant 12$, *we have* $|E(K_1 \vee (K_{n-3} \cup 2K_1))| = \binom{n-2}{2} + 2$ *and* $K_1 \vee (K_{n-3} \cup 2K_1)$ *has no fractional perfect matching.*

PROOF. It is straightforward to check the sizes of graphs $\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}$, $\overline{K_{(n+2)/2}} \vee K_{(n-2)/2}$ and $K_1 \vee (K_{n-3} \cup 2K_1)$. However, put $S = V(K_{(n-1)/2})$, $V(K_{(n-2)/2})$ and $V(K_1)$. By Lemma 2.5, $\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}$, $\overline{K_{(n+2)/2}} \vee K_{(n-2)/2}$ and $K_1 \vee (K_{n-3} \cup 2K_1))$ have no fractional perfect matching. □

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THEOREM 4.2. *Let n be a positive integer.*

(i) *For* $n = 4$, *we have* $\rho(\overline{K_3} \vee K_1) = \sqrt{3}$ *and* $\overline{K_3} \vee K_1$ *has no fractional perfect matching.*
(ii) *For* $n = 6$, *we have* $\rho(\overline{K_4} \vee K_2) = \frac{1}{2}(1 + \sqrt{33})$ *and* $\overline{K_4} \vee K_2$ *has no fractional perfect matching.*
(iii) *For* $n = 3, 5, 7, 9$, *we have* $\rho(\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}) = \xi_1(n)$ *and* $\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}$ *has no fractional perfect matching, where* $\xi_1(n)$ *is the largest root of the polynomial* $x^2 - \frac{1}{2}(n-3)x - \frac{1}{4}(n^2 - 1) = 0$.
(iv) *For* $n = 8$ *or* $n \geqslant 10$, *we have* $\rho(K_1 \vee (K_{n-3} \cup 2K_1)) = \xi_2(n)$, *and the graph* $K_1 \vee (K_{n-3} \cup 2K_1)$ *has no fractional perfect matching, where* $\xi_2(n)$ *is the largest root of the polynomial* $x^3 - (n-4)x^2 - (n-1)x + 2n - 8 = 0$.

PROOF. Here we only prove item (iv). A similar discussion shows items (i), (ii) and (iii).

According to the partition $V(2K_1) \cup V(K_1) \cup V(K_{n-3})$, the quotient matrix of $A(K_1 \vee (K_{n-3} \cup 2K_1))$ can be written as

$$B(K_1 \vee (K_{n-3} \cup 2K_1)) = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & n-3 \\ 0 & 1 & n-4 \end{pmatrix},$$

whose characteristic polynomial is $x^3 - (n-4)x^2 - (n-1)x + 2n - 8$. Since the vertex partition is equitable, the largest root $\xi_2(n)$ of $x^3 - (n-4)x^2 - (n-1)x + 2n - 8 = 0$ is equal to the spectral radius of $K_1 \vee (K_{n-3} \cup 2K_1)$. Let $S = V(K_1)$. By Lemma 2.5, $K_1 \vee (K_{n-3} \cup 2K_1)$ has no fractional perfect matching.

This completes the proof of item (iv). □

THEOREM 4.3. *Let n be a positive integer.*

(i) *For* $n = 3, 5, 7, 9, 11$, *we have* $\mu(\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}) = \zeta_1(n)$ *and* $\overline{K_{(n+1)/2}} \vee K_{(n-1)/2}$ *has no fractional perfect matching, where* $\zeta_1(n)$ *is the largest root of the polynomial* $x^2 - \frac{1}{2}(3n-5)x + \frac{1}{4}(n^2 - 8n + 7) = 0$.
(ii) *For* $n = 4, 6, 8$, *we have* $\mu(\overline{K_{(n+2)/2}} \vee K_{(n-2)/2}) = \zeta_2(n)$ *and* $\overline{K_{(n+2)/2}} \vee K_{(n-2)/2}$ *has no fractional perfect matching, where* $\zeta_2(n)$ *is the largest root of the polynomial* $x^2 - \frac{1}{2}(3n-4)x + \frac{1}{4}(n^2 - 8n + 4) = 0$.
(iii) *For* $n = 10$ *and* $n \geqslant 12$, *we have* $\mu(K_1 \vee (K_{n-3} \cup 2K_1)) = \zeta_3(n)$ *and the graph* $K_1 \vee (K_{n-3} \cup 2K_1)$ *has no fractional perfect matching, where* $\zeta_3(n)$ *is the largest root of* $x^3 - (n-2)x^2 - (7n-17)x - 4n + 10 = 0$.

PROOF. Here we only prove item (iii). A similar discussion shows items (i) and (ii).

According to the partition $V(2K_1) \cup V(K_1) \cup V(K_{n-3})$, the quotient matrix of $D(K_1 \vee (K_{n-3} \cup 2K_1))$ can be written as

$$B(K_1 \vee (K_{n-3} \cup 2K_1)) = \begin{pmatrix} 2 & 1 & 2n-6 \\ 2 & 0 & n-3 \\ 4 & 1 & n-4 \end{pmatrix},$$

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whose characteristic polynomial is $x^3 - (n-2)x^2 - (7n-17)x + -4n + 10$. Since the vertex partition is equitable, the largest root $\zeta_3(n)$ of $x^3-(n-2)x^2-(7n-17)x-4n+10$ is equal to the spectral radius of $K_1 \vee (K_{n-3} \cup 2K_1)$. Let $S = V(K_1)$. By Lemma 2.5, $K_1 \vee (K_{n-3} \cup 2K_1)$ has no fractional perfect matching.

This completes the proof of item (iii). □

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