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PREORDERS ON CANONICAL FAMILIES OF MODULES OF FINITE LENGTH

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Abstract

Let R be an artinian ring. A family, $\mathscr{M}$, of isomorphism types of R-modules of finite length is said to be *canonical* if every R-module of finite length is a direct sum of modules whose isomorphism types are in $\mathscr{M}$. In this paper we show that $\mathscr{M}$ is canonical if the following conditions are simultaneously satisfied: (a) $\mathscr{M}$ contains the isomorphism type of every simple R-module; (b) $\mathscr{M}$ has a preorder with the property that every nonempty subfamily of $\mathscr{M}$ with a common bound on the lengths of its members has a smallest type; (c) if M is a nonsplit extension of a module of isomorphism type II_1 by a module of isomorphism type II_2, with II_1, II_2 in $\mathscr{M}$, then M contains a submodule whose type II_3 is in $\mathscr{M}$ and II_1 does not precede II_3. We use this result to give another proof of Kronecker's theorem on canonical pairs of matrices under equivalence. If R is a tame hereditary finite-dimensional algebra we show that there is a preorder on the family of isomorphism types of indecomposable R-modules of finite length that satisfies Conditions (b) and (c).

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1. Precedence relations

With a few exceptions, all modules in this paper are unital right modules of finite length over an artinian ring R. Modules will often be used interchangeably with their (isomorphism) types, for example, the length of a type II is the length of a module whose type is II. A family, S, of types is said to be *bounded* (*by* m) if there is a positive integer m such that the length of every type in S is less than m. Proposition 1.1 generalizes [7, Proposition 4.7]. The proofs of both propositions are essentially the same.

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Proposition 1.1. *Suppose $\leq$ is a preorder (a reflexive and transitive relation) on a family, $\mathscr{M}$, of isomorphism types of R-modules of finite length, with the following properties:*

(a) *$\mathscr{M}$ contains the isomorphism type of every simple R-module;*

(b) *every bounded subfamily of $\mathscr{M}$ has a smallest type;*

(c) *if M is a nonsplit extension of a module of type Π_1 by a module of type Π_2, with Π_1, Π_2 in $\mathscr{M}$, then M contains a submodule whose type Π_3 is in $\mathscr{M}$ and Π_1 does not precede Π_3.*

Then every R-module, V, of finite length is a direct sum of submodules whose isomorphism types are in $\mathscr{M}$.

Proof. Let $l(V)$ denote the length of V. We shall prove, by induction on $l(V)$, that V satisfies the conclusion of the proposition. We may assume that $V \neq 0$. So it has a nonzero simple submodule. Hence by (a) the family $S = \{\text{type}(W) : W \subseteq V\} \cap \mathscr{M}$ is not empty. Since S is bounded by $l(V)$, there exists $\Pi_1 \in S$ such that $\Pi_1 \leq \Pi$ for every $\Pi \in S$. Let X be a submodule of V of type Π_1. If $X = V$, we would be done. So we may assume that X is a nonzero proper submodule of V. Therefore, $l(V/X) < l(V)$. By the induction hypothesis,

$$V/X = \sum{}^{\cdot}_{j \in J} U_j/X \tag{1}$$

with $\text{type}(U_j/X) \in \mathscr{M}$. Suppose X is not a direct summand of U_j. Then U_j is a nonsplit extension of X by U_j/X. So by (c), U_j contains a submodule Y (say) of type $\Pi_3 \in \mathscr{M}$ and Π_1 does not precede Π_3. Since Y is a submodule of V, $\Pi_3 \in S$. This contradicts the choice of Π_1. Therefore, X is a direct summand of U_j for each $j \in J$. This implies, from (1), that X is a direct summand of V. Applying the induction hypothesis to a direct complement of X in V gives us that V is a direct sum of submodules whose types are in $\mathscr{M}$.

A preorder which satisfies Conditions (a), (b), and (c) of Proposition 1.1 will be called a *precedence relation.* We use Proposition 1.1 to give a new exposition of Kronecker's theorem.

Let $A = (A_1, A_2, \ldots, A_n)$ and $B = (B_1, B_2, \ldots, B_n)$ be two n-tuples of $r \times s$ matrices. The n-tuple A is *equivalent* to B if there are invertible matrices P and Q such that

$$PA_iQ = B_i, \qquad i = 1, \ldots, n. \tag{2}$$

We are interested in the case $n = 2$. (For $n \geq 3$, see [8], and for $n = 1$, see [13].) We assume that the matrices have entries in an algebraically closed field, K. Following [3], we replace the pairs of matrices by pairs of linear transformations from an s-dimensional vector space V to an r-dimensional vector space W. By taking linear combinations, we get from each such pair of linear transformations a K-bilinear map, $\circ$, from $K^2 \times V$ to W. By linearity, it is enough to specify the map on a basis (a, b) of K^2 and on a basis of V. The pair (V, W) together with the bilinear map is a *Kronecker module.* The space W is called the *range* or *target* space, while V is the *domain* space of (V, W) of (V, W). Kronecker modules can be considered as modules over a finite-dimensional K-algebra, see, for example, [7, Proposition 0.1]. (This algebra is called a *Kronecker algebra.*)

Let $V = (V, W)$ be a Kronecker module. Each $e \in K^2$ gives rise to a linear map

(3) $$T_e: V \to W, \qquad T_e(v) = e \circ v \quad \text{for all } v \text{ in } V.$$

A module (X, Y) is a submodule of (V, W) precisely when X is a subspace of V, Y is a subspace of W and $T_e(X) \subset Y$ for all e in K^2. A *homomorphism* from a module (U, Z) to (V, W) is a pair of linear maps (φ, ψ) with φ a linear map from U to V and ψ a linear map from Z to W such that for each $e \in K^2$, $u \in U$, we have

(4) $$e \circ \varphi(u) = \psi(e \circ u).$$

In (4), $\circ$ on the left hand side is in (V, W) while $\circ$ on the right is in (U, Z).

To say that (U, Z) is *isomorphic* to (V, W) means that there is a homomorphism (φ, ψ) from (U, Z) to (V, W) with φ and ψ isomorphisms. This brings us back to (2), with $n = 2$, when φ and ψ are interpreted as matrices.

If (X, Y) is a submodule of (V, W), then $(V, W)/(X, Y) = (V/X, W/Y)$ is a module via

(5) $$e \circ (v + X) = e \circ v + Y$$

for all $v \in V$, all $e \in K^2$, where $e \circ v$ is from the bilinear map in (V, W).

Let (V, W) be a module in which, for some c in K^2, the linear map T_c (see (3)) is an isomorphism of V onto W. The map from $K^2 \times V$ to V that takes (e, v) to $T_c^{-1}(e \circ v)$ is bilinear and so makes (V, V) a module. Let id be the identity map on V. Then (id, T_c^{-1}) is an isomorphism from (V, W) to (V, V). Moreover, V is a $K[\zeta]$-module, ζ an indeterminate over K; see, for example, [13]. Conversely, let V be a $K[\zeta]$-module. We make (V, V) a Kronecker module as follows. Let (a, b) be a fixed basis of K^2. Given $e = \alpha a + \beta b \in K^2$, $v \in V$, set $e \circ v = (\alpha + \beta\zeta)v$. We summarize this discussion in Proposition 1.2.

PROPOSITION 1.2. *A Kronecker module* (V, W) *is isomorphic to a module that comes from a* $K[\zeta]$*-module if and only if for some* e *in* K^2, T_e *is an isomorphism of* V *onto* W.

If (U, Z) is an extension of (X, Y) by (V, W) and T_e is an isomorphism of X onto Y and V onto W then it is also an isomorphism of U onto Z.

The modules in Proposition 1.2 are said to be *nonsingular* or *regular*. We can call on the results in [9] when dealing with them.

Let θ be an element of K. A nonsingular module (V, V) is said to be a θ-module if for every v in V there exists a positive integer n—depending on v—such that $(\zeta - \theta)^n v = 0$. An extension of a θ-module by a θ-module is again a θ-module.

Let V^* be the vector space of linear functionals on a vector space, V. Let (V, W) be a Kronecker module. Then the *dual module* of $(V, W) = (W^\star, V^\star)$ is a Kronecker module with the bilinear map given by

$$(6) \qquad (e \circ w^\star)(v) = w^\star(e \circ v) \quad \text{for } e \in K^2,\ w^\star \in W^\star, \text{ and } v \in V.$$

If both V and W are finite-dimensional, then the double dual of (V, W) is naturally isomorphic to (V, W).

If S is a subset of a vector space V, then $[S]$ will denote the subspace of V spanned by S. The dimension of a vector space V will be denoted by $\dim V$. Let (P, P) denote the Kronecker module $(K[\zeta], K[\zeta])$ that comes from the $K[\zeta]$-module, $K[\zeta]$. For $n = 1, 2, \dots,$ let P_n denote the subspace of $K[\zeta]$ spanned by polynomials of degree strictly less than n; P_0 denotes the zero space. We have that (P_{n-1}, P_n) is a submodule of (P, P).

DEFINITIONS 1.3. (a) A module isomorphic to (P_{n-1}, P_n) is said to be of *type* III^n. Its dual is said to be of *type* I^n.

(b) A module is said to be of *type* II^n_∞ if it is isomorphic to $(P_n, P_{n+1})/(0, [1])$.

A module is said to be of *type* II^n_θ if it is isomorphic to $(P_n, P_{n+1})/(0, [(\zeta - \theta)^n])$. Modules of type II^n_θ, $\theta \in K \cup \{\infty\}$, are self-dual. If a nonzero element $e \in K^2$ is not a multiple of $b - \theta a$ then the map T_e in (3) is an isomorphism between the domain and target spaces of II^n_θ. A change of basis of K^2 transforms II^n_∞ to II^n_θ, $\theta \neq \infty$.

REMARK 1.4. From the definitions of the types, we get the following.

(a) If $n \geq m$, there are monomorphisms from III^m to III^n with II^{n-m}_∞ and II^{n-m}_0 as respective quotients. (The monomorphisms are respectively, the canonical injection and the pair of multiplications by ζ^{n-m}.)

(b) There is an epimorphism $(\varphi, \psi): \mathrm{II}_\theta^n \to I^n$ with φ monic and $\ker \psi$ one-dimensional.

(c) If $n \geq m$, there is an isomorphism from I^n to I^m.

(d) If $n \geq m$, then II_θ^m is a submodule of II_θ^n.

EXAMPLE 1.5. We now define the following *preorder* on $\mathscr{M} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$, where

$$\mathrm{I} = \{\mathrm{I}^m : m = 1, 2, \ldots\},$$
$$\mathrm{II} = \{\mathrm{II}_\theta^m : \theta \in K \cup \{\infty\},\ m = 1, 2, \ldots\},$$
$$\mathrm{III} = \{\mathrm{III}^m : m = 1, 2, \ldots\}.$$

(a) Every type in I procedes every type in $\mathrm{II} \cup \mathrm{III}$, while $\mathrm{I}^m \leq \mathrm{I}^n$ if and only if $m \leq n$.

(b) Every type in II precedes every type in III. For a fixed n, every type in $\{\mathrm{II}_\theta^n : \theta \in K \cup \{\infty\}\}$ precedes every type in $\{\mathrm{II}_\theta^m : \theta \in K \cup \{\infty\},\ m \leq n\}$.

(c) $\mathrm{III}^n \leq \mathrm{III}^m$ if and only if $n \geq m$.

Types III^1 and I^1 are the only isomorphism types of simple Kronecker modules. It is easy to verify that the above preorder on $\mathscr{M}$ satisfies Condition (b) of Proposition 1.1. In order to show that it is a precedence relation, we need only check Condition (c) of Proposition 1.1. We need to know for which types $\mathrm{II}_1, \mathrm{II}_2$ in $\mathscr{M}$ is $\mathrm{Ext}(\mathrm{II}_2, \mathrm{II}_1) \neq 0$. The next proposition is a special case of a formula in [16] whose easy proof belies its importance; see "Note added in proof" of [16]. One can also prove the formula for Kronecker modules using the fact that the indecomposable projective types are III^1 and III^2.

PROPOSITION 1.6. *Let (V, W) and (X, Y) be finite-dimensional Kronecker modules. Then*

$$\begin{aligned}(7)\quad \dim \mathrm{Ext}((V, W), (X, Y)) &= \dim \mathrm{Hom}((V, W), (X, Y)) \\ &\quad - \dim V \dim X - \dim W \dim Y + 2 \dim V \dim Y.\end{aligned}$$

From Proposition 1.2 and [9, Section 52D], we get that $\dim \mathrm{Ext}((\mathrm{II}_\theta^m, \mathrm{II}_\eta^n)$ is the minimum of m and n, if $\eta = \theta$; otherwise it is 0. By Proposition 1.6, we know $\dim \mathrm{Ext}((V, W), (X, Y))$ once we know $\dim \mathrm{Hom}((V, W), (X, Y))$. In computing the latter for $\mathscr{M}$ we use, without further comment, previously verified values of the former. The next lemma is easily deduced from the definitions of the types in 1.3.

LEMMA 1.7. $\mathrm{Hom}(\mathrm{II}_2, \mathrm{II}_1)$ *is* 0 *in the following cases*:

(a) $\mathrm{II}_2 \in I$ *while* $\mathrm{II}_1 \in \mathrm{II} \cup \mathrm{III}$;

(b) $\mathrm{II}_2 \in \mathrm{II}$ *while* $\mathrm{II}_1 \in \mathrm{III}$;

(c) II_2 *and* II_1 *are respectively of types* II_θ^m *and* II_η^n, $\eta \neq \theta$.

By duality, $\dim \operatorname{Hom}(\mathrm{I}^n, \mathrm{I}^m)$ and $\dim \operatorname{Hom}(\mathrm{II}_0^n, \mathrm{I}^m)$ are respectively equal to $\dim \operatorname{Hom}(\mathrm{III}^m, \mathrm{III}^n)$ and $\dim \operatorname{Hom}(\mathrm{III}^m, \mathrm{II}_\theta^n)$.

PROPOSITION 1.8. (a) $\dim \operatorname{Hom}(\mathrm{III}^n, \mathrm{III}^m) = \max(0, m - n + 1)$.

(b) $\dim \operatorname{Hom}(\mathrm{III}^n, \mathrm{II}_\eta^m) = m$.

(c) $\dim \operatorname{Ext}(\mathrm{II}_\eta^n, \mathrm{I}^m) = 0$ *for every* $\eta \in K \cup \{\infty\}$ *and every positive integer* n.

(d) $\dim \operatorname{Hom}(\mathrm{III}^n, \mathrm{I}^m) = n + m - 2$.

PROOF. We shall use the modules described in 1.3.

(a) It follows from (4) that a homomorphism (φ, ψ) from III^n to III^m is given by multiplications by the polynomial $f = \psi(1)$. Therefore, the degree of f must be less than $m-n+1$. So, $f = 0$, if $m-n+1 \leq 0$. Conversely, the pair of multiplications given by such an f form a homomorphism from III^n to III^m.

(b) We do an induction on n. If $n = 1$, (b) holds because the dimension is that of the target space of II_η^m. For $n > 1$, we have, by 1.4(a), a short exact sequence

$$(8) \qquad 0 \to \mathrm{III}^{n-1} \to \mathrm{III}^n \to \mathrm{II}_0^1 \to 0.$$

This leads to the exact sequence

$$\operatorname{Hom}(\mathrm{II}_0^1, \mathrm{II}_\eta^m) \to \operatorname{Hom}(\mathrm{III}^n, \mathrm{II}_\eta^m) \to \operatorname{Hom}(\mathrm{III}^{n-1}, \mathrm{II}_\eta^m) \to \operatorname{Ext}(\mathrm{II}_0^1, \mathrm{II}_\eta^m).$$

If $\eta \neq 0$, the first and last terms are zero. So the two middle terms have the same dimension. If $\eta = 0$, we replace (8) by a similar short exact sequence involving II_∞^1 instead of II_0^1.

(c) By duality and (b), $\dim \operatorname{Hom}(\mathrm{II}_\eta^n, \mathrm{I}^m) = n$. So, (c) follows from the formula in Proposition 1.6.

(d) From (8) we get the exact sequence

$$0 \to \operatorname{Hom}(\mathrm{II}_0^1, \mathrm{I}^m) \to \operatorname{Hom}(\mathrm{III}^n, \mathrm{I}^m) \to \operatorname{Hom}(\mathrm{III}^{n-1}, \mathrm{I}^m) \to \operatorname{Ext}(\mathrm{II}_0^1, \mathrm{I}^m).$$

By (c) the last term is 0; while duality and (b) give us that $\dim \operatorname{Hom}(\mathrm{II}_0^1, \mathrm{I}^m) = 1$. It follows that for $n > 1$, $\dim \operatorname{Hom}(\mathrm{III}^n, \mathrm{I}^m) = \operatorname{Hom}(\mathrm{III}^{n-1}, \mathrm{I}^m) + 1$. Since $\dim \operatorname{Hom}(\mathrm{III}^1, \mathrm{I}^m) =$ the dimension of the target space of $\mathrm{I}^m = m-1$, the formula follows by induction on n.

Using 1.7, 1.8, the intervening remarks, and 1.6, we obtain the following proposition. (X, Y) has the horizontal types.

PROPOSITION 1.9.

$\dim(\mathrm{Ext}((V, W), (X, Y)))$

$(V, W), (X, Y)$	I^m	II^m_η	III^m
I^n	$\max(0, m-n-1)$	m	$m+n$
II^n_θ	0	$\min(n, m)\delta_{\theta\eta}$	n
III^n	0	0	$\max(0, n-m-1)$

2. Canonical families have precedence relations

The first task in this section is the completion of the verification that the preorder defined on $\mathscr{M}$ in Section 1.5 is a precedence relation. We shall need a polynomial-free description of the types. Let (a, b) be a basis of K^2. An element $\theta \in K$ is said to be an *eigenvalue* of a Kronecker module (V, W) if $(b - \theta a) \circ v = 0$ for some nonzero vector $v \in V$. If $a \circ v = 0$ for some nonzero vector $v \in V$, we say ∞ is an eigenvalue of (V, W). A change of basis of K^2 results in a Möbius transform of the eigenvalues of a module.

PROPOSITION 2.1. *Let (V, W) be a finite-dimensional Kronecker module. Suppose V and W have the same dimension. Then (V, W) has an eigenvalue.*

PROOF. Let (a, b) be any basis of K^2. If $a \circ v = 0$ for some nonzero vector v in V, then ∞ is an eigenvalue of (V, W). So we may assume that the linear map

$$T_a: V \to W, \qquad T_a(v) = a \circ v, \tag{9}$$

is an isomorphism of V onto W. By Proposition 1.2, (V, W) is isomorphic to (V, V). Let the bilinear map in (V, V) be denoted by $\circ_1$. It is given by

$$e \circ_1 v = T_a^{-1}(e \circ v) \quad \text{for } e \in K^2, \ v \in V. \tag{10}$$

Let $T_b: V \to V$ be the linear map given by (3) with $e = b$, that is, $T_b(v) = b \circ v$. Since K is algebraically closed, the endomorphism $T_a^{-1}T_b$ of V has an eigenvector v belonging to an eigenvalue $\theta \in K$. From (10) we get that $(b - \theta a) \circ_1 v = 0$. So θ is an eigenvalue of (V, V). So (V, W) has an eigenvalue.

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We note, from the definitions in 1.3, that modules of respective types II_θ^n, I^n have eigenvalues. Conversely, if a Kronecker module has an eigenvalue, then it has a submodule of type II_θ^1 or I^1. Modules of type III^m have no eigenvalues.

Let $(\varphi, \psi): (P_{n-1}, P_n) \to (V, W)$ be a homomorphism. Let $\varphi(\zeta^i) = v_{i+1}$, $i = 0, \ldots, n-2$; $\psi(\zeta^i) = w_{i+1}$, $i = 0, \ldots, n-1$. Then, by the definition of a homomorphism

(11)

$$a \circ v_1 = w_1; \qquad b \circ v_i = w_{i+1} = a \circ v_{i+1}, \quad i = 1, \ldots, n-2; \qquad b \circ v_{n-1} = w_n.$$

DEFINITION 2.2. A pair of sequences $((v_i)_{i=1}^{n-1}, (w_i)_{i=1}^n)$ that satisfies (11) is said to be a *chain of type* III^n *with respect to the basis* (a, b). The module $(V_1, W_1) = (\varphi, \psi)P_{n-1}$ is said to be *spanned by the chain.* It is, therefore, of type III^n as defined in 1.3 if, in addition, $\dim V_1 = n - 1 = \dim W_1 - 1$.

Chains of types II_θ^n and I^n are defined in a similar manner; we use the quotient modules in 1.3. More precisely, a pair of sequences $((v_i)_{i=1}^n, (w_i)_{i=1}^n)$ is said to be a chain of type II_θ^n if

$$(12) \qquad \begin{aligned} b_\theta \circ v_1 = 0; \qquad b_\theta \circ v_{i+1} &= a \circ v_i = w_i, \quad i = 1, \ldots, n-1, \\ a \circ v_n = w_n; &\qquad b_\theta = b - \theta a. \end{aligned}$$

The submodule, (V_1, W_1), of (V, W) spanned by the chain (12) is of type II_θ^n if $\dim V_1 = \dim W_1 = n$. In that case, the homomorphism $(\varphi, \psi): (P_n, P_{n+1}) \to (V_1, W_1)$ given by $\varphi(\zeta - \theta)^i = v_{n-i}$, $\psi(\zeta - \theta)^i = w_{n-i}$, $i = 0, \ldots, n-1$, $\psi(\zeta - \theta)^n = 0$, induces an isomorphism from $P_n/(0, [(\zeta - \theta)^n])$ onto (V_1, W_1).

If w_n in (12) is replaced by 0, the resulting chain is of type I^n. If (V, W) is of type I^n then $\dim V = n = \dim W + 1$. A change of basis of K^2 takes II_θ^n to II_η^n, where η is some Möbius transform of θ. On the other hand, we show in Lemma 2.3 that if (V, W) is of type III^m with respect to a basis (a, b) it remains of that type with respect to any other basis of K^2. Since I^m is the dual of III^m, the same remark applies to it.

LEMMA 2.3 [3, Lemma 2.5]. *Suppose that* (V, W) *is a module of type* III^n *with respect to a basis* (a, b). *Then it is of type* III^n *with respect to any other basis* (c, d) *of* K^2.

PROOF. When $n = 1$, $(V, W) = (0, [w_1])$ and the basis of K^2 plays no role. Since (V, W) is isomorphic to (P_{n-1}, P_n), it is enough to prove the lemma for the latter. Recall that a and b act respectively as inclusion and multiplication by ζ from P_{n-1} to P_n. Let $c = \alpha a + \beta b$, $d = \gamma a + \delta b$ for some $\alpha, \beta, \gamma, \delta \in K$. Put $v_i = (\alpha+\beta\zeta)^{n-i-1}(\gamma+\delta\zeta)^{i-1}$, $i = 1, \dots, n-1$; $w_i = (\alpha+\beta\zeta)^{n-i}(\gamma+\delta\zeta)^{i-1}$, $i = 1, \dots, n$. The relations (11) are immediately verified with (c, d) in place of (a, b). The sets $\{v_1, \dots, v_{n-1}\}$, $\{w_1, \dots, w_n\}$ are linearly independent over K. For let

$$w = \sum_{l=1}^{n} \alpha_i w_i = (\gamma+\delta\zeta)^{n-1} \sum_{i=1}^{n} \alpha_i (\alpha+\beta\zeta)^{n-i}/(\gamma+\delta\zeta)^{n-i}.$$

Since (c, d) is linearly independent, $\alpha\delta - \beta\gamma \neq 0$, and therefore the map $\zeta \mapsto (\alpha+\beta\zeta)/(\gamma+\delta\zeta)$ is a field automorphism of $K(\zeta)$. So if $w = 0$, the scalars $\alpha_i, \dots, \alpha_n$ are 0. Similarly, $\{v_1, \dots, v_{n-1}\}$ is linearly independent.

If a module is spanned by a chain of type T, the module need not be of type T because the vectors defining the chain may not be linearly independent. However when T is I^n we have the following lemma.

LEMMA 2.4. *Suppose a Kronecker module* (U, Z) *contains a nonzero submodule spanned by a chain of type* I^n. *Then* (U, Z) *contains a submodule of type* I^m *for some positive integer* $m \leq n$.

PROOF. Let m be the least positive integer such that (U, Z) contains a nonzero submodule (U', Z') spanned by a chain of type I^m. So $m \leq n$. Say $U' = [u_1, u_2, \dots, u_m]$, $Z' = [z_2, \dots, z_m]$ and for some basis (c, d) of K^2

(13)
$$c \circ u_1 = 0; \qquad c \circ u_{i+1} = d \circ u_i = z_{i+1}, \quad i = 1, \dots, m-1; \qquad d \circ u_m = 0.$$

We claim that the sets $\{u_1, u_2, \dots, u_m\}$, $\{z_2, \dots, z_m\}$ are linearly independent. Linear dependence of the former set implies, from (13), linear dependence of the latter set. Suppose $\{z_2, \dots, z_m\}$ is linearly dependent. Let ℓ be some positive integer such that, for some scalars $\alpha_2, \dots, \alpha_{\ell-1}$, $z_\ell = \sum_{j=2}^{\ell-1} \alpha_j z_j$. We now construct a chain of type $\mathrm{I}^{\ell-1}$. Since $\ell - 1 < m$, this would contradict the minimality of m. Let $u_1' = u_1$. For $i = 2, \dots, \ell-1$, let

$$u_i' = u_i - \sum_{j=1}^{i-1} \alpha_{\ell-i+j} u_j.$$

Let $z_2' = z_2$. For $i = 3, \dots, \ell-1$ let

$$z_i' = z_i - \sum_{j=2}^{i-1} \alpha_{\ell-i+j} z_j.$$

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From (13), we get that $c \circ u'_1 = 0$, $c \circ u'_i = z'_i$ and $d \circ u'_i = z'_{i+1}$, $i = 2, \dots, \ell - 2$, and $d \circ u'_{\ell-1} = z_\ell - \sum_{j=2}^{\ell-1} \alpha_j z_j = 0$; that is, we have a chain of type $\mathrm{I}^{\ell-1}$. This chain spans a nonzero submodule. Indeed, if we had $u'_1 = u_1 = 0$, then $z_2 = 0$, and $((u_i)_{i=2}^m, (z_i)_{i=3}^m)$ would be a chain of type I^{m-1} spanning a nonzero submodule.

COROLLARY 2.5. *Let (φ, ψ) be a nonzero homomorphism to any module, (V, W), from a θ-module (X, X). If ψ is not monic, but φ is monic, then the image of (φ, ψ) contains a submodule of type I^m for some positive integer m.*

PROOF. By Lemma 2.4, it is enough to show that the image of (φ, ψ) has a submodule spanned by a chain of type I^ℓ for some positive integer ℓ.

For every $x \in X$, there is a positive integer ℓ with $(\zeta - \theta)^\ell x = 0$, because (X, X) is a θ-module. With $v_1 = (\zeta - \theta)^{\ell-1} x$, we get as in (12), a chain of type II_θ^ℓ. If $x \neq 0$ and $\psi(x) = 0$, the image in (V, W) of such a chain spans a nonzero submodule spanned by a chain of type I^ℓ.

THEOREM 2.6. *The preorder in Example 1.5 is a precedence relation.*

PROOF. Condition (c) of Proposition 1.1 is all there is left to check. let

$$0 \to (X, Y) \xrightarrow{(\mu, \nu)} (U, Z) \xrightarrow{(\sigma, \tau)} (V, W) \to 0 \tag{14}$$

be a nonsplit extension with type $(X, Y) = \mathrm{II}_1$, type$(V, W) = \mathrm{II}_2$ and $\mathrm{II}_1, \mathrm{II}_2 \in \mathscr{M}$.

Case (i), $\mathrm{II}_1 = \mathrm{I}^m$. By Proposition 1.9, $m \geq 3$ and $\mathrm{II}_2 = \mathrm{I}^n$, $n < m - 1$. Let (V_1, W_1) be a module of type I^{m-1}. By Remark 1.4(c), there is a map (φ, ψ) from (V_1, W_1) onto (V, W). Combining this map with the sequence (14), we get from pullback the exact sequence

$$0 \to (X, Y) \xrightarrow{(\mu_1, \nu_1)} (U_1, Z_1) \xrightarrow{(\sigma_1, \tau_1)} (V_1, W_1) \to 0 \tag{15}$$

with a map $(\varphi_1, \psi_1): (U_1, Z_1) \to (U, Z)$ whose kernel is isomorphic to $\ker(\varphi, \psi)$. By Proposition 1.9, (U_1, Z_1) is of type $\mathrm{I}^m \oplus \mathrm{I}^{m-1}$. Now, $(\varphi_1, \psi_1)\mathrm{I}^{m-1}$ is a nonzero module spanned by a chain of type I^{m-1}. By Lemma 2.4, (U, Z) has a submodule of type I^ℓ, $\ell \leq m - 1$. From 1.5, we see that I^m does not precede I^ℓ.

Case (ii), $\mathrm{II}_1 = \mathrm{II}_\theta^m$. By Proposition 1.9, II_2 is either I^n or II_θ^m. In the latter case, (U, Z) must have a submodule of type II_θ^{n+1}, by Proposition 1.2, and Section 15 of [9]. From 1.5, we see that II_θ^m does not precede II_θ^{m+1}.

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So let (V, W) in (14) be of type I^n. Let (V_1, W_1) be a module of type II_θ^n. By 1.4(b), there is an epimorphism $(\varphi, \psi): (V_1, W_1) \to (V, W)$ with φ monic and ψ has a one-dimensional kernel. Using this epimorphism, we proceed as in Case (i) to obtain (15). This time, (U_1, Z_1) is a θ-module. By Corollary 2.5, (U, Z) has a submodule of type I^ℓ for some positive integer ℓ.

Case (iii), $\mathrm{II}_1 = \mathrm{III}^m$. If (U, Z) has an eigenvalue, then, as remarked after 2.1, (U, Z) has a submodule of type II_θ^1 or I^1. Since these types are not preceded by III^m we may assume that (U, Z) has no eigenvalues. This precludes the possibility that $\mathrm{II}_2 = \mathrm{I}^n$ because in that case $\dim U = \dim X + \dim V = m - 1 + n = \dim Y + \dim Z$, which implies by Proposition 2.1 that (U, Z) has an eigenvalue. If $\mathrm{II}_2 = \mathrm{III}^n$, then by Proposition 1.9, $n > m+1$. By 1.4(a), (V, W) has a submodule $(U_1, Z_1)/(X, Y)$ of type III^{m+1}. By Proposition 1.9, we have a decomposition $(U_1, Z_1) = (X, Y) \oplus (U_2, Z_2)$ with $(U_2, Z_2) \subset (U_1, Z_1) \subset (U, Z)$ and $\operatorname{type}(U_2, Z_2) = \mathrm{III}^{m+1}$. This is not preceded by III^m. There remains the case $\mathrm{II}_2 = \mathrm{II}_\theta^n$. By 1.4(d), (V, W) contains a submodule $(U_1, Z_1)/(X, Y)$ of type II_θ^1. The extension

$$(16) \qquad 0 \to (X, Y) \to (U_1, Z_1) \to (U_1, Z_1)/(X, Y) \to 0$$

does not split because otherwise (U_1, Z_1) would contain a submodule of type II_θ^1. And so (U_1, Z_1), hence (U, Z), would have an eigenvalue. By Lemma 2.3, we can describe the modules of the III^ℓ, ℓ an arbitrary positive integer, using the basis $(b - \theta a, a)$ of K^2. (If $\theta = \infty$ we replace $(b - \theta a, a)$ by (a, b).) From this and 1.4(a), we get an extension

$$(17) \qquad 0 \to \mathrm{III}^m \to \mathrm{III}^{m+1} \to \mathrm{II}_\theta^1 \to 0.$$

By Proposition 1.8(a), $\dim \operatorname{End} \mathrm{III}^{m+1} = 1$. So III^{m+1} is indecomposable. Therefore, (17) does not split. By Proposition 1.9, $\dim \operatorname{Ext}(\mathrm{II}_\theta^1, \mathrm{III}^m) = 1$. Hence the sequence (16) is a multiple of the sequence (17). Hence, the submodule (U_1, Z_1) of (U, Z) is of type III^{m+1}, which is not preceded by III^m. This completes the proof of the theorem. So every finite-dimensional Kronecker module is a direct sum of modules whose types are in $\mathscr{M} = \mathrm{I} \cup \mathrm{II} \cup \mathrm{III}$.

Remark on Case (ii). When (X, Y) and (V, W) are of type II_θ^m and II_θ^n we referred to section 15 of [9] thereby implicitly relying on the structure of finitely generated torsion $K[\zeta]$-modules. Since the latter is part of Kronecker's theorem, it is interesting that the following argument avoids a reference to [9].

A nonsplit extension of II_θ^m by II_θ^n contains a submodule of type II_θ^{n+1}. To prove this, we may by Proposition 1.2 assume that we have the following nonsplit extension of $K[\zeta]$-modules

$$0 \to X \to U \to V \to 0 \tag{18}$$

where $X \cong K[\zeta]/(\zeta-\theta)^m$, $V \cong K[\zeta]/(\zeta-\theta)^n$ with respective generators x and v.

Let $u + X = v$. If $(\zeta-\theta)^n u = 0$, then the map $p(\zeta)v \mapsto p(\zeta)u$ is well-defined and gives a splitting of (18). Since (18) does not split, we have $(\zeta-\theta)^n u = \alpha_{m-k}(\zeta-\theta)^{m-k}x + \cdots + \alpha_{m-1}(\zeta-\theta)^{m-1}x$, where $m-k \geq 0$ and $\alpha_{m-k} \neq 0$, $k \geq 1$. Therefore, $\langle u\rangle \cong K[\zeta]/(\zeta-\theta)^{n+k}$.

If $n+k \geq m+1$, then we are done. Otherwise, the element $u' = u - \{\alpha_{m-k}(\zeta-\theta)^{m-(n+k)}x + \cdots + \alpha_{m-1}(\zeta-\theta)^{m-(n+1)}x\}$ gives $\langle u'\rangle \cong V$ and the map $v \mapsto u'$ gives a splitting of (18). Hence $n+k \geq m+1$ as required.

REMARK 2.7. There are many other proofs of Kronecker's theorem on canonical pairs of matrices under equivalence, for example [3], [5], [6], [10], [11], [14], [16], [17], and [18]. Some applications of the theorem can be found in [1], [2], [10], and [12].

A Kronecker algebra is an example of a *tame finite-dimensional hereditary algebra* as defined in [15]. For the rest of the paper, R is a tame finite-dimensional hereditary algebra over an algebraically closed field K. In [15] it is shown that there are precisely three families of finite-dimensional indecomposable R-modules: $\mathscr{P} = (P_n)_{n=1}^\infty$, $(S_\theta^n)_{n=1}^\infty$, for each $\theta \in K \cup \{\infty\}$, and $\mathscr{I} = (I_n)_{n=1}^\infty$. In [15, p. 350], it is shown that the indexing on $\mathscr{P}$ can be done to ensure that

$$\mathrm{Hom}(P_i, P_j) \neq 0 \Rightarrow i \leq j. \tag{19}$$

Similarly the indexing on $\mathscr{I}$ is chosen to have the property

$$\mathrm{Hom}(I_i, I_j) \neq 0 \Rightarrow i \geq j. \tag{20}$$

The families $\mathscr{P}$ and $\mathscr{I}$ are closed under indecomposable submodules and indecomposable quotients respectively [15, Propositions 2.7 and 3.4]. So, from (19) and (20), $\mathrm{End}\, M = K$ for each M with type $M \in \mathscr{P} \cup \mathscr{I}$.

Each S_θ^n may be considered as a module over a discrete valuation ring [15, Section 4] and hence amenable to the same treatment as II_θ^n. Corresponding to III^n and I^n are P_n and I_n respectively. With these correspondences in mind, we define a preorder, $\leq$, on $\mathscr{F}$, the family of isomorphism classes of finite-dimensional indecomposable R-modules, exactly as 1.5. To show that $\leq$ is a precedence relation we shall proceed as in the proof of Theorem 2.6 with the simplification that we know that $\mathscr{F}$ is canonical.

THEOREM 2.8. *The family $\mathscr{F}$ of finite-dimensional indecomposable isomorphism types over a tame finite-dimensional hereditary algebra has a precedence relation.*

PROOF. We shall show that the preorder, defined above, on $\mathscr{F}$ is a precedence relation. Condition (b) is readily checked. To check Condition (c), we let

$$0 \to L \to M \to N \to 0 \tag{21}$$

be a nonsplit sequence of R-modules.

Case (i), $L = \mathrm{I}_m$. By [15, Corollary 3.5], $N = \mathrm{I}_n$ for some positive integer n. Moreover, by [15, Proposition 3.4], $M = \bigoplus \mathrm{I}_{n_j}$, a finite direct sum of modules in $\mathscr{I}$. It follows from (20) and the nonsplitting of (21) that each $n_j < m$.

Case (ii), $L = S_\theta^n$. By [15, Section 4], $N = \mathrm{I}_n$ or S_θ^n. The latter case is handled in the same way as the corresponding case in Theorem 2.6. If $N = \mathrm{I}_n$, then by [15, Proposition 4.2], M must have a direct summand in $\mathscr{I}$.

Case (iii), $L = P_m$. If M has a submodule isomorphic to I_n or S_θ^n we would be done because P_m does not precede those types. So, by [15, Section 4.1], we may assume that $M = \bigoplus P_{n_j}$, a finite direct sum of modules in $\mathscr{P}$. It follows from (19) and the nonsplitting of (21) that each $n_j > m$.

We do not know if there are other artinian rings of infinite type for which Theorem 2.8 holds.

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