THE BULLETIN OF SYMBOLIC LOGIC
Volume 20, Number 4, December 2014

ERDŐS AND SET THEORY

AKIHIRO KANAMORI

Paul Erdős (26 March 1913–20 September 1996) was a mathematician *par excellence* whose results and initiatives have had a large impact and made a strong imprint on the doing of and thinking about mathematics. A mathematician of alacrity, detail, and collaboration, Erdős in his six decades of work moved and thought quickly, entertained increasingly many parameters, and wrote over 1500 articles, the majority with others. His *modus operandi* was to drive mathematics through cycles of problem, proof, and conjecture, ceaselessly progressing and ever reaching, and his *modus vivendi* was to be itinerant in the world, stimulating and interacting about mathematics at every port and capital.

Erdős' main mathematical incentives were to count, to estimate, to bound, to interpolate, and to get at the extremal or delimiting, and his main emphases were on elementary and random methods. These had a broad reach across mathematics but was particularly synergistic with the fields that Erdős worked in and developed. His mainstays were formerly additive and multiplicative number theory and latterly combinatorics and graph theory, but he ranged across and brought in probability and ergodic theory, the constructive theory of functions and series, combinatorial geometry, and set theory. He had a principal role in establishing probabilistic number theory, extremal combinatorics, random graphs, and the partition calculus for infinite cardinals.

Against this backdrop, this article provides an account of Erdős' work and initiatives in set theory with stress put on their impact on the subject. Erdős importantly contributed to set theory as it became a broad, sophisticated field of mathematics in two dynamic ways. In the early years, he established results and pressed themes that would figure pivotally in formative advances. Later and throughout, he followed up on combinatorial initiatives that became part and parcel of set theory. Emergent from combinatorial thinking, Erdős' results and initiatives in set theory had the feel of being simple and basic yet rich and pivotal, and so accrued into the subject as seminal at first, then formative, and finally central. Proceeding chronologically, we work to draw all this out as well as make connections with Erdős' larger work and thinking, to bring out how it is all of a piece.

Received July 31, 2014.

1079-8986/14/2004-0002/$3.40
DOI:10.1017/bsl.2014.38

Paul Erdős and His Mathematics, in two volumes Halász *et al.* (2002a) and Halász *et al.* (2002b), emanated from a 1999 celebratory conference, and it surveys Erdős' work, provides reminiscences, and contains research articles. *The Mathematics of Paul Erdős*, in two volumes Graham *et al.* (2013a) and Graham *et al.* (2013b), is the second edition of a 1997 compendium brought out soon after his "leaving", and it provides reminiscences and extended expository articles. And *Erdős Centennial*, Lovász *et al.* (2013), on the occasion of the 100th anniversary of his birth, provides summary expository articles emphasizing impact and late developments. This present account of Erdős' work in set theory and its impact bears an evident debt to the two previous accounts, Hajnal (1997) and Kunen (2013), as well as to the history Larson (2012) of infinite combinatorics. The details herein about Erdős' life, which are not otherwise documented, can be found in the biography by Béla Bollobás (2013).

We do proceed chronologically in general, taking up topics according to when the main thrusts for them occurred. On the other hand, within a section later developments and ramifications may be pursued, this to bring out the relevance and impact of the work. Section 1 recapitulates Erdős' mathematical beginnings, emphasizing anticipations of his later set-theoretic work. Section 2 describes Erdős' pioneering work on transfinite Ramsey theory. Section 3 sets out the Erdős–Tarski work on inaccessible cardinals, work of considerable import for the development of set theory. Section 4 follows through on a persistent theme in Erdős' early work, free sets for set mappings, a topic to become of broad reach. Section 5 takes up Erdős' work with Rado on the partition calculus, which will become a large part of set theory and be Erdős' main imprint on the subject. Section 6 focuses on Erdős' first joint work with Hajnal and the emergent Ramsey and Erdős cardinals. Section 7 is devoted to a basic, property $\mathcal{B}$ for families of sets, and works through the details of the joint article with Hajnal on the subject. Section 8 takes up the 1960s Erdős-Hajnal development of the partition calculus, the most consequential topic being square-brackets partition relations. Section 9 attends to early appeals to Erdős' work in model theory. Section 10 describes how close Erdős *et al.* came to Silver's Theorem on singular cardinal arithmetic. Section 11 charts out how the compactness of chromatic number of infinite graphs became much addressed in set theory. And finally, Section 12 quickly reviews the set-theoretic work of his later years and ventures some panoptic remarks. A list of Erdős' 121 publications in set theory is presented at the end, not only to be able to cite extensively from his body of work as we proceed, but also to provide a visual, quantitative sense of its extent.

To fix some terminology, a *tree* is to be a partially ordered set with a minimum element such that the predecessors of any element are well-ordered; the *αth level* of a tree is the set of elements whose predecessors have order type α; and the *height* of a tree is the least α such that the αth level is empty. A *chain* is a linearly ordered subset.

$\kappa, \lambda, \ldots$ denote infinite cardinals and $\mathrm{cf}(\kappa)$ the cofinality of κ, so that κ is singular *iff* $\mathrm{cf}(\kappa) < \kappa$. In his problems and results parametrized with

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

cardinals Erdős would generally proceed with the $\aleph_\alpha$'s, this in the Cantorian tradition of taking the infinite cardinals as autonomous numbers. With his investigations extending to ordered sets and order types, it became fitting to make the identification of $\aleph_\alpha$ with the initial (von Neumann) ordinal ω_α. We proceed in the modern vein of taking the ordinals as given and emphasizing the cardinal aspect of ω_α with $\aleph_\alpha$, this being coherent with Erdős' original intent of parametrizing counting with cardinal numbers.

§1. Salad Days. Erdős was a child prodigy in mathematics, quick at calculations and enthusiastic about properties of numbers and proofs, and notably, he learned about Cantor and set theory from his father, a high school teacher.[1]

In 1930 Erdős at the age of 17 entered Pámány Péter Tudományegyetem, the scientific university of Budapest; wrote his doctoral dissertation when he was a second-year undergraduate; and received his Ph.D. in 1934. During this period, Erdős interacted with many other students; began his long collaboration with Paul Turán; and assimilated a great deal of mathematics from his teachers, particularly Lipót Fejér and Dénes Kőnig, and especially from László Kalmár.[2] Kőnig is now remembered for his tree, or "infinity", lemma, the first result about infinite graphs, and Kalmár was then the Hungarian principal in mathematical logic, best known today for the Kalmár hierarchy of number-theoretic functions.

Erdős quickly established results and fostered approaches during this period that would anticipate his long-standing initiatives and preoccupations. In his first year, he found a remarkably simple proof of Chebyshev's 1850 result that between any $n > 1$ and $2n$ there is a prime, "Bertrand's postulate".[3] Kalmár wrote up the result for Erdős' first publication Erdős (1932).[4] Erdős in later years would talk about The Book, in which God

[1](Hajnal, 1997, p.352): "Paul told me that he learned the basics of set theory from his father, a well educated high-school teacher, and he soon became fascinated with 'Cantor's paradise'."

(Vazsonyi, 1996, p.1), describing the first encounter with Erdős, when the author was 14 and Erdős 17:

> My father had one of the top shoe shops in Budapest and I was sitting at the back of the shop. Erdos knocked at the door and entered. "Give me a four digit number," he said. "2532," I replied.
>
> "The square of it is 6, 411, 024. Sorry, I am getting old and cannot tell you the cube," said he. (During his entire life, even in youth, he referred to his old age, his old bones.)
>
> "How many proofs of the Pythagorean Theorem do you know?" "One," I said. "I know 37. Did you know that the points of a straight line do not form a denumerable set?" He proceeded to show me Cantor's proof of using the diagonal. "I must run," and he left.

[2]Erdős said in an interview ((Sós, 2002, p.87)): "I learned a lot from Lipót Fejer and very probably, I learned the most from László Kalmár."

[3](Turán, 2002, p.57).

[4](Vazsonyi, 1996, p.2).

keeps the perfect proofs of theorems, and this proof entered in an earthly rendition.[5] Starting with his dissertation of a year later clarifying issues raised by Isaai Schur of Berlin, Erdős would not only come up with many simple proofs in number theory,[6] but also elementary proofs where only analytic proofs had existed, and in the late 1940s he and Atle Selberg would famously provide an elementary proof of the Prime Number Theorem itself, the circumstances prompting a well-known priority dispute.

Also during his first year, Erdős observed that the recently proved Menger's theorem on connectivity in graphs also holds for infinite graphs. Kőnig had raised the issue in his graph theory course, and he published Erdős' argument as the very last in his monograph Kőnig (1936). A few years later, Erdős [1][2] with Tibor Gallai and Endre Vázsonyi provided a criterion for having an Euler path for infinite graphs; this of course extended to the infinite the original, "seven bridges of Königsberg" result of graph theory.

In his final university year 1934, Erdős with György Szekeres proved: *For any positive n, there is a integer $N(n)$ such that in any set of $N(n)$ points in the plane, no three of which are collinear, there are n points that form a convex polygon.*[7] In an article Erdős–Szekeres (1935) seminal for several reasons, they provided two proofs, one involving the (finite) Ramsey Theorem and the other, the "ordered pigeon-hole principle".[8] For both of these propositions, they provided paradigmatic proofs, these to spawn subjects in the emergent field of combinatorics. With the second proof, they conjectured that the least possibility for $N(n)$ is $\binom{2n-4}{n-2} + 1$, thus initiating the study of extremal possibilities. At the end of Erdős–Szekeres (1935), they pointed out that Kőnig's "infinity" lemma provides a "pure existence-proof" of the existence of the $N(n)$'s—an adumbration of later compactness arguments in graph theory.

In these early years Cantor was Erdős' hero, and his letters to friends ended with "let the spirit of Cantor be with you", soon shortened to "C with you".[9] Erdős' enthusiasm for Cantor had a substantive correlative, in that the infinite for Erdős was of a piece with the finite, particularly with propositions

---

[5] cf. Aigner and Ziegler (2013).

[6] (Turán, 2002, p.57).

[7] Eszter Klein, wife-to-be of Szekeres, had originally come up with a neat argument to show that $N(4)$ can be taken to be 5 and conjectured the general situation.

[8] For positive integers i, k, any sequence of $(i-1)(k-1)+1$ distinct integers has either an increasing sequence of i elements or a decreasing sequence of k elements.

[9] See Sós (2002) with letters from Erdős to Turán, specifically p.100,109,114: "... only the spirit of Cantor knows whether the theorem remains true or not"; "The spirit of Cantor was with me for some time during the last few days, the results of our encounters are the following:"; "... sad news, the spirit of Cantor took Landau ... The spirit of Cantor avoids me, yesterday I was thinking of number theory a lot, besides a few conjectures I had no success".

Nicolaas de Bruijn (2002) in reminiscences wrote: "Once, during a walk in 1954, I said that I wondered why he [Erdős] was such an excellent discoverer and solver of problems, and not a builder of theories. In a way it hit him, and he said that he would have liked so much to have been the first to discover Cantor's set theory."

parametrized according to cardinality to be entertained when those parameters are transfinite. The seamless transition from the finite to the transfinite was very much a part of the "spirit of Cantor",[10] and Erdős was the first prominent mathematician to engage counting and mathematical concepts over a broad range as *intrinsically* involving the infinite. Set theory was thus an inherent part of Erdős' field of play, not only the transfinite cardinals but infinite structures generally.[11] In this operative engagement, there was no particular difference in "ontological commitment" between the finite and the infinite. Moreover, set theory was enriched and influenced in its development by Erdős' initiatives from the finite. Infinite parametrization appeared early in Erdős' work, starting with the Menger-theorem and Euler-path results. Both propositions applied in the two proofs of Erdős–Szekeres (1935) would soon be extended into the transfinite.

By 1934, Erdős was in correspondence with mathematicians in England, the prominent Louis Mordell and the young Harold Davenport and Richard Rado.[12] On finishing university, Erdős took up research fellowships for four years arranged by Mordell at Manchester, where he was bringing in many emigré mathematicians. During this period, Erdős established his *modus operandi* of driving mathematics through cycles of problem, proof, and conjecture, ever punctuating and parametrizing mathematical concepts and procedures, and drawing in collaborators through increasing travel and interaction. He started to generate articles, almost all on number theory then, at a prodigious rate, a rate that would only double in the decades to come.

In 1938, Erdős took up a fellowship at the recently established Institute for Advanced Study at Princeton in the United States, and with the war in Europe he would not return for the next decade. During this period, Erdős settled into his *modus vivendi* of itinerant travel, having no fixed residence but traveling to do mathematics with an ever increasing array of collaborators. He continued to generate many articles in number theory, but now some in the constructive and interpolation theory of polynomials, and soon, in set theory.

§2. Transfinite Ramsey Theory. In the 1940s, Erdős began in earnest to consider infinite parametrizations, this naturally in the open-ended framework of graph theory. After securing initial footholds, he increasingly took

[10]Michael Hallett (1984) emphasized this, as Cantor's "finitism".

[11]Paul Bateman (2002) in reminiscences wrote, ingenuously: "Another early paper of mine which owes a lot to Erdős is 'A remark on finite groups,' *Amer. Math. Monthly*, **57** (1950), 623–624; after I had obtained the assertion of the paper for a denumerable group, Erdős pointed out to me that my proof worked for an infinite group if I merely used the concept of a limit ordinal."

[12]Rado, a student of Schur at Berlin, had emigrated to England. Erdős [110], on his joint work with Rado, wrote: "In one of my first letters to Richard early in 1934, I posed the following question: Let S be an infinite set of power m. Split the countable subsets of S into two classes. Is it true that there always exists an infinite subset S_1 of S all of whose countable subsets are in the same class? This, if true, would be a far reaching generalization of Ramsey's theorem. Almost by return mail, Rado found the now well-known counterexample using the axiom of choice." See Section 5.

on the transfinite landscape as the setting for intrinsically interesting problems. As a result, the transfinite became newly elaborated and articulated, infinite sets and cardinals becoming differentiated by combinatorial features.

Erdős was involved in the first avowedly transfinite result of graph theory. Dushnik and Miller (1941) broke ground with: *For an infinite cardinal κ, every graph on κ vertices without an independent (i.e., pairwise nonadjacent) set of vertices of cardinality κ has a complete (i.e., all vertices adjacent) infinite subgraph.* As acknowledged (1941, n.6), Erdős provided the discerning argument for singular κ, this setting the precedent for his attention to singular cardinals.

Soon afterwards, Erdős in a seminal 1942 article [3] established formative results for transfinite Ramsey theory. For stating coming propositions succinctly, we affirm the "arrow" notation of the later, 1950s partition calculus.

For a set X of ordinals, $[X]^\gamma = \{y \subseteq X \mid y \text{ has order type } \gamma\}$. The "ordinary" partition relation

$$\beta \longrightarrow (\alpha)^\gamma_\delta$$

asserts that for any partition $f : [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ *homogeneous* for f, i.e., $|f``[H]^\gamma| \leq 1$. Colorfully put, for any coloring of the order type γ subsets of β with δ colors there is an $H \subseteq \beta$ of order type α all of whose order type γ subsets are of the same color. For the case $\delta = 2$, the elaborated, "unbalanced" relation

$$\beta \longrightarrow (\alpha_0, \alpha_1)^\gamma$$

asserts that for any $f : [\beta]^\gamma \to 2$, there is an $i < 2$ and an $H \in [\beta]^{\alpha_i}$ such that $f``[H]^\gamma = \{i\}$. Negations of such relations are indicated with a $\not\longrightarrow$ replacing the $\longrightarrow$.

The finite Ramsey Theorem asserts that for any $0 < r, k, m < \omega$, there is an $n < \omega$ such that

$$n \longrightarrow (m)^r_k\,,$$

with the least possibility for n, the extremal Ramsey number $R_r(m; k)$, still unknown in general. The infinite Ramsey Theorem asserts that for $0 < r, k < \omega$,

$$\aleph_0 \longrightarrow (\aleph_0)^r_k\,.$$

Finally, the Dushnik–Miller Theorem, translated from graphs, was the first, unbalanced extension of the infinite Ramsey Theorem: For infinite cardinals κ,

$$\kappa \longrightarrow (\kappa, \aleph_0)^2\,.$$

Erdős [3] newly established for infinite cardinals κ:

(a) $(2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa$, and

(b) If $2^\kappa = \kappa^+$, then $(\kappa^{++}) \longrightarrow (\kappa^{++}, \kappa^+)^2$,

(a) decisively incorporated cardinal exponentiation, and (b) modulated it to establish a sharpening of Dushnik–Miller. Erdős' argument for (b) actually showed

$$(2^\kappa)^+ \longrightarrow ((2^\kappa)^+, \kappa^+)^2$$

outright, a sharpening of (a) in the case of two colors.

(a) is the best possible in the sense that both $2^\kappa \not\to (3)^2_\kappa$ and $2^\kappa \not\to (\kappa^+)^2_2$.[13] Consequently, it is readily seen that in the transfinite the extremal possibilities, what the Ramsey numbers are, for superscript 2 has been solved with one swoop. The later "Erdős–Rado Theorem" would provide the extremal possibilities for all superscripts r; often, however, the term is often used to refer just to (a), the case that became basic to set theory through its many applications.

For both [3] results Erdős made inaugural use of the *ramification*, or tree, argument, an argument to become the signature method for getting homogeneous sets in the next several decades. In brief, suppose that $f : [X]^2 \to \delta$. Choose an "anchor" $a_0 \in X$; the sets $Q_\xi = \{b \mid f(\{a_0, b\}) = \xi\}$ partition the rest of X into δ parts. Next, for each $\xi < \delta$ choose an anchor $a_1^\xi \in Q_\xi$, and again partition the rest of Q_ξ according to what f does. At limit stages, take intersections of $\supset$-chains of sets, and if nonempty, continue again starting with an anchor in there. By this means, one generates a tree of sets under $\supset$. Note that the anchors corresponding to any $\supset$-chain form an *end-homogeneous* set, in that for anchors a, b, c appearing in that order, $f(\{a, b\}) = f(\{a, c\})$. One can check that the αth level of the tree has size at most $|\delta|^{|\alpha|}$, so that a sufficiently large $|X|$ ensures a substantial $\supset$-chain through the tree. Finally, with $|\delta|$ less than the size of the $\supset$-chain, the corresponding anchors can be thinned out to get a genuinely homogeneous set for f.

This is in the manner of a "pure existence proof", in that cardinality considerations alone provide for a homogeneous set which otherwise has no particular definition. Through his results and initiatives, and especially with his "probabilistic method" in number theory and graph theory, Erdős would make *nonconstructive existence arguments* conspicuous in mathematics as a matter of style and procedure, and this resonated in set theory through infinite cardinality.

It is a notable historical happenstance that Đuro Kurepa was one contextual step away from earlier establishing the Erdős [3] results for two colors.[14] As part of his penetrating work on partial orders, Kurepa (1939) had established a pivotal cardinal inequality for partial orders, his "fundamental relation". In 1950, Kurepa (1953) recast this relation for graphs, showing in effect: $(\mu^\nu)^+ \longrightarrow (\mu^+, \nu^+)^2$ for infinite cardinals μ and ν. With $\mu = 2^\kappa$ and $\nu = \kappa$, one has the Erdős $(2^\kappa)^+ \longrightarrow ((2^\kappa)^+, \kappa^+)^2$.

The deliberate appeal in (b) to $2^\kappa = \kappa^+$ was the first of Erdős' many appeals to instances of the Generalized Continuum Hypothesis (GCH) in his theorems. For Erdős it would be less about what is true, but what can be proved, how enough structuring would lead to neat theorems. For GCH

[13] $2^\kappa \not\to (3)^2_\kappa$ was pointed out by Erdős in [3] and actually accredited by him to Kurt Gödel in [5]; the quickly seen counterexample is $F : [{}^\kappa 2]^2 \to \kappa$ given by $F(\{f, g\}) =$ the least α such that $f(\alpha) \neq g(\alpha)$. $2^\kappa \not\to (\kappa^+)^2_2$ is attributable to Sierpiński (1933); the simply put, straightforward counterexample is $G : [{}^\kappa 2]^2 \to 2$ given by $G(\{f, g\}) = 0$ *iff* for the least α such that $f(\alpha) \neq g(\alpha)$, $f(\alpha) < g(\alpha)$.

[14] cf. the commentary by Stevo Todorcevic in (Kurepa, 1996, Sect. C), from which the following remarks are derived.

itself, there was actually a direct antecedent at its provenance: Hausdorff (1908) had first formulated GCH, and assumed it to establish that for every infinite cardinal κ there is a universal linear order, a linear order of size κ into which every linear order of size κ embeds. For CH, Wacław Sierpiński had recently brought out a monograph Sierpiński (1934a) on the Continuum Hypothesis (CH), and Kurt Gödel had recently established the relative consistency of GCH. Mathematical investigation had transmuted CH from a primordial hypothesis about cardinality to an enumeration principle for the reals. Continuing the conversation, Erdős readily used CH and GCH, come what may. In this Erdős anticipated and contributed to the predisposition to assume set-theoretic hypotheses to prove theorems, whether Martin's Axiom or large cardinals.

Also in 1942, Erdős with Shizuo Kakutani [5] provided a characterization of $\aleph_1$ in terms of graphs, and used this to show that CH is equivalent to the reals having a partition into countably many sets each consisting of rationally independent reals. And extending a result of Sierpiński (1934b), Erdős established the now well-acknowledged Erdős–Sierpiński Duality: *Assuming* CH*, there is a bijection of the reals into itself that interchanges the* (*Lebesgue*) *null sets with the* (*Baire*) *meager sets.* This appeared in the 1943 article "Some remarks on set theory" [6], the first of eventually eleven articles of that title, mostly co-authored, which recorded Erdős' ongoing set-theoretic problems, proofs, and conjectures (Section 4). It is still open whether having such a duality is consistent with $\neg$CH.

§**3. Inaccessible Cardinals.** Erdős' work most salient for the early development of set theory appeared in the concluding section of his 1943 joint article [4] with Alfred Tarski and was later elaborated in their 1961 article [32]. Tarski in set theory had done considerable work on cardinal numbers vis-à-vis the Axiom of Choice; was becoming known for his set-theoretic definition of truth; and in Sierpiński and Tarski (1930) and Tarski (1938) had studied the (strongly) inaccessible cardinals. Like Erdős, with the war Tarski had been itinerant in the United States.[15] In their [4] they first brought forth the inaccessible numbers as part of a fabric of wider set-theoretic issues, and this would foster the integration of large cardinal hypotheses into set theory.

A cardinal κ is *weakly inaccessible iff* it is a regular, uncountable limit cardinal, and is (*strongly*) *inaccessible iff* it is a regular, uncountable cardinal which is a *strong limit*: If $\lambda < \kappa$, then $2^\lambda < \kappa$. The concluding section of [4] dealt with inaccessible cardinals, but most of the paper had to do with fields of sets in which weakly inaccessible cardinals figured in a result notable for its modern resonance.

Proceeding in present parlance, for a partially ordered set $\langle P, \leq\rangle$, an $A \subseteq P$ is an *antichain iff* it consists of pairwise incompatible elements, i.e., for

[15] In a letter of 23 September 1940 to Turán ((Sós, 2002, p.133)), Erdős' father wrote about Erdős: "He travelled together with a Pole called Tarski, whose wife and children stayed still in Warsaw."

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

distinct $p, q \in A$, there is no r such that $r \leq p$ and $r \leq q$. The *Suslin number* $S(P)$ is the least cardinal κ such that there is *no* antichain of size κ. Erdős and Tarski, in different terms, characterized the Suslin numbers as follows: *A cardinal* κ *is regular and uncountable iff* $\kappa = S(P)$ *for some partially ordered* $\langle P, < \rangle$. In modern forcing, the Levy collapse of κ (to ω_1) is a canonical, universal example of such a partially ordered set, and Erdős and Tarski essentially gave this example two decades before the advent of forcing! This would surely be the first appearance of the Levy collapse, which in the substantive case of κ being weakly inaccessible is now standard fare for getting relative consistency results.[16] This illustrates the kind of prescient thinking involved in asking the "right questions" that became a hallmark of Erdős' initiatives.

The concluding section of [4], "General Remarks on Inaccessible Numbers", presented six problems involving inaccessible cardinals, some problems stating properties for the first time that would become enduring in the theory of large cardinals. The first three problems, related to investigations of Tarski, stated properties equivalent to either the now well-known strong compactness or measurability of cardinals. For the latter, the now-standard formulation emanating from the thesis of Stansław Ulam (1930) is that κ is *measurable iff* κ is uncountable and there is an ultrafilter over κ which is nonprincipal (contains no singletons) and is κ-complete (closed under intersections of fewer than κ sets).

The last three problems, evidently arising from Erdős' work, stated the properties:

(a) $\kappa \longrightarrow (\kappa)^2_2$.
(b) Every linearly ordered set of size κ has a subset of size κ which is either well-ordered by the ordering or well-ordered by the converse of the ordering.
(c) Every tree of height κ each of whose levels has size less than κ has a chain of size κ.

These several properties hold for $\kappa = \aleph_0$, and are evidently extensions to the transfinite of propositions from Erdős' earliest days. Erdős–Szekeres (1935) had the Ramsey Theorem and the ordered pigeon-hole principle of which (b) is a transfinite extension, and (c) is the direct generalization of Kőnig's infinity lemma. In 1934 Nathan Aronszajn had shown that there is a counterexample to (c) for $\kappa = \aleph_1$, and generally a counterexample at κ is now called a κ-*Aronszajn tree*.

(a) quickly implies (b), and by Erdős' ramification argument from his [3], (c) with κ already inaccessible implies (a). Also, that ramification argument can be effected assuming the measurability of κ, and so, the measurability of κ implies (a) and hence also (b) and (c). One sees here how Erdős was pursuing direct combinatorial generalizations from the finite and $\aleph_0$, the proofs being the engine.

[16](Kanamori, 2009, p.126ff).

Despite the various connections made, Erdős and Tarski could not ascertain the extent of these properties. They all, except for (c), imply the inaccessibility of κ, but could the inaccessibility of κ actually imply any of them? In considering their problems Erdős and Tarski took an open-ended, empirical approach to ostensibly strong propositions about sets and cardinals. They wrote (p.428ff):

> The difficulties which we meet in attempting to solve the problems under consideration do not seem to depend essentially on the nature of inaccessible numbers. In most cases the difficulties seem to arise from lack of devices which enables us to construct maximal sets which are closed under certain infinite operations. It is quite possible that a complete solution of these problems would require new axioms which would differ considerably in their character not only from the usual axioms of set theory, but also from those hypotheses whose inclusion among the axioms has previously been discussed in the literature and mentioned previously in this paper (e.g., the existential axioms which secure the existence of inaccessible numbers, or from hypotheses like that of Cantor which establish arithmetical relations between the cardinal numbers).

In the years hence, Tarski, ensconced at the University of California at Berkeley, worked broadly across mathematical logic, and Erdős, ever itinerant, pursued mathematics across a broad range, mostly number theory but also the development of the partition calculus in set theory. Erdős for his part would incorporate inaccessible cardinals and measure into his problems and proofs, this in ways that stimulated important developments. In some articles (Sections 6 and 7), he simply took on a central question of [4] as a hypothesis $(**)$, that all inaccessible cardinals are measurable (!), to push induction through such cardinals, and this led to significant results about measurability. In his second "Some remarks on set theory" article [11], he addressed a question raised by Ulam ("oral communication") by presenting a joint observation with Leonidas Alaoglu: *If κ is less than the least inaccessible cardinal, then one cannot have a family of $\aleph_0$ countably additive $\{0,1\}$-valued measures defined for the subsets of κ (with singletons measured 0 and κ itself measured 1) such that every subset of κ is measured by at least one of the measures.* The Erdős–Alaoglu Theorem would be seminal for a wide range of developments in set theory about such weakenings of having a measurable cardinal.[17] In a prominent investigation of rings of continuous functions with Leonard Gillman and Melvin Hendriksen [16], Erdős developed a useful characterization of certain real-closed fields of size less than the least measurable cardinal.[18]

[17]With forcing, Prikry (1972) established the relative consistency of the above proposition with both $\aleph_0$ and κ replaced by $\aleph_1$. See Taylor (1980) for subsequent developments under the rubric of "Ulam's problem".

[18]For X a completely regular space, $C(X)$ the ring of continuous real-valued functions on X, and M a maximal ideal over $C(X)$, $C(X)/M$ is a real-closed field. For discrete X of

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

Latterly in 1958–9, the issues in that concluding section of [4], set aside for so many years, were revisited and elaborated in a seminar conducted at Berkeley by Tarski with his first student from Poland, Andrzej Mostowski. The propositions corresponding to the six problems were elaborated, and implications among them, only announced in a footnote to [4], were worked out, all this soon to appear in a new joint Erdős–Tarski article [32].

By the mid-1960s, it would become well-known that properties (a), (b), and (c) together with inaccessibility each characterize the large cardinal property of the *weak compactness* of κ, and (c) would become very well-known as the *tree property* for κ, that there are no κ-Aronszajn trees, a substantial large cardinal property that could consistently hold at accessible cardinals. There are many weakly compact cardinals below a measurable cardinal, and there are many inaccessible cardinals below a weakly compact cardinal.[19] It is remarkable that propositions from Erdős' earliest days pursued by him into the transfinite would become such prominent large cardinal hypotheses.

§4. Free Sets for Set Mappings. With free sets for set mappings, Erdős developed a particular set-theoretic theme, through cycles of problem, proof, and conjecture, that would become substantive in itself and in connection with the later partition calculus. His work in this direction started in 1940; eventually continued with his set-theoretic collaborator András Hajnal, and set the ground for later work of set theorists with and without forcing. Free sets for set mappings have since become in and of themselves a significant part of combinatorial set theory. In what follows, we chronicle Erdős' work on set mappings and subsequent developments, particularly to illustrate his dynamic engagement with a theme through several articles and ramifications.

A function $f : X \to \mathcal{P}(X)$ from some set X into its power set is a *set mapping iff* $x \notin f(x)$ for every $x \in X$. Such a function is *of order* λ *iff* $|f(x)| < \lambda$ for every $x \in X$. In terms of graphs, a set mapping of order λ amounts to a loop-free directed graph having out-degrees all less than λ. Finally, $S \subseteq X$ is *free* (or "independent" in the early articles) for f *iff* for any $x, y \in S$, $y \notin f(x)$. Turán in 1935 originally asked, in interpolation theory, whether there are infinite sets free for set mappings on the unit interval of order $\aleph_0$. After it was shown that there are in fact size $2^{\aleph_0}$ such free sets, Stanisław Ruziewicz in 1936 conjectured: *For cardinals* $\lambda < \kappa$ *with* κ *infinite, any set mapping* : $\kappa \to \mathcal{P}(\kappa)$ *of order* λ *has a free subset of size* κ.

After several partial results by others, Erdős in 1940 established this under GCH.[20] The appeal to GCH was a typically blanket one; only the case of κ being singular had been left, and Erdős established it with the assumption

size less than the least measurable cardinal, Erdős (p.550) characterized those M such that $C(X)/M$ properly contains the reals.

[19]See (Kanamori, 2009, Sects. 4 and 7) for the foregoing large cardinal theory.

[20]cf. (Sós, 2002, p.133f). Erdős' father wrote to Turán that Erdős proved this on 25 August 1940, and that "[i]t won Gödel's highest appreciation."

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

$\lambda^{\text{cf}(\kappa)} < \kappa$. Erdős mentioned his result at the end of his 1942 article [3] and his proof was collected into his 1950, 2nd "Some remarks on set theory" article [11].

In [11], Erdős also considered (p.137) set mappings of finite order. He observed that if a set mapping is on a finite set X and has order $k \in \omega$, then (with $|X| \geq 2k - 1$) X is the disjoint union of $2k - 1$ free sets. He astutely noted that this then holds for countable X by the Kőnig infinity lemma and that he had conjectured that it would hold for all X as a consequence of a compactness assertion, one then proved by Nicolaas de Bruijn. A set mapping on a set X rendering it a disjoint union of r free sets corresponds to a graph on X having an r-*coloring*, i.e., a labeling of its vertices by r colors so that no adjacent vertices get the same color. de Bruijn proved: *If every finite subgraph of a graph G has an r-coloring, then so does G itself.* In their joint [12], these various results were described.

Compactness arguments soon became common fare in graph theory as a bridge from the finite to the infinite, very much in the spirit of Erdős' initiatives toward the infinite. Today, the de Bruijn theorem is viewed as a simple consequence of the Compactness Theorem for Propositional Logic. Be that as it may, it is notable that a specific problem about set mappings stimulated a compactness strategy, and that there seems no way to get to the specific result about $2k - 1$ free sets other than by passing from the finite through compactness.

Erdős continued in his "Some remarks on set theory" series with free sets for set mappings. In the 3rd "remarks" article [15], Erdős returned to the original Turán context and initiated a new direction by considering measure and category. Erdős showed e.g., that if f is a set mapping on the reals such that $f(x)$ is always nowhere dense, then there is an infinite free set, and leadingly mentioned that he was "unable to establish a stronger conclusion." In the 5th "remarks" [20], with Géza Fodor, Erdős considered set-theoretic, parametrized variations. In the 6th "remarks" [21], also with Fodor, Erdős continued with measure and category, weaving in weakly inaccessible cardinals. The last theorem therein extended Erdős' GCH result on Ruziewicz's conjecture, and notably the singular case is attributed to Hajnal.

Erdős visited Hungary for the first time after the war in 1955–6, and at the University of Szeged he met Fodor and Hajnal, the latter then a student there of Kalmár. Hajnal would soon become Erdős' main collaborator in set theory, with the second largest number of joint papers with him.[21] With more to be said about their early collaboration below (Section 6), we mention here that the 8th "remarks" article [29], joint with Hajnal, continued with set mappings involving measure and category. Soon afterwards in 1960, Hajnal (1961) proved Ruziewicz's conjecture outright, not conditional on GCH, so that it is now the Hajnal Set Mapping Theorem. This theorem would stand as a landmark and find significant applications e.g., in a proof of the Galvin–Hajnal extension (Galvin and Hajnal, 1975, p.497) of Silver's Theorem on singular cardinals (Section 10).

[21]Hajnal had 56 joint papers with Erdős, and András Sárközy, 62.

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

Over a decade later, nowhere-dense set mappings, i.e., set mappings f on the reals such that $f(x)$ is always nowhere dense, would again be taken up, with Erdős' [15] result that they have infinite free sets the benchmark. Frederick Bagemihl (1973) showed that they have everywhere dense free sets. The 8th "remarks" [29] had raised the question of whether they have uncountable free sets. Stephen Hechler (1972) observed that assuming CH, there is a set mapping f on the reals with no uncountable free sets such that $f(x)$ is an ω-sequence converging to x (so quite nowhere dense). Later Uri Abraham (1981) showed that Martin's Axiom MA_{ω_1} is consistent with all nowhere-dense set mappings having uncountable free sets.

In his ceaseless questing, Erdős himself with collaborators would take up the motif of set mappings in the later light of forcing and large cardinals. In the 1973 [71], with Hajnal and Attila Máté, structural restrictions are imposed on the range of set mappings, in a way typical for Erdős, and various results are thereby achieved, e.g., with Martin's Axiom, as well as a characterization of weak compactness under $V = L$. In the 1974, 11th and last "remarks" article [77], with Hajnal, it is shown that *for uncountable κ, κ is weakly compact iff whenever $F \subseteq [\kappa]^{<\kappa}$ with $|F| = \kappa$ and $x \not\subseteq y$ for distinct $x, y \in F$, there is a $G \subseteq F$ with $|G| = \kappa$ such that $|\kappa - \bigcup G| = \kappa$.*

Still later, Chris Freiling (1986) in the mid-1980s considered "axioms of symmetry" based on intuitions "about throwing darts at the real number line". Whether couched in new terms and philosophical rationales, these axioms were but propositions once again about set mappings and free sets. For example, his first axiom $\mathrm{A}_{\aleph_0}$ amounts to: Every set mapping on the reals of order $\aleph_1$ has a free set of size two. Freiling showed that $\mathrm{A}_{\aleph_0}$ is equivalent to $\neg$CH, but the simple arguments had been traversed long before by Erdős e.g., in his [15].[22] One sees in Freiling's further axioms and arguments more opaque interplay with Erdős' early work. Such eternal returns corroborate the significance of astutely formulated mathematical concepts.

§5. Erdős–Rado Partition Calculus. The partition calculus, an extension of Erdős' initial work on transfinite Ramsey theory (Section 2), is the most conspicuous and significant subject in set theory to result from his initiatives. Rado was Erdős' main collaborator in this direction in the 1950s, and Hajnal, in the 1960s. As in the finite, Erdős pursued increasingly parametrized problems in the direction of transfinite partitions and homogeneous sets, and this led for quite some time to self-fueling developments. When these gained a new significance in connection with strong hypotheses broached in the 1943 Erdős–Tarski work (Section 3), the partition calculus achieved a permanent place of prominence in set theory. In what follows, relatively few results,

[22]If $\langle r_\alpha \mid \alpha < \omega_1 \rangle$ well-orders the reals, then the set mapping f on the reals given by $f(r_\alpha) = \{r_\beta \mid \beta < \alpha\}$ has no free set of size two. (cf. [15, thm.1].) Conversely, suppose that CH fails and f is a set mapping on the reals of order $\aleph_1$. Let A be a set of reals of size $\aleph_1$. Then $\bigcup f``A$ is a set of size at most $\aleph_1$, so let r be a real not in this set. Since $f(r)$ is at most countable, there is an $s \in A$ such that $s \notin f(r)$. Hence, $\{r, s\}$ is a free set of size two (cf. [15, thm.4]).

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

details, and ramifications are given in favor of imparting the historical thrust, and this necessarily belies the extent of the theory developed.

Already in his final university year 1934, Erdős had asked about the possibility, in the later arrow notation, of $\kappa \longrightarrow (\aleph_0)^{\omega}_2$ as a "far reaching generalization of Ramsey's theorem", and Rado had responded forthwith with a counterexample using the Axiom of Choice (AC).[23] This would delimit their further work on the partition calculus. Also, as with several AC constructions, much would be done getting positive such relations if one restricts to e.g., Borel functions. Finally, strong extensions would become pivotal in the investigation of the Axiom of Determinacy.[24]

Erdős and Rado started their collaborative work in earnest in 1950, when they coincided in London.[25] They would ultimately be involved in 18 joint articles, and the first [10] was on an extension of Ramsey's Theorem $\aleph_0 \longrightarrow (\aleph_0)^r_k$. Allowing the number of colors k to be infinite, they established a self-refinement, the Canonical Ramsey Theorem, which would foreshadow a wide range of such Ramsey-type results.[26]

In their 1952, broad-ranging [13], Erdős and Rado articulated and expanded the emerging Ramsey theory with better bounds for the (finite) Ramsey numbers, consideration of real and rational order types, and, at the end, delimitations to possible transfinite generalizations. After presenting the 1934 $\kappa \not\longrightarrow (\aleph_0)^{\omega}_2$, they presciently considered the still possible relations for partitions of all finite subsets. For a set X of ordinals, $[X]^{<\omega} = \bigcup_{r\in\omega}[X]^r$, the set of finite subsets of X. The arrow notation

$$\beta \longrightarrow (\alpha)^{<\omega}_{\delta}$$

asserts that for any partition $f : [\beta]^{<\omega} \to \delta$, there is an $H \in [\beta]^{\alpha}$ *homogeneous* for f, i.e., for every $r \in \omega$, $|f``[H]^r| \leq 1$. Erdős and Rado (p.418) asked whether for infinite κ, $\kappa \not\longrightarrow (\aleph_0)^{<\omega}_2$, and observed (p.435f) that this holds for $\kappa = \aleph_0$ and $\kappa = 2^{\aleph_0}$. Within a decade, partitions of all finite subsets would figure centrally in set theory when it became infused with emerging model-theoretic techniques.

In their succeeding [14], Erdős and Rado first broached the arrow notation that we have been using with "$a \longrightarrow (b_1, b_2)^2$" as a "convenient abbreviation", this for application to a positive result about linearly ordered sets

[23] See the last footnote of Section 1. Rado's argument:

Let $\prec$ well-order $[\kappa]^{\omega}$ and define $f : [\kappa]^{\omega} \to 2$ by $f(s) = 0$ *iff* every $t \in [s^{\omega}] - \{s\}$ satisfies $s \prec t$. Then no $x \in [\kappa]^{\omega}$ can be homogeneous for f: If y is the $\prec$-least member of $[x]^{\omega}$, then $f(y) = 0$. However, for any infinite $\subset$-increasing chain $x_0 \subset x_1 \subset x_2 \subset \cdots$ in $[x]^{\omega}$, $f(x_n) = 0$ for every $n \in \omega$ would imply that $\cdots \prec x_2 \prec x_1 \prec x_0$, contrary to $\prec$ being a well-ordering.

[24] cf. (Kanamori, 2009, p.382,432ff).

[25] Rado was at King's College London, and Erdős spent the year at University College London.

[26] The Canonical Ramsey Theorem asserts that for any $0 < r < \omega$ and $f : [\omega]^r \to k$ with k possibly infinite, there is an infinite $H \subseteq \omega$ and a $v \subseteq r$ such that whenever $x_0 < x_1 < \cdots < x_{r-1}$ and $y_0 < y_1 < \cdots < y_{r-1}$ are all in H, $f(\{x_0, x_1, \ldots, x_{r-1}\}) = f(\{y_0, y_1, \ldots, y_{r-1}\})$ *iff* $x_i = y_i$ for $i \in v$. It is evident that if k is finite, then v must be empty so that H is homogeneous for f, and so one does indeed have a selfrefinement.

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

having large sets well-ordered by the ordering or by the converse of the ordering. This can be seen as an accessible version of the ordering problem of the 1943 Erdős–Tarski article (cf. (b) of Section 3), one ultimately having finite provenance in the seminal Erdős–Szekeres (1935).

The past to be prologue, Erdős and Rado in their 1956, 60-page "A partition calculus in set theory" [19] comprehensively set out the emergent theory in the broad context of order types and with their arrow notation now in full parametrization. After incorporating the previous results and establishing basic connections among the elaborated partition relations, they settled first into the study of countable order types and then of the real order type. The investigations initiated here, especially of countable ordinals, would stimulate a cottage industry of work to the present day.

Taking on the parameter r of "r-tuples", they (p.467f) with ramification established the first instance of a "positive stepping up lemma", which shows how a positive partition relation for r-tuples leads to one for $(r+1)$-tuples. With this, they extended the Erdős [3] result $(2^\kappa)^+ \longrightarrow (\kappa^+)^2_\kappa$ from 2-tuples to r-tuples. They formulated their result with a GCH-type hypothesis, but to proceed without, let $\beth_0(\kappa) = \kappa$, and $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$. Then we have the *Erdős–Rado Theorem*:

$$\text{For infinite cardinals } \kappa \text{ and } r \in \omega, \quad \beth_r(\kappa)^+ \longrightarrow (\kappa^+)^{r+1}_\kappa.$$

This is extremal, in that $\beth_r(\kappa)^+$ cannot be replaced by any smaller cardinal. This was subsequently shown by Hajnal in 1957, using "a negative stepping up lemma" starting from $2^\kappa \not\longrightarrow (\kappa^+)^2_2$.[27] Erdős and Rado [19, p.464ff] had established the first instance of such a lemma, from which one can show the optimality of $\beth_r(\kappa)^+$, but only assuming GCH. In any case, the Erdős–Rado Theorem, definitive in providing the exact Ramsey numbers for the transfinite, would henceforth become a mainstay of set theory.

Erdős and Rado at the end of their [19] introduced the *polarized partition relation*. In a simple case,

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} \longrightarrow \begin{pmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\end{pmatrix}$$

asserts that for any partition $f : \alpha \times \beta \to 2$, there is an $i < 2$ and sets $A \subseteq \alpha$ and $B \subseteq \beta$ with order types α_i and β_i respectively such that $f``[A \times B] = \{i\}$. In terms of graphs, this is an assertion about partitions of a complete bipartite graph of a certain sort having a complete bipartite subgraph of specified sort in one of the parts. Erdős and Rado showed that this is a distinctive relation of separate interest. For example, they proved that

$$\begin{pmatrix}\aleph_0\\ \aleph_1\end{pmatrix} \longrightarrow \begin{pmatrix}\aleph_0 & \aleph_0\\ \aleph_1 & \aleph_0\end{pmatrix}$$

and noted that Sierpiński had in effect established with CH that

$$\begin{pmatrix}\aleph_0\\ \aleph_1\end{pmatrix} \not\longrightarrow \begin{pmatrix}\aleph_0 & \aleph_0\\ \aleph_1 & \aleph_1\end{pmatrix}.$$

[27] (Hajnal, 1997, p.361).

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

With all these various results, Erdős and Rado's [19] established the partition calculus as a new combinatorics of the transfinite, a topic that newly informed and variegated the Cantorian terrain of infinite cardinals and order types.

§**6. Free Sets, Ramsey and Erdős Cardinals.** It would be Erdős' *first* joint work with Hajnal, bearing on partitions of all finite sets, that would veer closest to central developments of the 1960s in set theory, these being in the investigation of large cardinal hypotheses. Erdős and Hajnal provided the context, spurred the possibilities, and got enticingly close to a transformative result. In what follows, we pursue an arc that begins at their first joint article, goes through some subsequent mainstream set theory results, and then drops back to the topic of that article, now newly seen.

As mentioned in Section 4, Hajnal collaborated with Erdős on set mappings. Actually, their first joint work was on set mappings of high "type", a topic broached by Hajnal in their first encounter.[28] With $[\kappa]^{<\lambda} = \{y \subseteq \kappa \mid |y| < \lambda\}$, a function $f : [\kappa]^{\mu} \rightarrow [\kappa]^{<\lambda}$ satisfying $f(s) \cap s = \emptyset$ for every $s \in [\kappa]^{\mu}$ is said to be a *set mapping of order* λ *and type* μ, and a set $S \subseteq \kappa$ is *free for* f *iff* $f(s) \cap S = \emptyset$ for every $s \in [S]^{\mu}$. μ is thus a new "type" parameter, with $\mu = 1$ corresponding to the former set mappings. Erdős saw the applicability of the Erdős–Rado Theorem to finite-type set mappings, and he and Hajnal in their [22] worked out when there would be large free sets, freely invoking GCH to get orderly results. Notable was that, working up to inaccessible cardinals, they invoked a hypothesis $(**)$: inaccessible cardinals are measurable (!). With his experience with measures, Erdős readily pushed through inaccessibility here by using a two-valued measure as given by measurability.

A crucial connection was soon made to the Erdős–Rado [19] problem of whether for infinite κ, $\kappa \not\rightarrow (\aleph_0)_2^{<\omega}$. Set mappings of type less than ω were seen to be closely connected to partitions of all finite sets, and Erdős and Hajnal [22] got (theorem 9a), with underlying hypothesis $(**)$, a counterexample, one that translates to: *If* κ *is measurable, then* $\kappa \longrightarrow (\kappa)_2^{<\omega}$. This was the first new brick inserted into the edifice of "problems" erected by the 1943 Erdős–Tarski article [4]. As a historical happenstance from this result, cardinals satisfying $\kappa \longrightarrow (\kappa)_2^{<\omega}$ are now known as *Ramsey*. On the other hand, for any ordinal α the least κ satisfying $\kappa \longrightarrow (\alpha)_2^{<\omega}$ is the *Erdős cardinal* $\kappa(\alpha)$, so that the solecism "$\kappa(\alpha)$ exists" amounts to asserting that there is some λ satisfying $\lambda \longrightarrow (\alpha)_2^{<\omega}$. Ramsey cardinals are just the fixed points of Erdős cardinals.

Soon after that 1958–9 Berkeley seminar on the Erdős–Tarski work (Section 3), William Hanf, a student of Tarski, established a result transformative for the theory. He showed that a weakly compact cardinal has below it, applying infinitary languages and their compactness (and hence the term), many inaccessible cardinals in a strong hierarchical sense. At a 1960 conference,

[28] (Hajnal, 1997, Sects. 8 and 9) informatively discusses that encounter and their work then.

Tarski (1962) pointed out the implications, e.g., that *a fortiori* the least measurable cardinal has, after all, a wide class of inaccessible cardinals below it, and H. Jerome Keisler (1962) sketched how the recently developed theory of ultraproducts can be applied to get Hanf's hierarchical results.

Within a year, having heard about these dramatic advances, Erdős and Hajnal [36] themselves pointed out that the least inaccessible cardinal not being measurable could have easily been seen by 1958, when they were using the countervailing $(**)$: (1) measurable cardinals are Ramsey ([22], theorem 9a); (2) the least inaccessible cardinal t_1 is at most $\kappa(\omega)$ ([22], theorem 9b); and (3) in general terms $\omega \leq \alpha < \beta$ implies that $\kappa(\alpha) < \kappa(\beta)$ by a simple argument ([36], theorem 3). Quite a missed opportunity! Had they come to this in 1958, they would have showcased Ramsey cardinals and contextually set out beforehand the combinatorial underpinnings of coming results. As things transpired, once the large cardinals "problems" from the 1943 Erdős–Tarski work were hierarchically systematized, Ramsey and Erdős cardinals would nonetheless figure in central advances made concerning Gödel's constructible universe L. We summarize these, in the briefest of terms, vis-à-vis the cardinals:[29]

Gödel's construction of the inner model L through which he established the relative consistency of GCH stood as a high watermark for set theory for over two decades. In 1961, Dana Scott, taking an ultrapower of the universe V, dramatically established that *if there is a measurable cardinal, then* $V \neq L$. Then Frederick Rowbottom in his 1964 thesis established that partition properties alone provide the model-theoretic means to establish that V and L are locally far apart. For example, *if* κ *is Ramsey, then for any infinite* $\lambda < \kappa$ *there are just* λ *many subsets of* λ *in* L, and e.g., *if* $\kappa(\omega_1)$ *exists, then there are just countably many subsets of* ω *in* L. He also showed that with Scott's notion of a normal ultrafilter over a measurable cardinal, measurability implies Ramseyness intrinsically in that homogeneous sets can always found in the normal ultrafilter. This led in particular to the result that Ramseyness is strictly weaker than measurability. In 1964, Hajnal lectured at Berkeley on the partition calculus, including $\kappa \longrightarrow (\lambda)_2^{<\omega}$, with Jack Silver in the audience.[30] In his 1966 thesis, Silver got to the essence of the transcendence over L by showing that having $\kappa(\omega_1)$ implies the existence of a closed unbounded class of indiscernibles for L, i.e., any two increasing n-tuples from the class satisfy the same n-free-variable formulas over L. The corresponding theory can be coded by a set of integers, the Silver-Solovay set $0^\sharp$, the existence of which is then tantamount to having a proper class of indiscernibles with which L can be uniformly generated. With $\kappa(\omega_1)$ thus enthroned, the Erdős cardinals gained in importance vis-à-vis L, and Silver showed that if $\kappa \longrightarrow (\alpha)_2^{<\omega}$ and $\alpha < \omega_1^L$ (the least uncountable cardinal in the sense of L), then κ has that same property in the sense of L—so that ω_1 is a sharp divide for transcendence over L. A decade later, "generalized Erdős cardinals" sensitive to the corresponding

[29] See Kanamori (2009), mainly chapter 2, for details and references.
[30] (Hajnal, 1997, p.362).

theories of indiscernibles were developed in Baumgartner–Galvin (1978) and contextually sharp implications provided for $0^\sharp$ itself.

All in all, it is remarkable that Erdős' early speculations about partitions of all finite sets as a combinatorial problem became transmuted to central concerns of set theory with the infusion of model-theoretic techniques. Notably, the higher-type set mappings of that first Erdős–Hajnal paper themselves resurfaced in the early 1970s, in the new light. James Baumgartner in his thesis Baumgartner (1970) showed that if $V = L$, then every set mapping $f : [\kappa]^{<\omega} \to \kappa$ has an infinite free set exactly when in fact $\kappa \longrightarrow (\aleph_0)_2^{<\omega}$. Then Devlin and Paris (1973) showed that just having free subsets can provide indiscernibles, and in particular that if every set mapping $f : [\kappa]^{<\omega} \to [\kappa]^{<\omega}$ has an uncountable free set, then $0^\sharp$ exists.

A decade later, an arc was completed back to 1956. On the very first day that they had met, Erdős and Hajnal had come up with their first joint problem, the plausibility of: Every set mapping: $[\aleph_\omega]^{<\omega} \to \aleph_\omega$ has an infinite free set.[31] Peter Koepke in his thesis (cf. Koepke (1984)) proved that this proposition is actually equi-consistent with the existence of a measurable cardinal.

§7. **Property $\mathcal{B}$.** Erdős and Hajnal's second major article [30] investigated a property of an infinite family of sets, having a set "cut through" it, of evident significance. Their contextualizing work established sharp results and raised basic issues; informed on topological compactness and stimulated interest in combinatorial compactness; and soon inspired a finite counterpart.

A family $\mathcal{F}$ of sets has the property $\mathcal{B}$ *iff* there is a set B such that $F \cap B \neq \emptyset$ and $F \not\subseteq B$ for every $F \in \mathcal{F}$. By happenstance, Erdős took up an old paper of Edwin Miller (1937) on this property, and stimulated by possibilities in the transfinite, he and Hajnal made an incisive study.[32] In a formulation essentially as in their [30], for $\kappa \leq \lambda$, $M(\lambda, \kappa, \mu) \longrightarrow \mathcal{B}$ asserts that whenever $\mathcal{F}$ is a family of λ sets each of size κ which is μ-almost disjoint (i.e., $|X \cap Y| < \mu$ for distinct $X, Y \in \mathcal{F}$), $\mathcal{F}$ has the property $\mathcal{B}$.

Miller had coined "Property $\mathcal{B}$" in honor of Felix Bernstein, who in 1908 had made conspicuous use of the Axiom of Choice to show that the family of perfect sets of reals has the Property $\mathcal{B}$, thus affirming that uncountable sets of reals do not necessarily have perfect subsets. Bernstein enumerated the $2^{\aleph_0}$ perfect sets of reals and recursively chose from each both a real in and a real out. By this argument, $M(\kappa, \kappa, \kappa^+) \longrightarrow \mathcal{B}$ for any κ, the κ^+ here signaling a vacuous almost-disjointness condition. On the other hand, any $\mathcal{F}$ consisting of pairwise disjoint sets each having at least two members trivially has the property $\mathcal{B}$. Focusing on the degree of almost disjointness, Miller proved that $M(2^{\aleph_0}, \aleph_0, \aleph_0) \not\longrightarrow \mathcal{B}$ while for any λ and $n \in \omega$, $M(\lambda, \aleph_0, n) \longrightarrow \mathcal{B}$. For the latter result, Miller proceeded by induction on the cardinality λ, constructing a cutting set B by what can be now be seen as an elementary chain construction.

[31] (Hajnal, 1997, p.378).

[32] (Hajnal, 1997, p.370f).

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

Erdős and Hajnal [30] generalized the Miller construction to get positive results for families of uncountable sets, the next-level case being that under CH, $M(\lambda, \aleph_1, \aleph_0) \longrightarrow \mathcal{B}$ for any $\lambda \leq \aleph_\omega$, with the inductive argument relying on $\aleph_n^{\aleph_0} = \aleph_n$, which breaks down at $\aleph_\omega^{\aleph_0} > \aleph_\omega$. Again, the cycle of problem, proof, and conjecture would kick in, here with modern set theory eventually taking up the challenge. As set out by Hajnal, István Juhász and Saharon Shelah in their Hajnal *et al.* (1986, 2000), a range of results clarified the situation and showed in particular that *if* $V = L$*, then* $M(\lambda, \aleph_1, \aleph_0) \longrightarrow \mathcal{B}$ *for every* λ, yet *if there is a supercompact cardinal, then in a forcing extension* $M(\aleph_{\omega+1}, \aleph_1, \aleph_0) \not\longrightarrow \mathcal{B}$. Very recently, Kojman (2015), in light of Shelah's celebrated *pcf* theory and revised GCH, established strong ZFC theorems extending $M(\lambda, \kappa, \mu) \longrightarrow \mathcal{B}$ in various directions, particularly to all μ and sufficiently large κ relative to μ.

Ever parametrizing, Erdős further considered $M(\lambda, \kappa, \mu) \longrightarrow \mathcal{B}(s)$, that the requisite set B moreover satisfy $0 < |F \cap B| < s$ for every $F \in \mathcal{F}$. Surprisingly, Erdős and Hajnal [30] proved under GCH the sharp results that for $r, n \in \omega$, $M(\aleph_{\alpha+n}, \aleph_\alpha, r) \longrightarrow \mathcal{B}((r-1)(n+1)+2)$ yet $M(\aleph_{\alpha+n}, \aleph_\alpha, r) \not\longrightarrow \mathcal{B}((r-1)(n+1)+1)$. With this they were able to inform on compactness in the just-developing set-theoretic topology. A topological space is κ*-compact iff* every family of closed sets with empty intersection has a subfamily of size less than κ with empty intersection. In particular, the $\aleph_0$-compact spaces are the compact spaces and the $\aleph_1$-compact spaces are the Lindelöf spaces. $T(\lambda, \nu) \longrightarrow \kappa$ asserts that the product of λ discrete ν-compact spaces is κ-compact. In particular, Tychonoff's Theorem, equivalent to the Axiom of Choice, asserts that $T(\lambda, \aleph_0) \longrightarrow \aleph_0$ for every λ. Erdős and Hajnal pointed out that under GCH, $T(\aleph_{\alpha+n}, \aleph_{\alpha+1}) \not\longrightarrow \aleph_{\alpha+n}$ for every α and $n \in \omega$. In brief, they took a family $\mathcal{F}$ affirming $M(\aleph_{\alpha+n}, \aleph_\alpha, r) \not\longrightarrow \mathcal{B}((r-1)(n+1)+1)$ and used $M(\aleph_{\alpha+n-1}, \aleph_\alpha, r) \longrightarrow \mathcal{B}((r-1)n+2)$ to show that the topological product of the members of $\mathcal{F}$ construed as discrete spaces affirms $T(\aleph_{\alpha+n}, \aleph_{\alpha+1}) \not\longrightarrow \aleph_{\alpha+n}$.

On the topic of compactness, Erdős and Hajnal next made deductions notable for both approach and result. Invoking the hypothesis $(**)$ from their [22], that inaccessible cardinals are measurable (!), they established for such κ that $T(\kappa, \kappa) \longrightarrow \kappa$. Specifically, they used the inaccessibility of κ together with the Erdős 1943 property of trees having long chains ((c) of Section 3). As with their previous appeal to the false $(**)$, one sees the content, here that *if* κ *is measurable then* $T(\kappa, \kappa) \longrightarrow \kappa$. Jerzy Łoś (1959) had recently shown that if λ is less than the least measurable cardinal, then $T(\lambda^+, \aleph_1) \not\longrightarrow \lambda$. Hence, for any λ less than the least measurable cardinal there are products of Lindelöf spaces which are not λ-compact, while if κ is measurable, then every product of κ Lindelöf spaces is κ-compact.

Erdős and Hajnal [30] offered up a wide range of problems. One was whether $T(\aleph_\omega, \aleph_1) \longrightarrow \aleph_\omega$, which they [34] soon showed to be false. Another had to do with graphs of size $\aleph_2$ and compactness of chromatic number, which they answered in the negative (Section 11). The final problems of [30] had to do with the possibilities for property $\mathcal{B}$ in the finite. Erdős himself would in subsequent papers Erdős (1963, 1964, 1969), initiate the finite theory, the focus being mainly on the extremal function $m(n)$ = the least m such that any family of m sets each of size n does not have property $\mathcal{B}$. A lingering question is whether there is an asymptotic formula for $m(n)$. With a substantial theory emerging, Erdős and Lovász (1975) recorded and extended the developments. The study of Property $\mathcal{B}$ is a singular instance of one initially undertaken in infinite parametrization reverberating into the finite.

§8. Erdős–Hajnal Partition Calculus. In the 1960s, Erdős continued with Hajnal to advance the partition calculus, eventually to render it a full-fledged, broad-based subject of set theory. With Hajnal having come to negative stepping up lemmas (Section 5), it was agreed around 1957–8 that he together with Erdős and Rado would engage in the next leap forward for the subject, to write what Hajnal later termed the Giant Triple Paper, or GTP.[33] By 1960, the manuscript was almost complete, but the paper [42], which amounted to 104 pages, only appeared in 1965. In what follows, we pursue the progression of [42] while bringing in related and subsequent developments that particularly bear on the impact of this work. As to [42] itself, relatively few of its results and details are imparted, and this inevitably belies its impressive extent.

After setting out several partition relations in full parametrization, [42] focused on providing a far-reaching extremal analysis under GCH of the unbalanced partition relation $\kappa \longrightarrow (\lambda, \mu)^r$ for cardinals. Positive stepping up lemmas secured partition relations by induction on r, and negative stepping up lemmas provided delimitations by induction on r.

For partition relations at singular strong limit cardinals κ, *canonization*, a transfinite generalization of the Erdős–Rado [10] Canonical Ramsey Theorem, was worked out and applied. Recall (Section 2) that Erdős had provided the singular cardinal case for the seminal Dushnik–Miller Theorem $\kappa \longrightarrow (\kappa, \aleph_0)^2$; for the next level, from $\aleph_0$ to $\aleph_1$, Erdős [3, p.366] had noted the simple $\aleph_\omega \not\longrightarrow (\aleph_\omega, \aleph_1)^2$. Erdős first come to canonization in the process of showing that $\aleph_{\omega_1} \not\longrightarrow (\aleph_{\omega_1}, \aleph_1)^2$ and $\aleph_{\omega_2} \longrightarrow (\aleph_{\omega_2}, \aleph_1)^2$.[34] Coming to the scene years later, Shelah (1975b, 1981) would provide a new type of canonization from which further partition relations for singular cardinals can be derived. What still remains is a characterization of those singular κ such that $\kappa \longrightarrow (\kappa, \aleph_1)^2$.[35]

[33] (Hajnal, 1997, p.361, 363), [110, p.53].

[34] cf. (Hajnal, 1997, p.364), also for subsequent remarks on canonization below.

[35] Late developments illustrate the immanence of partition relations in modern set theory: Erdős and Hajnal came to a focal question that they could not answer for a long time: With $c = 2^{\aleph_0}$, does CH together with $\lambda^{\aleph_0} < \aleph_{c^+}$ for every $\lambda < \aleph_{c^+}$ imply $\aleph_{c^+} \longrightarrow (\aleph_{c^+}, \aleph_1)$? Shelah and Stanley (1987) showed that this is consistently false. Erdős and Hajnal did show that if $\aleph_{c^+}$ is a strong limit cardinal, then the partition relation holds. Shelah and Stanley (1993) eventually showed that if there are c^+ measurable cardinals, then in a forcing extension there is a canonization which entails the consistency of the partition property even though $\aleph_{c^+}$ is not a strong limit cardinal.

While on the topic of such unbalanced partition relations, we describe an incisive Erdős–Hajnal elucidation for countable linear order types. In their [33], they provided a complete analysis of such order types. A linear order type is *scattered iff* it has no densely ordered subset. Erdős and Hajnal classified the countable scattered order types into hierarchy: $\mathcal{O}_0$ consists of the empty and one-element order types; $\mathcal{O}_\alpha$ consists of sums $\Sigma_{i\in\varphi}\varphi_i$ (the order type resulting from replacing each i in its place in φ by φ_i) where each $\varphi_i \in \bigcup_{\beta<\alpha}\mathcal{O}_\beta$ and φ is either ω or its converse ω^*; and $\mathcal{O} = \bigcup_{\alpha<\omega_1}\mathcal{O}_\alpha$, shown to contain all the countable scattered order types. Actually, this hierarchical analysis had appeared long ago in (Hausdorff, 1908, Sects. 10 and 11), but Erdős and Hajnal were not aware of this at the time.[36] They then used this analysis to show that every nonscattered countable order type is a sum $\Sigma_{i\in\varphi}\varphi_i$ where φ is densely ordered and each φ_i is nonempty and scattered.

Freestanding as this analysis is, Erdős and Hajnal [33] applied it to characterize the possibilities for countable order types with respect to a partition relation. Erdős and Rado [13, thm.4] had established that $\eta \longrightarrow (\eta, \aleph_0)^2$ for the rational order type η. This implies forthwith that $\varphi \longrightarrow (\varphi, \aleph_0)^2$ for any countable order type φ having a dense subset, since η is embedded in φ and φ is embedded in η. Also, $\omega \longrightarrow (\omega, \aleph_0)^2$ and $\omega^* \longrightarrow (\omega^*, \aleph_0)^2$ by Ramsey's Theorem. Erdős and Hajnal proceeded by induction up the hierarchy of countable scattered order types to show that for any such order type other than ω or ω^*, $\varphi \not\longrightarrow (\varphi, \aleph_0)^2$.

[42, p.144] considered a new, "square-brackets" partition relation, the basic case of which is

$$\beta \longrightarrow [\alpha]^\gamma_\delta$$

asserting that for any partition $f : [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ such that $f``[H]^\gamma \neq \delta$. That is, f on $[H]^\gamma$ omits at least one value, a far weaker conclusion than for the ordinary partition relation. Presciently, Erdős already in 1956 had generalized Sierpiński's $2^{\aleph_0} \not\longrightarrow (\aleph_1)^2_2$ (Section 2) under CH to $2^{\aleph_0} \not\longrightarrow [\aleph_1]^2_3$.[37] With a prominent incentive being the articulation of strong counterexamples to ordinary partition relations, [42] presented a thorough-going analysis of the square-brackets partition relation. The simplest instance of a nice result, proved with canonization, is: *If κ is a strong limit cardinal of cofinality ω, then $\kappa \longrightarrow [\kappa]^2_3$.*[38]

While the earlier partition relations had figured in the formative developments of large cardinals (Sections 3 and 6), the square-brackets partition relations would of themselves be the subject of investigation in modern set theory. Stevo Todorcevic (1987) established in ZFC the remarkable and unexpected $\kappa^+ \not\longrightarrow [\kappa^+]^2_{\kappa^+}$ for all regular κ, developing "minimal walks" for ordinals, which in his hands would become a much elaborated and applied method of uncountable combinatorics.[39] Building on this, Shelah (1988)

[36] [33, n.1].
[37] (Hajnal, 1997, p.365).
[38] [42, p.148].
[39] cf. the book Todorcevic (2007).

established, nicely complementing Erdős' 1956 $2^{\aleph_0} \not\longrightarrow [\aleph_1]^2_3$ under CH, that *if the* ω_1*-Erdős cardinal exists, then in a forcing extension* $2^{\aleph_0} \longrightarrow [\aleph_1]^2_3$. In wide-ranging work, Shelah (1992, 2000) subsequently pursued this theme of large cardinals effecting such positive partition relations for accessible cardinals. Venturing from exponent 2 to 3, Todorcevic (1994) established, again surprisingly in ZFC, the best possible $\aleph_2 \not\longrightarrow [\aleph_1]^3_{\aleph_1}$. In terms of the sophistication of methods brought to bear and range of results established, the investigation of square-brackets partition relations, among all of Erdős' initiatives in set theory, has arguably been the most broad-ranging and consequential.

With square-brackets partition relations one also gets to the version with partitions of infinite subsets and partitions of all finite subsets, but notably the progression was in reverse order than for the ordinary partition relations. When Hajnal was lecturing at Berkeley in 1964, he heard from Tarski of his student Bjarni Jónsson's problem: For a cardinal κ, is there a size κ algebra with countable many finitary operations having no proper subalgebra of size κ? Such an algebra is a *Jónsson algebra*, and a cardinal κ with *no* Jónsson algebra of size κ is a *Jónsson cardinal*. Upon returning to Hungary, Hajnal and Erdős quickly got results on Jónsson's problem which appeared in their [43], the first article on the subject.[40] They showed that if $2^\kappa = \kappa^+$, then κ^+ is not Jónsson, and that no $\aleph_n$ is Jónsson for $n < \omega$. In the decades to come Jónsson's problem would gain increasing prominence, and whether $\aleph_\omega$ can be Jónsson would remain a focal open problem in set theory about possible consistency low in the cumulative hierarchy. In the 1990s, Jónsson cardinals became a testing ground for Shelah (1994) in his development of his celebrated *pcf* theory; he showed that the least regular Jónsson cardinal is highly inaccessible.

It is straightforward that κ is Jónsson *iff* $\kappa \longrightarrow [\kappa]^{<\omega}_\kappa$, with the expected meaning about partitions of all finite subsets. For partitions of infinite subsets, the very early Rado result $\kappa \longrightarrow (\aleph_0)^\omega_2$ had precluded any substantive possibility for ordinary partition relations. Erdős and Hajnal at the end of [43] established, building on a [22] set mapping result and making conspicuous use of the Axiom of Choice, that $\kappa \not\longrightarrow [\kappa]^\omega_\kappa$ for any κ. Having proceeded lastly to partitions of infinite subsets for square-brackets partition relations, reversing the order for ordinary partition relations, Erdős and Hajnal had actually reached a pivotal point as set theory would unfold.

In 1970, Kenneth Kunen (1971) dramatically applied $\kappa \not\longrightarrow [\kappa]^\omega_\kappa$ to establish in ZFC that there is no elementary embedding $j: V \to V$ of the universe into itself. Stronger and stronger large cardinal hypotheses had been devised approaching this possibility, and Kunen decisively delimited the emerging hierarchy. As perhaps befits a result denying a proffered possibility, Kunen's argument had a simple, basic feel, and has not since been bettered in terms of getting a sharper inconsistency. With that, the Erdős–Hajnal $\kappa \not\longrightarrow [\kappa]^\omega_\kappa$ has become a conceptual landmark about the fullness of partitions of infinite subsets even to the scrutiny of the role of the Axiom of Choice.

[40](Hajnal, 1997, p.366).

The last major topic of [42] was the polarized partition relation, which was given a substantial airing for the first time. The focus was on the simple case described at the end of Section 5, and using set mappings a number of articulating results were established under GCH. What is notable here is that one of the basic problems raised inspired the first forcing consistency result for the partition calculus.

Karel Prikry (1972) established the consistency of

$$\binom{\aleph_2}{\aleph_1} \not\to \begin{bmatrix}\aleph_0\\ \aleph_1\end{bmatrix}_{\aleph_1},$$

i.e., there is a function $F : \omega_2 \times \omega_1 \to \omega_1$ such that for any countable $S \subseteq \omega_2$ and uncountable $T \subseteq \omega_1$, $F``S \times T = \omega_1$. He actually established with forcing the consistency of the following, *Prikry's Principle*:

> There is a family $\{f_\alpha \mid \alpha < \omega_2\}$ of functions $\omega_1 \to \omega_1$ such that for any countable $S \subseteq \omega_2$ and $\psi : S \to \omega_1$, $\{\xi < \omega_1 \mid \forall\alpha \in S(f_\alpha(\xi) \neq \psi(\alpha))\}$ is countable.

$F : \omega_2 \times \omega_1 \to \omega_1$ defined by $F(\alpha, \xi) = f_\alpha(\xi)$ gives the negative partition relation. Prikry evidently educed his principle trying to forcing the negative partition relation, the idea being to construct ω_2 functions: $\omega_1 \to \omega_1$ so that if any guesses are made at values for countably many of them, then for sufficiently large $\xi < \omega_1$ at least one guess is attained at ξ. Jensen had devised his morasses in L to get at such phenomena, and he soon established with a morass that Prikry's Principle holds in L. There would be several more "morass-level" propositions to arising in combinatorial set theory, shown consistent first by forcing, and then seen to hold in L.[41] Several years later, Richard Laver (1978, 1982) showed that if a very strong large cardinal hypothesis holds, then in a forcing extension the positive polarized partition holds in a strong sense. Prikry's result was one of the first addressing a problem from a stimulating and influential list of problems:

In the summer of 1967, a three-week conference in set theory was held at the University of California at Los Angeles. Set theory had newly been transformed, largely by the advent of Cohen's method of forcing, and this was by all accounts one those rare, exhilarating conferences that summarized the recent progress and focused the energy of a new field opening up. Erdős was asked to write up all the difficult problems that had emerged in his set-theoretic work, and he and Hajnal soon came up with a list of 82 problems.[42] This list was distributed at the conference, and appeared four years later in the proceedings [63]. While Erdős had taken to publishing problems and bringing them up at conferences in ongoing fields, the 1967 list was particularly timely both because a new generation was being drawn into set theory and because of a mushrooming of methods becoming available.

Most of the problems had to do with partition relations in all their variety, and the rest on set mappings, the property $\mathcal{B}$, transversals, and infinite

[41] See Kanamori (1983) for a systematic account.

[42] (Hajnal, 1997, p.378).

graphs. For the partition relation problems, connections were soon made with forcing, large cardinals, and $V = L$, and this was duly described in a follow-up Erdős–Hajnal article [79] for a 1971 symposium commemorating Tarski's 70th birthday. On the topics of the rest of the problems, increasingly regarded as in "infinite combinatorial analysis", Erdős described the progress in [83], for a 1973 conference commemorating his 60th birthday.

Erdős, since the Prikry consistency result and the like, became less interested and involved in problems and results that may have to do with consistency, via forcing or large cardinals. The 1967 list did have a range of problems on ordinal and order type partition relations that would have to be decidable. In an 1980 article [100,104], Erdős fully followed up on results and problems in "infinite combinatorial analysis", this time including ordinal partition relations. With such stimulations, even to the point of cash prizes offered, the study of ordinal partition relations has continued to the present day.[43]

In the fullness of time, the four-authored book [106] came out on the partition calculus for cardinals. It presented the theory without GCH in Byzantine detail, incorporating the later work of Shelah and others. It would be the only monograph having Erdős as an author, this indicative of a particular importance of the partition calculus in his corpus.

§9. In Model Theory. Two of Erdős' results about sets were applied in the 1960s in model theory, when it was developing into a modern, sophisticated subject interacting with set theory, particularly in the hands of Tarski and his students at Berkeley. Although straightforward, we briefly describe these applications to illustrate the broad reach of Erdős' combinatorics.

In 1962, Michael Morley famously established his Categoricity Theorem (Morley (1965a)). A theory is *κ-categorical* if all models of size κ are isomorphic. Morley established: *If a theory in a countable first-order language is κ-categorical for some uncountable κ, then it is λ-categorical for every uncountable λ.* In the process, Morley drew in Ramsey's Theorem via its role of providing sets of indiscernibles, which actually was how Ramsey originally applied his theorem to "a problem of formal logic". For a structure $\mathcal{M}$ for a language $\mathcal{L}$, a subset of the domain linearly ordered by a relation $<$ (not necessarily interpreting an $\mathcal{L}$ symbol) is a *set of indiscernibles for $\mathcal{M}$ iff* for each $n \in \omega$ all increasing n-tuples satisfy the same n-free-variable $\mathcal{L}$ formulas in $\mathcal{M}$. Ehenfeucht and Mostowski (1956) established, with Ramsey's Theorem and the Compactness Theorem, that if T is a theory in a countable first-order language with infinite models and $\langle X, < \rangle$ is a linearly ordered set, then there is a model of T for which X is a set of indiscernibles. This result was underlying Silver's work on $0^\sharp$ (Section 6); while the result provides models with indiscernibles, Silver saw that having an Erdős cardinal implies that any model of that size already has in it a set of indiscernibles. With

[43] cf. (Hajnal and Larson, 2010, Sects. 9 and 10).

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

Ehrenfeucht–Mostowski, Morley established, toward his theorem, that if a theory T is κ-categorical for some uncountable κ then it is *totally transcendental* (or equivalently, ω-*stable*) in terms of what is now known as *Morley rank*.

Morley's next, 1963 result Morley (1965b) was an omitting types theorem which has as a corollary that the Hanf number of $\mathcal{L}_{\omega_1\omega}$ is $\beth_{\omega_1}$.[44] Morley used the Erdős–Rado Theorem to construct *un*countable models omitting a type. This was presumably the first use of the theorem outside of Erdős' circle, and it was in a basic role of generating sets of indiscernibles.

Morley's work inspired a large subject in model theory, Classification Theory and Stability Theory, which means to classify models up to isomorphism according to invariants like Morley rank. See e.g., Shelah (1990a); with set-theoretic constructions of models a primary concern, the Erdős–Rado Theorem is a basic ingredient.

Erdős' penultimate "remarks" article [46], with Michael Makkai and consisting of just three pages, introduced a combinatorial property of sets that would play a significant role in the new Stability Theory. For a set A, G a family of subsets of A, and $f : \omega \to A$, G *is strongly cut by* f *iff* there are $X_n \in G$ for $n \in \omega$ such that for every $i \in \omega$, $f(i) \in X_n$ *iff* $i < n$. That is, X_0 has none of the $f(i)$s, X_1 has just $f(0)$, X_2 has just $f(0), f(1)$, and so forth. [46] established that if A is infinite and G a set of subsets of A with $|G| > |A|$, then there is an $f : \omega \to A$ such that either G or $\{A - X \mid X \in G\}$ is strongly cut by f.

Shelah (1971) settled questions from [46], which also appeared on the 1967 problem list, as well as generalized formulations. With this, he provided characterizations of unstable theories in infinitary languages in terms their models having n-tuples with a strongly-cut, order property contradistinctive to being indiscernible. Shelah subsequently established (Shelah (1974)) a generalization of Morley's Categoricity Theorem to uncountable first-order theories and provided (Shelah (1972)) a first broad development of stability theory. In this work, the Erdős–Rado theorem and a strongly cut property were part of the combinatorial underpinnings. The Erdős–Makkai [46] result itself was used by Shelah to study the possibilities of the stability function; Keisler (1976) used it a second time to establish that there are exactly six possibilities for the stability function.

§10. To Silver's Theorem. Silver's 1974 result on singular cardinal arithmetic veritably reoriented set theory with new incentives and goals. Remarkably, Erdős *et al.* already in 1965 were but one step away in their ongoing combinatorial work. We describe this near miss, not only to bring out again the general relevance of Erdős' combinatorial work but its potency in its relative simplicity.

[44] The *Hanf number* of a language is the least cardinal such that if a sentence of the language has a model of that cardinality, then it has models of arbitrarily large cardinality; $\mathcal{L}_{\omega_1\omega}$ is like first-order logic except that conjunctions of countably many formulas are allowed; and the Beth numbers are defined by: $\beth_0 = \aleph_0$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\gamma = \sup_{\alpha<\gamma} \beth_\alpha$ for limit γ.

Silver's result is about singular cardinals of *un*countable cofinality, for which it is still substantive to consider whether or not a subset is stationary, i.e., meets every closed unbounded subset. Silver established Silver (1975): *If* $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$ *and* $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ *is stationary, then* $2^\kappa = \kappa^+$. This result starkly and unexpectedly brought on how at such cardinals there is a strong constraint on the power set. Previously, it was presumed that one can render 2^κ large even for singular κ without disturbing power set cardinalities below, and there had been some progress about countable cofinality κ depending on large cardinals. Silver was able to illuminate the "singular cardinals problem" because of possibilities afforded at *un*countable cofinalities, specifically the well-ordered ranking of functions in a "generic ultrapower". This set in motion in the ensuing years a wide range of results about delimiting power set cardinalities for singular cardinals. Shelah would latterly take up this theme, eventually developing his *pcf* theory subsuming countable cofinalities.[45] Most striking in terms of alacrity with depth, Ronald Jensen was spurred to establish, within a year, the Covering Theorem for L about $0^\sharp$ and the distance between V and L, easily the most prominent advance in set theory in the 1970s and the beginning of core model theory.

Through its proof Silver's Theorem was seen to have higher emanations, the 2nd case being: *If* $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$ *and* $\{\alpha < \kappa \mid 2^\alpha \leq \alpha^{++}\}$ *is stationary, then* $2^\kappa \leq \kappa^{++}$. What is remarkable is that there was a 0th case established a decade earlier, when the singular cardinals problem was first being entertained with the advent of forcing. In 1965, Erdős and Hajnal, working with Eric Milner on transversals of sets, established:[46]

> Suppose that $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$ and $\lambda^{\mathrm{cf}(\kappa)} < \kappa^+$ for $\lambda < \kappa$; $S \subseteq \kappa$ is stationary; and $\mathcal{F}$ is a family of functions: $S \to \kappa$ satisfying (a) $f \in \mathcal{F}$ implies that $f(\alpha) < \alpha$ for $\alpha \in S$, and (b) for distinct $f, g \in \mathcal{F}$, $|\{\alpha \in S \mid f(\alpha) = g(\alpha)\}| < \kappa$. Then $|\mathcal{F}| \leq \kappa$.

This is indeed the 0th case! Suppose that $\aleph_0 < \mathrm{cf}(\kappa) < \kappa$ and $S \subseteq \kappa$ is stationary with $\tau_\alpha \colon \mathcal{P}(\alpha) \to \alpha^+$ a bijection for every $\alpha \in S$. Then for any $X \subseteq \kappa$, $f_X \colon S \to \kappa$ given by $f_X(\alpha) = \tau_\alpha(X \cap \alpha) < \alpha^+$ satisfies that for distinct $X, Y \subseteq \kappa$, $|\{\alpha \in S \mid f_X(\alpha) = f_Y(\alpha)\}| < \kappa$. So one has the hypotheses of Erdős–Hajnal–Milner with α in its (a) replaced by α^+, and the Silver conclusion will be that $2^\kappa = |\{f_X \mid X \subseteq \kappa\}| \leq \kappa^+$. Just as the Silver argument can proceed from the 1st to the 2nd case, so also from the 0th to the 1st case. As soon as Silver's result appeared, Baumgartner and Prikry (1976) contextualized it in just this manner, providing a direct combinatorial proof.

With their focus on transversals, Erdős, Hajnal, and Milner did not take the straightforward step from α to α^+ and hence from their result to Silver's. Had they done so in 1965, it would have been a fitting correlative to Erdős' attention to and interest in singular cardinals. The impact would have been dramatic, even more so than the near miss by two years in connection with inaccessibility vs. measurability (Section 6). Presumably, the very next

[45] cf. Shelah (1994).

[46] cf. (Hajnal, 1997, p.374) and [52, thm.6].

move after Silver's would have been made then: Fred Galvin and Hajnal independently, using the idea of linearly ranking functions and Hajnal his Set Mapping Theorem, established in Galvin and Hajnal (1975): *If* $\aleph_\alpha$ *is a singular strong limit cardinal of uncountable cofinality, then* $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$.

§11. Compactness of Chromatic Number. In a continuing study of infinite graphs, Erdős and Hajnal in their [50] established focal results about compactness of chromatic number. This theme so forwarded as having an immediacy about the transfinite, [50] would have the distinction of raising questions that would be addressed the most extensively and with methods of the most depth, not only of the theory of infinite graphs, but of all Erdős' work in combinatorial analysis.

Erdős considered infinite graphs from the beginning (Section 1) and engaged with the basic concept of graph coloring through compactness (Section 4). In his 1961 article [22] with Hajnal (Section 7), he came to transfinitely parametrized graph compactness. A graph has *chromatic number* κ *iff* κ is the least number of colors with which its vertices can labeled so that no adjacent vertices get the same color. In this language, if $r < \omega$ and every subgraph of a graph has chromatic number at most r, then so does the entire graph (Section 4). [22, p.118] asked about countably many colors: If a graph has size $\aleph_2$ and every subgraph of smaller size has chromatic number at most $\aleph_0$, then does the entire graph?[47]

In 1966, building on previous work Erdős and Hajnal [50] answered this in the negative if one assumes CH: *There is a size* $(2^{\aleph_0})^+$ *graph with chromatic number at least* $\aleph_1$ *all of whose subgraphs of smaller size have chromatic number at most* $\aleph_0$. They asked forthwith in [50], and in the 1967 problem list and elsewhere, whether with GCH there is a size $\aleph_2$ graph with chromatic number $\aleph_2$ such that every graph of smaller size has chromatic number at most $\aleph_0$.

In the fullness of time, Baumgartner (1984) established that this proposition is consistent with forcing. On the other hand, Foreman and Laver (1988) showed that if there is a huge cardinal, a strong large cardinal hypothesis, then in a forcing extension GCH holds and there is *no* $\aleph_2$ size graph as above. Then Shelah (1990b) effected the Baumgartner consistency direction with combinatorial principles instead of forcing to establish: If $V = L$, then for regular, nonweakly compact κ, there is a κ size graph with chromatic number κ all of whose subgraphs of smaller size had chromatic number at most $\aleph_0$. Very recently, Shelah (2013) provided general constructions, under combinatorial assumptions, of graphs with uncountable chromatic number all of whose subgraphs have chromatic number at most $\aleph_0$.

Erdős and Hajnal in [50] also offered a compelling universal graph for chromatic consideration. In the first significant case, the Erdős–Hajnal graph $G(\omega_2, \omega)$ consists of the functions $\omega_2 \to \omega$ with f and g connected *iff* $|\{\alpha < \omega_2 \mid f(\alpha) = g(\alpha)\}| < \aleph_2$. They proved two simple yet striking results:

[47]It is immediate that if $\aleph_2$ is replaced by $\aleph_1$ here, then the complete graph on $\aleph_1$ vertices is a counterexample.

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

(a) $G(\omega_2, \omega)$ has the property that every subgraph of size $\aleph_1$ has chromatic number at most $\aleph_0$, and (b) any graph of size $\aleph_2$ with this property is embeddable into $G(\omega_2, \omega)$. What is the chromatic number of $G(\omega_2, \omega)$? The answer with CH is that it is at least $\aleph_1$ because of the first graph construction in [50], and the results of Baumgartner and Shelah above consistently got the chromatic number to be $\aleph_2$. Nevertheless, the situation remained mysterious for three decades.

Into the 1990s, Péter Komjáth (1991) observed that $2^{\aleph_0} \leq \aleph_2$ still implies that the chromatic number of $G(\omega_2, \omega)$ is at least $\aleph_1$ and showed that it is consistent with GCH that it is the largest possible value $\aleph_3$. Then Foreman (1998) established that if there is a huge cardinal, then in a forcing extension ultrapowers of form ω^{ω_2}/U can have size $\aleph_1$ and so the chromatic number of $G(\omega_2, \omega)$ is (exactly) $\aleph_1$. (In particular, the Foreman–Laver result above is subsumed.) Finally, any lingering speculations about the chromatic number being possibly $\aleph_0$ were put to rest by (Todorcevic, 1997, prop.4), who showed that even if $2^{\aleph_0} > \aleph_2$, the chromatic number of $G(\omega_2, \omega)$ is at least $\aleph_1$.

§**12. Envoi.** Having cast a net far and wide across Erdős' work of significance for and having impact on modern set theory and its development, we bring matters to a close here as well as venture a few panoptic remarks. Into the 1970s and beyond, Erdős continued to work across a broad range of "combinatorial analysis", addressing both new and old issues and problems. It would be that Erdős' last work with reverberations into modern set theory would notably be kindred to his early successes with singular cardinals.

Erdős and Hechler [81] considered maximal almost disjoint (MAD) families of sets at κ, i.e., families F of κ size subsets of κ such that for distinct $X, Y \in F$, $|X \cap Y| < \kappa$ and moreover for any κ size subset A of κ there is a $Z \in F$ such that $|A \cap Z| = \kappa$. With the simple observation that no MAD family at κ can have size $\mathrm{cf}(\kappa)$ because of a diagonalization argument, let $\mathrm{MAD}(\kappa)$ be the set of cardinals $\mu > \mathrm{cf}(\kappa)$ such that μ is the size of a MAD family at κ. With arguments akin to Erdős' from his early days, [81] showed that $\mathrm{MAD}(\kappa)$ is closed under singular limits, i.e., if $\mu_\alpha \in \mathrm{MAD}(\kappa)$ for $\alpha < \nu$ and $\nu < \mu_0$, then $\sup_\alpha \mu_\alpha \in \mathrm{MAD}(\kappa)$, and with this, that if κ is singular and $\lambda < \kappa$ implies $\lambda^{\mathrm{cf}\kappa} < \kappa$, then $\kappa \in \mathrm{MAD}(\kappa)$. κ itself can be the size of a MAD family at κ! But then [81] could not come up with a singular κ not in $\mathrm{MAD}(\kappa)$, and even conjectured that $2^{\aleph_0} > \aleph_\omega$ together with Martin's Axiom would imply $\aleph_\omega \notin \mathrm{MAD}(\aleph_\omega)$.

Three decades later, Menachem Kojman, Wiesław Kubiś and Shelah in their Kojman *et al.* (2004) newly approached $\mathrm{MAD}(\kappa)$ in light of the latter's *pcf* theory. They affirmed the [81] conjecture above; generalized its closure result to show that if κ is singular and (just) $2^{\mathrm{cf}(\kappa)} < \kappa$, then $\kappa \in \mathrm{MAD}(\kappa)$; and with *pcf* showed that for singular κ, all the cardinals from the minimum element of $\mathrm{MAD}(\kappa)$ up to a "bounding" cardinal larger than κ belong to $\mathrm{MAD}(\kappa)$. Erdős and Hechler's [81] was a fitting coda to Erdős' study of singular cardinal phenomena, and the Kojman–Kubiś–Shelah Kojman *et al.* (2004), a fitting response invoking *pcf* theory to extend the analysis.

In the last two decades of his life, Erdős published fully one-third of his articles with collaborators that would make up his set-theoretic corpus. There was episodic elaboration of themes from earlier years, but also a substantial development of structural Ramsey theory, a partition calculus for infinite graphs. While this work certainly falls under the umbrella of set theory, it would remain self-fueling and autonomously internal. Komjáth (2013) provides a detailed account of Erdős' work on infinite graphs, to which we defer. During this period, the main initiatives of modern set theory would be elsewhere, in the direction of the investigation of strong large cardinal hypotheses, inner models, and advanced forcing techniques, axioms, and results. While there would be continuing attention to issues emerging from "combinatorial analysis", the increasing preoccupation has been on consistency results and models of set theory.

For putting Erdős' set-theoretic work and initiatives into a large perspective, it is worth looking to Felix Hausdorff, rather than Cantor himself, for historical antecedence and affinity. Hausdorff was the first developer of the transfinite after Cantor, the one whose work first suggested the rich possibilities for a mathematical investigation of the higher transfinite. He first formulated the distinction between regular and singular cardinals, and even considered the possibility of a regular limit cardinal. He routinely carried out transfinite recursion and induction both with ordinal numbers and cardinal numbers. And he first formulated the Generalized Continuum Hypothesis (GCH), and assumed it to get uniform existence results for all infinite cardinals. With all this to become integral and conspicuous in Erdős' work, one sees "the spirit of Hausdorff" very much at work, with a particular hallmark being Erdős' attention to and grasp of singular cardinals in inductive arguments. Moreover, with Hausdorff's broad context including linear ordering and order types, the Erdős–Hajnal study [33] of countable scattered order types was actually a point of intersection with studies of the old master (Section 8).

Particular to Erdős would be his *modus operandi* of proceeding through cycles of problem, proof, and conjecture with collaborators, and particular to his work in set theory would be his combinatorial attitude, of sets providing an expansive playing field for raising and solving problems about infinite complexes and counting. In all this Erdős evinced an anti-foundationalist attitude about set theory, much as Hausdorff did. Erdős' work on inaccessible cardinals (Sections 3 and 6) turned out to be important for theory of large cardinals and questions of the consistency, but for Erdős it was evident that he was considering direct generalizations of properties of $\aleph_0$ and the play of possible implications.

Erdős' work the most consequential for and having the most impact on set theory occurred in the late 1950s and through the 1960s, all in collaboration with András Hajnal. There was the work leading to Ramsey and Erdős cardinals, including the near miss on inaccessibility vs. measurability (Section 6); the work on Property $\mathcal{B}$ (Section 7); the development of square brackets partition relations, including Jónsson's problem and the $\kappa \not\rightarrow [\kappa]^{\omega}_{\kappa}$ pivotal for large cardinals (Section 8); and the near miss for Silver's

https://doi.org/10.1017/bsl.2014.38 Published online by Cambridge University Press

Theorem (Section 10). In all this the hand of Hajnal is evident, and they would continue their collaboration for two more decades.

When set theory was transformed in the mid-1960s by the advent of forcing, Erdős' cycles of problem, proof, and conjecture were newly modulated by the possibility of consistency. A range of propositions which for Erdős had remained as problems were "solved", as we have documented, by establishing their consistency in forcing extensions in interesting ways. In thus inspiring new mathematical activity of a high order, the work achieved a new prominence for richly populating the landscape of set theory.

For Erdős himself, however, the new consistency results were antithetical to his combinatorial incentives and initiatives. When Prikry in the early 1970s established the consistency of a negative partition relation with forcing (Section 8), Erdős rued the situation. A basic, simple question about the transfinite cannot be directly decided? Especially with so much more to do in number theory and combinatorics, Erdős would not follow the new set-theoretic work involving forcing, and remain on the firmament of "combinatorial analysis".

In several ways, the baton would pass to Saharon Shelah for modern set theory. Starting in model theory, Shelah saw the importance and applicability of a couple of Erdős' combinatorial results (Section 9). His joint paper [68] with Erdős on some combinatorics of Property $\mathcal{B}$ was a direct handshake, and his notable work on singular cardinal compactness (Shelah (1975a)) had inspirations from Erdős' questions about compactness of chromatic number. Subsequently, as we have partially documented, Shelah variously appealed to large cardinal hypotheses or used forcing to establish the relative consistency of a range of propositions put forth by Erdős. Finally, Shelah's *pcf* theory has resonances with Erdős' early attention to singular cardinals and can be seen as a vast combinatorial edifice that emerged on the fertile ground that Erdős first broke and tilled.

Returning to Erdős, his contributions to and impact on set theory had to do mainly with a fortunate timeliness, an engaging concreteness, and an accessible simplicity. Early in his long career, Erdős lifted into set theory themes and results that would play important roles at a formative stage. Variegating the transfinite, Erdős' concrete approach with problems and proofs set in motion a continuing engagement with the specifics of the backdrop. And increasingly, the relative simplicity of his conceptualizations allowed for their easy assimilation to become part of the basic furniture of set theory. As across mathematics, Erdős brought in a certain way of doing and thinking about set theory.

Publications of Paul Erdős in Set Theory

Set theory is rather arbitrarily construed here as having to do with the *interactions* of various infinite sets, e.g., in the study of infinite graphs. A list of Erdős' publications appears in (Graham *et al.*, 2013b, p.497–604). Almost all of Erdős' papers to 1989 are available at http://www.

renyi.hu/˜p_erdos/Erdos.html, and for such papers listed below, their labels at the website are provided in square brackets.

[1] (with Tibor Grünwald (Gallai) and Endre Weiszfeld (Vázsonyi)) Végtelen gráfok Euler-vonalairól. *Matematikai és Fizikai Lapok*, 43:129–140, 1936. [1936-11]

[2] (with Tibor Grünwald (Gallai) and Endre Vázsonyi) Über Euler-Linen unendlicher Graphen. *Journal of Mathematics and Physics*, 17:59–75, 1938. German translation of [1]. [1938-15]

[3] Some set-theoretical properties of graphs. *Revista, Universidad Nacional de Tucumán, Serie A*, 3:363–367, 1942. [1942-06]

[4] (with Alfred Tarski) On families of mutually exclusive sets. *Annals of Mathematics*, 44:315–329, 1943. [1943-04]

[5] (with Shizuo Kakutani) On non-denumerable graphs. *Bulletin of the American Mathematical Society*, 49:457–461, 1943. [1943-05]

[6] Some remarks on set theory. *Annals of Mathematics*, 44:643–646, 1943. [1943-08]

[7] Some remarks on connected sets. *Bulletin of the American Mathematical Society*, 50:442–446, 1944. [1944-06]

[8] On the Hausdorff dimension of some sets in Euclidean space. *Bulletin of the American Mathematical Society*, 52:107–109, 1946. [1946-07]

[9] (with Nicolaas G. de Bruijn) On a combinatorial problem. *Koninklijke Nederlandse Akademie van Wetenschappen*, 51:1277–1279, 1948. [1948-01]

[10] (with Richard Rado) A combinatorial theorem. *Journal of the London Mathematical Society*, 25:249–255, 1950. [1950-01]

[11] Some remarks on set theory. *Proceedings of the American Mathematical Society*, 1:127–151, 1950. [1950-13]

[12] (with Nicolaas G. de Bruijn) A colour problem for infinite graphs and a problem in the theory of relations. *Koninklijke Nederlandse Akademie van Wetenschappen*, 54:369–373, 1951. [1951-01]

[13] (with Richard Rado) Combinatorial theorems on classifications of subsets of a given set. *Proceedings of the London Mathematical Society*, 2:417–439, 1952. [1952-02]

[14] (with Richard Rado) A problem on ordered sets. *Journal of the London Mathematical Society*, 28:426–438, 1953. [1953-01]

[15] Some remarks on set theory, III. *Michigan Mathematical Journal*, 2:51–57, 1954. [1954-08]

[16] (with Leonard Gillman and Melvin Henriksen) An isomorphism theorem for real-closed fields. *Annals of Mathematics*, 61:542–554, 1955. [1955-01]

[17] (with John C. Oxtoby) Partitions of the plane into sets having positive measure in every non-null measurable product set. *Transactions of the American Mathematical Society*, 79:91–102, 1955. [1955-10]

[18] Some remarks on set theory, IV. *Michigan Mathematical Journal*, 2:169–173, 1955. [1955-14]

[19] (with Richard Rado) A partition calculus in set theory. *Bulletin of the American Mathematical Society*, 62:427–489, 1956. [1956–02]

[20] (with Géza Fodor) Some remarks on set theory, V. *Acta Scientiarum Mathematicarum (Szeged)*, 17:250–260, 1956. [1956-18]

[21] (with Géza Fodor) Some remarks on set theory, VI. *Acta Scientiarum Mathematicarum (Szeged)*, 18:243–260, 1957. [1957-12]

[22] (with András Hajnal) On the structure of set-mappings. *Acta Mathematica Academiae Scientiarum Hungaricae*, 9:111–131, 1958. [1958-12]

[23] (with Richard Rado) A theorem on partial well-ordering of sets of vectors. *Journal of the London Mathematical Society*, 34:222–224, 1959. [1959-02]

[24] (with Géza Fodor and András Hajnal) On the structure of inner set mappings. *Acta Scientiarum Mathematicarum (Szeged)*, 20:81–90, 1959. [1959-18]

[25] (with Richard Rado) Partition relations connected with the chromatic number of graphs. *Journal of the London Mathematical Society*, 34: 63–72, 1959. [1959-19]

[26] (with Richard Rado) A construction of graphs without triangles having pre-assigned order and chromatic number. *Journal of the London Mathematical Society*, 35:445–448, 1960. [1960-01]

[27] (with Richard Rado) Intersection theorems for systems of sets. *Journal of the London Mathematical Society*, 35:85–90, 1960. [1960-04]

[28] (with András Hajnal) Some remarks on set theory, VII. *Acta Scientiarum Mathematicarum (Szeged)*, 21:154–163, 1960. [1960-18]

[29] (with András Hajnal) Some remarks on set theory, VIII. *Michigan Mathematical Journal*, 7:187–191, 1960. [1960-19]

[30] (with András Hajnal) On a property of families of sets. *Acta Mathematica Academiae Scientiarum Hungaricae*, 12:87–123, 1961. [1961-11]

[31] (with Ernst Specker) On a theorem in the theory of relations and a solution of a problem of Knaster. *Colloquium Mathematicum*, 8:19–21, 1961. [1961-12]

[32] (with Alfred Tarski) On some problems involving inaccessible cardinals. In Bar-Hillel *et al.*, editors, *Essays on the Foundations of Mathematics*, pages 50–82. Magnes Press, Jerusalem, 1961. [1961-14]

[33] (with András Hajnal) On a classification of denumerable order types and an application to the partition calculus. *Fundamenta Mathematicae*, 51:117–129, 1962. [1962-06]

[34] (with András Hajnal) On the topological product of discrete λ-compact spaces. In Josef Novák, editor, *General Topology and its Relations to Modern Analysis and Algebra (Prague, 1961)*, pages 148–151. Academic Press, New York, 1962. [1962-15]

[35] (with János Czipszer and András Hajnal) Some extremal problems on infinite graphs. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences*, 7:441–457, 1962. [1962-19]

[36] (with András Hajnal) Some remarks concerning our paper "On the structure of set-mappings". Non-existence of a two-valued σ-measure for the first uncountable inaccessible cardinal. *Acta Mathematica Academiae Scientiarum Hungaricae*, 13:223–226, 1962. [1962-20]

[37] On some properties of Hamel bases. *Colloquium Mathematicum*, 10:267–269, 1963. [1963-09]

[38] (with Samuel J. Taylor) The Hausdorff measure of the intersection of sets of positive Lebesgue measure. *Mathematika* 10:1–9, 1963. [1963-19]

[39] An interpolation problem associated with the continuum hypothesis. *Michigan Mathematical Journal*, 11:9–10, 1964. [1964-04]

[40] (with András Hajnal) On complete topological subgraphs of certain graphs. *Annales Universitatis Scientiarum Budapestinensis Eötvös, Sectio Mathematica* 7:143–149, 1964. [1964-12]

[41] (with András Hajnal) Some remarks on set theory, IX. *Michigan Mathematical Journal*, 11:107–127, 1964. [1964-28]

[42] (with András Hajnal and Richard Rado) Partition relations for cardinal numbers. *Acta Mathematica Academiae Scientiarum Hungaricae*, 16:93–196, 1965. [1965-14]

[43] (with András Hajnal) On a problem of B. Jónsson. *Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques*, 14:19–23, 1966. [1966-05]

[44] (with András Hajnal) On chromatic number of graphs and set-systems. *Acta Mathematica Academiae Scientiarum Hungaricae*, 17:61–99, 1966. [1966-07]

[45] (with András Hajnal and Eric C. Milner) On the complete subgraphs of graphs defined by systems of sets. *Acta Mathematica Academiae Scientiarum Hungaricae*, 17:159–229, 1966. [1966-12]

[46] (with Michael Makkai) Some remarks on set theory, X. *Studia Scientiarum Mathematicarum Hungarica* 1:157–159, 1966. [1966-19]

[47] (with András Hajnal) On decomposition of graphs. *Acta Mathematica Academiae Scientiarum Hungaricae*, 18:359–377, 1967. [1967-11]

[48] (with Richard Rado) Partition relations and transitivity domains of binary relations. *Journal of the London Mathematical Society*, 42: 624–633, 1967. [1967-19]

[49] Hilbert térben levõ ponthalmazok néhány geometriai és halmazelméleti tulajdonságárol. *Matematikai Lapok*, 19:255–258, 1968. [1968-2]

[50] (with András Hajnal) On chromatic number of infinite graphs. In Paul Erdős and Gyula Katona, editors, *Theory of Graphs, Proceedings of the Colloquium held at Tihany, Hungary, September 1966*, pages 83–98. Academic Press, New York, 1968. [1968-04]

[51] (with Stanislaw Ulam) On equations with sets as unknowns. *Proceedings of the National Academy of Sciences U.S.A.*, 60:1189–1195, 1968. [1968-06]

[52] (with András Hajnal and Eric C. Milner) On sets of almost disjoint subsets of a set. *Acta Mathematica Academiae Scientiarum Hungaricae*, 19:209–218, 1968. [1968-08]

[53] (with András Hajnal and Eric C. Milner) A problem on well ordered sets. *Acta Mathematica Academiae Scientiarum Hungaricae*, 20: 323–329, 1969. [1969-01]

[54] (with Richard Rado) Intersection theorems for systems of sets (II). *Journal of the London Mathematical Society*, 44:467–479, 1969. [1969-02]

[55] (with Arthur H. Stone) On the sum of two Borel sets. *Proceedings of the American Mathematical Society*, 25:304–306, 1970. [1970-15]

[56] (with András Hajnal) Problems and results in finite and infinite combinatorial analysis. *Annals of the New York Academy of Sciences*, 175:115–124, 1970. [1970-16]

[57] Problems in combinatorial set theory. In Richard Guy *et al.*, editors, *Combinatorial Structures and their Applications. Proceedings of the Calgary International Conference*, pages 97–100. Gordon and Breach, New York, 1970. [1970-17]

[58] (with András Hajnal and Eric C. Milner) Set mappings and polarized partition relations. In Paul Erdős, Alfred Rényi, and Vera T. Sós, editors, *Combinatorial Theory and its Applications, Balatonfüred, 1969*, volume 4 of *Colloquia Mathematica Societatis János Bolyai*, pages 327–363. North-Holland, Amsterdam, 1970. [1970-19]

[59] (with András Hajnal) Some results and problems on certain polarized partitions. *Acta Mathematica Academiae Scientiarum Hungaricae*, 21:369–392, 1970. [1970-25]

[60] (with András Hajnal) Ordinary partition relations for ordinal numbers. *Periodica Mathematica Hungarica*, 1:171–185, 1971. [1971-15]

[61] (with András Hajnal and Eric C. Milner) Polarized partition relations for ordinal numbers. In Leon Mirsky, editor, *Studies in Pure Mathematics (Presented to Richard Rado)*, pages 63–87. Academic Press, London, 1971. [1971-17]

[62] (with András Hajnal and Eric C. Milner) Partition relations for η_α sets. *Journal of the London Mathematical Society*, 3:193–204, 1971. [1971-16]

[63] (with András Hajnal) Unsolved problems in set theory. In Dana S. Scott, editor, *Axiomatic Set Theory*, volume 13, part 1 of *Proceedings of Symposia in Pure Mathematics*, pages 17–48. American Mathematical Society, Providence, 1971. [1971-28]

[64] (with Eric C. Milner) A theorem in the partition calculus. *Canadian Mathematical Bulletin*, 15:501–505, 1972. [1972-03]

[65] (with Saharon Shelah) On problems of Moser and Hanson. In Yousef Alavi, Don R. Lick, and Alexander T. White, editors, *Graph Theory and Applications (Western Michigan University, Kalamazoo, 1971)*, volume 303 of *Lecture Notes in Mathematics*, pages 75–79. Springer, Berlin, 1972. [1972-09]

[66] (with András Hajnal) On Ramsey like theorems, problems and results. In Dominic J. A. Welsh and Douglas R. Woodall, editors, *Combinatorics: Being the Proceedings of the Conference on Combinatorial Mathematics held at the Mathematical Institute, Oxford*, pages

123–140. Institute of Mathematics and Its Applications, Southend-on-Sea, 1972. [1972-10]

[67] (with András Hajnal and Eric C. Milner) Partition relations for η_α and for $\aleph_\alpha$-saturated models. In Günter Asser, Jürgen Flachsmeyer and Willi Rinow, editors, *Theory of Sets and Topology: In Honor of Felix Hausdorff, 1868–1942*, pages 95–108. VEB Deutscher Verlag, Berlin, 1972. [1972-19]

[68] (with Saharon Shelah) Separability properties of almost-disjoint families of sets. *Israel Journal of Mathematics*, 12:207–214, 1972. [1972-21]

[69] (with András Hajnal and Eric C. Milner) Simple one-point extensions of tournaments. *Mathematika*, 19:57–62, 1972. [1972-22]

[70] (with Ervin Fried, András Hajnal and Eric C. Milner) Some remarks on simple tournaments. *Algebra Universalis*, 2:238–245, 1972. [1972-24]

[71] (with András Hajnal and Attila Máté) Chain conditions on set mappings and free sets. *Acta Scientiarum Mathematicarum (Szeged)*, 34:69–79, 1973. [1973-03]

[72] (with András Hajnal and Bruce L. Rothschild) On chromatic number of graphs and set systems. In Adrian R. D. Mathias, editor, *Cambridge Summer School in Mathematical Logic*, volume 337 of *Lecture Notes in Mathematics*, pages 531–538. Springer, Berlin, 1973. [1973-13]

[73] (with Eric C. Milner) A theorem in the partition calculus. Corrigendum. *Canadian Mathematical Bulletin*, 17:305, 1974. [1974-07]

[74] (with Eric C. Milner and Richard Rado) Intersection theorems for systems of sets (III). *Journal of the Australian Mathematical Society*, 18:22–40, 1974. [1974-11]

[75] (with András Hajnal and Saharon Shelah) On some general properties of chromatic numbers. In Ákos Császár, editor, *Topics in Topology, Keszthely (Hungary), 1972*, volume 8 of *Colloquia Mathematica Societatis János Bolyai*, pages 243–255. North-Holland, Amsterdam, 1974. [1974-17]

[76] Remark on a theorem of Lindström. *Journal of Combinatorial Theory, Series A*, 17:129–130, 1974. [1974-26]

[77] (with András Hajnal) Some remarks on set theory XI. *Fundamenta Mathematicae*, 81:261–265, 1974. [1974-33]

[78] (with Robert Bonnet) The chromatic index of an infinite complete hypergraph: A partition theorem. In Claude Berge and Dijen Ray-Chaudhuri, editors, *Hypergraph Seminar (Ohio State University, 1972)*, volume 411 of *Lecture Notes in Mathematics*, pages 54–60. Springer, Berlin, 1974. [1974-35]

[79] (with András Hajnal) Unsolved and solved problems in set theory. In Leon Henkin, editor, *Proceedings of the Tarski Symposium*, volume 25 of *Proceedings of Symposia in Pure Mathematics*, pages 269–287. American Mathematical Society, Providence, 1974. [1974-36]

[80] (with Mary Ellen Rudin) A non-normal box product. In András Hajnal, Richard Rado, and Vera T. Sós, editors, *Infinite and Finite*

Sets, Keszthely (Hungary), 1973, volume II, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 629–631. North-Holland, Amsterdam, 1975. [1975-01]

[81] (with Stephen H. Hechler) On maximal almost-disjoint families over singular cardinals. In András Hajnal, Richard Rado, and Vera T. Sós, editors, *Infinite and Finite Sets, Keszthely (Hungary), 1973*, volume I, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 597–604. North-Holland, Amsterdam, 1975. [1975-21]

[82] (with Fred Galvin and András Hajnal) On set-systems having large chromatic number and not containing prescribed subsystems. In András Hajnal, Richard Rado, and Vera T. Sós, editors, *Infinite and Finite Sets, Keszthely (Hungary), 1973*, volume I, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 425–513. North-Holland, Amsterdam, 1975. [1975-24]

[83] Problems and results on finite and infinite combinatorial analysis. In András Hajnal, Richard Rado, and Vera T. Sós, editors, *Infinite and Finite Sets, Keszthely (Hungary), 1973*, volume I, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 403–424. North-Holland, Amsterdam, 1975. [1975-32]

[84] Problems and results on finite and infinite graphs. In Miroslav Fiedler, editor, *Recent Advances in Graph Theory, Proceedings of the Second Czechoslovak Symposium, 1974*, pages 183–192. Academia, Prague, 1975. [1975-33]

[85] (with András Hajnal and Lajos Pósa) Strong embeddings of graphs into colored graphs. In András Hajnal, Richard Rado, and Vera T. Sós, editors, *Infinite and Finite Sets, Keszthely (Hungary), 1973*, volume I, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 585–595. North-Holland, Amsterdam, 1975. [1975-44]

[86] (with Menachem Magidor) A note on regular methods of summability and the Banach-Saks property. *Proceedings of the American Mathematical Society*, 59:232–234, 1976. [1976-01]

[87] (with David Preiss) Decomposition of spheres in Hilbert spaces. *Commentationes Mathematicae Universitatis Carolinae*, 17:791–795, 1976. [1976-08]

[88] (with Eric C. Milner and Richard Rado) Families of sets whose pairwise intersections have prescribed cardinals or order types. *Mathematical Proceedings of the Cambridge Philosophical Society*, 80:215–221, 1976. [1976-14]

[89] (with Marshall L. Cates, Neil Hindman, and Bruce L. Rothschild) Partition theorems for subsets of vector spaces. *Journal of Combinatorial Theory, Series A*, 20:279–291, 1976. [1976-31]

[90] (with R. Daniel Mauldin) The nonexistence of certain invariant measures. *Proceedings of the American Mathematical Society*, 59:321–322, 1976. [1976-47]

[91] (with Eric C. Milner and Richard Rado) Families of sets whose pairwise intersections have prescribed cardinals or order types.

Corrigenda. *Mathematical Proceedings of the Cambridge Philosophical Society*, 81:523, 1997. [1977-07]

[92] (with András Hajnal) Embeddings theorems for graphs establishing negative partition relations. *Periodica Mathematica Hungarica*, 9: 205–230, 1978. [1978-09]

[93] (with András Hajnal and Eric C. Milner), On set systems having paradoxical covering properties. *Acta Mathematica Academiae Scientiarum Hungaricae*, 31:89–124, 1978. [1978-25]

[94] (with György Elekes and András Hajnal) On some partition properties of families of sets. *Studia Scientiarum Mathematicarum Hungarica*, 13:151–155, 1978.

[95] (with Frank S. Cater and Fred Galvin) On the density of λ-box products. *General Topology and its Applications*, 9:307–312, 1978. [1978-27]

[96] Set-theoretic, measure-theoretic, combinatorial, and number-theoretic problems concerning point sets in Euclidean space. *Real Analysis Exchange*, 4:113–138, 1978. [1978-40]

[97] (with James E. Baumgartner, Fred Galvin and Jean A. Larson) Colorful partitions of cardinal numbers. *Canadian Journal of Mathematics*, 31:524–541, 1979. [1979-03]

[98] Some remarks on subgroups of real numbers. *Colloquium Mathematicum*, 42:119–120, 1979. [1979-20]

[99] (with Fred Galvin and Richard Rado) Transversals and multitransversals. *Journal of the London Mathematical Society*, 20:387–395, 1979. [1979-28]

[100] Problems and results on finite and infinite combinatorial analysis II, *l'En seignement Mathématique*, 27:163–176, 1981. [1981-22]

[101] (with Vance Faber and Jean A. Larson) Sets of natural numbers of positive density and cylindric set algebras of dimension 2. *Algebra Universalis*, 12:81–92, 1981. [1981-26]

[102] (with Kenneth Kunen and R. Daniel Mauldin) Some additive properties of sets of real numbers. *Fundamenta Mathematicae*, 113:187–199, 1981. [1981-28]

[103] (with András Hajnal and Endre Szemerédi) On almost bipartite large chromatic graphs. In Alexander Rosa, Gert Sabidussi and Jean Turgeon, editors, *Theory and Practice of Combinatorics*, volume 12 of *Annals of Discrete Mathematics*, pages 117–123. North-Holland, Amsterdam, 1982. [1982-11]

[104] Problems and results on finite and infinite combinatorial analysis II. In *Logic and Algorithmic (Zurich, 1980)*, volume 30 of *Monographies de l'Enseignement Mathématique*, pages 131–144. Université de Genève, Genève, 1982. A reprint of [100]. [1982-24]

[105] (with James E. Baumgartner and Denis A. Higgs) Cross-cuts in the power set of an infinite set. *Order*, 1:139–145, 1984. [1984-03]

[106] (with András Hajnal, Attila Máté, and Richard Rado), *Combinatorial Set Theory: Partition Relations for Cardinals*, volume 106 of

Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1984. [1984-25]

[107] (with András Hajnal) Chromatic number of finite and infinite graphs and hypergraphs. *Discrete Mathematics*, 53:281–285, 1985. [1985-08]

[108] Problems and results on chromatic numbers in finite and infinite graphs. In Yousef Alavi *et al.*, editors, *Graph Theory with Applications to Algorithms and Computer Science (Kalamazzo, 1984)*, pages 201–213. Wiley, New York, 1985. [1985-26]

[109] (with Zolán Füredi, András Hajnal, Péter Komjáth, Vojtěch Rödl and Ákos Seress) Coloring graphs with locally few colors. *Discrete Mathematics*, 59:21–34, 1986. [1986-03]

[110] My joint work with Richard Rado. In C. A. Whitehead, editor, *Surveys in Combinatorics 1987*, volume 123 of *London Mathematics Society Lecture Note Series*, pages 53–80. Cambridge University Press, Cambridge, 1987. [1987-12]

[111] Some problems on finite and infinite graphs. In Stephen G. Simpson, editor, *Logic and Combinatorics (Arcata 1985)*, volume 65 of *Contemporary Mathematics*, pages 223–228. American Mathematical Society, Providence, 1988. [1987-28]

[112] (with Chris D. Godsil, Steven G. Krantz, and Torrence Parsons) Intersection graphs for families of balls in $\mathbb{R}^n$. *European Journal of Combinatorics*, 9:501–505, 1988. [1988-13]

[113] (with Péter Komjáth) Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Discrete and Computational Geometry*, 5:325–331, 1990.

[114] (With Jean A. Larson) Matchings from a set below to a set above. *Discrete Mathematics*, 95:169–182, 1991.

[115] (with Fred Galvin) Some Ramsey-type theorems. *Discrete Mathematics*, 87:261–269,1991.

[116] (with Andreas Blass and Alan D. Taylor) Diverse homogeneous sets. *Journal of Combinatorial Theory, Series A*, 59:312–317, 1992.

[117] (with Fred Galvin) Monochromatic infinite paths. *Discrete Mathematics*, 113:59–70, 1993.

[118] (with András Hajnal and Jean A. Larson) Ordinal partition behavior of finite powers of cardinals. In Norbert W. Sauer, Robert Woodrow, and Bill Sands, editors, *Finite and Infinite Combinatorics in Sets and Logic (Banff 1991)*, volume 411 of *NATO Science Series C*, pages 97–115. Kluwer, Dordrecht, 1993.

[119] (With Steve Jackson and R. Daniel Mauldin) On partitions of lines and space. *Fundamental Mathematicae*, 145:101–119, 1994.

[120] On some problems in combinatorial set theory. *Publications de l'Institut Mathématique de Belgrade*, 57:61–65, 1995.

[121] (with Steve Jackson and R. Daniel Mauldin) On infinite partitions of lines and space. *Fundamenta Mathematicae*, 152:75–95, 1997.

REFERENCES

Uri Abraham (1981). *Free sets for nowhere-dense set mappings*. ***Israel Journal of Mathematics***, 39:167–176.

Martin Aigner and Günter Ziegler (2009). ***Proofs from THE BOOK***. Springer, Berlin, 2013. Corrected printing of the 4th edition.

Frederick Bagemihl (1973). *The existence of an everywhere dense independent set*. ***Michigan Mathematical Journal***, 20:1–2.

Paul T. Bateman (2002). *Some personal memories of Paul Erdős*. In Gábor Halász, Lázló Lovász, Miklós Simonovits, and Vera T. Sós, editors, ***Paul Erdős and His Mathematics I***, volume 11 of *Bolyai Society Mathematical Studies*, pages 9–12. Springer, Berlin.

James E. Baumgartner (1970). ***Results and Independence Proofs in Combinatorial Set Theory***. PhD thesis, University of California at Berkeley.

———, (1984). *Generic graph construction*. ***The Journal of Symbolic Logic***, 49:234–240.

James E. Baumgartner and Fred Galvin (1978). *Generalized Erdős cardinals and* $0^{\#}$. ***Annals of Mathematical Logic***, 15:289–311.

James E. Baumgartner and Karel L. Prikry (1976). *On a theorem of Silver*. ***Discrete Mathematics***, 14:17–21.

Béla Bollobás (2013). *Paul Erdős: Life and work*. In Ronald L. Graham, Jaroslav Nešetřil, and Steve Butler, editors, ***The Mathematics of Paul Erdős I***. Springer, Berlin.

Nicolaas G. de Bruijn (2002). *Remembering Paul Erdős*. In Gábor Halász, Lázló Lovász, Miklós Simonovits, and Vera T. Sós, editors, ***Paul Erdős and His Mathematics I***, volume 11 of *Bolyai Society Mathematical Studies*, pages 13–16. Springer, Berlin.

Keith J. Devlin and Jeffrey B. Paris (1973). *More on the free subset problem*. ***Annals of Mathematical Logic***, 5:327–336.

Ben Dushnik and Edwin W. Miller (1941). *Partially ordered sets*. ***American Journal of Mathematics***, 63:606–610.

Andrzej Ehenfeucht and Andrzej M. Mostowski (1956). *Models of axiomatic theories admitting automorphisms*. ***Fundamenta Mathematicae***, 43:50–68.

Paul Erdős (1932). *Beweis eines Satzes von Tschebyschef*. ***Acta Universitatis Szegediensis***, 5:194–198.

———, (1963). *On a combinatorial problem*. ***Nordisk Matematisk Tidskrift***, 11:5–10.

———, (1964). *On a combinatorial problem. II*. ***Acta Mathematica Academiae Scientiarum Hungaricae***, 15:445–447.

———, (1969). *On a combinatorial problem III*. ***Canadian Mathematical Bulletin***, 12:413–416.

Paul Erdős and Lázló Lovász (1975). *Problems and results on 3-chromatic hypergraphs and some related questions*. In András Hajnal, Richard Rado, and Vera T. Sós, editors, ***Infinite and Finite Sets, Keszthely (Hungary), 1973, volume II***, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 609–627. North-Holland, Amsterdam.

Paul Erdős and György Szekeres (1935). *A combinatorial problem in geometry*. ***Compositio Mathematica***, 2:463–470.

Matthew Foreman (1998). *An* $\aleph_1$*-dense ideal on* $\aleph_2$. ***Israel Journal of Mathematics***, 108:253–290.

Matthew Foreman and Richard Laver (1988). *Some downwards transfer properties for* $\aleph_2$. ***Advances in Mathematics***, 67:230–238.

Chris Freiling (1986). *Axioms of symmetry: Throwing darts at the real number line*. ***The Journal of Symbolic Logic***, 51:190–200.

Fred Galvin and András Hajnal (1975). *Inequalities for cardinal powers*. ***Annals of Mathematics***, 101:491–498.

Ronald L. Graham, Jaroslav Nešetřil, and Steve Butler, editors (2013a). ***The Mathematics of Paul Erdős I***. Springer, Berlin, 2013. Second, expanded edition; first edition 1997.

———, (2013b). ***The Mathematics of Paul Erdős II***. Springer, Berlin, 2013. Second, expanded edition; first edition 1997.

András Hajnal (1961). *Proof of a conjecture of S. Ruziewicz*. ***Fundamenta Mathematicae***, 50:123–128.

———, (1997). *Paul Erdős' set theory*. In Ronald L. Graham and Jaroslav Nešetřil, editors, ***The Mathematics of Paul Erdős II***, volume 14 of *Algorithms and Combinatorics*, pages 352–393. Springer, Berlin.

András Hajnal, István Juhász, and Saharon Shelah (1986). *Splitting strongly almost disjoint families*. ***Transactions of the American Mathematical Society***, 295:369–387.

———, (2000). *Strongly almost disjoint famil[i]es, revisited*. ***Fundamental Mathematicae***, 163:13–23.

András Hajnal and Jean A. Larson (2010). *Partition relations*. In Matthew Foreman and Akihiro Kanamori, editors, ***Handbook of Set Theory***, volume 1, pages 129–213. Springer, Dordrecht.

Gábor Halász, Lázló Lovász, Miklós Simonovits, and Vera T. Sós, editors (2002a). ***Paul Erdős and His Mathematics I***, volume 11 of *Bolyai Society Mathematical Studies*. Springer, Berlin.

———, (2002b). ***Paul Erdős and His Mathematics II***, volume 11 of *Bolyai Society Mathematical Studies*. Springer, Berlin.

Michael Hallett (1984). ***Cantorian Set Theory and Limitation of Size***, volume 10 of *Oxford Logic Guides*. Clarendon Press, Oxford.

Felix Hausdorff (2005). *Grundzüge einer Theorie der geordneten Mengen*. ***Mathematische Annalen***, 65:435–505, 1908. Translated with an introduction in Jacob Plotkin, editor, ***Hausdorff on Ordered Sets***, American Mathematical Society, pages 181–258.

Stephen H. Hechler (1972). *Directed graphs over topological spaces: some set theoretical aspects*. ***Israel Journal of Mathematics***, 11:231–248.

Akihiro Kanamori (1983). *Morasses in combinatorial set theory*. In A. R. D. Mathias, editor, ***Surveys in Set Theory***, volume 87 of *London Mathematical Society Lecture Note Series*, pages 167–197. Cambridge University Press, Cambridge.

———, (2009). ***The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings***. Springer, Berlin, 2009. Second edition; first edition, 1994.

H. Jerome Keisler (1962). *Some applications of the theory of models to set theory*. In Ernest Nagel, Patrick Suppes, and Alfred Tarski, editors, ***Logic, Methodology and Philosophy of Science. Proceedings of the 1960 International Congress***, pages 80–86. Stanford University, Stanford.

———, (1976). *Six classes of theories*. ***Journal of the Australian Mathematical Society***, 21:257–266.

Peter Koepke (1984). *The consistency strength of the free-subset property for ω_ω*. ***The Journal of Symbolic Logic***, pages 1198–1204.

Menachem Kojman (2015). *Splitting families of sets in ZFC*. ***Advances in Mathematics***, Volume 269, 10 January 2015, Pages 707–725.

Menachem Kojman, Wiesław Kubiś, and Saharon Shelah (2004). On two problems of Erdős and Hechler: New methods in singular madness. ***Proceedings of the American Mathematical Society***, 132:3357–3365.

Péter Komjáth (1991). *The chromatic number of some uncountable graphs*. In Gábor Halász, editor, ***Sets, Graphs, and Numbers: A Birthday Salute to Vera T. Sós and András Hajnal***, volume 60 of *Colloquia Mathematica Societatis János Bolyai*, pages 439–444. North-Holland, Amsterdam.

———, (2013). *Erdős's work on infinite graphs*. In Lázló Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors, *Erdős Centennial*, volume 25 of *Bolyai Society Mathematical Studies*, pages 325–345. Springer, Berlin.

Dénes Kőnig (1936). ***Theorie der endliche und unendlichen Graphen***. Akademische Verlagsgesellschaft MBG, Leipzig.

Kenneth Kunen (1971). *Elementary embeddings and infinitary combinatorics*. ***The Journal of Symbolic Logic***, 36:407–413.

———, (2013). The impact of Paul Erdős on set theory. In Lázló Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors, ***Erdős Centennial***, volume 25 of *Bolyai Society Mathematical Studies*, pages 347–363. Springer, Berlin.

Đuro Kurepa (1939). Sur la puissance des ensembles partiellement ordonnés. ***Sprawozdania Towarzystwo Naukowe Warszawa, Mat.-Fiz***, 32:61–67.

———, (1953). *On reflexive symmetric relations and graphs*. ***Slovenska Akademija Znanosti in Umetinosti. Razred za Matematične, Fizikalne in Tehnične Vede. Serija A***, 4:65–92.

———, (1996). ***Selected Papers of Đuro Kurepa***. Matematicki Institute SANU.

Jean A. Larson (2012). *Infinite combinatorics*. In Akihiro Kanamori, John H. Woods, and Dov Gabbay, editors, ***Sets and Extensions in the Twentieth Century***, volume 6 of *Handbook of the History of Logic*. Elsevier, Amsterdam.

Richard Laver (1978). *Strong saturation properties of ideals (abstract)*. ***The Journal of Symbolic Logic***, 43:371.

———, (1982). *An* $(\aleph_2, \aleph_2, \aleph_0)$*-saturated ideal on* ω_1. In Dirk van Dalen and Daniel Lascar aqnd Timothy J. Smiley, editors, ***Logic Colloquium '80***, volume 108 of *Studies in Logic and the Foundatiions of Mathematics*, pages 173–180. North-Holland, Amsterdam.

Jerzy Łoś (1959). *Linear equations and pure subgroups*. ***Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques,*** 7:13–18.

Lázló Lovász, Imre Z. Ruzsa, and Vera T. Sós, editors (2013). *Erdős Centennial*, volume 25 of *Bolyai Society Mathematical Studies*. Springer, Berlin.

Edwin W. Miller (1937). *On a property of families of sets*. ***Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie***, 30:31–38.

Michael Morley (1965a). *Categoricity in power*. ***Transactions of the American Mathematical Society***, 144:514–538.

———, (1965b). *Omitting classes of elements*. In John W. Addison, Leon Henkin, and Alfred Tarski, editors, ***The Theory of Models. Proceedings of the 1963 International Symposium at Berkeley***. North-Holland, Amsterdam.

Karel L. Prikry (1972). *On a problem of Erdős, Hajnal and Rado*. ***Discrete Mathematics***, 2:51–59.

Saharon Shelah (1971). *Stability, the f.c.p., and superstability; model theoretic properties of formulas in first order theory*. ***Annals of Mathematical Logic***, 3:271–362.

———, (1972). *A combinatorial problem; stability and order for models and theories in infinitary languages*. ***Pacific Journal of Mathematics***, 41:247–261.

———, (1974). *Categoricity of uncountable theories*. In Leon Henkin, editor, ***Proceedings of the Tarski Symposium***, volume 25 of *Proceedings of Symposia in Pure Mathematics*, pages 187–203. American Mathematical Society, Providence.

———, (1975). *A compactness theorem for singular cardinals, free algebra, Whitehead problem and transversals*. ***Israel Journal of Mathematics***, 62:319–349.

———, (1975). *Notes on partition calculus*. In András Hajnal, Richard Rado, and Vera T. Sós, editors, ***Infinite and Finite Sets, Keszthely (Hungary), 1973, volume II***, volume 10 of *Colloquia Mathematica Societatis János Bolyai*, pages 1257–1276. North-Holland, Amsterdam.

———, (1981). *Canonization theorems and applications*. ***The Journal of Symbolic Logic***, 46:345–353.

———, (1988). *Was Sierpiński right? I*. ***Israel Journal of Mathematics***, 62:355–380.

———, (1990). ***Classification Theory and the Number of Non-Isomorphic Models***. North-Holland, Amsterdam, revised edition.

———, (1990). *Incompactness for chromatic numbers of graphs*. In Alan Baker, Béla Bollobás, and András Hajnal, editors, ***A Tribute to Paul Erdős***, pages 361–371. Cambridge University Press, Cambridge.

———, (1992). *Strong partition relations below the power set: Consistency, Was Sierpiński right? II*. In Gábor Halász, editor, ***Sets, Graphs and Numbers: A Birthday Salute to Vera T. Sós and András Hajnal***, volume 60 of *Colloquia Mathematica Societatis János Bolyai*, pages 637–668. North-Holland, Amsterdam.

———, (1994). ***Cardinal Arithmetic***, volume 29 of *Oxford Logic Guides*. Clarendon Press, Oxford.

———, (2000). *Was Sierpiński right? IV*. ***The Journal of Symbolic Logic***, 65:1031–1054.

———, (2013). *On incompactness of chromatic number of graphs*. ***Acta Mathematica Hungarica***, 139:363–371.

SAHARON SHELAH and LEE J. STANLEY (1987). *A theorem and some consistency results in partition calculus*. ***Annals of Pure and Applied Logic***, 36:119–152.

———, (1993). *More consistency results in partition calculus*. ***Israel Journal of Mathematics***, 81:97–110.

WACŁAW SIERPIŃSKI (1933). *Sur un problème de la théorie des relations*. ***Annali della Scuola Normale Superiore di Pisa***, 2(2):285–287.

———, (1934a). ***Hypothèse du Continu***, volume 4 of *Monografie Matematyczne*. Seminarjum Matematyczne Uniwersytetu Warszawskiego, Warsaw. Second revised edition, Chelsea, New York, 1956.

———, (1934b). Sur la dualité entre la première catégorie et la mesure nulle. ***Fundamenta Mathematicae***, pages 276–280.

WACŁAW SIERPIŃSKI and ALFRED TARSKI (1930). *Sur une propriété caractéristique des nombres inaccessibles*. ***Fundamenta Mathematicae***, 15:292–300.

JACK H. SILVER (1975). *On the singular cardinals problem*. In ***Proceedings of the International Congress of Mathematicians***, pages 265–268. Canadian Mathematical Congress, Montreal.

VERA T. SÓS (2002). *Turbulent years: Erdős in his correspondence with Turán*. In Gábor Halász, Lázló Lovász, Miklós Simonovits, and Vera T. Sós, editors, ***Paul Erdős and His Mathematics I***, volume 11 of *Bolyai Society Mathematical Studies*, pages 85–146. Springer, Berlin.

ALFRED TARSKI (1938). *Über unerreichbare Kardinalzahlen*. ***Fundamenta Mathematicae***, 30:68–89.

———, (1962). *Some problems and results relevant to the foundations of set theory*. In Ernest Nagel, Patrick Suppes, and Alfred Tarski, editors, ***Logic, Methodology and Philosophy of Science. Proceedings of the 1960 International Congress***, pages 125–135. Stanford University, Stanford.

ALAN D. TAYLOR (1980). *On saturated sets of ideals and Ulam's problem*. ***Fundamenta Mathematicae***, 109:37–53.

STEVO TODORCEVIC (1987). *Partitioning pairs of countable ordinals*. ***Acta Mathematica***, 159:261–294.

———, (1994). *Some partition of three-dimensional combinatorial cubes*. ***Journal of Combiantorial Theory***, 68:410–437.

———, (1997). *Comparing the continuum with the first two uncountable cardinals*. In M.L. Dalla Chiara, editor, ***Logic and Scientific Methods***, volume 259 of *Synthese Library*, pages 145–155. Kluwer, Dordrecht.

———, (2007). *Walks on Ordinals and Their Characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser, Basel.

PAUL TURÁN (2002). *Pál Erdős is fifty*. In Gábor Halász, Lázló Lovász, Miklós Simonovits, and Vera T. Sós, editors, ***Paul Erdős and His Mathematics I***, volume 11 of *Bolyai Society Mathematical Studies*, pages 55–83. Springer, Berlin.

STANISŁAW M. ULAM (1930). *Zur Masstheorie in der allgemeinen Mengenlehre*. ***Fundamenta Mathematicae***, 16:140–150.

ANDREW VAZSONYI (1996). *Paul Erdös, the world's most beloved mathematical genius "leaves"*. ***Pure Mathematics and Applications***, 7:1–12. Anthologized at www.emis.de/classics/Erdos/textpdf/vazsonyi/genius.pdf.

BOSTON UNIVERSITY
111 CUMMINGTON MALL
BOSTON, MASSACHUSETTS 02215, USA
E-mail: aki@math.bu.edu