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ON A PROBLEM RELATED TO THE CONJECTURE OF SENDOV ABOUT THE CRITICAL POINTS OF A POLYNOMIAL

BY

Q. I. RAHMAN AND Q. M. TARIQ

ABSTRACT. Let P be a polynomial of degree n having all its zeros in the closed unit disk. Given that a is a zero (of P) of multiplicity k we *seek to determine* the radius $\rho(n; k; a)$ of the smallest disk centred at a containing at least k zeros of the derivative P'. In the case $k = 1$ the answer has been conjectured to be 1 and is known to be true for $n \leqq 5$. We prove that $\rho(n; k; a) \leqq 2k/(k + 1)$ for arbitrary $k \in \mathbf{N}$ and $n \leqq k + 4$.

1. **Introduction.** We denote by $D(z_0; R)$ the open disk $\{z \in \mathbf{C}: |z - z_0| < R\}$ and by $\bar{D}(z_0; R)$ its closure. While counting the zeros of a function we will always take multiplicity into account. Recently, the second named author considered the following problem:

"Let $a \in \bar{D}(0; 1)$ and $k \in \mathbf{N}$. Given an arbitrary polynomial $P(z) := c(z - a)^k \prod_{j=1}^{n-k} (z - z_j)$ of degree n $(>k)$ with $|z_j| \leqq 1$ for $j = 1, \ldots, n - k$, determine the radius $\rho(n; k; a)$ of the smallest (closed) disk centred at a containing at least k zeros of the derivative P'".

The case $k = 1$ of this problem has been investigated by several mathematicians under the title of Sendov's (or Iliev's) conjecture according to which "$\rho(n; 1; a) \leqq 1$" (for references see [6]; also see [1]). The example $P(z) := z^n - 1$ shows that $\sup_{0 \leqq |a| \leqq 1} \rho(n; 1; a) \geqq 1$. In general, for any $k \geqq 1$ the disk $D(a; 2k/(k + 1))$ may contain only $k - 1$ zeros of P', namely the $(k - 1)$-fold zero at a. For example, if $P(z) := (z + 1)(z - 1)^k$ then P' has a $(k - 1)$-fold zero at 1 and a simple zero at $-((k - 1)/(k + 1))$. As another example we may consider

$$P(z) := \left(z^2 + 2\frac{(k + 1)^2 - 2}{(k + 1)^2} z + 1\right)(z - 1)^k$$

whose derivative has a double zero at $-((k - 1)/(k + 1))$ in addition to a $(k - 1)$-fold zero at 1. The following result which was proved in [10] suggests that "$\rho(n; k; a) \leqq 2k/(k + 1)$" may hold for all $k \in \mathbf{N}$ and all $a \in \bar{D}(0; 1)$.

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THEOREM A. *Let* $|a| = 1$. *If* $P(z) := c(z - a)^k \prod_{j=1}^{n-k} (z - z_j)$ *is a polynomial of degree* n $(>k)$ *such that* $|z_j| \leqq 1$ *for* $j = 1, \ldots, n - k$, *then* P' *has at least* k *zeros in*

$$\bar{D}\left(\frac{a}{k+1}; \frac{k}{k+1}\right) \subset \bar{D}\left(a; \frac{2k}{k+1}\right).$$

Four different proofs of Theorem A are known in the case $k = 1$ ([3], [8], [4], [7]). In [11] it was shown that $\rho(n; k; a) \leqq 2k/(k + 1)$ for all $a \in \bar{D}(0:1)$ and all $k \in \mathbf{N}$ if $k + 1 \leqq n \leqq (k + 1)^2$ (and so if $k + 1 \leqq n \leqq k + 3$). The result says in particular that if $k = 1$ then $\bar{D}(a; 1)$ contains at least one zero of P' for $n \leqq 4$. However, more is known in the case $k = 1$. In fact, it was shown by Meir and Sharma [7] that if $k = 1$ and $n = 5$ then $\bar{D}(a; (|a| + \sqrt{2 - |a|^2})/2)$ contains at least one zero of P', i.e.

$$\rho(5:1:a) \leqq (|a| + \sqrt{2 - |a|^2})/2 \leqq 1. \tag{1}$$

The purpose of this paper is to prove the following extension of (1).

THEOREM 1. *Let* $a \in \bar{D}(0; 1)$ *and* k *an integer* $\geqq 2$. *If*

$$P(z) := c(z - a)^k \prod_{\nu=1}^{4} (z - z_\nu)$$

is a polynomial of degree $k + 4$ *such that* $|z_\nu| \leqq 1$ *for* $\nu = 1, \ldots, 4$, *then* P' *has at least* k *zeros in* $\bar{D}(a; \frac{2}{3}(|a| + \sqrt{2 - |a|^2}))$ *if* $k = 2$ *and in* $\bar{D}(a; (\sqrt{(k + 1)^2 - |a|^2(2k + 1)} + |a|k)/(k + 1))$ *if* $k \geqq 3$.

REMARK 1. Theorem 1 in conjunction with (1) implies that $\rho(k + 4; k; a) \leqq 2k/(k + 1)$ for all $k \in \mathbf{N}$.

2. **Auxiliary results.** For the proof of our theorem we require two lemmas (Lemmas 2 and 3) in addition to Theorem A. Lemma 1 which is a weak version of the well-known Cohn rule [2, p. 7] is needed for the proof of Lemma 3.

LEMMA 1. *If* $|\lambda_0| > |\lambda_n|$, *then the polynomial*

$$\Lambda(z) := \lambda_0 + \lambda_1 z + \ldots + \lambda_n z^n$$

(of degree $\leqq n$*) can have a zero in* $\bar{D}(0; 1)$ *only if the polynomial*

$$\Lambda_1(z) := \sum_{\nu=0}^{n-1} (\bar{\lambda}_0\lambda_\nu - \lambda_n\bar{\lambda}_{n-\nu})z^\nu$$

(of degree $\leqq n - 1$*) has one also.*

Here is a short proof which we do not claim to be new.

Let $\Lambda_1(z) \neq 0$ in $\bar{D}(0; 1)$. Then Λ cannot have a zero on $|z| = 1$. For if $\Lambda(e^{i\alpha}) = 0$, then

$$\Lambda_1(e^{i\alpha}) = \bar{\lambda}_0\Lambda(e^{i\alpha}) - \lambda_n e^{in\alpha}\overline{\Lambda(e^{i\alpha})} = 0.$$

Hence if $\Lambda^*(z) := z^n\overline{\Lambda(1/\bar{z})}$ then for $|z| = 1$ we have $|\Lambda(z)| = |\Lambda^*(z)| > 0$ and so $|\bar{\lambda}_0\Lambda(z)| > |\lambda_n\Lambda^*(z)|$. By Rouché's theorem Λ has the same number of zeros in $D(0; 1)$ as the function $\Lambda_1(z) = \bar{\lambda}_0\Lambda(z) - \lambda_n\Lambda^*(z)$ and so none. Thus Λ has zeros neither on $|z| = 1$ nor in $D(0; 1)$.

LEMMA 2. *From the given polynomials*

$$A(z) := \sum_{\mu=0}^{m} \binom{m}{\mu} a_\mu z^\mu, \quad B(z) := \sum_{\mu=0}^{m} \binom{m}{\mu} b_\mu z^\mu$$

let us form the third polynomial

$$(A * B)(z) := \sum_{\mu=0}^{m} \binom{m}{\mu} a_\mu b_\mu z^\mu.$$

If all the zeros of A lie in a circular region G, then every zero γ of $A * B$ has the form $\gamma = -\alpha\beta$ where α is a suitably chosen point in G and β is a zero of B.

Lemma 2 is known as the "composition theorem of Szegö"; for a proof see [9] or [5, Chapter IV].

LEMMA 3. *Let $p_4(k; z) := \sum_{\nu=0}^{4} \binom{4}{\nu}(1/(k + \nu))z^\nu$. Then for each $k \in \mathbf{N}$ there exists a number $R_k > 1$ such that $p_4(k; z) \neq 0$ for $z \in D(0; R_k)$. In fact, $p_4(1; z) \neq 0$ for $|z| < 2 \sin \pi/5$ and $p_4(2; z) \neq 0$ for $|z| \leqq 3/(2\sqrt{2})$.*

PROOF. The statement about $p_4(1; z)$ is well-known; it follows from the observation that

$$p_4(1; z) = \frac{1}{5z}\{(z + 1)^5 - 1\}.$$

Applying Lemma 1 to $\Lambda(z) := 2p_4(2; (3/(2\sqrt{2}))z)$ we see that $p_4(2; z)$ cannot vanish in $\bar{D}(0; 3/(2\sqrt{2}))$ if

$$\Lambda_1(z) := \frac{1}{32}\left(\frac{3367}{128} + \frac{1831}{40}\sqrt{2}z + \frac{999}{16}z^2 + \frac{81}{5}\sqrt{2}z^3\right)$$

does not vanish in $\bar{D}(0; 1)$. Again, by Lemma 1, Λ_1 cannot vanish in $\bar{D}(0; 1)$ if

$$\Lambda_2(z) := 68426377 + 78892880\sqrt{2}z + 65244744z^2$$

does not. But it can be easily checked that Λ_2 does not vanish in $\bar{D}(0{:}1)$. Hence the same can be said about Λ_1 and $p_4(2; (3/(2\sqrt{2}))z)$. Thus $p_4(2; z) \neq 0$ for $|z| \leqq 3/(2\sqrt{2})$.

By Lemma 1, $p_4(k; z)$ cannot vanish in $\bar{D}(0; 1)$ if

$$\frac{1}{k+4} + \frac{3k}{(k+1)(k+3)}z + \frac{3k}{(k+2)^2}z^2 + \frac{k}{(k+1)(k+3)}z^3$$

does not. Again, by Lemma 1 it is enough to check that

$$\frac{2k^2+8k+3}{(k+1)(k+3)(k+4)} + \frac{4k}{(k+2)^2}z + \frac{k(2k^2+8k+9)}{(k+1)(k+2)^2(k+3)}z^2$$

does not vanish in $\bar{D}(0; 1)$. This can be done either directly or by yet another application of Lemma 1.

REMARK 2. We must caution the reader against thinking that $p_n(k; z) := \sum_{\nu=0}^{n} \binom{n}{\nu}(1/(k+\nu))z^\nu \neq 0$ in $\bar{D}(0; 1)$ for all $n \in \mathbf{N}$ and all $k \in \mathbf{N}$. In fact, $p_5(1; z)$ has two zeros on $|z| = 1$ and for each $k \geqq 2$ the polynomial $p_5(k; z)$ has zeros both inside the (open) unit disk and outside the (closed) unit disk.

3. **Proof of Theorem 1.** Without loss of generality we may assume $0 < a < 1$. Let $P(z) = (z-a)^k q(z)$. Then $P'(z) = (z-a)^{k-1}\{kq(z) + (z-a)q'(z)\}$. Now let us suppose that the disk $\bar{D}(a; \eta)$ where $\eta > 1 - a$ contains only $k-1$ zeros of P'. Then

$$A(z) := kq(z+a) + zq'(z+a) = \sum_{\nu=0}^{4} \binom{4}{\nu}\frac{k+\nu}{\nu!}\frac{q^{(\nu)}(a)}{\binom{4}{\nu}}z^\nu$$

must have all its zeros in the circular region $G := \hat{\mathbf{C}}\backslash\bar{D}(0; \eta)$. Hence if $B(z) := \sum_{\nu=0}^{4}\binom{4}{\nu}(1/(k+\nu))z^\nu$ then Lemma 2 in conjunction with Lemma 3 implies that $q(z) := (A * B)(z-a)$ has all its zeros in

$$|z-a| > \begin{cases} \dfrac{3}{2\sqrt{2}}\eta & \text{if } k = 2 \\ \eta & \text{if } k \geqq 3. \end{cases}$$

(i) *The case* $k = 2$.

If $\eta := \frac{2}{3}(a + \sqrt{2-a^2})$ then the circle $|z-a| = (3/(2\sqrt{2}))\eta$ cuts the unit circle $|z| = 1$ in the points $((a - \sqrt{2-a^2})/2) \pm i((a + \sqrt{2-a^2})/2))$. Hence the zeros of q lie in the disk $D((a - \sqrt{2-a^2})/2; (a+\sqrt{2-a^2})/2)$ whose boundary passes through the point a. We may now apply Theorem A to conclude that P' has at least k zeros in $|z-a| \leqq \frac{4}{3}((a+\sqrt{2-a^2})/2) = \eta$ which contradicts the assumption that "$\bar{D}(a; \eta)$ contains only $k-1$ zeros of P'".

(ii) *The case* $k \geqq 3$.

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For $(1 - a)/2 < b < 1$, the circle $|z + b| = a + b$ intersects the unit circle $|z| = 1$ in the points $(a^2 + 2ab - 1)/2b \pm i\sqrt{1 - ((a^2 + 2ab - 1)/2b)^2}$ whose distance from the point a is $\sqrt{((1 - a^2)(a + b))/b}$ which is equal to $(\sqrt{(k + 1)^2 - a^2(2k + 1)} + ak)/(k + 1)$ if

$$b = (\sqrt{(k + 1)^2 - a^2(2k + 1)} - ak)/2k.$$

Hence if P' has only $k - 1$ zeros in

$$|z - a| \leqq (\sqrt{(k + 1)^2 - a^2(2k + 1)} + ak)/(k + 1)$$

then from the observation made above about the zeros of q it follows that they all lie inside the disk $D(-(\sqrt{(k + 1)^2 - a^2(2k + 1)} - ak)/2k;\ (\sqrt{(k + 1)^2 - a^2(2k + 1)} + ak)/2k)$ whose boundary passes through the point a. Now Theorem A implies that P' has at least k zeros in $\bar{D}(a;\ (\sqrt{(k + 1)^2 - a^2(2k + 1)} + ak)/(k + 1))$.

4. **Conclusion.** Putting together the result proved in [11] and Theorem 1 we now know that $\rho(n;\ k;\ a) \leqq 2k/(k + 1)$ for all $a \in \bar{D}(0;\ 1)$ and $k + 1 \leqq n \leqq k + 4$.

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Département de Mathématiques et de Statistique
Université de Montréal
Montréal, PQ H3C 3J7