
The First k -Regular Subgraph is Large

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Let $(G_m)_{0 \leq m \leq \binom{n}{2}}$ be the random graph process starting from the empty graph on vertex set $[n]$ and with a random edge added in each step. Let m_k denote the minimum integer such that G_{m_k} contains a k -regular subgraph. We prove that for all sufficiently large k , there exist two constants $\epsilon_k \geq \sigma_k > 0$, with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that asymptotically almost surely any k -regular subgraph of G_{m_k} has size between $(1 - \epsilon_k)|C_k|$ and $(1 - \sigma_k)|C_k|$, where C_k denotes the k -core of G_{m_k} .

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1. Introduction

Let $\mathcal{G}_{n,m}$ denote the probability space of random graphs on n vertices and m edges with the uniform distribution and let $\mathcal{G}(n, p)$ denote the binomial model of random graphs on vertex set $[n] := \{1, 2, \dots, n\}$, where each edge occurs independently with probability p . This paper relates to the research on the existence of k -regular subgraphs of $\mathcal{G}_{n, cn/2}$ (correspondingly $\mathcal{G}(n, c/n)$), when c is slightly greater than c_k , the threshold of the emergence of the k -core. Letzter [8] showed that the property of having a k -regular subgraph has a sharp threshold $p^*(n, k)$ in $\mathcal{G}(n, p)$ for all $k \geq 3$, where $p^*(n, k)$ is of order $\Theta(1/n)$. Whether the limit of $np^*(n, k)$ exists is not known. Let $\bar{c}_k = \limsup_{n \rightarrow \infty} np^*(n, k)$ and $\underline{c}_k = \liminf_{n \rightarrow \infty} np^*(n, p)$. It is obvious that a k -regular subgraph exists only if the k -core is non-empty, whereas the converse is not necessarily true except for $k \leq 2$. Thus, $\underline{c}_k \geq c_k$. Bollobás, Kim and Verstraëte [2] proved that \underline{c}_3 is strictly greater than c_3 . However, whether \underline{c}_k is strictly greater than c_k , for general $k \geq 4$, is not clear. Bollobás, Kim and Verstraëte conjectured that this strict inequality holds also for all $k \geq 4$. However, Pretti and Weigt [15] claimed the opposite: for all $k \geq 4$, c_k and \bar{c}_k are equal (thus $\underline{c}_k = \bar{c}_k = c_k$). Their analysis uses the (non-rigorous) cavity method and certain statistical physics techniques. (Basically they argued that \bar{c}_k is the critical point where a certain entropy becomes zero and that \bar{c}_k is a non-trivial solution of a certain equation, the solution of which coincides with c_k , for

every $k \geq 4$. In the application of the cavity method they assumed some hypothesis on the stability of replica symmetry, for which they only managed to provide numerical evidence instead of a mathematical proof.)

The most recent progress on this topic consists of improvements of upper bounds of \bar{c}_k . Prałat, Verstraëte and Wormald [14] proved that for all sufficiently large k , $\bar{c}_k \leq c_{k+2}$, by showing that for any c slightly greater than c_{k+2} , asymptotically almost surely (a.a.s.) the $(k + 2)$ -core contains a k -factor (or a k -regular subgraph that spans all but at most $k - 1$ vertices of the $(k + 2)$ -core). More recently, Chan and Molloy [4] improved it further to $\bar{c}_k \leq c_{k+1}$, for all sufficiently large k . Thus, the currently best known bounds of \underline{c}_k and \bar{c}_k are $c_k \leq \underline{c}_k \leq \bar{c}_k \leq c_{k+1}$, for large k .

In this paper, we study the size of the first k -regular subgraph in the graph evolution process defined as follows. Let $G_0, G_1, \dots, G_{\binom{n}{2}}$ be a graph process where G_0 is the empty graph on vertex set $[n]$ and for every $1 \leq i \leq \binom{n}{2}$, G_i is obtained by adding one edge x to G_{i-1} , where x is chosen uniformly at random from all edges not in G_{i-1} . Therefore, $G_0 \subseteq G_1 \subseteq \dots \subseteq G_{\binom{n}{2}}$, and for every $0 \leq m \leq \binom{n}{2}$, G_m is distributed exactly as $\mathcal{G}_{n,m}$. This random graph process was first introduced by Erdős and Rényi; for details, see [5]. Let m_k denote the minimum integer such that G_{m_k} contains a k -regular subgraph. What is the size of a typical k -regular subgraph in G_{m_k} ? Obviously, any k -regular subgraph of G_{m_k} is a subgraph of \mathcal{C}_k , the k -core of G_{m_k} . For $k = 3$, an observation by Pretti and Weigt [15] suggests that the first 3-regular subgraph contains around 24% of the vertices in G_{m_3} . (They claimed that they will identify the size of the first k -regular subgraph for general $k \geq 4$ in a following publication, but we have not found a paper on that.) In this paper, we prove that for all sufficiently large k , asymptotically almost surely (a.a.s.) the first k -regular subgraph that appears in the random graph process misses at most $\epsilon_k |\mathcal{C}_k|$ vertices of \mathcal{C}_k , where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. On the other hand, with a simple second moment argument, we will show that a.a.s. this k -regular subgraph must miss at least $\Omega(|\mathcal{C}_k|)$ vertices of \mathcal{C}_k .

It follows immediately as a corollary that for any c slightly greater than c_k , a.a.s. the sizes of all k -regular subgraphs of $\mathcal{G}_{n,cn/2}$ (or $\mathcal{G}(n, c/n)$), if there exists one, lie between $(1 - \epsilon_k)|\mathcal{C}_k|$ and $(1 - \sigma_k)|\mathcal{C}_k|$, where $\epsilon_k \geq \sigma_k > 0$ and $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. Note that this does not confirm the existence of a k -regular subgraph in $\mathcal{G}_{n,cn/2}$ and thus does not confirm or disprove that $\bar{c}_k = c_k$ for $k \geq 4$. Hence, either to rigorously prove Pretti and Weigt's prediction or to prove the conjecture by Bollobás, Kim and Verstraëte, we only need to restrict our future investigation to the existence of k -regular subgraphs with size in a narrow range $(1 - \epsilon_k)|\mathcal{C}_k|$ and $(1 - \sigma_k)|\mathcal{C}_k|$, where $\epsilon_k \geq \sigma_k > 0$ and $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

2. Main results

Let n, M and k be positive integers such that $M \geq kn$ is even. Let $\mathcal{M}(n, M, k)$ denote the probability space of random multigraphs on vertex set $[n]$ with $M/2$ edges whose end-vertices are independently and chosen uniformly at random (u.a.r.) from $[n]$, conditional on that each vertex has degree at least k . Let $\mathcal{H}(n, M, k)$ denote the probability space of $\mathcal{M}(n, M, k)$ restricted to simple graphs. It is well known (see [3]) that the k -core of $\mathcal{G}_{n,m}$ (and $\mathcal{G}(n, p)$) is distributed as $\mathcal{H}(n', 2m', k)$, conditional on the number of vertices and edges in the k -core being n' and m' .

Given a graph G and a positive integer k , let $C_k(G)$ denote the k -core of G . For a sequence of probability spaces indexed by n (e.g., $\mathcal{M}(n, M, k)$ and $\mathcal{H}(n, M, k)$), we say an event A_n is true asymptotically almost surely (a.a.s.) if the probability that A_n holds goes to 1 as $n \rightarrow \infty$. For two functions $f(n)$ and $g(n)$ of n , we write $f(n) = O(g(n))$ if there is a constant $C > 0$ such that $|f(n)| \leq C|g(n)|$ for large n . We write $f(n) = o(g(n))$ if $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$. All unspecified asymptotics refer to $n \rightarrow \infty$. Some asymptotics refer to $k \rightarrow \infty$. In the latter case, we will always specify it.

In places where the parameter under discussion is clearly integral (e.g., $m = cn/2$ should be an integer), we omit the floor function if the error caused by omitting it does not affect the analysis.

We will prove two results on the sizes of the k -regular subgraphs of $\mathcal{H}(n, M, k)$, when M/n is in a certain range.

Theorem 2.1. *Let $G \sim \mathcal{H}(n, M, k)$, where $d = M/n = k + o(k)$ (as $k \rightarrow \infty$) and $d \geq k + \frac{1}{2}\sqrt{k \log k}$. Then, for every $\epsilon > 0$, there exists $K > 0$ such that for all $k \geq K$, a.a.s. there is no k -regular subgraph of G whose size is between ϵn and $(1 - \epsilon)n$.*

Theorem 2.2. *Let $k \geq 3$ be a fixed integer. Let $G \sim \mathcal{H}(n, M, k)$, where $M = O(n)$. Then there exists $\epsilon > 0$, such that a.a.s. there is no k -regular subgraph of G whose size is more than $(1 - \epsilon)n$.*

Now we consider the random graph process $(G_m)_{0 \leq m \leq \binom{n}{2}}$ defined in Section 1, and recall that m_k denotes the minimum integer such that G_{m_k} contains a k -regular subgraph. In order to apply Theorems 2.1 and 2.2 and show that the same conclusions also hold for the k -core of G_{m_k} , we need to show that the average degree of the k -core of G_{m_k} satisfies the hypotheses of Theorems 2.1 and 2.2.

For $k \geq 3$, define

$$f_k(\mu) = \sum_{i \geq k} e^{-\mu} \frac{\mu^i}{i!}, \quad \text{and} \quad h_k(\mu) = \frac{\mu}{f_{k-1}(\mu)}, \tag{2.1}$$

and let

$$c_k = \inf\{h_k(\mu), \mu > 0\}.$$

For any $c \geq c_k$, define $\mu_{c,k}$ to be the larger solution of $h_k(\mu) = c$. (There are two solutions for any $c > c_k$ and there is a unique solution when $c = c_k$.) In particular, let $\mu_k = \mu_{c_k,k}$ for $k \geq 3$ and let $d_k = \mu_k f_{k-1}(\mu_k) / f_k(\mu_k)$. Pittel, Spencer and Wormald [12] determined the (a.a.s.) size and density of the k -core for any $c > c_k$. We cite their result as follows (in a less precise form).

Theorem 2.3. *Let $k \geq 3$ be fixed. Suppose that $c \leq c_k - n^{-1/3}$ and $m = cn/2$. Then a.a.s. $\mathcal{G}_{n,m}$ has an empty k -core. Suppose $c \geq c_k + n^{-1/3}$ and $m = cn/2$. Then, a.a.s. $\mathcal{G}_{n,m}$ has a non-empty k -core with $f_k(\mu_{c,k})n + o(n)$ vertices and $\frac{1}{2}\mu_{c,k}f_{k-1}(\mu_{c,k})n + o(n)$ edges. The same conclusions hold for $\mathcal{G}(n, p)$ with $p = c/n$.*

In the same paper, they estimated the size of the first k -core in the random graph process $(G_i)_{0 \leq i \leq \binom{n}{2}}$ (see [12, Theorems 1 and 3]), from which we can easily deduce the following lemma (by noting that $\mu f_{k-1}(\mu)/f_k(\mu)$ is an increasing function on $\mu > 0$).

Lemma 2.4. *Asymptotically almost surely the number of vertices in $\mathcal{C}_k(G_{m_k})$ is $f_k(\mu_k)n + o(n)$ and the average degree of $\mathcal{C}_k(G_{m_k})$ is $d_k + o(1)$.*

Pittel, Spencer and Wormald [12] also determined that

$$c_k = k + \sqrt{k \log k} + O(\sqrt{k/\log k}).$$

(The error term in [12] is $O(\log k)$, and was corrected in [14].) A more precise expression of c_k was given in [14, Lemma 1], from which we can easily deduce that $c_k < 3k$ for every $k \geq 3$. In the following lemma, we estimate d_k and μ_k and show that they are both close to c_k when k is large.

Lemma 2.5. *As $k \rightarrow \infty$,*

$$d_k = k + \sqrt{k \log k} + O(\sqrt{k/\log k}), \quad \mu_k = k + \sqrt{k \log k} + O(\sqrt{k/\log k}) \text{ and } f_k(\mu_k) \rightarrow 1.$$

By Lemmas 2.4 and 2.5, we see that a.a.s. the average degree of $\mathcal{C}_k(G_{m_k})$ is at least $k + \frac{1}{2}\sqrt{k \log k}$, as required by Theorem 2.1, whereas the condition of its average degree being at most $k + o(k)$ can be easily verified by noting that $\bar{c}_k \leq c_{k+1}$ for large k , proved by Chan and Molloy [4]. It is easy to prove that a.a.s. for all $k \geq 3$, $\mathcal{C}_k(G_{m_k})$ cannot have a k -regular subgraph with at most $\epsilon_0|\mathcal{C}_k|$ vertices, for some small fixed $\epsilon_0 > 0$ (by applying Lemma 5.1 in Section 5). Then, using Theorems 2.1 and 2.2 and by Lemma 2.5, we can prove the following main theorem for the sizes of the k -regular subgraphs of G_{m_k} .

Theorem 2.6. *For all sufficiently large k , there exist $\epsilon_k \geq \sigma_k > 0$ such that $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$ and a.a.s. all k -regular subgraphs of G_{m_k} have size between $(1 - \epsilon_k)|\mathcal{C}_k|$ and $(1 - \sigma_k)|\mathcal{C}_k|$, where $\mathcal{C}_k = \mathcal{C}_k(G_{m_k})$.*

We also have the following result for $\mathcal{G}_{n,m}$ and $\mathcal{G}(n,p)$.

Theorem 2.7. *Let $k \geq 3$. For every constant $c_k < c < 3k$ and $c = k + o(k)$ (with respect to $k \rightarrow \infty$), there exist two constants $\epsilon_k \geq \sigma_k > 0$, with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that a.a.s. all k -regular subgraphs of $\mathcal{G}_{n,cn/2}$ (and $\mathcal{G}(n,c/n)$), if they exist, have size between $(1 - \epsilon_k)|\mathcal{C}_k|$ and $(1 - \sigma_k)|\mathcal{C}_k|$.*

We will prove Theorem 2.1 in Section 3 and Theorem 2.2 in Section 4. We provide proofs of Lemma 2.5 and Theorems 2.6 and 2.7 in Section 5.

3. Proof of Theorem 2.1

Let \mathbf{Y} denote the degree sequence of $\mathcal{M}(n, M, k)$. Cain and Wormald [3] proved that the distribution of \mathbf{Y} is precisely the truncated multinomial $\text{Multi}(n, M, k)$ defined as follows.

Let $\mathcal{D}_{n,M,k}$ denote the set of vectors

$$\left\{ \mathbf{d} = (d_i)_{i=1}^n : \sum_{1 \leq i \leq n} d_i = M, \quad d_i \geq k, \forall 1 \leq i \leq n \right\}.$$

Then, for any $\mathbf{d} \in \mathcal{D}_{n,M,k}$,

$$\mathbb{P}(\mathbf{Y} = \mathbf{d}) = \frac{\binom{M}{d_1, \dots, d_n} / n^M}{\sum_{\mathbf{d} \in \mathcal{D}_{n,M,k}} \binom{M}{d_1, \dots, d_n} / n^M} = \frac{\prod_{i=1}^n 1/d_i!}{\sum_{\mathbf{d} \in \mathcal{D}_{n,M,k}} \prod_{i=1}^n 1/d_i!}. \tag{3.1}$$

The distribution of \mathbf{Y} can be approximated by n independent copies of a truncated Poisson random variable, defined as follows. Let $d = M/n$. Choose $\lambda > 0$ such that $\lambda f_{k-1}(\lambda) = d f_k(\lambda)$. It is easy to see that λ exists and is unique as long as $d > k$, by noting that $\lambda f_{k-1}(\lambda)/f_k(\lambda)$ is an increasing function of $\lambda > 0$. Define $Z_{\geq k}(\lambda)$ to be the random variable with probability function

$$\mathbb{P}(Z_{\geq k}(\lambda) = j) = \frac{e^{-\lambda} \lambda^j}{f_k(\lambda) j!}, \quad \text{for all } j \geq k, \tag{3.2}$$

where

$$f_k(\lambda) = \sum_{i \geq k} e^{-\lambda} \lambda^i / i!$$

is as defined in (2.1).

The following result on approximating \mathbf{Y} by independent copies of $Z_{\geq k}(\lambda)$ can be found in [6, Corollary 5.3].

Proposition 3.1. *Assume $d = M/n = O(1)$ and let λ be chosen such that $\lambda f_{k-1}(\lambda) = d f_k(\lambda)$. Let A_n be a subset of $\mathcal{D}_{n,M,k}$. Let $\mathbb{P}_{TP}(A_n)$ denote the probability that $(Z_1, \dots, Z_n) \in A_n$, where Z_i are independent copies of $Z_{\geq k}(\lambda)$ defined in (3.2) and let $\mathbb{P}_{\mathcal{M}}(A_n)$ denote the probability that $(Y_1, \dots, Y_n) \in A_n$ in $\mathcal{M}(n, M, k)$. Assume that $M - kn \rightarrow \infty$ as $n \rightarrow \infty$. Then*

$$\mathbb{P}_{\mathcal{M}}(A_n) = O(\sqrt{M}) \mathbb{P}_{TP}(A_n).$$

□

Given a degree sequence $\mathbf{d} \in \mathcal{D}_{n,M,k}$, let $g(\mathbf{d})$ denote the number of simple graphs with degree sequence \mathbf{d} . In particular, let $g_k(n)$ denote the number of k -regular graphs on n vertices. The estimation of $g(\mathbf{d})$ has been studied in a few research papers: see [9, 11, 10]. Here we cite the result by McKay [9]. Let $d_{\max} = \max\{d_i : 1 \leq i \leq n\}$. Recall that $\mathbf{d} \in \mathcal{D}_{n,M,k}$. Therefore $\sum_{i=1}^n d_i = M$. If $d_{\max} = o(M^{1/4})$, we have the following asymptotic estimate:

$$g(\mathbf{d}) = \frac{M!}{2^{M/2} (M/2)! \prod_{i=1}^n d_i!} \exp(-\varphi(\mathbf{d})), \tag{3.3}$$

where

$$\varphi(\mathbf{d}) = \sum_{i=1}^n d_i(d_i - 1)/2M + \left(\sum_{i=1}^n d_i(d_i - 1)/2M \right)^2 + O(d_{\max}^4/M). \tag{3.4}$$

The proof of (3.3) uses the configuration model, first introduced by Bollobás [1]. Consider each vertex i as a bin containing d_i points. Take a uniformly random matching of all M points and represent each pair in the matching as an edge in the resulting (multi)graph. Then the total number of matchings is $M!/2^{M/2}(M/2)!$. The resulting (random) graph is not necessarily simple. However, it is easy to see that every simple graph corresponds to exactly $\prod_{i=1}^n d_i!$ distinct matchings. Hence, the above estimate was obtained by proving that the probability that the resulting graph is simple is $\exp(-\varphi(\mathbf{d}))$, which holds when $d_{\max} = o(M^{1/4})$. Of course, if we ignore the probability of the resulting graph being simple, we have the following coarse upper bound of $g(\mathbf{d})$:

$$g(\mathbf{d}) \leq \frac{M!}{2^{M/2}(M/2)! \prod_{i=1}^n d_i!}, \tag{3.5}$$

which holds for any degree sequence \mathbf{d} .

Let $\mathbb{P}_{\mathcal{M}}(\cdot)$ and $\mathbb{E}_{\mathcal{M}}(\cdot)$ denote probability and expectation in the probability space $\mathcal{M}(n, M, k)$ and let $\mathbb{P}_{\mathcal{H}}(\cdot)$ and $\mathbb{E}_{\mathcal{H}}(\cdot)$ denote probability and expectation in $\mathcal{H}(n, M, k)$. Let $G \sim \mathcal{M}(n, M, k)$. Then, for any event A and any random variable X ,

$$\mathbb{P}_{\mathcal{H}}(A) = \frac{\mathbb{P}_{\mathcal{M}}(A \wedge (G \text{ simple}))}{\mathbb{P}_{\mathcal{M}}(G \text{ simple})}, \quad \mathbb{E}_{\mathcal{H}}(X) = \frac{\mathbb{E}_{\mathcal{M}}(XI\{G \text{ simple}\})}{\mathbb{P}_{\mathcal{M}}(G \text{ simple})},$$

as $\mathcal{H}(n, M, k)$ is $\mathcal{M}(n, M, k)$ restricted to simple graphs. The following proposition is a standard method of proving a.s. properties in $\mathcal{H}(n, M, k)$. Instead of proving that some property holds a.s. in $\mathcal{H}(n, M, k)$ directly, it is usually easier to prove that it holds a.s. in $\mathcal{M}(n, M, k)$ instead. This proposition can be found in many papers (e.g., [6]). (In fact, it can be easily deduced from (3.3) and Lemma 3.3 below.)

Proposition 3.2. *Assume $M = O(n)$. Then $\mathbb{P}_{\mathcal{M}}(G \text{ simple}) = \Omega(1)$, where $G \sim \mathcal{M}(n, M, k)$. \square*

Let λ be such that $\lambda f_{k-1}(\lambda) = df_k(\lambda)$, where $d = M/n$. Define

$$\zeta = \lambda^2 f_{k-2}(\lambda) / 2df_k(\lambda). \tag{3.6}$$

Define $\mathcal{D}_{n,M,k}^0$ to be the subset of $\mathcal{D}_{n,M,k}$ satisfying

$$d_{\max} \leq M^{1/4} / \log n, \quad \zeta - 1 / \log n \leq \sum_{i=1}^n d_i(d_i - 1) / 2M \leq \zeta + 1 / \log n. \tag{3.7}$$

Lemma 3.3. *Assume $M = O(n)$. Then $\mathbb{P}_{\mathcal{M}}(\mathbf{Y} \notin \mathcal{D}_{n,M,k}^0) = o(1)$.*

Proof. Let Z_1, \dots, Z_n be n independent copies of $Z_{\geq k}(\lambda)$, where $\lambda f_{k-1}(\lambda) = df_k(\lambda)$ and $d = M/n$. Then $\mathbb{E}Z_i = \lambda^2 f_{k-2}(\lambda) / f_k(\lambda)$. First we bound the probability

$$\mathbb{P}\left(\max_{1 \leq i \leq n} Z_i \geq M^{1/4} / \log n\right).$$

By the definition of $Z_{\geq k}(\lambda)$ in (3.2) and putting $t = M^{1/4}/\log n$, we immediately have

$$\mathbb{P}\left(\max_{1 \leq i \leq n} Z_i \geq t\right) \leq \sum_{i=1}^n \mathbb{P}(Z_i \geq t) = O(ne^{-\lambda} \lambda^t / f_k(\lambda)t!) = O(n(e\lambda/t)^t) = O(e^{-t}). \tag{3.8}$$

Next, we bound the probability that

$$\left| \sum_{i=1}^n Z_i(Z_i - 1)/2M - \zeta \right| > 1/\log n.$$

Let $W_i = Z_i I\{Z_i \leq \log n\}$. Then, for every $1 \leq i \leq n$,

$$\begin{aligned} \mathbb{P}(Z_i \neq W_i) &= \mathbb{P}(Z_i > \log n) = O(e^{-\lambda} \lambda^{\log n} / (\log n)!) < \exp(-\lambda + \log n \log(e\lambda / \log n)) \\ &< \exp\left(-\frac{1}{2} \log n \log \log n\right) = n^{-\log \log n/2}. \end{aligned}$$

We have

$$\mathbb{E}W_i(W_i - 1) = \lambda^2 f_{k-1}(\lambda) / f_k(\lambda) = 2d\zeta.$$

By the Azuma–Hoeffding inequality [7, Theorem 2.25] and noting that $M = dn$ and $|W_i| \leq \log n$ for all i , we have

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{i=1}^n W_i(W_i - 1)/2M - \zeta \right| > \frac{1}{\log n}\right) \\ &= \mathbb{P}\left(\left| \sum_{i=1}^n (W_i(W_i - 1) - \mathbb{E}W_i(W_i - 1)) \right| > \frac{2M}{\log n}\right) \\ &\leq 2 \exp\left(-\frac{2(2M/\log n)^2}{(\log n)^2 n}\right) = 2 \exp\left(-\frac{8d^2 n}{(\log n)^4}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}\left(\left| \sum_{i=1}^n Z_i(Z_i - 1)/2M - \zeta \right| > \frac{1}{\log n}\right) \\ &\leq \mathbb{P}\left(\left| \sum_{i=1}^n W_i(W_i - 1)/2M - \zeta \right| > \frac{1}{\log n}\right) + n\mathbb{P}(Z_1 \neq W_1) \\ &\leq 2 \exp\left(-\frac{8d^2 n}{(\log n)^4}\right) + n^{1-\log \log n/2} = o(1/M). \end{aligned}$$

Hence, by Proposition 3.1, $\mathbb{P}_{\mathcal{M}}(\mathbf{Y} \notin \mathcal{D}_{n,M,k}^0) = o(1)$. □

In the rest of this section, let $\epsilon > 0$ be a small but fixed constant and let B_ϵ denote the event that there is a k -regular subgraph whose size is between ϵn and $(1 - \epsilon)n$. Given a set S , let A_S denote the event that there is a k -regular subgraph on S . Then, by Proposition 3.2

and Lemma 3.3 and by symmetry,

$$\begin{aligned}
 \mathbb{P}_{\mathcal{H}}(B_\epsilon) &\leq \frac{\mathbb{P}_{\mathcal{M}}(B_\epsilon \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0) + \mathbb{P}_{\mathcal{M}}(\mathbf{Y} \notin \mathcal{D}_{n,M,k}^0)}{\mathbb{P}_{\mathcal{M}}(G \text{ simple})} \\
 &= O(\mathbb{P}_{\mathcal{M}}(B_\epsilon \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0)) + o(1) \\
 &= O\left(\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \sum_{S \subseteq [n]: |S|=s} \mathbb{P}_{\mathcal{M}}(A_S \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0)\right) + o(1) \\
 &= O\left(\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \mathbb{P}_{\mathcal{M}}(A_{[s]} \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0)\right) + o(1). \tag{3.9}
 \end{aligned}$$

Let $X_{[s]}$ denote the number of k -regular subgraphs on $S = [s]$. For an arbitrary subset \mathcal{D} of $\mathcal{D}_{n,M,k}^0$, let $\bar{\mathcal{D}} = \mathcal{D}_{n,M,k}^0 \setminus \mathcal{D}$. Then, by the Markov inequality,

$$\begin{aligned}
 \mathbb{P}_{\mathcal{M}}(A_{[s]} \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0) &= \mathbb{P}_{\mathcal{M}}(A_{[s]} \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}) + O(\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})) \\
 &\leq \mathbb{E}_{\mathcal{M}}(X_{[s]} I\{(G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}\}) + O(\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})) \\
 &= \sum_{\mathbf{d} \in \mathcal{D}} \mathbb{E}_{\mathcal{M}}(X_{[s]} \mid (G \text{ simple}) \wedge \mathbf{Y} = \mathbf{d}) \mathbb{P}_{\mathcal{M}}(G \text{ simple} \mid \mathbf{Y} = \mathbf{d}) \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) + O(\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})) \\
 &\leq \sum_{\mathbf{d} \in \mathcal{D}} \mathbb{E}_{\mathcal{H}}(X_{[s]} \mid \mathbf{Y} = \mathbf{d}) \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) + O(\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})) \\
 &\leq \sum_{\mathbf{d} \in \mathcal{D}} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) + O(\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})), \tag{3.10}
 \end{aligned}$$

where $\mathbf{d}' = (d'_i)_{i=1}^n$ is defined by $d'_i = d_i I\{i > s\} + (d_i - k) I\{i \leq s\}$ for all $1 \leq i \leq n$. Note that the last inequality above holds because the number of k -regular graphs on S is $g_k(s)$, whereas the probability that a given k -regular graph on S is a k -regular subgraph of $\mathcal{H}(n, M, k)$ on S , conditional on the degree sequence of $\mathcal{H}(n, M, k)$ being \mathbf{d} , is at most $g(\mathbf{d}')/g(\mathbf{d})$.

In what follows, we will choose appropriate \mathcal{D} so that $\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})$ is sufficiently small, and we will upper-bound

$$\sum_{\mathbf{d} \in \mathcal{D}} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}). \tag{3.11}$$

Let $M_s \geq ks$ and let $\mathcal{M}(n, M, M_s, k)$ denote the probability space of $\mathcal{M}(n, M, k)$ conditioned to $\sum_{i \leq s} Y_i = M_s$. Then the distribution of the degree sequence Y_1, \dots, Y_s , a subsequence of \mathbf{Y} , is precisely the truncated multinomial $\text{Multi}(s, M_s, k)$. Let

$$\mathcal{D}_{n,M,M_s,k} = \left\{ \mathbf{d} \in \mathcal{D}_{n,M,k}^0 : \sum_{i \leq s} d_i = M_s \right\}.$$

Let $\sigma > 0$ be fixed. Define

$$\mathcal{D}_\sigma = \bigcup_{ks \leq M_s \leq (1+\sigma)ds} \mathcal{D}_{n,M,M_s,k}.$$

We will estimate (3.11) and $\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}})$ with $\mathcal{D} = \mathcal{D}_\sigma$. First we upper-bound $\mathbb{P}_{\mathcal{M}}(\bar{\mathcal{D}}_\sigma)$.

Lemma 3.4. Assume $d \geq k + \frac{1}{2}\sqrt{k \log k}$ and $d = k + o(k)$ (as $k \rightarrow \infty$). Take μ which satisfies $\mu f_{k-1}(\mu) = df_k(\mu)$. Then $k \leq \mu = k + o(k)$ for all large k and $\mu/d \rightarrow 1$, as $k \rightarrow \infty$.

Proof. Let $\hat{\mu} = k$. Then $f_k(\hat{\mu}) \geq 1/2$. Hence, for all large k ,

$$\frac{\hat{\mu} f_{k-1}(\hat{\mu})}{f_k(\hat{\mu})} = \hat{\mu} \left(1 + \frac{e^{-\hat{\mu}} \hat{\mu}^{k-1}}{(k-1)! f_k(\hat{\mu})} \right) \leq k + C\sqrt{k} \leq d,$$

for some positive constant C , by Stirling’s approximation. It is easy to check that $xf_{k-1}(x)/f_k(x)$ is an increasing function of x and so we have that $\mu > \hat{\mu} = k$. We also observe that

$$\mu = df_k(\mu)/f_{k-1}(\mu) \leq d = k + o(k).$$

So $\mu/d \rightarrow 1$ as $k \rightarrow \infty$. □

For a set S of vertices, let $\text{deg}(S)$ denote the sum of degrees of vertices in S . The following lemma is a standard concentration result on the degree sum of a set of vertices that is not too small. See [6, Corollary 5.4] for a detailed proof.

Lemma 3.5. Let $G \sim \mathcal{M}(n, M, k)$, where $M = O(n)$ and $d = M/n \geq k$. Take μ which satisfies $\mu f_{k-1}(\mu) = df_k(\mu)$. Assume $\mu/d \rightarrow 1$ as $k \rightarrow \infty$. Then, for every $\sigma > 0$, there exist $K > 0$ and $0 < \alpha < 1$ such that, for all $k > K$, and for any $S \subset [n]$ with $|S| \geq \log^2 n$,

$$\mathbb{P}_{\mathcal{M}}(|\text{deg}(S) - d|S|| > \sigma d|S|) = \alpha^{d|S|}.$$

By Lemmas 3.4 and 3.5, for any fixed $\sigma > 0$, we have

$$\mathbb{P}_{\mathcal{M}}(\overline{\mathcal{D}}_\sigma) \leq \beta^{ds}, \quad \text{for some } 0 < \beta < 1. \tag{3.12}$$

Next, we estimate (3.11). Note that for every $\sigma > 0$, we have $1 + \sigma \geq k/d$, as $d \geq k$. Thus, \mathcal{D}_σ is non-empty. We will prove the following lemma.

Lemma 3.6. Let $G \sim \mathcal{M}(n, M, k)$, where $M = O(n)$ and $d = M/n \geq k + \frac{1}{2}\sqrt{k \log k}$. For every $\sigma > 0$ and $\eta > 1$, define

$$f(\eta, \sigma) = \frac{(1 - 1/\eta)^{\eta/2}}{\sqrt{\eta - 1}} \left(\frac{e^{-\mu}}{f_k(\mu)} \right)^{1/k} \mu^{(1+\sigma)d/k} \left(\frac{e}{(1 + \sigma)d - k} \right)^{(1+\sigma)d/k-1}, \tag{3.13}$$

where μ is the root of

$$\mu f_{k-1}(\mu) = df_k(\mu).$$

Then there is an absolute constant $C > 0$ (independent of k) such that, for every $\sigma > 0$, and for every $\epsilon n \leq s \leq (1 - \epsilon)n$,

$$\sum_{\mathbf{d} \in \mathcal{D}_\sigma} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) \leq 2dsM((1 + C(\sigma + d/k - 1))f(M/ks, \sigma))^{ks}.$$

We leave Lemma 3.6 to be proved in Section 3.1. In this section, we complete the proof of Theorem 2.1 by applying Lemma 3.6.

Let $\delta = s/n$. By (3.10) and Lemma 3.6, for any fixed $\sigma > 0$,

$$\begin{aligned} &\mathbb{P}_{\mathcal{M}}(A_{[s]} \wedge (G \text{ simple}) \wedge \mathbf{Y} \in \mathcal{D}_{n,M,k}^0) \\ &\leq 2dsM((1 + C(\sigma + d/k - 1))f(d/k\delta, 1 + \sigma))^{ks} + O(\mathbb{P}_{\mathcal{M}}(\overline{\mathcal{D}}_\sigma)). \end{aligned}$$

Since $\binom{n}{s} \leq (\delta^{-\delta}(1 - \delta)^{-1+\delta})^n$ eventually for every $\epsilon n \leq s \leq (1 - \epsilon)n$, by (3.9), we have

$$\begin{aligned} \mathbb{P}_{\mathcal{H}}(B_\epsilon) &= O\left(\sum_{\epsilon n \leq s \leq (1-\epsilon)n} 2dsM(\delta^{-1/k}(1 - \delta)^{-\frac{1-\delta}{k\delta}}(1 + C(\sigma + d/k - 1)))f(d/k\delta, \sigma)^{ks}\right) \\ &\quad + O\left(\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \mathbb{P}_{\mathcal{M}}(\overline{\mathcal{D}}_\sigma)\right) + o(1). \end{aligned} \tag{3.14}$$

Lemma 3.7. Assume $d \geq k + \frac{1}{2}\sqrt{k \log k}$. For every $\epsilon > 0$ and every $\sigma > 0$, there exists a $K > 0$ such that, for all $k > K$,

$$\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \mathbb{P}_{\mathcal{M}}(\overline{\mathcal{D}}_\sigma) = o(1).$$

Proof. By (3.12), there exists $0 < \beta < 1$ depending only on σ such that

$$\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \mathbb{P}_{\mathcal{M}}(\overline{\mathcal{D}}_\sigma) \leq \sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \beta^{ds} \leq \sum_{s \geq \epsilon n} \left(\frac{\epsilon n \beta^d}{s}\right)^s \leq \sum_{s \geq \epsilon n} \left(\frac{\epsilon \beta^d}{\epsilon}\right)^s.$$

Choose K sufficiently large that $\epsilon \beta^d / \epsilon < 1$. Then the above summation is $o(1)$. □

Recall the definition of f in (3.13), where μ is the root of $\mu f_{k-1}(\mu) = d f_k(\mu)$.

Lemma 3.8. Assume $d \geq k + \frac{1}{2}\sqrt{k \log k}$ and $d = k + o(k)$ (as $k \rightarrow \infty$). Then, for every $\epsilon > 0$, there exist sufficiently large $K > 0$ and sufficiently small constants $\sigma > 0$ and $\hat{\epsilon} > 0$ such that, for all $k > K$ and for all $\epsilon \leq \delta \leq 1 - \epsilon$,

$$\delta^{-1/k}(1 - \delta)^{-\frac{1-\delta}{k\delta}}(1 + C(\sigma + d/k - 1))f(\eta, \sigma) < 1 - \hat{\epsilon},$$

where $\eta = d/k\delta$.

Proof. By Lemma 3.4, $\mu \geq k$ and so $f_k(\mu) \geq 1/2$. Then, clearly, as $k \rightarrow \infty$,

$$\delta^{-1/k}(1 - \delta)^{-\frac{1-\delta}{k\delta}} \rightarrow 1, \quad \frac{e^{1/k}}{k!^{1/k}} = (1 + o(1))e/k, \quad \frac{\mu}{k} \rightarrow 1, \quad f_k(\mu)^{1/k} \rightarrow 1.$$

Define

$$g(\eta, \sigma) = \delta^{-1/k}(1 - \delta)^{-\frac{1-\delta}{k\delta}}(1 + C(\sigma + d/k - 1))f(\eta, \sigma).$$

(Note that $\delta = d/k\eta$ depends only on η .) Then, as $k \rightarrow \infty$ and $\sigma \rightarrow 0$,

$$g(\eta, \sigma) \rightarrow (\eta - 1)^{\eta/2-1/2} \eta^{-\eta/2} e^{-\mu/k+(1+\sigma)d/k} \left(\frac{\mu}{(1+\sigma)d-k} \right)^{(1+\sigma)d/k-1},$$

since $\sigma + d/k - 1 \rightarrow 0$ as $k \rightarrow \infty$ and $\sigma \rightarrow 0$. Next, we show that

$$e^{-\mu/k+(1+\sigma)d/k} \left(\frac{\mu}{(1+\sigma)d-k} \right)^{(1+\sigma)d/k-1} \rightarrow 1,$$

as $k \rightarrow \infty$ and $\sigma \rightarrow 0$. Let $x_\mu = \mu - k$ and $x_d = d - k$. Then, $x_\mu = o(k)$ and $x_d = o(k)$ by the assumption that $d = k + o(k)$ and Lemma 3.4. So,

$$\begin{aligned} & e^{-\mu/k+(1+\sigma)d/k} \left(\frac{\mu}{(1+\sigma)d-k} \right)^{(1+\sigma)d/k-1} \\ &= e^{(x_d-x_\mu)/k+\sigma(1+x_d/k)} \left(\frac{k+x_\mu}{x_d+\sigma(k+x_d)} \right)^{x_d/k+\sigma(1+x_d/k)} \\ &= \exp((1+o(1))\sigma) \left(\frac{k+x_\mu}{x_d+\sigma(k+x_d)} \right)^{x_d/k+\sigma(1+x_d/k)} \\ &= \exp((1+o(1))\sigma) \exp\left(\left(\frac{x_d}{k} + \sigma(1+x_d/k) \right) \log\left(\frac{1+x_\mu/k}{x_d/k+\sigma(1+x_d/k)} \right) \right), \end{aligned}$$

where the asymptotics above refers to $k \rightarrow \infty$. Since $x_\mu/k \rightarrow 0$ and $x_d/k + \sigma(1+x_d/k) \rightarrow 0$, as $k \rightarrow \infty$ and $\sigma \rightarrow 0$, the above can be arbitrarily small by choosing sufficiently large k and sufficiently small σ . Hence, $g(\eta, \sigma) \rightarrow (\eta - 1)^{\eta/2-1/2} \eta^{-\eta/2}$. It is easy to prove that $(\eta - 1)^{\eta/2-1/2} \eta^{-\eta/2}$ is a strictly decreasing function on $\eta > 1$ with limit 1 as η approaches 1 from above. Since $d > k$ and $\delta \leq 1 - \epsilon$, we have $\eta = d/k\delta \geq 1/(1 - \epsilon)$. Thus, there exists $\epsilon' > 0$, such that $(\eta - 1)^{\eta/2-1/2} \eta^{-\eta/2} < 1 - \epsilon'$ and so $g(\eta, \sigma) < 1 - \hat{\epsilon}$, for some small constant $\hat{\epsilon} > 0$, by choosing sufficiently small σ and sufficiently large k (so that x_d/k and x_μ/k are sufficiently small). Therefore, for any $\epsilon > 0$, there exist sufficiently large $K > 0$ and sufficiently small $\sigma > 0$, such that, for all $k > K$,

$$\delta^{-1/k} (1 - \delta)^{-\frac{1-\delta}{k\delta}} (1 + C(\sigma + d/k - 1)) f(\eta, \sigma) < 1 - \hat{\epsilon}$$

for all $\epsilon \leq \delta \leq 1 - \epsilon$. □

Now we complete the proof of Theorem 2.1. By Lemma 3.8, for every $\epsilon > 0$, there exist constants $\sigma > 0$ and $K > 0$, such that, for all $k > K$,

$$\begin{aligned} & \sum_{\epsilon n \leq s \leq (1-\epsilon)n} 2dsMn (\delta^{-1/k} (1 - \delta)^{-\frac{1-\delta}{k\delta}} (1 + C(\sigma + d/k - 1)) f(\eta, \sigma))^{ks} \\ & < \sum_{\epsilon n \leq s \leq (1-\epsilon)n} \beta^{ks} = o(1), \end{aligned}$$

for some constant $0 < \beta < 1$. Let $\sigma > 0$ be chosen so that the above holds. Then, by Lemma 3.7, provided k is sufficiently large,

$$\sum_{\epsilon n \leq s \leq (1-\epsilon)n} \binom{n}{s} \mathbb{P}_{\mathcal{M}}(\overline{D}_\sigma) = o(1).$$

By (3.14), $\mathbb{P}_{\mathcal{H}}(B_\epsilon) = o(1)$, and so a.a.s. there is no k -regular subgraph in $\mathcal{H}(n, M, k)$ with size between ϵn and $(1 - \epsilon)n$.

3.1. Proof of Lemma 3.6

Recall that for any $\mathbf{d} \in \mathcal{D}_{n, M, M_s, k}$, \mathbf{d}' is defined as

$$d'_i = d_i I\{i > s\} + (d_i - k) I\{i \leq s\}.$$

By (3.5) and (3.3),

$$\begin{aligned} g_k(s) &\leq \frac{(ks)!}{2^{ks/2}(ks/2)!k!^s}, \\ g(\mathbf{d}) &= \frac{M!}{2^{M/2}(M/2)! \prod_{i=1}^n d_i!} \exp(-\varphi(\mathbf{d}) + o(d_{\max}^4/M)), \\ g(\mathbf{d}') &\leq \frac{(M - ks)!}{2^{(M-ks)/2}((M - ks)/2)! \prod_{i=1}^n (d_i - k)!}, \end{aligned}$$

where $\varphi(\mathbf{d})$ is defined in (3.4). By the definition of $\mathcal{D}_{n, M, k}^0$ in (3.7), and the definition of ζ in (3.6), for any $\mathbf{d} \in \mathcal{D}_{n, M, M_s, k} \subseteq \mathcal{D}_{n, M, k}^0$ we have

$$\varphi(\mathbf{d}) = \zeta + \zeta^2 + o(1).$$

Then, for some constant $C > 0$,

$$\begin{aligned} \sum_{\mathbf{d} \in \mathcal{D}_\sigma} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) &= \sum_{M_s=ks}^{(1+\sigma)ds} \sum_{\mathbf{d} \in \mathcal{D}_{n, M, M_s, k}} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) \tag{3.15} \\ &\leq \frac{(ks)! \exp(\zeta + \zeta^2 + o(1))}{2^{ks/2}(ks/2)!k!^s} \frac{2^{M/2}(M/2)!(M - ks)!}{M!2^{(M-ks)/2}((M - ks)/2)!} \\ &\quad \times \sum_{M_s=ks}^{(1+\sigma)ds} \sum_{\mathbf{d} \in \mathcal{D}_{n, M, M_s, k}} \prod_{1 \leq i \leq s} [d_i]_k \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) \\ &\leq C \left(\frac{ks}{M - ks}\right)^{ks/2} \frac{1}{k!^s} \left(1 - \frac{ks}{M}\right)^{M/2(1+\sigma)ds} \sum_{M_s=ks} \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{1 \leq i \leq s} [d_i]_k \mathbb{P}_{\mathcal{M}}(\tilde{\mathbf{Y}} = \mathbf{d}), \end{aligned}$$

where $\tilde{\mathbf{Y}} = (Y_1, \dots, Y_s)$ is a subsequence of \mathbf{Y} . Since the distribution of $\tilde{\mathbf{Y}}$ is $\text{Multi}(s, M_s, k)$, by (3.1),

$$\begin{aligned} \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{1 \leq i \leq s} [d_i]_k \mathbb{P}_{\mathcal{M}}(\tilde{\mathbf{Y}} = \mathbf{d}) &= \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \frac{\prod_{i \leq s} [d_i]_k \prod_{i=1}^s 1/d_i!}{\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s 1/d_i!} \\ &= \frac{1}{\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s 1/d_i!} \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s \frac{1}{(d_i - k)!}. \end{aligned}$$

Obviously, for every $M_s \geq ks$, $\prod_{i=1}^s 1/d_i!$ for $\mathbf{d} \in \mathcal{D}_{s, M_s, k}$ is minimized when

$$\mathbf{d} = (k, k, \dots, k, k + \phi),$$

where $\phi = M_s - ks$. Thus, we immediately have the following lemma.

Lemma 3.9. For every $M_s \geq ks$,

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s \frac{1}{d_i!} \geq k!^{-s+1} (k + \phi)!^{-1} = k!^{-s} \frac{k!}{(k + \phi)!} \geq \frac{k!^{-s}}{(k + \phi)^\phi},$$

where $\phi = M_s - ks$.

When $M_s - ks \rightarrow \infty$, we have a fairly precise estimate of $\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s 1/d_i!$, stated in the following lemma.

Lemma 3.10. Suppose $M_s - ks \rightarrow \infty$ as $n \rightarrow \infty$. Then

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s 1/d_i! = \Theta(1/\sqrt{M_s}) e^{\mu s} \mu^{-M_s} f_k(\mu)^s,$$

where μ satisfies

$$\frac{\mu f_{k-1}(\mu)}{f_k(\mu)} = \frac{M_s}{s}.$$

Proof. Let $Z_i, i \leq s$ be independent copies of the truncated Poisson variables $Z_{\geq k}(\mu)$, defined in (3.2). Then,

$$\mathbb{P}\left(\sum_{i \leq s} Z_i = M_s\right) = \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i \leq s} \frac{e^{-\mu} \mu^{d_i}}{f_k(\mu) d_i!} = \frac{e^{-\mu s} \mu^{M_s}}{f_k(\mu)^s} \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i \leq s} \frac{1}{d_i!}.$$

By the definition of μ ,

$$\mathbb{P}\left(\sum_{i \leq s} Z_i = M_s\right) = \Theta(1/\sqrt{M_s})$$

by [13, Theorem 4(a)]. Hence,

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i \leq s} \frac{1}{d_i!} = \Theta(1/\sqrt{M_s}) e^{\mu s} \mu^{-M_s} f_k(\mu)^s. \quad \square$$

In the next lemma we deduce an upper bound of

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s 1/(d_i - k)!$$

Lemma 3.11. For every $M_s \geq ks$,

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s \frac{1}{(d_i - k)!} \leq \left(\frac{e}{M_s/s - k}\right)^{M_s - ks}.$$

Proof. First we observe that

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s, k}} \prod_{i=1}^s \frac{1}{(d_i - k)!} = \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s - ks, 0}} \prod_{i=1}^s \frac{1}{d_i!}.$$

If $M_s = ks$, the inequality holds trivially as both sides are equal to 1 (by defining $0^0 = 1$). Suppose $M_s > ks$. Let $Z_i, i \leq s$ be independent copies of the Poisson variable $\text{Po}(\lambda)$, where $\lambda = M_s/s - k$. Then

$$\mathbb{P}\left(\sum Z_i = M_s - ks\right) = \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s - ks, 0}} \prod_{i \leq s} \frac{e^{-\lambda} \lambda^{d_i}}{d_i!} = e^{-M_s + ks} \lambda^{M_s - ks} \sum_{\mathbf{d} \in \mathcal{D}_{s, M_s - ks, 0}} \prod_{i \leq s} \frac{1}{d_i!}.$$

Since $\mathbb{P}(\sum Z_i = M_s - ks) \leq 1$, we have

$$\sum_{\mathbf{d} \in \mathcal{D}_{s, M_s - ks, 0}} \prod_{i \leq s} \frac{1}{d_i!} \leq \frac{e^{M_s - ks}}{\lambda^{M_s - ks}}. \quad \square$$

By Lemmas 3.9, 3.10 and 3.11 and (3.15) and using Stirling’s formula, we have

$$\begin{aligned} \sum_{\mathbf{d} \in \mathcal{D}_\sigma} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) &\leq \phi + M \left(\frac{ks}{M - ks}\right)^{ks/2} \frac{1}{k!^s} \left(1 - \frac{ks}{M}\right)^{M/2} \\ &\times \sum_{M_s = ks + \log n}^{(1+\sigma)ds} \left(\frac{e^{-\bar{\mu}}}{f_k(\bar{\mu})}\right)^s \bar{\mu}^{M_s} \left(\frac{e}{M_s/s - k}\right)^{M_s - ks}, \end{aligned} \quad (3.16)$$

where $\bar{\mu}$ satisfies

$$\frac{\bar{\mu} f_{k-1}(\bar{\mu})}{f_k(\bar{\mu})} = \frac{M_s}{s},$$

and

$$\phi = (\log n) M \left(\frac{ks}{M - ks}\right)^{ks/2} \frac{1}{k!^s} \left(1 - \frac{ks}{M}\right)^{M/2} k!^s (k + \log n)^{\log n} \frac{e^{M_s - ks}}{(M_s/s - k)^{M_s - ks}}. \quad (3.17)$$

Note that ϕ corresponds to the contribution to the summation (3.16) from $ks \leq M_s < ks + \log n$.

We will prove that the summand in (3.16) maximizes at $M_s = (1 + \sigma)ds$. First, we prove two technical lemmas.

Lemma 3.12. *Let $\rho > k$ and let $\mu = \mu(\rho)$ be the root of $\mu f_{k-1}(\mu)/f_k(\mu) = \rho$. Then $\mu > \rho - k$ for all $\rho > k$.*

Proof. Let $\hat{\mu} = \rho - k > 0$. Since $f_k(\hat{\mu}) > e^{-\hat{\mu}} \hat{\mu}^k/k!$, we have

$$\frac{\hat{\mu} f_{k-1}(\hat{\mu})}{f_k(\hat{\mu})} = \hat{\mu} + \frac{\hat{\mu} e^{-\hat{\mu}} \hat{\mu}^{k-1}}{(k-1)! f_k(\hat{\mu})} = \hat{\mu} + \frac{k e^{-\hat{\mu}} \hat{\mu}^k}{k! f_k(\hat{\mu})} < \hat{\mu} + k = \rho.$$

Since $\mu f_{k-1}(\mu)/f_k(\mu)$ is an increasing function of μ , we have $\mu > \hat{\mu} = \rho - k$. This completes the proof of the lemma. □

Lemma 3.13. Let $\mu = \mu(\rho)$ be the root of $\mu f_{k-1}(\mu)/f_k(\mu) = \rho$, for $\rho > k$. Then

$$\mu^\rho \frac{e^{-\mu}}{f_k(\mu)} \left(\frac{e}{\rho - k} \right)^{\rho - k}$$

is an increasing function on $\rho > k$.

Proof. Define $g(x) = x f_{k-1}(x)/f_k(x)$. Then $g(x)$ is an increasing function on $x > 0$ and $g(x) \rightarrow k$ as $x \rightarrow 0$. Let $h_1(\rho) = \rho \log \mu - \mu - \log f_k(\mu)$. Then

$$\mu^\rho \frac{e^{-\mu}}{f_k(\mu)} = \exp(h_1(\rho)).$$

Taking the derivative of h_1 with respect to ρ , we have

$$h'_1(\rho) = \log \mu + \frac{\rho}{\mu} \mu'(\rho) - \mu'(\rho) - \frac{f'_k(\mu)}{f_k(\mu)} \mu'(\rho) = \log \mu + \mu'(\rho) \left(\frac{\rho}{\mu} - 1 - \frac{f'_k(\mu)}{f_k(\mu)} \right).$$

Since $f'_k(\mu) = f_{k-1}(\mu) - f_k(\mu)$ for all $k \geq 1$, we have

$$h'_1(\rho) = \log \mu + \mu'(\rho) \left(\frac{\rho}{\mu} - \frac{f_{k-1}(\mu)}{f_k(\mu)} \right).$$

Since $\rho = g(\mu)$, we have $\rho/\mu = f_{k-1}(\mu)/f_k(\mu)$, which implies that $h'_1(\rho) = \log \mu$.

On the other hand,

$$\left(\frac{e}{\rho - k} \right)^{\rho - k} = \exp((\rho - k)(1 - \log(\rho - k))).$$

Let $h_2(\rho) = (\rho - k)(1 - \log(\rho - k))$. Then the derivative of h_2 is $-\log(\rho - k)$.

Thus,

$$\mu^\rho \frac{e^{-\mu}}{f_k(\mu)} \left(\frac{e}{\rho - k} \right)^{\rho - k} = \exp(h_1(\rho) + h_2(\rho)),$$

and

$$h'_1(\rho) + h'_2(\rho) = \log \mu - \log(\rho - k) = \log \left(\frac{\mu}{\rho - k} \right).$$

By Lemma 3.12, the above is greater than 0 for all $\rho > k$. Hence,

$$\mu^\rho \frac{e^{-\mu}}{f_k(\mu)} \left(\frac{e}{\rho - k} \right)^{\rho - k}$$

is an increasing function of ρ on $\rho > k$. □

Now, we upper-bound the summation in (3.16). By Lemma 3.13,

$$\left(\frac{e^{-\bar{\mu}}}{f_k(\bar{\mu})} \right)^s \bar{\mu}^{M_s} \frac{e^{M_s - ks}}{(M_s/s - k)^{M_s - ks}} = \left(\mu^\rho \frac{e^{-\bar{\mu}}}{f_k(\bar{\mu})} \left(\frac{e}{\rho - k} \right)^{\rho - k} \right)^s$$

in (3.16) (taking $\rho = M_s/s$) is maximized at $M_s = (1 + \sigma)ds$. Let $\eta = M/k_s$. Define

$$\hat{f}(\eta, \sigma) = \frac{(1 - 1/\eta)^{\eta/2}}{\sqrt{\eta - 1}} \frac{1}{k!^{1/k}} \left(\frac{e^{-\hat{\mu}}}{f_k(\hat{\mu})} \right)^{1/k} \hat{\mu}^{(1+\sigma)d/k} \left(\frac{e}{(1 + \sigma)d - k} \right)^{(1+\sigma)d/k - 1},$$

where $\widehat{\mu}$ is the root of

$$\widehat{\mu}f_{k-1}(\widehat{\mu})/f_k(\widehat{\mu}) = (1 + \sigma)d.$$

Compare \widehat{f} with f in (3.13). They are almost the same except that μ and $\widehat{\mu}$ are defined differently. By (3.16),

$$\sum_{\mathbf{d} \in \mathcal{D}_\sigma} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) \leq \phi + (1 + \sigma)dsM\widehat{f}(\eta, \sigma)^{ks},$$

where ϕ is defined in (3.17). Now we complete the proof of Lemma 3.6. Comparing the terms in (3.17) with terms in (3.16), it is straightforward to verify that eventually

$$\begin{aligned} &k!^{1/k}(k + \log n)^{\log n/ks} \frac{e^{M_s/ks-1}}{(M_s/s - k)^{M_s/ks-1}} \\ &\leq \left(\frac{e^{-\widehat{\mu}}}{f_k(\widehat{\mu})} \right)^{1/k} \widehat{\mu}^{(1+\sigma)d/k} \frac{e^{(1+\sigma)d/k-1}}{((1 + \sigma)d - k)^{(1+\sigma)d/k-1}} \end{aligned}$$

for every $ks \leq M_s \leq ks + \log n$, since $s \rightarrow \infty$ as $n \rightarrow \infty$. Thus,

$$\sum_{\mathbf{d} \in \mathcal{D}_\sigma} g_k(s) \frac{g(\mathbf{d}')}{g(\mathbf{d})} \mathbb{P}_{\mathcal{M}}(\mathbf{Y} = \mathbf{d}) \leq 2dsM\widehat{f}(\eta, \sigma)^{ks},$$

since $(1 + \sigma)ds + \log n \leq 2ds$ eventually.

By the definition of \widehat{f} , we only need to show that there exists $C > 0$ such that $\widehat{f}(\eta, \sigma) \leq (1 + C(\sigma + d/k - 1))f(\eta, \sigma)$ for every $\sigma > 0$ and $d = k + o(k)$. By Lemma 3.4 and the assumption that $d \geq k + \frac{1}{2}\sqrt{k \log k}$, we have $\mu \geq k$. By the definition of $\widehat{\mu}$, we have

$$\mu \leq \widehat{\mu} = (1 + \sigma)df_k(\widehat{\mu})/f_{k-1}(\widehat{\mu}) \leq (1 + \sigma)d.$$

Thus,

$$1 \leq \frac{\widehat{\mu}}{\mu} \leq (1 + \sigma)\frac{d}{k} \leq 1 + C'(\sigma + d/k - 1),$$

for some constant $C' > 0$ that does not depend on k or σ , since $d = k + o(k)$. Thus,

$$\begin{aligned} \widehat{\mu}^{(1+\sigma)d/k} &= (1 + O(\sigma + d/k - 1))\mu^{(1+\sigma)d/k}, \\ \frac{e^{-\widehat{\mu}/k}}{f_k(\widehat{\mu})^{1/k}} &= (1 + O(\sigma + d/k - 1))\frac{e^{-\mu/k}}{f_k(\mu)^{1/k}}, \end{aligned}$$

where the constants involved in the asymptotics above are independent of σ and k . This completes the proof of Lemma 3.6.

4. Proof of Theorem 2.2

Lemma 4.1. *Let $M = O(n)$ and $M \geq kn$. For every $k \geq 3$, a.a.s. there exists $\Omega(n)$ vertices in $\mathcal{H}(n, M, k)$ whose degrees are at least $k + 1$ and whose neighbours all have degree exactly k .*

Proof. As $M = O(n)$, a.a.s. the numbers of vertices with degree k and $k + 1$ are both $\Omega(n)$ by applying Proposition 3.1, that is, there exist positive constants $C_1 < C_2$ and $D_1 < D_2$ such that a.a.s.

$$C_1n \leq N_k \leq C_2n, \quad D_1n \leq N_{k+1} \leq D_2n, \tag{4.1}$$

where N_i denote the number of vertices with degree $i \in \{k, k + 1\}$. Recall the definition of $\mathcal{D}_{n,M,k}^0$ in (3.7). Let $\mathcal{D}_{n,M,k}^1$ denote the subset of $\mathcal{D}_{n,M,k}^0$ such that

$$C_1n \leq \sum_{i:d_i=k} 1 \leq C_2n \quad \text{and} \quad D_1n \leq \sum_{i:d_i=k+1} 1 \leq D_2n.$$

Let $\mathbf{d} \in \mathcal{D}_{n,M,k}^1$. Conditional on the degree sequence of $\mathcal{H}(n, M, k)$ being \mathbf{d} , $\mathcal{H}(n, M, k)$ can be generated by the configuration model, as described below (3.4). Let $\mathcal{H}_{\mathbf{d}}$ denote the random (multi)graph generated by the configuration model. We prove that the claim in this lemma holds in $\mathcal{H}_{\mathbf{d}}$ for every $\mathbf{d} \in \mathcal{D}_{n,M,k}^1$. Then the lemma follows by (4.1) and Lemma 3.3 and the fact that the probability that $\mathcal{H}_{\mathbf{d}}$ is simple is $\Omega(1)$ (see [9]). Now consider $\mathcal{H}_{\mathbf{d}}$ where $\mathbf{d} \in \mathcal{D}_{n,M,k}^1$. Let $n_k = \sum_{i:d_i=k} 1$ and $n_{k+1} = \sum_{i:d_i=k+1} 1$. Let X denote the number of vertices with degree $k + 1$, whose neighbours all have degree k . There are n_{k+1} ways to choose a vertex v with degree $k + 1$ and $\binom{n_k}{k+1}$ ways to choose $k + 1$ vertices as neighbours of v . For each of these $k + 1$ chosen vertices, there are k ways to choose a point inside the vertex. The number of ways to match those $k + 1$ points to the $k + 1$ points inside v is $(k + 1)!$. Then,

$$\mathbb{E}X = n_{k+1} \binom{n_k}{k+1} k^{k+1} (k + 1)! \prod_{i=0}^k \frac{1}{M - 1 - 2i} = \Omega(n),$$

where $\prod_{i=0}^k \frac{1}{M - 1 - 2i}$ is the probability that a given set of $k + 1$ pairs occurs in the random matching over M points. Next, we compute $\mathbb{E}X(X - 1)$, that is, we compute the expected number of ordered pairs of distinct vertices, both with degree $k + 1$, and both with all neighbours having degree k . We first count pairs (v_1, v_2) such that v_1 is not adjacent to v_2 and $N(v_1) \cap N(v_2) = \emptyset$, where $N(v)$ denotes the set of neighbours of v . There are

$$n_{k+1}(n_{k+1} - 1)$$

ways to choose (v_1, v_2) and

$$\binom{n_k}{k+1} \binom{n_k - k - 1}{k+1}$$

ways to choose the neighbours of v_1 and v_2 . Hence, the expected number of such pairs is

$$\begin{aligned} n_{k+1}(n_{k+1} - 1) \binom{n_k}{k+1} \binom{n_k - k - 1}{k+1} k^{2(k+1)} (k + 1)!^2 \prod_{i=0}^{2k+1} \frac{1}{M - 1 - 2i} \\ = (1 + o(1))(\mathbb{E}X)^2. \end{aligned} \tag{4.2}$$

Next, we bound the number of pairs (v_1, v_2) such that v_1 is adjacent to v_2 . There are

$$n_{k+1}(n_{k+1} - 1)$$

ways to choose (v_1, v_2) and at most

$$\binom{n_k}{k} \binom{n_k}{k}$$

ways to choose the neighbours of v_1 and v_2 . Hence, the expected number of such pairs is at most

$$O\left(n_{k+1}^2 n_k^{2k} \prod_{i=0}^{2k} \frac{1}{M-1-2i}\right) = O(n). \tag{4.3}$$

Last, we bound the number of pairs (v_1, v_2) such that v_1 is not adjacent to v_2 and

$$|N(v_1) \cap N(v_2)| = j, \quad j \geq 1.$$

There are

$$\binom{n_k}{j} \binom{n_k - j}{k+1-j} \binom{n_k - k - 1}{k+1-j} \leq n_k^{2k+2-j}$$

ways to choose the neighbours of v_1 and v_2 and hence the expected number of such pairs is

$$O\left(n_{k+1}^2 n_k^{2k+2-j} \prod_{i=0}^{2k+1} \frac{1}{M-1-2i}\right) = O(n^{2-j}) = O(n). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4), we have $\mathbb{E}X(X-1) = (1+o(1))(\mathbb{E}X)^2$. It then follows that the variance of X is

$$\mathbb{E}X(X-1) + \mathbb{E}X - (\mathbb{E}X)^2 = o((\mathbb{E}X)^2).$$

By Chebyshev's inequality, it follows that a.s. $X = \Omega(n)$ and the claim of the lemma follows thereby. □

Proof of Theorem 2.2. Let $G \in \mathcal{H}(n, M, k)$. A vertex in G has property B if it has degree at least $k+1$ and all its neighbours have degree exactly k . If v has property B , then any k -regular subgraph H of G must miss at least one of its neighbours. By Lemma 4.1, a.s. there are $\Omega(n)$ vertices having property B . On the other hand, each vertex with degree k can be adjacent to at most k vertices that have property B . Hence, any k -regular subgraph of G must miss $\Omega(n)$ vertices of G . □

5. Proofs of Theorems 2.6 and 2.7

Before approaching Theorems 2.6 and 2.7, we first give the proof of Lemma 2.5.

Proof of Lemma 2.5. By the definition of μ_k , $h'_k(\mu_k) = 0$. Since

$$h'_k(x) = \frac{f_{k-1}(x) - xf'_{k-1}(x)}{f_{k-1}(x)^2},$$

and $f'_k(x) = f_{k-1}(x) - f_k(x)$ for all $k \geq 1$, we have

$$f_{k-1}(\mu_k) = \mu_k(f_{k-2}(\mu_k) - f_{k-1}(\mu_k)),$$

that is,

$$\frac{f_{k-2}(\mu_k)}{f_{k-1}(\mu_k)} = \frac{1 + \mu_k}{\mu_k} = 1 + \frac{1}{\mu_k}.$$

On the other hand,

$$f_{k-2}(\mu_k) = f_{k-1}(\mu_k) + e^{-\mu_k} \frac{\mu_k^{k-2}}{(k-2)!}.$$

It follows immediately that

$$\frac{\mu_k e^{-\mu_k} \mu_k^{k-2}}{(k-2)! f_{k-1}(\mu_k)} = 1.$$

Multiplying both sides by $1/(k-1)$, we get

$$\frac{e^{-\mu_k} \mu_k^{k-1}}{(k-1)! f_{k-1}(\mu_k)} = \frac{1}{k-1}. \tag{5.1}$$

The left-hand side is a decreasing function of μ_k on $[k, +\infty)$. Moreover, taking $\hat{\mu} = k$, we have $f_{k-1}(\hat{\mu}) \geq 1/2$, and so

$$\frac{e^{-\hat{\mu}} \hat{\mu}^{k-1}}{(k-1)! f_{k-1}(\hat{\mu})} \leq 2 \frac{e^{-k} k^k}{k!} \leq C/\sqrt{k},$$

for some constant $C > 0$, by Stirling's approximation. Since C/\sqrt{k} is larger than $1/(k-1)$ for all large k , we have $\mu_k \geq \hat{\mu} = k$ for all large k . So, $f_{k-1}(\mu_k) \geq 1/2$.

Let $x = \mu_k - k$ ($x \geq 0$ as $\mu_k \geq k$). Then, using Stirling's formula, we have

$$\begin{aligned} f_{k-1}(\mu_k) &= \mu_k \frac{e^{-\mu_k} \mu_k^{k-2}}{(k-2)!} \sim \frac{(k+x)e^{-x-2}}{\sqrt{2\pi k}} \left(1 + \frac{x+2}{k-2}\right)^{k-2} \\ &= \frac{(k+x)e^{-x-2}}{\sqrt{2\pi k}} \exp\left(x+2 - \frac{(x+2)^2}{2(k-2)} + O(x^3/k^2)\right) \\ &= \frac{\sqrt{k}(1+O(x/k))}{\sqrt{2\pi}} \exp\left(-\frac{(x+2)^2}{2(k-2)} + O(x^3/k^2)\right). \end{aligned}$$

Thus,

$$-\frac{(x+2)^2}{2(k-2)} + O(x^3/k^2) = -\frac{1}{2} \log k + \log(\sqrt{2\pi} f_{k-1}(\mu_k)(1+O(x/k))).$$

It follows immediately that $\mu_k = k + \sqrt{k \log k} + O(\sqrt{k/\log k})$. Then, by (5.1) and the fact that $f_{k-1}(\mu_k), f_k(\mu_k) = \Omega(1)$,

$$\begin{aligned} d_k &= \mu_k \left(1 + \frac{e^{-\mu_k} \mu_k^{k-1}}{(k-1)! f_k(\mu_k)} \right) = \mu_k (1 + O(1/k)) \\ &= \mu_k + O(1) = k + \sqrt{k \log k} + O(\sqrt{k/\log k}). \end{aligned}$$

Since $\mu_k = k + \sqrt{k \log k} + O(\sqrt{k/\log k})$, it is very easy to verify that $f_k(\mu_k) \rightarrow 1$ as $k \rightarrow \infty$. □

Lemma 5.1. *Let $k \geq 3$ and $\epsilon_0 = 1/30e^5$. Assume $p \leq 3k/n$. Then, a.a.s. all subgraphs of $\mathcal{G}(n, p)$ with size $s \leq \epsilon_0 n$ have less than $ks/2$ edges.*

Proof. Given s , let X_s denote the number of sets S with $|S| = s$ that contain at least $ks/2$ edges. There are $\binom{n}{s}$ ways to choose a set of s vertices, and given S with $|S| = s$, the probability that there are at least $ks/2$ edges inside S is at most

$$\binom{\binom{s}{2}}{ks/2} p^{ks/2} \leq \left(\frac{es^2 p}{ks} \right)^{ks/2} \leq (3es/n)^{3s/2}.$$

Hence, using $\binom{n}{s} \leq (en/s)^s$,

$$\mathbb{E}X_s \leq \binom{n}{s} (3es/n)^{3s/2} \leq (3e^2)^s (3es/n)^{s/2} = (27e^5 s/n)^{s/2}.$$

By the choice of ϵ_0 , the expected number of sets S with $|S| \leq \epsilon_0 n$ that contain at least $k|S|/2$ edges is at most

$$\sum_{s \leq \log n} (O(s/n))^{s/2} + \sum_{\log n < s \leq \epsilon_0 n} 3e^2 \left(\frac{3}{4} \right)^{s/2} = o(1).$$

The claim follows by the first moment method. □

Recall that m_k is the minimum integer such that G_{m_k} contains a k -regular subgraph in the graph evolution process. We will use the following theorem to upper-bound m_k .

Theorem 5.2 (Chan and Molloy [4]). *For all sufficiently large k , and for every $c_{k+1} < c < c_{k+1} + 2\sqrt{k \log k}$, a.a.s. the $(k+1)$ -core of $\mathcal{G}(n, c/n)$ contains a k -factor or a k -regular subgraph that expands all but one vertex.*

Proof of Theorem 2.6. By Theorems 2.3 and 5.2 and the monotonicity that if G_i contains a k -regular subgraph then so does G_j for all $j \geq i$, for all sufficiently large k , a.a.s. $\hat{m}_1 \leq m_k \leq \hat{m}_2$, where $\hat{m}_1 = (c_k - n^{-1/3})n/2$ and $\hat{m}_2 = (c_{k+1} + 1)n/2$. Thus, a.a.s. $G_{\hat{m}_1} \subseteq G_{m_k} \subseteq G_{\hat{m}_2}$. Let $\mathcal{C}_k = \mathcal{C}_k(G_{m_k})$. Then \mathcal{C}_k is non-empty. By Lemma 2.4, the average degree of \mathcal{C}_k is at least $d_k + o(1)$ (asymptotics referring to $n \rightarrow \infty$), which is greater than $k + \frac{1}{2}\sqrt{k \log k}$ by Lemma 2.5. Since a.a.s. $G_{m_k} \subseteq G_{\hat{m}_2} \sim \mathcal{G}_{n, \hat{m}_2}$, the average degree of \mathcal{C}_k is

a.a.s. $k + o(k)$ by Theorem 2.3 (by noting that $\mu f_{k-1}(\mu)/f_k(\mu)$ is an increasing function on $\mu > 0$). Then, by Theorem 2.1, there are $(\epsilon_k)_{k \geq 3}$ with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that a.a.s. there is no k -regular subgraph of \mathcal{C}_k with size between $\epsilon_k |\mathcal{C}_k|$ and $(1 - \epsilon_k) |\mathcal{C}_k|$. Since a.a.s. the number of edges in $\mathcal{G}(n, 3k/n)$ is at least \widehat{m}_2 by the definition of \widehat{m}_2 , we can couple $G_{\widehat{m}_2}$ and $\mathcal{G}(n, 3k/n)$ so that a.a.s. $G_{\widehat{m}_2} \subseteq \mathcal{G}(n, 3k/n)$. For details of the coupling, we refer readers to Janson, Łuczak and Ruciński [7]. So, a.a.s. $G_{m_k} \subseteq \mathcal{G}_{n, \widehat{m}_2} \subseteq \mathcal{G}(n, 3k/n)$. By Lemma 5.1, a.a.s. there is no k -regular subgraph of $\mathcal{G}(n, 3k/n)$ (and hence of G_{m_k}) with size at most $\epsilon_0 |\mathcal{C}_k| \leq \epsilon_0 n$, where $\epsilon_0 = 1/4e$. Since $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we have $\epsilon_0 \geq \epsilon_k$ for all sufficiently large k . Then, for all large k , there is no k -regular subgraph with size at most $(1 - \epsilon_k) |\mathcal{C}_k|$. By Theorem 2.2, there exists $\sigma_k > 0$, with $\sigma_k \rightarrow 0$ as $k \rightarrow \infty$, such that a.a.s. there is no k -regular subgraph of \mathcal{C}_k with size greater than $(1 - \sigma_k) |\mathcal{C}_k|$. This completes the proof for Theorem 2.6. \square

Proof of Theorem 2.7. Let $G \sim \mathcal{G}(n, c/n)$ or $G \sim \mathcal{G}_{n, cn/2}$. Consider constant $c_k < c < 3k$ and $c = k + o(k)$. We can couple G and $\mathcal{G}(n, 3k/n)$ so that $G \subseteq \mathcal{G}(n, 3k/n)$ since $c < 3k$. Then, by Lemma 5.1, a.a.s. there is no k -regular subgraph of G with size at most $\epsilon_0 |\mathcal{C}_k| \leq \epsilon_0 n$, where $\epsilon_0 = 1/30e^5$. Let μ be the larger solution of $\mu/f_{k-1}(\mu) = c$. Then $\mu > \mu_k$ and $\mu = k + o(k)$ and $f_{k-1}(\mu) \rightarrow 1$ as $k \rightarrow \infty$. Then, by Theorem 2.1 and Lemma 5.1, there are $(\epsilon_k)_{k \geq 3}$ with $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, such that a.a.s. there is no k -regular subgraph of \mathcal{C}_k with size between $\epsilon_k |\mathcal{C}_k|$ and $(1 - \epsilon_k) |\mathcal{C}_k|$ (by setting $\epsilon_k = 1 - \epsilon_0$ for small values of k that are not considered in Theorem 2.1). Since $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$, we have $\epsilon_0 \geq \epsilon_k$ for all but finitely many k . For each k such that $\epsilon_0 < \epsilon_k$, redefine $\epsilon_k = 1 - \epsilon_0$. Then, for all $k \geq 3$, there is no k -regular subgraph with size at most $(1 - \epsilon_k) |\mathcal{C}_k|$. By Theorem 2.2, there exists $\sigma_k > 0$ with $\sigma_k \rightarrow 0$ as $k \rightarrow \infty$, such that a.a.s. there is no k -regular subgraph of \mathcal{C}_k with size greater than $(1 - \sigma_k) |\mathcal{C}_k|$. This completes the proof for Theorem 2.7. \square

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