INFINITE SERIES AND THE DERIVED SET
OF THE AGGREGATE OF THE FRACTIONAL PARTS
OF ITS PARTIAL SUMS: ADDENDUM

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In the author's paper [1] it was proved that the fractional parts of
the partial sums of an infinite series (of real terms) diverging to $+\infty$ or
$-\infty$, *in which the general term tends to zero*, are everywhere dense in the
closed unit interval. This result was extended to series of infinite
oscillation (see Remark 4.1 of the said paper) on the argument that a
sequence of partial sums having infinite oscillation has a subsequence that
diverges to $+\infty$ or $-\infty$. However, it remains to be shown that this
subsequence is the sequence of partial sums of some infinite series
diverging to $+\infty$ or $-\infty$, *in which the* nth *term tends to zero*; and the
purpose of this addendum is to prove this result in the form of the
following:

THEOREM. *Let* (S_n) *be a sequence of real terms such that*
$S_n - S_{n-1} \to 0$. *If* (S_n) *oscillates between finite and infinite limits or*
between $+\infty$ *and* $-\infty$, *then* (S_n) *has a subsequence* $\left(S_{m_i}\right)_{i=1}^{\infty}$ *which*
diverges to $+\infty$ *or* $-\infty$ *and is such that* $S_{m_i} - S_{m_{i-1}} \to 0$.

Proof. We prove the theorem when the sequence (S_n) oscillates
between a positive number a and $+\infty$. The proofs in other cases are
similar.

We know (see [2], Part I, Chapter 3, Problem 100, p. 23) that for an

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oscillatory series in which the nth term tends to zero the sequence of partial sums is everywhere dense between its limit superior and limit inferior. Thus the terms of (S_n) are everywhere dense between a and ∞ .

Consider the sequence (T_n) where

$$T_n = a + (a/2) + (a/3) + \ldots + (a/n) .$$

This sequence diverges to ∞ and is such that $T_n - T_{n-1} \to 0$. Let (Δ_n) be a sequence of *non-overlapping* open intervals such that $T_n \in \Delta_n$ for every n . Since $a \leq T_n$ for each n , every member of (T_n) can be realised as the limit of a subsequence of (S_n) . Now, since each neighbourhood Δ_n of T_n contains infinitely many members of (S_n) , we can construct a subsequence $(S_{m_i})_{i=1}^{\infty}$ of (S_n) such that $S_{m_i} \in \Delta_i$ for each i . Clearly the subsequence (S_{m_i}) is an increasing one and diverges to ∞ . Also,

$$0 < S_{m_i} - S_{m_{i-1}} < T_{i+1} - T_{i-2} ,$$

so that $S_{m_i} - S_{m_{i-1}} \to 0$.

The proof of the theorem is now complete.

References

[1]   S. Audinarayana Moorthy, "Infinite series and the derived set of the aggregate of the fractional parts of its partial sums", *Bull. Austral. Math. Soc.* 21 (1980), 253-264.

[2]   G. Pólya, G. Szegö, *Problems and theorems in analysis.* Volume I: *Series, integral calculus, theory of functions* (translated by D. Aeppli. Die Grundlehren der mathematischen Wissenschaften, 193. Springer-Verlag, Berlin, Heidelberg, New York, 1972).

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