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REPETITIVE EQUIVALENCES AND TILTING THEORY

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Abstract. Let R be a ring and T be a good Wakamatsu-tilting module with $S = \mathrm{End}(T_R)^{op}$. We prove that T induces an equivalence between stable repetitive categories of R and S (i.e., stable module categories of repetitive algebras $\hat{R}$ and $\hat{S}$). This shows that good Wakamatsu-tilting modules seem to behave in Morita theory of stable repetitive categories as that tilting modules of finite projective dimension behave in Morita theory of derived categories.

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§1. Introduction

Tilting theory plays an important role in the representation theory of Artin algebras. The classical tilting modules were introduced in the early 1980s by Brenner–Butler [6], Bongartz [5] and Happel and Ringel [19]. Beginning with Miyashita [23] and Happel [19], the defining conditions for a classical tilting module were relaxed to tilting modules of arbitrary finite projective dimension, and further were relaxed to arbitrary rings and infinitely generated modules by many authors such as Colby and Fuller [12], Colpi and Trlifaj [14], Angeleri-Hügel and Coelho [1], Bazzoni [4], etc.

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One important result in tilting theory is the famous Brenner–Butler Theorem which shows that a classical tilting module induces a torsion theory counter equivalence (later named in [11, 13]). Precisely, if T is a classical tilting R-module with the endomorphism algebra S, then there is a torsion pair $(\mathcal{B}, \mathcal{A})$ in the R-module category and a torsion pair $(\mathcal{G}, \mathcal{K})$ in the S-module category such that there is an equivalence between $\mathcal{A}$ and $\mathcal{G}$ and an equivalence between $\mathcal{K}$ and $\mathcal{B}$. In this sense, tilting theory may be viewed as a far-reaching way of generalization of the Morita theory of equivalences between module categories. More interesting, when considering the derived category of an algebra, which contains the module category of the algebra as a full subcategory, Happel [18] and later Cline, Parshall and Scott [10] proved that a tilting module of finite projective dimension induces an equivalence between the bounded derived category of the ordinary algebra and the derived category of the endomorphism algebra of the tilting module. This leads to the study of the Morita theory for derived categories, which were completely solved by Rickard [24] through the notion of tilting complexes and by Keller [21] through dg-categories.

A further generalization of tilting modules to tilting modules of possibly infinite projective dimension was given by Wakamatsu [25]. Following [17], such tilting modules of possibly infinite projective dimension are called Wakamatsu-tilting modules. It is known that Wakamatsu-tilting modules also induce some equivalences between certain subcategories of module categories [26]. But Wakamatsu-tilting modules do not induce derived equivalences in general.

However, we will show in this paper that Wakamatsu-tilting modules make more sense when we consider a more general category than the derived category of an algebra, namely, the stable module category of the repetitive algebra of an algebra. Let us call the latter category the stable repetitive category of the algebra. The stable repetitive category is a triangulated category. Moreover, by Happel's result [18], for an Artin algebra R, there is a fully faithful triangle embedding of the bounded derived category of R into the stable repetitive category of R. Moreover, this embedding is an equivalence if and only if the global dimension of R is finite; see [28] for a generalization and a simple proof of this result.

We say that two algebras are repetitive equivalent if there is an equivalence between their stable repetitive categories. It should be noted that repetitive equivalences are more general than derived equivalences. In fact, by results in [2, 7, 24], etc., if two algebras are derived equivalent, then their repetitive algebras are derived equivalent, and hence stably equivalent. Thus, derived equivalences always induce repetitive equivalences. However, repetitive equivalences need not be derived equivalences (see Example 5.3).

The following is the main theorem of this paper.

THEOREM 1. *Let R be an Artin algebra. If T is a good Wakamatsu-tilting R-module with $S = \mathrm{End}(T_R)^{op}$, that is, bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between a complete hereditary cotorsion pair $(\mathcal{B}, \mathcal{A})$ in* mod$R$ *and a complete hereditary cotorsion pair $(\mathcal{G}, \mathcal{K})$ in* mod$S$*, then R and S are repetitive equivalent. The equivalence can be chosen to restrict to the equivalence between $\mathcal{A}$ and $\mathcal{G}$.*

Remark.

(1) The definition of good Wakamatsu-tilting modules is given in Section 3.2. It is still a question for us whether or not all Wakamatsu-tilting modules are good Wakamatsu-tilting modules in general. For algebras of finite representation type, the answer is affirmative.

(2) In a subsequent paper [9] collaborated with Chen, we can prove that the equivalence in the main theorem is indeed a triangle equivalence.

(3) The result shows that good Wakamatsu-tilting modules seem to behave in the Morita theory of stable repetitive categories as that tilting modules of finite projective dimension behave in the Morita theory of derived categories. We hope that this paper could give some spark on the study of the Morita theory of stable repetitive categories, which is clearly a new area and far from being solved.

Conversely, we have the following result.

PROPOSITION 2. *Let R and S be Artin algebras. Assume that there is a triangle equivalence between their stable repetitive categories and that this equivalence restricts to an equivalence between a covariantly finite coresolving subcategory $\mathcal{A}$ in* mod$R$ *and a contravariantly finite resolving subcategory $\mathcal{G}$ in* mod$S$. *Let T be the preimage in* mod$R$ *of S. Then T is a good Wakamatsu-tilting R-module with $S \simeq \mathrm{End}(T_R)^{op}$.*

The paper is organized as follows. After the introduction, we provide basic knowledge on Wakamatsu-tilting modules and repetitive categories in Section 2. Then in Section 3, we introduce good Wakamatsu-tilting modules through cotorsion pair counter equivalences. Some properties and characterizations of good Wakamatsu-tilting modules are presented. Section 4 is devoted to the proof of the main theorem and the proposition in Section 1. Though the proof of the theorem is a little complicated, the main idea is inspired by constructions in [18, Lemma 4.1 in Chap. 3] and [26, Section 1]. Finally, we provide some examples in the last section. In particular, it is shown that every Wakamatsu-tilting module over an algebra of finite representation type is a good Wakamatsu-tilting module and hence induces a repetitive equivalence. We also provide an example of repetitive equivalences but not derived equivalences.

Conventions. Throughout this paper, we always work over Artin algebras and finitely generated right modules unless we claim otherwise. For an algebra R, we denote by mod$R$ the category of all finitely generated R-modules and by proj$R$ (resp., inj$R$) the category of finitely generated projective (resp., injective) R-modules. We denote the usual duality over an Artin algebra R by D.

Let $\mathcal{A}$ be an additive category and $T \in \mathcal{A}$, we use $\mathrm{add}_{\mathcal{A}}T$ to denote the additive closure of T in $\mathcal{A}$, that is, the class of all objects in $\mathcal{A}$ which is isomorphic to a direct summand of finite direct sums of some copies of T.

For two functors $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ and $\mathrm{G} : \mathcal{B} \to \mathcal{C}$, we use GF to denote their composition. While we use $f \cdot g$, or simply just fg, to denote the composition of two homomorphisms $f : A \to B$ and $g : B \to C$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories and $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ be an additive functor, we use KerF to denote the subcategory of $A \in \mathcal{A}$ such that $\mathrm{F}(A) = 0$. Moreover, if $\mathrm{F}_i : \mathrm{A} \to \mathcal{B}$, $i \in I$, is a class of functors, we denote $\mathrm{KerF}_I = \bigcap_{i \in I} \mathrm{KerF}_i$. For instance, $\mathrm{KerExt}_R^{\geqslant 1}(T, -)$ is the subcategory of all $M \in \mathrm{mod}R$ such that $\mathrm{Ext}_R^i(T, M) = 0$ for all $i \geqslant 1$.

We write the elements of direct sums as row vectors.

§2. Wakamatsu-tilting modules and repetitive categories

2.1 Wakamatsu-tilting modules

Recall that an R-module $T \in \text{mod}R$ is *Wakamatsu-tilting* [25] provided that

(1) $\text{End}({}_S T)^{op} \simeq R$, where $S := \text{End}(T_R)^{op}$, and
(2) $\text{Ext}^i_R(T, T) = 0 = \text{Ext}^i_S(T, T) = 0$ for all $i > 0$.

These two conditions are also equivalent to the following two conditions [25, Proposition 3.5]:

(1) $\text{Ext}^i_R(T, T) = 0$ for all $i > 0$ and
(2) there is an exact sequence $0 \to R \to T_0 \to T_1 \to \cdots$, where $T_i \in \text{add}_{\text{mod}R} T$ for all i, which stays exact after applying the functor $\text{Hom}_R(-, T)$.

Note that if T is Wakamatsu-tilting and $S = \text{End}(T_R)^{op}$, then ${}_S T$ is a Wakamatsu-tilting left S-module. In this case, we say that T is a Wakamatsu-tilting S-R-bimodule. It is easy to see that DT is a Wakamatsu-tilting R-S-bimodule in the mean time.

2.1.1 Auslander–Reiten class and co-Auslander–Reiten class

Let $T \in \text{mod}R$ be a Wakamatsu-tilting module with $S = \text{End}(T_R)^{op}$. There are the following two interesting classes associated with Wakamatsu-tilting modules.

The *Auslander–Reiten class* in mod$R$ with respect to the Wakamatsu-tilting module T_R, denoted by $\mathcal{X}_T$, is defined as follows [3].

$\mathcal{X}_T := \{M \in \text{mod}R|$ there is an infinite exact sequence $0 \to M \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots$ such that $\text{Im} f_i \in \text{KerExt}^{\geqslant 1}_R(-, T)$ for each $i \geqslant 0$, where $T_i \in \text{add}_{\text{mod}R} T$ for all $i\}$.

Obviously, it holds that $\mathcal{X}_T \subseteq \text{KerExt}^{\geqslant 1}_R(-, T)$. Moreover, these two classes coincide with each other provided that T is a cotilting R-module.

Dually, the *co-Auslander–Reiten class* in mod$R$ with respect to the Wakamatsu-tilting R-module T, denoted by ${}_T\mathcal{X}$, is defined as follows.

${}_T\mathcal{X} := \{M \in \text{mod}R|$ there is an infinite exact sequence $\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0$ such that $\text{Im} f_i \in \text{KerExt}^{\geqslant 1}_R(T, -)$ for each $i \geqslant 0$, where $T_i \in \text{add}_{\text{mod}R} T$ for all $i\}$.

Similarly, we have that ${}_T\mathcal{X} \subseteq \text{KerExt}^{\geqslant 1}_R(T, -)$ and they coincide with each other provided that T is a tilting R-module.

The following result gives some properties about the Auslander–Reiten class and the co-Auslander–Reiten class for a Wakamatsu-tilting module [3, 22, 26, 27].

Proposition. Let T be a Wakamatsu-tilting R-module with $S = \text{End}(T_R)^{op}$.

(1) *The Auslander–Reiten class $\mathcal{X}_T$ is a resolving subcategory, that is, it contains all projective R-modules and is closed under extensions, kernels of epimorphisms and direct summands.*
(2) *The co-Auslander–Reiten class ${}_T\mathcal{X}$ is a coresolving subcategory, that is, it contains all injective R-modules and is closed under extensions, cokernels of monomorphisms and direct summands.*
(3) $\text{KerExt}^1_R(\mathcal{X}_T, -) = \text{KerExt}^{\geqslant 1}_R(\mathcal{X}_T, -) \subseteq {}_T\mathcal{X}$.
(4) $\text{KerExt}^1_R(-, {}_T\mathcal{X}) = \text{KerExt}^{\geqslant 1}_R(-, {}_T\mathcal{X}) \subseteq \mathcal{X}_T$.
(5) $\text{Hom}_R(T, -)$ *and* $- \otimes_S T$ *induce an (additive) equivalence between the co-Auslander–Reiten class* ${}_T\mathcal{X}$ *in* mod$R$ *and the Auslander–Reiten class* $\mathcal{X}_{DT}$ *in* mod$S$. *The*

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equivalence restricts to an (additive) equivalence between the class $\mathrm{KerExt}_R^{\geqslant 1}(\mathcal{X}_T, -)$ *and the class* $\mathrm{KerExt}_S^{\geqslant 1}(-, {}_{DT}\mathcal{X})$.

Proof. (1) and (2) follow from [3, Section 5]; see also [22].
(3) and (4) follow from [27, Lemma 1.4 and Proposition 1.6].
(5) follows from [26, Proposition 2.14]. ∎

We remark that in case $T = R$, the class $\mathcal{X}_T = \mathcal{X}_R$ is just the class of all Gorenstein projective R-modules. Dually, in case $T = DR$, the class ${}_T\mathcal{X} = {}_{DR}\mathcal{X}$ is just the class of all Gorenstein injective modules. We refer to [15] for more on Gorenstein projective and Gorenstein injective modules.

2.1.2 The following is a characterization of the Auslander–Reiten class and the co-Auslander–Reiten class, by [26, Section 2].

Lemma. Let T *be a Wakamatsu-tilting* R*-module with* $S = \mathrm{End}(T_R)^{op}$. *Assume* $X \in \mathrm{mod}R$.

(1) $X \in {}_T\mathcal{X}$ *if and only if* $X \in \mathrm{KerExt}_R^{>0}(T, -)$, $\mathrm{Hom}_R(T, X) \otimes_S T \simeq X$ *and* $\mathrm{Hom}_R(T, X) \in \mathrm{KerTor}_{>0}^S(-, T)$ *canonically.*
(2) $X \in \mathcal{X}_T$ *if and only if* $X \in \mathrm{KerExt}_R^{>0}(-, T)$, $\mathrm{Hom}_S(\mathrm{Hom}_R(X, T), T) \simeq X$ *and* $\mathrm{Hom}_R(X, T) \in \mathrm{KerExt}_S^{>0}(-, T)$ *canonically.*

2.1.3 Useful isomorphisms

Let T be a Wakamatsu-tilting S-R-bimodule. Then we have the following isomorphisms of bimodules:

$$ {}_SDS_S \simeq {}_ST \otimes_R DT_S \qquad \text{and} \qquad {}_RDR_R \simeq {}_RDT \otimes_S T_R. $$

Given an adjoint pair (F, G) of functors, we denote by $\mathbf{\Gamma}$ the natural adjoint isomorphism

$$ \mathbf{\Gamma} : \mathrm{Hom}(\mathrm{F}(-), -) \simeq \mathrm{Hom}(-, \mathrm{G}(-)). $$

Moreover, for a homomorphism $f : \mathrm{F}(X) \to Y$, we denote by $\mathbf{\Gamma}(f) : X \to \mathrm{G}(Y)$ the image of f under the isomorphism $\mathbf{\Gamma}$. We denote by η and ϵ the unit and counit of this adjoint pair, respectively, that is,

$$ \eta_X = \mathbf{\Gamma}^T(1_{F(X)}) : X \to GF(X) \qquad \text{and} $$
$$ \epsilon_Y = (\mathbf{\Gamma}^T)^{-1}(1_{G(Y)}) : FG(Y) \to Y. $$

In particular, associated with an S-R-bimodule T, we have the following adjoint isomorphism:

$$ \mathbf{\Gamma}^T : \mathrm{Hom}_R(- \otimes_S T, -) \simeq \mathrm{Hom}_S(-, \mathrm{Hom}_R(T, -)). $$

We denote by η^T and ϵ^T the unit and counit of this adjoint pair, respectively, that is, for $X \in \mathrm{mod}S$ and $Y \in \mathrm{mod}R$, respectively,

$$ \eta_X^T = \mathbf{\Gamma}^T(1_{X \otimes_S T}) : X \to \mathrm{Hom}_R(T, X \otimes_S T) \qquad \text{and} $$
$$ \epsilon_Y^T = (\mathbf{\Gamma}^T)^{-1}(1_{\mathrm{Hom}_R(T,Y)}) : \mathrm{Hom}_R(T, Y) \otimes_S T \to Y. $$

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By the naturality of the isomorphism $\mathbf{\Gamma}$, for all homomorphisms $f : X_1 \to X_2$, $g : \mathrm{F}(X_2) \to Y_1$ and $h : Y_1 \to Y_2$, it holds that $\mathbf{\Gamma}(\mathrm{F}(f) \cdot g \cdot h) = f \cdot \mathbf{\Gamma}(g) \cdot \mathrm{G}(h)$.

In particular, for a morphism $g : F(X) \to Y$, by applying $\mathbf{\Gamma}$ to the composition $F(X) \overset{1_{F(X)}}{\to} F(X) \overset{g}{\to} Y$, we have that $\mathbf{\Gamma}(g) = \mathbf{\Gamma}(1_{F(X)}) \cdot \mathrm{G}(g) = \eta_X \cdot \mathrm{G}(g)$. Dually, for a morphism $f : X \to G(Y)$, we have that $\mathbf{\Gamma}^{-1}(f) = F(f)\epsilon_Y$.

2.2 Repetitive algebras and repetitive categories

2.2.1 We recall some basic facts on repetitive algebras mainly from [18].

Let R be an Artin algebra. The repetitive algebra $\widehat{R}$ of R was first introduced in [20] and is defined to be the direct sum $\widehat{R} = \bigoplus_{n\in\mathbb{Z}} R \oplus \bigoplus_{n\in\mathbb{Z}} DR$ with the multiplication given by

$$(a_n, \varphi_n)(b_n, \psi_n)_n = (a_n b_n, a_{n+1}\psi_n + \varphi_n b_n)_n.$$

The repetitive algebra $\widehat{R}$ can be interpreted as the following infinite matrix algebra (without the identity):

$$\begin{pmatrix} \ddots & & & & \\ \ddots & R & & & \\ & DR & R & & \\ & & DR & R & \\ & & & \ddots & \ddots \end{pmatrix}.$$

2.2.2 Consider the following two categories:

(1) $\mathcal{RC}^{\otimes}(R) := \{X = \{X_i, \delta_i^{\otimes}(X)\}_{i\in\mathbb{Z}} \mid X_i \in \mathrm{mod}R$ such that almost all X_i are 0 and $\delta_i^{\otimes}(X) : X_i \otimes_R DR \to X_{i-1}$ satisfying $(\delta_{i+1}^{\otimes}(X) \otimes_R DR) \cdot \delta_i^{\otimes}(X) = 0$, for each $i\}$, where a morphism between two objects X and Y is given by $f = \{f_i : X_i \to Y_i\}$ such that $\delta_i^{\otimes}(X) \cdot f_{i-1} = f_i \otimes_R DR \cdot \delta_i^{\otimes}(Y)$ for all i.

(2) $\mathcal{RC}^{\mathrm{H}}(R) := \{X = \{X_i, \delta_i^{\mathrm{H}}(X)\}_{i\in\mathbb{Z}} \mid X_i \in \mathrm{mod}R$ such that almost all X_i are 0 and $\delta_i^{\mathrm{H}}(X) : X_i \to \mathrm{Hom}_R(DR, X_{i-1})$ satisfying $\delta_{i+1}^{\mathrm{H}}(X) \cdot \mathrm{Hom}_R(DR, \delta_i^{\mathrm{H}}(X)) = 0$, for each $i\}$, where a morphism between two objects X and Y is given by $f = \{f_i : X_i \to Y_i\}$ such that $\delta_i^{\mathrm{H}}(X) \cdot \mathrm{Hom}_R(DR, f_{i-1}) = f_i \cdot \delta_i^{\mathrm{H}}(Y)$ for all i.

One can check that these two categories $\mathcal{RC}^{\otimes}(R)$ and $\mathcal{RC}^{\mathrm{H}}(R)$ are both abelian categories. Moreover, they are indeed equivalent to each other as abelian categories, via the adjoint pair $(-\otimes_R DR, \mathrm{Hom}_R(DR, -))$. Indeed, an object $X = \{X_i, \delta_i^{\otimes}(X)\} \in \mathcal{RC}^{\otimes}(R)$ is equivalent to an object $X = \{X_i, \mathbf{\Gamma}^{DR}(\delta_i^{\otimes}(X))\} \in \mathcal{RC}^{\mathrm{H}}(R)$. We will freely use this equivalence. In particular, we often view objects X in these two categories being of the following form with almost all terms $X_i = 0$

$$\cdots \overset{\delta_{i+1}}{\rightsquigarrow} X_i \overset{\delta_i}{\rightsquigarrow} X_{i-1} \overset{\delta_{i-1}}{\rightsquigarrow} \cdots,$$

where δ_i means $\delta_i^{\otimes}(X)$ (resp., $\delta_i^{\mathrm{H}}(X)$) if $X \in \mathcal{RC}^{\otimes}(R)$ (resp., $X \in \mathcal{RC}^{\mathrm{H}}(R)$). We call it a (bounded chain) repe-complex with the repe-difference δ and denote by $\mathcal{RC}(R)$ the category of all such repe-complexes and call it the repetitive category over R. Note that there is an obvious automorphism $[1] : \mathcal{RC}(R) \to \mathcal{RC}(R)$ defined by $(X[1])_i = X_{i-1}$ for each i.

Note that if $X = \{X_i\}$ is a repe-complex, then $\delta_i^{\otimes}(X) \cdot \delta_{i-1}^{\mathrm{H}}(X) = 0$ since $\mathbf{\Gamma}^{DR}(\delta_i^{\otimes}(X) \cdot \delta_{i-1}^{\mathrm{H}}(X)) = \delta_i^{\mathrm{H}}(X) \cdot \mathrm{Hom}_R(DR, \delta_{i-1}^{\mathrm{H}}(X)) = 0$.

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We say that a repe-complex $X = \{X_i, \delta_i\} \in \mathcal{RC}(R)$ is trivial if each $\delta_i = 0$. The full subcategory of all trivial repe-complexes is denoted by $\mathcal{RC}^{\mathrm{tr}}(R)$. Note that there is a natural forgetful functor from $\mathcal{RC}(R)$ to $\mathcal{RC}^{\mathrm{tr}}(R)$ by forgetting the repe-difference.

Let $\mathcal{C}$ be a class of R-modules, we denote by $\mathcal{RC}(\mathcal{C})$ the class of repe-complexes with terms in $\mathcal{C}$. The notation $\mathcal{RC}^{\mathrm{tr}}(\mathcal{C})$ is defined similarly.

The connection between the repetitive category and the algebra $\widehat{R}$ is that the repetitive category is equivalent to $\mathrm{mod}\widehat{R}$, that is, the module category of the algebra $\widehat{R}$, as abelian categories; see [18] for details. Thus, we may identify $\mathcal{RC}(R) = \mathrm{mod}\widehat{R}$.

2.2.3 As shown in [18], $\widehat{R}$ is a self-injective algebra and then the category $\mathcal{RC}(R)(= \mathrm{mod}\widehat{R})$ is a Frobenius category, where the projective (and also injective) objects are of the form

$$\cdots \overset{\delta_{i+1}}{\rightsquigarrow} P_i \oplus I_i \overset{\delta_i}{\rightsquigarrow} P_{i-1} \oplus I_{i-1} \overset{\delta_{i-1}}{\rightsquigarrow} \cdots ,$$

where $P_i \in \mathrm{proj}R$, $I_i \in \mathrm{inj}R$ and $\delta_i = \left(\begin{smallmatrix} 0 & \delta'_i \\ 0 & 0 \end{smallmatrix}\right)$ such that $\delta'_i : P_i \otimes_R DR \to I_{i-1}$ is an isomorphism (considered in $\mathcal{RC}^{\otimes}(R)$) or, equivalently, $\delta'_i : P_i \to \mathrm{Hom}_R(DR, I_{i-1})$ is an isomorphism (considered in $\mathcal{RC}^{\mathrm{H}}(R)$). Thus, its stable category $\underline{\mathcal{RC}}(R)$ is a triangulated category. We will call it the *stable repetitive category* of $\mathrm{mod}R$ (or simply, of R).

It was shown in [18] that there is a fully faithful triangle embedding from the derived category $\mathcal{D}^b(\mathrm{mod}R)$ to the stable repetitive category $\underline{\mathcal{RC}}(R)$. Moreover, there is a triangle equivalence between $\mathcal{D}^b(\mathrm{mod}R)$ and $\underline{\mathcal{RC}}(R)$ if and only if R has finite global dimension. We note that this result was generalized in [28] and also a simple proof of this result was presented there.

For basic knowledge on triangulated categories, derived categories and the tilting theory, we refer to [18].

§3. Cotorsion pairs and good Wakamatsu-tilting modules

3.1 Cotorsion pair counter equivalences

A pair of subcategories $(\mathcal{B}, \mathcal{A})$ in $\mathrm{mod}R$ is called a cotorsion pair, if $\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{A})$ and $\mathcal{A} = \mathrm{KerExt}^1_R(\mathcal{B}, -)$. A cotorsion pair $(\mathcal{B}, \mathcal{A})$ is called hereditary provided that $\mathcal{B}$ is resolving, or equivalently, $\mathcal{A}$ is coresolving. Moreover, a cotorsion pair $(\mathcal{B}, \mathcal{A})$ is called complete provided that, for each $X \in \mathrm{mod}R$, there exist exact sequences $0 \to X \to A \to B \to 0$ and $0 \to A' \to B' \to X \to 0$ for some $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$. We refer to the book [16] for general results on cotorsion pairs.

Let $(\mathcal{B}, \mathcal{A})$ be a cotorsion pair in $\mathrm{mod}R$ and $(\mathcal{G}, \mathcal{K})$ be a cotorsion pair in $\mathrm{mod}S$. Similar to torsion theory counter equivalences in the Brenner–Butler theorem (see [11, 13]), we say that there is a *cotorsion pair counter equivalence* between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{K})$ provided that there is an equivalence $\mathrm{H} : \mathcal{A} \rightleftarrows \mathcal{G} : \mathrm{T}$ and an equivalence $\mathrm{H}' : \mathcal{K} \rightleftarrows \mathcal{B} : \mathrm{T}'$, all as additive categories. Moreover, we say that two bimodules ${}_SV_R$ and ${}_RV'_S$ represent the cotorsion pair counter equivalence if $\mathrm{H} = \mathrm{Hom}_R(V, -)$, $\mathrm{T} = - \otimes_S V$ and $\mathrm{H}' = \mathrm{Hom}_S(V', -)$, $\mathrm{T}' = - \otimes_R V'$.

There are close relations between Wakamatsu-tilting modules and cotorsion pair counter equivalences, as shown in the following proposition.

Proposition. Let T be a Wakamatsu-tilting R-module with $S = \mathrm{End}(T_R)^{op}$.

(1) *Both pairs $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ and $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ are hereditary cotorsion pairs.*

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(2) *The bimodules* ${}_ST_R$ *and* ${}_RDT_S$ *represent a cotorsion pair counter equivalence between the cotorsion pair* $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ *in* $\mathrm{mod}R$ *and the cotorsion pair* $(\mathcal{X}_{DT}, \mathrm{KerExt}^1_S(\mathcal{X}_{DT}, -))$ *in* $\mathrm{mod}S$.
(3) *The bimodules* ${}_RDT_S$ *and* ${}_ST_R$ *represent a cotorsion pair counter equivalence between the cotorsion pair* $(\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ *in* $\mathrm{mod}S$ *and the cotorsion pair* $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ *in* $\mathrm{mod}R$.

Proof.

(1) follows from [22, Proposition 3.1] and Proposition 2.1.1.
(2) follows from Proposition 2.1.1(5).
(3) is obtained from (2) by replacing ${}_ST_R$ with ${}_RDT_S$. ∎

3.2 Good Wakamatsu-tilting modules

3.2.1 In general, the two cotorsion pairs in Proposition 3.1(1) are not complete. For instance, consider the case $T = R$. Then $\mathcal{X}_R$ is the class of all Gorenstein projective modules (note that we only consider finitely generated modules). It is well known that this class is not a precovering class in general; see, for instance, [29]. Thus, the cotorsion pair $(\mathcal{X}_R, \mathrm{KerExt}^1_R(\mathcal{X}_R, -))$ cannot be complete. Dually, the cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_{DR}\mathcal{X}), {}_{DR}\mathcal{X})$ in case $T = DR$ is not complete in general.

However, the other cotorsion pair of the two cotorsion pairs in Proposition 3.1(1), that is, the cotorsion pair

$$(\mathrm{KerExt}^1_R(-, {}_R\mathcal{X}), {}_R\mathcal{X})(= (\mathrm{proj}R, \mathrm{mod}R))$$

for $T = R$ and the cotorsion pair

$$(\mathcal{X}_{DR}, \mathrm{KerExt}^1_R(\mathcal{X}_{DR}, -))(= (\mathrm{mod}R, \mathrm{inj}R))$$

for $T = DR$, respectively, is clearly complete. This leads to the following general definition.

Definition. A Wakamatsu-tilting bimodule ${}_ST_R$ is said to be **good** if the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between a complete hereditary cotorsion pair $(\mathcal{B}, \mathcal{A})$ in $\mathrm{mod}R$ and a complete hereditary cotorsion pair $(\mathcal{G}, \mathcal{K})$ in $\mathrm{mod}S$. Furthermore, an R-module T is said to be a good Wakamatsu-tilting module if ${}_ST_R$ is a good Wakamatsu-tilting bimodule with $S = \mathrm{End}(T_R)^{op}$.

For example, R and DR are good Wakamatsu-tilting modules. In general, if ${}_ST_R$ is a good Wakamatsu-tilting bimodule, then ${}_RDT_S$ is also a good Wakamatsu-tilting bimodule by the definition and the fact that $DDT = T$.

In general, we do not know if the following question has an affirmative answer.

Question. Are all Wakamatsu-tilting modules good Wakamatsu-tilting modules?

However, we will see in Section 5 that the answer to the above question is ‘yes’ for algebras of finite representation type.

3.2.2 By the definition, we have the following property of Wakamatsu-tilting bimodules.

Proposition. Let ${}_ST_R$ *be a Wakamatsu-tilting bimodule. Assume that* $(\mathcal{B}, \mathcal{A})$ *is a hereditary cotorsion pair in* $\mathrm{mod}R$ *and* $(\mathcal{G}, \mathcal{K})$ *is a hereditary cotorsion pair in* $\mathrm{mod}S$ *such that the bimodules* ${}_ST_R$ *and* ${}_RDT_S$ *represent a cotorsion pair counter equivalence between*

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them. Then

(1) $\mathcal{B} \subseteq \mathcal{X}_T$, $\mathcal{A} \subseteq {}_T\mathcal{X}$ *and* $\mathcal{G} \subseteq \mathcal{X}_{DT}$, $\mathcal{K} \subseteq {}_{DT}\mathcal{X}$
(2) $\mathrm{add}_{\mathrm{mod}R}T = \mathcal{B} \bigcap \mathcal{A}$ *and* $\mathrm{add}_{\mathrm{mod}S}DT = \mathcal{G} \bigcap \mathcal{K}$

Proof. (1) First, we show that $\mathrm{add}_{\mathrm{mod}R}T \subseteq \mathcal{B} \bigcap \mathcal{A}$ and $\mathrm{add}_{\mathrm{mod}S}DT \subseteq \mathcal{G} \bigcap \mathcal{K}$.

Note that all the involved subcategories in (1) are closed under finite direct sums and direct summands. Since $\mathcal{G}$ is resolving, we have that $S \in \mathcal{G}$. By the equivalence between $\mathcal{A}$ and $\mathcal{G}$, we obtain that $T = S \otimes_S T \in \mathcal{A}$. It follows that $\mathrm{add}_{\mathrm{mod}R}T \subseteq \mathcal{A}$. Dually, since $\mathcal{K}$ is coresolving, we have that $DS \in \mathcal{K}$. It follows from the equivalence between $\mathcal{B}$ and $\mathcal{K}$ that $T = \mathrm{Hom}_S(S, T) = \mathrm{Hom}_S(DT, DS) \in \mathcal{B}$. Hence, $\mathrm{add}_{\mathrm{mod}R}T \subseteq \mathcal{B}$ too. Thus, we obtain that $\mathrm{add}_{\mathrm{mod}R}T \subseteq \mathcal{B} \bigcap \mathcal{A}$. Dually, one also has $\mathrm{add}_{\mathrm{mod}S}DT \subseteq \mathcal{G} \bigcap \mathcal{K}$.

Clearly, $\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{A}) = \mathrm{KerExt}^{\geqslant 1}_R(-, \mathcal{A}) \subseteq \mathrm{KerExt}^{\geqslant 1}_R(-, T)$ follows from $\mathrm{add}_{\mathrm{mod}R}$ $T \subseteq \mathcal{B} \bigcap \mathcal{A}$ and the fact that $(\mathcal{B}, \mathcal{A})$ is a hereditary cotorsion pair. Take any $B \in \mathcal{B}$, then $B \otimes_R DT \in \mathcal{K}$. Take an exact sequence $0 \to B \otimes_R DT \to I \to Y \to 0$ with $I \in \mathrm{inj}S = \mathrm{add}_{\mathrm{mod}S}DS$. Since $\mathcal{K}$ is coresolving, we have that $I, Y \in \mathcal{K}$ too. Applying the functor $\mathrm{Hom}_S(DT, -)$, we obtain an induced exact sequence $0 \to \mathrm{Hom}_S(DT, B \otimes_R DT) \to \mathrm{Hom}_S(DT, I) \to \mathrm{Hom}_S(DT, Y) \to 0$ since $\mathcal{K} = \mathrm{KerExt}^1_S(\mathcal{G}, -) \subseteq \mathrm{KerExt}^1_S(DT, -)$ by the fact that $DT \in \mathcal{G}$. Note that $B \simeq \mathrm{Hom}_S(DT, B \otimes_R DT)$, $\mathrm{Hom}_S(DT, I) \in \mathrm{add}_{\mathrm{mod}R}T$ and $\mathrm{Hom}_S(DT, Y) \in \mathcal{B}$, so one can easily see that $B \in \mathcal{X}_T$. Thus, $\mathcal{B} \subseteq \mathcal{X}_T$. By the equivalence in Proposition 3.1(3), we also obtain that $\mathcal{K} \subseteq {}_{DT}\mathcal{X}$.

Now considering the Wakamatsu-tilting module ${}_RDT_S$ and applying the above result, we can obtain that $\mathcal{G} \subseteq \mathcal{X}_{DT}$ and that $\mathcal{A} \subseteq {}_T\mathcal{X}$.

(2) If $X \in \mathcal{B} \bigcap \mathcal{A}$, then $X \in \mathcal{B}$. Following the proof of (1), we obtain that there is an exact sequence $0 \to X \to T_X \to X' \to 0$ with $T_X \in \mathrm{add}_{\mathrm{mod}R}T$ and $X' \in \mathcal{B}$. Since $X \in \mathcal{A}$ too, we have that $\mathrm{Ext}^1_R(X', X) = 0$. It follows that the exact sequence splits. Hence, $X \in \mathrm{add}_{\mathrm{mod}R}T$. Together with the first claim in the proof of (1), we obtain that $\mathrm{add}_{\mathrm{mod}R}T = \mathcal{B} \bigcap \mathcal{A}$. Dually, we also have that $\mathrm{add}_{\mathrm{mod}S}DT = \mathcal{G} \bigcap \mathcal{K}$. ∎

3.2.3 Recall that a subcategory $\mathcal{A} \subseteq \mathrm{mod}R$ is covariantly finite (or a preenveloping class) if for any $X \in \mathrm{mod}R$, there is an object $A_X \in \mathcal{A}$ and a homomorphism $u_X : X \to A_X$ such that $\mathrm{Hom}_R(u_X, A)$ is surjective for any object $A \in \mathcal{A}$; see, for instance, [3]. Dually, a subcategory $\mathcal{B} \subseteq \mathrm{mod}R$ is contravariantly finite (or a precovering class) if for any $X \in \mathrm{mod}R$, there is an object $B_X \in \mathcal{B}$ and a homomorphism $v_X : B_X \to X$ such that $\mathrm{Hom}_R(B, v_X)$ is surjective for any object $B \in \mathcal{B}$.

A cotorsion pair $(\mathcal{B}, \mathcal{A})$ is complete if and only if $\mathcal{A}$ is covariantly finite, if and only if $\mathcal{B}$ is contravariantly finite; see [3, Proposition 1.9].

Let $\mathcal{A}$ be a subcategory of $\mathrm{mod}R$. An R-module T is said to be Ext-projective in $\mathcal{A}$ if $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(-, \mathcal{A})$. Moreover, it is said to be an Ext-projective generator in $\mathcal{A}$ if, for any $A \in \mathcal{A}$, there exists an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_{\mathrm{mod}R}T$ and $A' \in \mathcal{A}$. Dually, an R-module T is said to be an Ext-injective cogenerator in $\mathcal{A}$ if $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(\mathcal{A}, -)$ and, for any $A \in \mathcal{A}$, there exists an exact sequence $0 \to A \to T_A \to A' \to 0$ with $T_A \in \mathrm{add}_{\mathrm{mod}R}T$ and $A' \in \mathcal{A}$.

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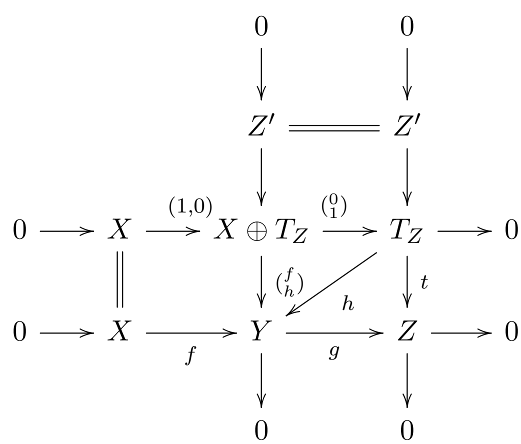
Lemma. Let $\mathcal{A}$ be a subcategory closed under extensions and direct summands.

(1) *Assume that $\mathcal{A}$ has an Ext-projective generator T. If $0 \to X \to Y \to Z \to 0$ is an exact sequence which stays exact after applying the functor $\mathrm{Hom}_R(T, -)$, where $Y, Z \in \mathcal{A}$, then $X \in \mathcal{A}$ too.*

(2) *Assume that $\mathcal{A}$ has an Ext-injective cogenerator T. If $0 \to X \to Y \to Z \to 0$ is an exact sequence which stays exact after applying the functor $\mathrm{Hom}_R(-, T)$, where $X, Y \in \mathcal{A}$, then $Z \in \mathcal{A}$ too.*

Proof. (1) By the assumptions, we can construct the following commutative diagram, where $T_Z \in \mathrm{add}_{\mathrm{mod}R} T$ and $Z' \in \mathcal{A}$:



Since $\mathcal{A}$ is closed under extensions and direct summands, we have that $X \in \mathcal{A}$ from the middle column.

(2) Dually. ∎

3.2.4 *Lemma. Let T be a Wakamatsu-tilting R-module, $S = \mathrm{End}(T_R)^{op}$. Assume that $\mathrm{Hom}_R(T, -) : \mathcal{A} \rightleftarrows \mathcal{G} : - \otimes_S T$ define an equivalence. Then the following are equivalent:*

(1) *$\mathcal{A}$ is coresolving and T is an Ext-projective generator in $\mathcal{A}$.*

(2) *$\mathcal{G}$ is resolving and DT is an Ext-injective cogenerator in $\mathcal{G}$.*

Proof. (1) $\Rightarrow$ (2) The condition that T is an Ext-projective generator in $\mathcal{A}$ means that $T \in \mathcal{A} \subseteq \mathrm{KerExt}_R^1(T, -)$ and that every $A \in \mathcal{A}$ admits an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_{\mathrm{mod}R} T$ and $A' \in \mathcal{A}$. This implies that $\mathcal{A} \subseteq \mathrm{KerExt}_R^{\geqslant 1}(T, -)$. In particular, $\mathcal{A} \subseteq {}_T\mathcal{X}$.

Note that, for any $X \in \mathcal{A}$, there is an exact sequence $0 \to X \to I \to X' \to 0$ with $I \in \mathrm{inj}R \subseteq \mathcal{A}$ and $X' \in \mathcal{A}$ since $\mathcal{A}$ is coresolving. Applying the functor $\mathrm{Hom}_R(T, -)$, we have an exact sequence $0 \to \mathrm{Hom}_R(T, X) \to \mathrm{Hom}_R(T, I) \to \mathrm{Hom}_R(T, X') \to 0$. Since $\mathrm{Hom}_R(T, I) \in \mathrm{add}_{\mathrm{mod}S} DT$ and $\mathrm{Ext}_S^1(\mathrm{Hom}_R(T, X), DT) \simeq \mathrm{Ext}_S^1(\mathrm{Hom}_R(T, X), \mathrm{Hom}_R(T, DR)) = 0$, we obtain that DT is an Ext-injective cogenerator in $\mathrm{Hom}_R(T, \mathcal{A}) = \mathcal{G}$.

It is clear that $\mathcal{G}$ is closed under direct summands. Assume now there is an exact sequence $(\flat) : 0 \to X \to Y \xrightarrow{g} Z \to 0$ with $Z \in \mathcal{G}$, then $Z \in \mathrm{Hom}_R(T, \mathcal{A}) \subseteq \mathrm{KerTor}_1^S(-, T)$. Applying the functor $- \otimes_S T$, we obtain an induced exact sequence $(\flat \otimes_S T) : 0 \to X \otimes_S T \to Y \otimes_S T \xrightarrow{g \otimes_S T} Z \otimes_S T \to 0$.

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Assume first $X \in \mathcal{G}$ too, then $X \otimes_S T \in \mathcal{A}$. It follows that $Y \otimes_S T \in \mathcal{A}$ too since $\mathcal{A}$ is closed under extensions. Note now that there is an exact sequence $0 \to \mathrm{Hom}_R(T, X \otimes_S T) \to \mathrm{Hom}_R(T, Y \otimes_S T) \to \mathrm{Hom}_R(T, Z \otimes_S T) \to 0$, so we have that $\mathrm{Hom}_R(T, Y \otimes_S T) \simeq Y$ since $\mathrm{Hom}_R(T, X \otimes_S T) \simeq X$ and $\mathrm{Hom}_R(T, Z \otimes_S T) \simeq Z$. Thus, $Y \in \mathrm{Hom}_R(T, \mathcal{A}) = \mathcal{G}$. This shows that $\mathcal{G}$ is closed under extensions.

Assume now $Y \in \mathcal{G}$, then $\mathrm{Hom}_R(T, g \otimes_S T) \simeq g$. In particular, we have that $\mathrm{Hom}_R(T, X \otimes_S T) \simeq X$ and the homomorphism $\mathrm{Hom}_R(T, g \otimes_S T)$ is surjective. It follows that the exact sequence $(\flat \otimes_S T)$ stays exact after applying the functor $\mathrm{Hom}_R(T, -)$. By Lemma 3.2.3, we obtain that $X \otimes_S T \in \mathcal{A}$. Hence, $X \in \mathrm{Hom}_R(T, \mathcal{A}) = \mathcal{G}$. This shows that $\mathcal{G}$ is closed under kernels of epimorphisms. Then we see that $\mathcal{G}$ is resolving.

(2) ⇒ (1) Dually. ∎

3.2.5 *Proposition. Let $_S T_R$ be a Wakamatsu-tilting module. Assume that $(\mathcal{B}, \mathcal{A})$ is a hereditary cotorsion pair in* mod$R$ *and that T is an Ext-projective generator in $\mathcal{A}$, then $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ is a hereditary cotorsion pair in* mod$S$. *In particular, the bimodules $_S T_R$ and $_R DT_S$ represent a cotorsion pair counter equivalence between $(\mathcal{B}, \mathcal{A})$ and $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ in this case.*

Proof. Since T is an Ext-projective generator in $\mathcal{A}$, we see that $T \in \mathrm{KerExt}^1_R(-, \mathcal{A}) \bigcap \mathcal{A} = \mathcal{B} \bigcap \mathcal{A}$ and that, for any $A \in \mathcal{A}$, there is an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_{\mathrm{mod}R} T$ and $A' \in \mathcal{A}$. In particular, for any $X \in \mathcal{A} \bigcap \mathcal{B}$, there is an exact sequence $0 \to X' \to T_X \to X \to 0$ with $T_X \in \mathrm{add}_{\mathrm{mod}R} T$ and $X' \in \mathcal{A}$, which is clearly split. Hence, $X \in \mathrm{add}_{\mathrm{mod}R} T$. It follows that $\mathrm{add}_{\mathrm{mod}R} T = \mathcal{B} \bigcap \mathcal{A}$. Moreover, by an argument similar to the one used in the proof of [22, Proposition 2.13(b)], we have that T is also an Ext-injective cogenerator in $\mathcal{B}$. Note that these facts imply that $\mathcal{A} \subseteq {}_T\mathcal{X}$ and that $\mathcal{B} \subseteq \mathcal{X}_T$. In particular, $\mathrm{Hom}_R(T, -) : \mathcal{A} \rightleftarrows \mathrm{Hom}_R(T, \mathcal{A}) : - \otimes_S T$ define an equivalence and $\mathrm{Hom}_S(DT, -) : \mathcal{B} \otimes_R DT \rightleftarrows \mathcal{B} : - \otimes_R DT$ define an equivalence, by Proposition 2.1.1.

The above arguments, in particular, show that Lemma 3.2.4 can be applied to $\mathcal{A}$ (considering the Wakamatsu-tilting bimodule $_S T_R$) and $\mathcal{B}$ (considering the Wakamatsu-tilting bimodule $_R DT_S$); thus, we see that $\mathrm{Hom}_R(T, \mathcal{A})$ is resolving and that $\mathcal{B} \otimes_R DT$ is coresolving. It is also clear that the bimodules $_S T_R$ and $_R DT_S$ represent a counter equivalence between two pairs $(\mathcal{B}, \mathcal{A})$ and $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$, by assumptions. So, it just remains to show that $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ is a cotorsion pair.

We divide the remaining proof into three steps.

Step 1. $\mathrm{Ext}^i_S(\mathrm{Hom}_R(T, A), B \otimes_R DT) = 0$, for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$ and for any $i \geqslant 0$.

Note that there is a natural isomorphism

$$D\mathrm{Hom}_S(S, B \otimes_R DT) \simeq \mathrm{Hom}_R(B, S \otimes_S T).$$

It induces a natural isomorphism, for any $S_i \in \mathrm{add}_{\mathrm{mod}S} S$,

$$(\P) \qquad D\mathrm{Hom}_S(S_i, B \otimes_R DT) \simeq \mathrm{Hom}_R(B, S_i \otimes_S T).$$

Now take $A \in \mathcal{A}$; since T is an Ext-projective generator, there is a long exact sequence

$$(\dagger) \qquad \cdots \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0,$$

where each $T_i \in \mathrm{add}_{\mathrm{mod}R} T$ and each image in $\mathcal{A}$. Here we consider the sequence (†) as a (cochain) complex with the term A at the first position.

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Since $B \in \mathrm{KerExt}_R^1(-, \mathcal{A})$, we have the following induced exact sequence $\mathrm{Hom}_R(B, \dagger)$:

$$\cdots \to \mathrm{Hom}_R(B, T_n) \to \cdots \to \mathrm{Hom}_R(B, T_1) \to \mathrm{Hom}_R(B, T_0) \to \mathrm{Hom}_R(B, A) \to 0.$$

On the other hand, by applying the functor $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, -), B \otimes_R DT)$, we have a complex $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT)$:

$$\cdots \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_n), B \otimes_R DT) \to \cdots \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_1), B \otimes_R DT)$$
$$\to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_0), B \otimes_R DT) \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), B \otimes_R DT) \to 0.$$

Since $\mathrm{Hom}_R(T, \dagger)$ is exact, the functor $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, -), B \otimes_R DT)$ is right exact. By the above isomorphism (¶), we obtain the following isomorphisms of complexes:

$$D\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT) \simeq \mathrm{Hom}_R(B, \mathrm{Hom}_R(T, \dagger) \otimes_S T) \simeq \mathrm{Hom}_R(B, \dagger).$$

But the later is exact, so we obtain that, for $i \geqslant 1$,

$$\mathrm{Ext}_S^i(\mathrm{Hom}_R(T, A), B \otimes_R DT) \simeq \mathrm{H}^i(\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT))$$
$$\simeq D\mathrm{H}^{-i}(D\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT)) \simeq D\mathrm{H}^{-i}(\mathrm{Hom}_R(B, \dagger)) = 0.$$

Thus, Step 1 is established. In particular, we obtain that $\mathrm{Hom}_R(T, \mathcal{A}) \subseteq \mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT)$ and that $\mathcal{B} \otimes_R DT \subseteq \mathrm{KerExt}_S^1(\mathrm{Hom}_R(T, \mathcal{A}), -)$ due to the arbitrarity of $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Step 2. $\mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT) \subseteq \mathrm{Hom}_R(T, \mathcal{A})$.

Take any $Y \in \mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT)$ and a projective resolution of Y, where we consider as a (cochain) complex with the term Y at the zeroth position:

$$(\sharp) \qquad \cdots \xrightarrow{f_{n+1}} S_n \xrightarrow{f_n} \cdots \xrightarrow{f_3} S_2 \xrightarrow{f_2} S_1 \xrightarrow{f_1} S_0 \xrightarrow{f_0} Y \to 0.$$

Note that $DT = R \otimes_R DT \in \mathcal{B} \otimes_R DT$ and that $\mathcal{B} \otimes_R DT$ is coresolving, so we obtain that

$$Y \in \mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT) = \mathrm{KerExt}_S^{\geq 0}(-, \mathcal{B} \otimes_R DT)$$
$$\subseteq \mathrm{KerExt}_S^{\geq 0}(-, DT) = \mathrm{KerTor}^S_{\geq 0}(-, T).$$

Then we have an induced exact sequence:

$$(\sharp \otimes_S T) \qquad \cdots \to S_n \otimes_S T \to \cdots \to S_2 \otimes_S T \to S_1 \otimes_S T \to S_0 \otimes_S T \to Y \otimes_S T \to 0.$$

For any $B \in \mathcal{B}$, applying the left exact functor $\mathrm{Hom}_R(B, -)$, we obtain a complex $\mathrm{Hom}_R(B, \sharp \otimes_S T)$:

$$\cdots \to \mathrm{Hom}_R(B, S_n \otimes_S T) \to \cdots \to \mathrm{Hom}_R(B, S_2 \otimes_S T) \to \mathrm{Hom}_R(B, S_1 \otimes_S T)$$
$$\to \mathrm{Hom}_R(B, S_0 \otimes_S T) \to \mathrm{Hom}_R(B, Y \otimes_S T) \to 0.$$

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Applying the right exact functor $D\mathrm{Hom}_S(-, B \otimes_R DT)$ to the sequence ($\sharp$), we obtain a complex $D\mathrm{Hom}_S(\sharp, B \otimes_R DT)$:

$$\cdots \to D\mathrm{Hom}_S(S_n, B \otimes_R DT) \to \cdots \to D\mathrm{Hom}_S(S_2, B \otimes_R DT)$$
$$\to D\mathrm{Hom}_S(S_1, B \otimes_R DT) \to D\mathrm{Hom}_S(S_0, B \otimes_R DT) \to D\mathrm{Hom}_S(Y, B \otimes_R DT) \to 0,$$

which is indeed exact since $S_i, Y \in \mathrm{KerExt}_S^{\geqslant 0}(-, \mathcal{B} \otimes_R DT)$.

By the natural isomorphism (¶) in Step 1 again, we have isomorphisms of truncated complexes

$$(D\mathrm{Hom}_S(\sharp, B \otimes_R DT))^{<0} \simeq (\mathrm{Hom}_R(B, \sharp \otimes_S T))^{<0},$$

where $(-)^{<0}$ denotes the truncated complex of a complex by replacing the ith term with 0 for all $i \geqslant 0$.

Since $B \in \mathrm{KerExt}_R^1(-, S_i \otimes_S T)$ and $\mathrm{Hom}_R(B, -)$ is left exact, we obtain that, for $i \geqslant 4$, $\mathrm{Ext}_R^1(B, Y_i \otimes_S T) \simeq \mathrm{H}^{-i+2}(\mathrm{Hom}_R(B, \sharp \otimes_S T))$, where $Y_i = \mathrm{Im} f_i$. But the latter homology is 0 by the above isomorphism of truncated complexes and by the fact that the complex $D\mathrm{Hom}_S(\sharp, B \otimes_R DT)$ is exact. This shows that $Y_i \otimes_S T \in \mathrm{KerExt}_R^1(\mathcal{B}, -) = \mathcal{A}$, for all $i \geqslant 4$.

Now consider the exact sequence obtained from ($\sharp \otimes_S T$):

$$0 \to Y_4 \otimes_S T \to S_3 \otimes_S T \to S_2 \otimes_S T \to S_1 \otimes_S T \to S_0 \otimes_S T \to Y \otimes_S T \to 0.$$

As also each $S_i \otimes_S T \in \mathrm{add}_{\mathrm{mod} R} T \subseteq \mathcal{A}$ and $\mathcal{A}$ is coresolving, we obtain that each $Y_i \otimes_S T \in \mathcal{A}$, where $Y_i = \mathrm{Im} f_i$ for $0 \leqslant i \leqslant 3$. Thus, the exact sequence $\sharp \otimes_S T$ is indeed in $\mathcal{A}$. Then we have an induced exact sequence $\mathrm{Hom}_R(T, \sharp \otimes_S T)$:

$$\cdots \to \mathrm{Hom}_R(T, S_n \otimes_S T) \to \cdots \to \mathrm{Hom}_R(T, S_1 \otimes_S T)$$
$$\to \mathrm{Hom}_R(T, S_0 \otimes_S T) \to \mathrm{Hom}_R(T, Y \otimes_S T) \to 0.$$

Since $S_i \simeq \mathrm{Hom}_R(T, S_i \otimes_S T)$ for each i, it follows that

$$Y \simeq \mathrm{Hom}_R(T, Y \otimes_S T) \in \mathrm{Hom}_R(T, \mathcal{A}).$$

This shows that $\mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT) \subseteq \mathrm{Hom}_R(T, \mathcal{A})$. Together with Step 1, we obtain that $\mathrm{KerExt}_S^1(-, \mathcal{B} \otimes_R DT) = \mathrm{Hom}_R(T, \mathcal{A})$.

Step 3. $\mathrm{KerExt}_S^1(\mathrm{Hom}_R(T, \mathcal{A}), -) \subseteq \mathcal{B} \otimes_R DT$.

Note that there is an isomorphism

$$\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), DS) \simeq D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, DS), A),$$

for any $A \in \mathrm{mod} R$ and that it induces an isomorphism

$$\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), I_i) \simeq D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, I_i), A),$$

for any $I_i \in \mathrm{add}_{\mathrm{mod} S} DS$.

Now take any $X \in \mathrm{KerExt}_S^1(\mathrm{Hom}_R(T, \mathcal{A}), -)$ and consider an injective resolution of X:

$$(\natural) \qquad 0 \to X \xrightarrow{g_0} I_0 \xrightarrow{g_1} I_1 \xrightarrow{g_2} I_2 \xrightarrow{g_3} \cdots \xrightarrow{g_n} I_n \xrightarrow{g_{n+1}} \cdots ;$$

here, we consider ($\natural$) as a (cochain) complex with the term X at the zeroth position. Since $DT \simeq \mathrm{Hom}_R(T, DR) \in \mathrm{Hom}_R(T, \mathcal{A})$ and $\mathrm{Hom}_R(T, \mathcal{A})$ is resolving, we obtain that

$$X \in \mathrm{KerExt}_S^1(\mathrm{Hom}_R(T, \mathcal{A}), -) = \mathrm{KerExt}_S^{\geqslant 0}(\mathrm{Hom}_R(T, \mathcal{A}), -) \subseteq \mathrm{KerExt}_S^{\geqslant 0}(DT, -).$$

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Thus, for any $A \in \mathcal{A}$, applying the functor $\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), -)$, we have an induced exact complex $\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), \natural)$:

$$\begin{aligned}0 \to \mathrm{Hom}_S(\mathrm{Hom}_R(T, A), X) &\longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T, A), I_0)\\ \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T, A), I_1) &\longrightarrow \cdots \longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T, A), I_n) \longrightarrow \cdots .\end{aligned}$$

On the other hand, by applying the functor $D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, -), A)$, we also have the following induced complex $D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, \natural), A)$:

$$\begin{aligned}0 \to D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, X), A) &\longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, I_0), A)\\ \longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, I_1), A) &\longrightarrow \cdots \longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, I_n), A) \longrightarrow \cdots .\end{aligned}$$

Since $\mathrm{Hom}_S(DT, I_i) \in \mathrm{add}_{\mathrm{mod}R}\mathrm{Hom}_S(DT, DS) = \mathrm{add}_{\mathrm{mod}R}T$ and $A \in \mathrm{KerExt}^1_R(T, -)$ and since $\mathrm{Hom}_S(DT, \natural)$ is exact and $\mathrm{Hom}_R(-, A)$ is left exact, we can obtain that, for any $i \geqslant 2$,

$$\mathrm{Ext}^1_R(\mathrm{Hom}_S(DT, X_{i+1}), A) \simeq \mathrm{H}^{-i}(\mathrm{Hom}_R(\mathrm{Hom}_S(DT, \natural), A)),$$

where $X_i := \mathrm{Im} g_i$. However, by the above-mentioned isomorphism $(*)$ and the fact that $\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), \natural)$ is exact, we further have that

$$\begin{aligned}\mathrm{H}^{-i}(\mathrm{Hom}_R(\mathrm{Hom}_S(DT, \natural), A)) &\simeq D\mathrm{H}^i(D\mathrm{Hom}_R(\mathrm{Hom}_S(DT, \natural), A))\\ &\simeq D\mathrm{H}^i(\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), \natural)) = 0.\end{aligned}$$

It follows that $\mathrm{Hom}_S(DT, X_{i+1}) \in \mathrm{KerExt}^1_R(-, \mathcal{A}) = \mathcal{B}$ for any $i \geqslant 2$. Since $\mathcal{B}$ is resolving and $\mathrm{Hom}_S(DT, I_i) \in \mathrm{add}_{\mathrm{mod}R}T \subseteq \mathcal{B}$, we also obtain that each $\mathrm{Hom}_S(DT, X_i) \in \mathcal{B}$ for each $0 \leqslant i \leqslant 2$, where $X_0 := X$ and $X_i := \mathrm{Im} g_i$ for $i = 1, 2$, from the exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_S(DT, X) &\to \mathrm{Hom}_S(DT, I_0) \to \mathrm{Hom}_S(DT, I_2)\\ &\to \mathrm{Hom}_S(DT, I_3) \to \mathrm{Hom}_S(DT, X_3) \to 0.\end{aligned}$$

The above arguments show that the exact sequence $\mathrm{Hom}_S(DT, \natural)$ is indeed in $\mathcal{B}$. Then we have an induced exact sequence $\mathrm{Hom}_S(DT, \natural) \otimes_R DT$ as follows since $\mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, T) = \mathrm{KerTor}^R_1(-, DT)$:

$$\begin{aligned}0 \to \mathrm{Hom}_S(DT, X) \otimes_R DT &\to \mathrm{Hom}_S(DT, I_0) \otimes_R DT\\ \to \mathrm{Hom}_S(DT, I_1) \otimes_R DT &\to \cdots \to \mathrm{Hom}_S(DT, I_n) \otimes_R DT \to \cdots .\end{aligned}$$

It follows that

$$X \simeq \mathrm{Hom}_S(DT, X) \otimes_R DT \in \mathcal{B} \otimes_R DT$$

since $I_i \simeq \mathrm{Hom}_S(DT, I_i) \otimes_R DT$ for each i. This shows that

$$\mathrm{KerExt}^1_S(\mathrm{Hom}_R(T, \mathcal{A}), -) \subseteq \mathcal{B} \otimes_R DT.$$

Together with Step 1, we obtain that $\mathrm{KerExt}^1_S(\mathrm{Hom}_R(T, \mathcal{A}), -) = \mathcal{B} \otimes_R DT$.

Altogether, we obtain that $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ is a hereditary cotorsion pair. $\square$

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3.2.6 *Corollary. Let* $T \in \mathrm{mod}R$ *be Wakamatsu-tilting with* $S = \mathrm{End}(T_R)^{op}$*. Assume that the functor* $\mathrm{Hom}_R(T, -)$ *gives an equivalence between a covariantly finite coresolving subcategory* $\mathcal{A}$ *in* $\mathrm{mod}R$ *and a contravariantly finite resolving subcategory* $\mathcal{G}$ *in* $\mathrm{mod}S$*. If* T *is an Ext-projective generator in* $\mathcal{A}$*, then* T *is a good Wakamatsu-tilting module.*

Proof. Since $\mathcal{G}$ is a contravariantly finite resolving subcategory in $\mathrm{mod}S$, there is a cotorsion pair $(\mathcal{G}, \mathrm{KerExt}^1_S(\mathcal{G}, -))$ in $\mathrm{mod}S$, by [3, Proposition 1.10]. Dually, there is a cotorsion pair $(\mathrm{KerExt}^1_R(-, \mathcal{A}), \mathcal{A})$ in $\mathrm{mod}R$ since $\mathcal{A}$ is a covariantly finite coresolving subcategory in $\mathrm{mod}R$. Note that both cotorsion pairs are complete and hereditary by [3, Proposition 3.3 and the Remark after Proposition 3.4]. Since T is an Ext-projective generator in $\mathcal{A}$, by Proposition 3.2.5, the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the above two cotorsion pairs. Hence, T is a good Wakamatsu-tilting module by the definition. ∎

§4. The proof of main results

The whole section will be devoted to the proof of the two results mentioned in Section 1.

Let R be an Artin algebra and T be a good Wakamatsu-tilting module with $S = \mathrm{End}(T_R)^{op}$. Then ${}_ST_R$ is a good Wakamatsu-tilting bimodule. Assume that $(\mathcal{B}, \mathcal{A})$ is a complete hereditary cotorsion pair in $\mathrm{mod}R$ and $(\mathcal{G}, \mathcal{K})$ is a complete hereditary cotorsion pair in $\mathrm{mod}S$ such that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between these two cotorsion pairs.

The sketch of our proof of Theorem 1 is as follows.

First, we construct a functor $\mathrm{L}_T : \mathcal{RC}^{\mathrm{tr}}(R) \to \mathcal{RC}(S)$ and a functor $-\hat{\otimes} DT : \mathcal{RC}(R) \to \mathcal{RC}(S)$. Then we give a natural homomorphism

$$l^X_Y : \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X, Y) \to \mathrm{Hom}_{\mathcal{RC}(S)}(X \hat{\otimes} DT, \mathrm{L}_T(Y))$$

which is functorial in both variables. After this, associated with an object $X \in \mathcal{RC}(R)$, we use the condition that $(\mathcal{B}, \mathcal{A})$ is a complete cotorsion pair in $\mathrm{mod}R$ to obtain an object $A_X \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and establish a homomorphism $u_X \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X, A_X)$. We then show that the assignment $X \longmapsto \mathrm{Cok}(l(u_X))$ induces our desired functor $\mathbf{S}_T : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$. We use the dual method to construct another desired functor $\mathbf{Q}_{DT} : \underline{\mathcal{RC}}(S) \to \underline{\mathcal{RC}}(R)$. Then we prove that there are natural isomorphisms $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$ and $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$.

4.1 From $\mathcal{RC}(R)$ to $\mathcal{RC}(S)$: the functor $\mathbf{S}_T$

4.1.1 The functor $\mathrm{L}_T : \mathcal{RC}^{\mathrm{tr}}(R) \to \mathcal{RC}(S)$

Let $X = \{X_i\} \in \mathcal{RC}^{\mathrm{tr}}(R)$. We define $\mathrm{L}_T(X) \in \mathcal{RC}(S)$ as follows:

(l1) the underlying module $\mathrm{L}_T(X)_i = \mathrm{Hom}_R(T, X_{i-1}) \oplus X_i \otimes_R DT$ and

(l2) the structure map $\delta^{\otimes}_i(\mathrm{L}_T(X)) : \mathrm{L}_T(X)_i \otimes_S DS \to \mathrm{L}_T(X)_{i-1}$ is given by $\begin{pmatrix} 0 & \delta_{L_i} \\ 0 & 0 \end{pmatrix}$, where δ_{L_i} is the composition:

$$\mathrm{Hom}_R(T, X_{i-1}) \otimes_S DS \xrightarrow{\simeq} \mathrm{Hom}_R(T, X_{i-1}) \otimes_S T \otimes_R DT \xrightarrow{\epsilon^T_{X_{i-1}} \otimes_R DT} X_{i-1} \otimes_R DT.$$

From the functor property of $\mathrm{Hom}_R(T, -)$ and $-\otimes_R DT$, one can easily see that L_T is a functor from $\mathcal{RC}^{\mathrm{tr}}(R)$ to $\mathcal{RC}(S)$.

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Remark.

(1) If $X \in \mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_{\mathrm{mod}R}T)$, that is, $X = \{X_i\}$ with each $X_i \in \mathrm{add}_{\mathrm{mod}R}T$, then $\mathrm{Hom}_R(T, X_{i-1}) \in \mathrm{add}_{\mathrm{mod}S}S$ and δ_{L_i} defined above is an isomorphism for each i. It follows that $\mathrm{L}_T(X)$ is a projective object in $\mathcal{RC}(S)$ in this case.
(2) As a special case, if $T = R$, then we obtain the functor $\mathrm{L}_R : \mathcal{RC}^{\mathrm{tr}}(R) \to \mathcal{RC}(R)$ which sends objects in $\mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_{\mathrm{mod}R}R)$ to a projective object in $\mathcal{RC}(R)$.

4.1.2 The functor $-\hat{\otimes}DT : \mathcal{RC}(R) \to \mathcal{RC}(S)$

Let $Y = \{Y_i, \delta_i^{\otimes}(Y)\} \in \mathcal{RC}(R)$. We define $Y \hat{\otimes} DT \in \mathcal{RC}(S)$ by setting

(t1) the underlying module is $(Y \hat{\otimes} DT)_i = Y_i \otimes_R DT$ and
(t2) the structure map $\delta_i^{\otimes}(Y \hat{\otimes} DT)$ is given by the composition

$$\begin{aligned} Y_i \otimes_R DT \otimes_S DS &\xrightarrow{\simeq} Y_i \otimes_R DT \otimes_S T \otimes_R DT \\ &\xrightarrow{\simeq} Y_i \otimes_R DR \otimes_R DT \xrightarrow{\delta_i^{\otimes}(Y)\otimes_R DT} Y_{i-1} \otimes_R DT. \end{aligned}$$

From the functor property of $-\otimes_R DT$, one can see that $-\hat{\otimes}DT$ is a functor from $\mathcal{RC}(R)$ to $\mathcal{RC}(S)$.

4.1.3 The homomorphism $l_Y^X : \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X, Y) \to \mathrm{Hom}_{\mathcal{RC}(S)}(X \hat{\otimes} DT, \mathrm{L}_T(Y))$

Recall that we have a forgetful functor from $\mathcal{RC}(R)$ to $\mathcal{RC}^{\mathrm{tr}}(R)$. For any $X \in \mathcal{RC}(R)$ and $Y \in \mathcal{RC}^{\mathrm{tr}}(R)$, there is a canonical homomorphism

$$l_Y^X : \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{RC}(S)}(X \hat{\otimes} DT, \mathrm{L}_T(Y))$$

which is functorial in both variables, defined by

$$l_Y^X : u = \{u_i\} \longmapsto f = \{f_i\}, \quad \text{with} f_i = \big(-\theta_{l_i}, \quad u_i \otimes_R DT\big),$$

where θ_{l_i} is given by the composition

$$\begin{aligned} X_i \otimes_R DT &\xrightarrow{\eta^T_{X_i\otimes_R DT}} \mathrm{Hom}_R(T, X_i \otimes_R DT \otimes_S T) \xrightarrow{\simeq} \mathrm{Hom}_R(T, X_i \otimes_R DR) \\ &\xrightarrow{\mathrm{Hom}_R(T,\delta_i^{\otimes}(X))} \mathrm{Hom}_R(T, X_{i-1}) \xrightarrow{\mathrm{Hom}_R(T,u_{i-1})} \mathrm{Hom}_R(T, Y_{i-1}). \end{aligned}$$

Remark. Using the fact that ${}_R DR_R \simeq {}_R(DT \otimes_S T)_R$ and the adjoint isomorphism

$$\mathbf{\Gamma}^T : \mathrm{Hom}_R(X_i \otimes_R DR, Y_{i-1}) \simeq \mathrm{Hom}_S(X_i \otimes_R DT, \mathrm{Hom}_R(T, Y_{i-1})),$$

one can easily check that θ_{l_i} is just the image of the natural homomorphism $\delta_i^{\otimes}(X) \cdot u_{i-1}$ under Γ^T, that is, $\theta_{l_i} = \mathbf{\Gamma}^T(\delta_i^{\otimes}(X) \cdot u_{i-1})$.

In the following, we simply write l instead of l_Y^X.

It is easy to see that, for any commutative diagram in $\mathcal{RC}(R)$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{u'} & Y', \end{array}$$

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there is an induced commutative diagram in $\mathcal{RC}(S)$

$$\begin{array}{ccc} X \mathbin{\hat{\otimes}} DT & \xrightarrow{l(u)} & \mathrm{L}_T(Y) \\ \downarrow{\scriptstyle x \mathbin{\hat{\otimes}} DT} & & \downarrow{\scriptstyle L_T(y)} \\ X' \mathbin{\hat{\otimes}} DT & \xrightarrow{l(u')} & \mathrm{L}_T(Y'). \end{array}$$

4.1.4 A monomorphism $u_X : X \to A_X$ in $\mathcal{RC}^{\mathrm{tr}}(\mathcal{R})$ with $A_X \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$, for $X \in \mathcal{RC}(R)$

Let $X = \{X_i\} \in \mathcal{RC}(R)$. Since $(\mathcal{B}, \mathcal{A})$ is a complete cotorsion pair in mod$R$, there are exact sequences $0 \to X_i \xrightarrow{(u_X)_i} (A_X)_i \xrightarrow{(\pi_X)_i} (B_X)_i \to 0$ with $(A_X)_i \in \mathcal{A}$ and $(B_X)_i \in \mathcal{B}$, for each i. This gives an exact sequence $0 \to X \xrightarrow{u_X} A_X \xrightarrow{\pi_X} B_X \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_X = \{(A_X)_i\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_X = \{(B_X)_i\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$.

Now let $Y = \{Y_i\} \in \mathcal{RC}(R)$ and $h = \{h_i\} \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$. Then we have an exact sequence $0 \to Y \xrightarrow{u_Y} A_Y \xrightarrow{\pi_Y} B_Y \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_Y = \{(A_Y)_i\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_Y = \{(B_Y)_i\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$, as above. Using that $\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{A})$, it is easy to see that there is a homomorphism $h_A \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(A_X, A_Y)$ and further $h_B \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(B_X, B_Y)$ such that the following diagram in $\mathcal{RC}^{\mathrm{tr}}(R)$ is commutative with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u_X} & A_X & \xrightarrow{\pi_X} & B_X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h_A} & & \downarrow{\scriptstyle h_B} & & \\ 0 & \longrightarrow & Y & \xrightarrow{u_Y} & A_Y & \xrightarrow{\pi_Y} & B_Y & \longrightarrow & 0. \end{array}$$

4.1.5 The cokernel $\mathrm{Cok}(l(u_X))$

Applying the functor $-\otimes_R DT$ to the exact sequences

$$0 \to X_i \xrightarrow{(u_X)_i} (A_X)_i \xrightarrow{(\pi_X)_i} (B_X)_i \to 0$$

above, we obtain induced exact sequences

$$0 \longrightarrow X_i \otimes_R DT \xrightarrow{(u_X)_i \otimes_R DT} (A_X)_i \otimes_R DT \longrightarrow (B_X)_i \otimes_R DT \longrightarrow 0$$

since $(B_X)_i \in \mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, T) = \mathrm{KerTor}^R_1(-, DT)$ for each i. It follows that, by applying the homomorphism l in 4.1.3 to the homomorphism u_X in 4.1.4, there is an induced exact sequence

$$0 \longrightarrow X \mathbin{\hat{\otimes}} DT \xrightarrow{l(u_X)} \mathrm{L}_T(A_X) \xrightarrow{\pi_{l_X}} \mathrm{Cok}(l(u_X)) \longrightarrow 0.$$

Remark. From the definition of $\mathrm{Cok}(l(u_X))$, one sees that, for each i, $\mathrm{Cok}(l(u_X))_i$ is given by the pushout

$$\begin{array}{ccc} X_i \otimes_R DT & \xrightarrow{\theta_{l_i}} & \mathrm{Hom}_R(T, (A_X)_{i-1}) \\ {\scriptstyle (u_X)_i \otimes_R DT}\downarrow & & \downarrow \\ (A_X)_i \otimes_R DT & \longrightarrow & \mathrm{Cok}(l(u_X))_i. \end{array}$$

Moreover, for $Y \in \mathcal{RC}(R)$ and $h \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$, by applying the homomorphism l to the left square in the commutation diagram in 4.1.4, we obtain the following commutative diagram in $\mathcal{RC}(S)$, for some h_{Cok}:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \hat{\otimes} DT & \xrightarrow{l(u_X)} & \mathrm{L}_T(A_X) & \xrightarrow{\pi_{l_X}} & \mathrm{Cok}(l(u_X)) & \longrightarrow & 0 \\ & & {\scriptstyle h \hat{\otimes} DT}\downarrow & & {\scriptstyle \mathrm{L}_T(h_A)}\downarrow & & {\scriptstyle h_{\mathrm{Cok}}}\downarrow & & \\ 0 & \longrightarrow & Y \hat{\otimes} DT & \xrightarrow{l(u_Y)} & \mathrm{L}_T(A_Y) & \xrightarrow{\pi_{l_Y}} & \mathrm{Cok}(l(u_Y)) & \longrightarrow & 0 \end{array}$$

4.1.6 The assignment $\mathbf{S}_T : \mathcal{RC}(R) \to \underline{\mathcal{RC}}(S)$ given by $X \longmapsto \mathrm{Cok}(l(u_X))$ is a functor

By 4.1.5, it is sufficient to prove that $\mathbf{S}_T(h) := h_{\mathrm{Cok}} = 0$ in $\underline{\mathcal{RC}}(S)$ provided $h = 0$. We divide the proof into two steps.

Step 1: Consider each piece in the commutative diagram in 4.1.4. If $h = \{h_i\} = 0$, then $h_i = 0$ for each i. Thus, we have that $(u_X)_i(h_A)_i = 0$ and, consequently, $(h_A)_i = (\pi_X)_i g_i$ for some $g_i : (B_X)_i \to (A_Y)_i$. Since $(B_X)_i \in \mathcal{B} \subseteq \mathcal{X}_T$ for each i and T is an Ext-injective cogenerator in $\mathcal{B}$ (see the first part in the proof of Proposition 3.2.5), there are exact sequences $0 \to (B_X)_i \xrightarrow{b_i} T_{(B_X)_i} \to (B'_X)_i \to 0$ with $T_{(B_X)_i} \in \mathrm{add}_{\mathrm{mod} R} T$ and $(B'_X)_i \in \mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, \mathcal{A})$. It follows that there exists $t_i \in \mathrm{Hom}_R(T_{(B_X)_i}, (A_Y)_i)$ such that $g_i = b_i t_i$. Altogether, we obtain the following commutative diagram:

$$\begin{array}{ccc} (A_X)_i & \xrightarrow{(\pi_X)_i} & (B_X)_i \\ {\scriptstyle (h_A)_i}\downarrow & \swarrow {\scriptstyle g_i} & \downarrow {\scriptstyle b_i} \\ (A_Y)_i & \xleftarrow[t_i]{} & T_{(B_X)_i}. \end{array}$$

This induces the following commutative diagram in $\mathcal{RC}^{\mathrm{tr}}(R)$, where $T_{B_X} = \{T_{(B_X)_i}\}$:

$$\begin{array}{ccc} A_X & \xrightarrow{\pi_X} & B_X \\ {\scriptstyle h_A}\downarrow & \swarrow {\scriptstyle g} & \downarrow {\scriptstyle b} \\ A_Y & \xleftarrow[t]{} & T_{B_X}. \end{array}$$

Set $k := \pi_X b$. Then $\mathrm{L}_T(h_A) = \mathrm{L}_T(kt) = \mathrm{L}_T(k)\mathrm{L}_T(t)$.

Step 2: Consider the commutative diagram in 4.1.5. Since $(u_X)_i(\pi_X)_i = 0$, it holds that $\mathrm{Hom}_R(T, (u_X)_i) \cdot \mathrm{Hom}_R(T, (\pi_X)_i) = 0$ and $(u_X)_i \otimes_R DT \cdot (\pi_X)_i \otimes_R DT = 0$. Then we see that $l(u_X)\mathrm{L}_T(k) = l(u_X)\mathrm{L}_T(\pi_X)\mathrm{L}_T(b) = 0 \cdot \mathrm{L}_T(b) = 0$ by the definition of the functor L_T in 4.1.1 and the morphism $l(u_X)$ in 4.1.3. Hence, there is some $\theta \in \mathrm{Hom}_{\mathcal{RC}(S)}(\mathrm{Cok}(l(u_X)), \mathrm{L}_T(T_{B_X}))$ such that $\mathrm{L}_T(k) = \pi_{l_X}\theta$. Consequently, we have that $\mathrm{L}_T(h_A) = \mathrm{L}_T(k)L(t) = \pi_{l_X}\theta\mathrm{L}_T(t)$. Now we obtain that $\pi_{l_X}h_{\mathrm{Cok}} = \mathrm{L}_T(h_A)\pi_{l_Y} = \pi_{l_X}\theta\mathrm{L}_T(t)\pi_{l_Y}$. Since π_{l_X} is epic, we get that $h_{\mathrm{Cok}} = \theta\mathrm{L}_T(t)\pi_{l_Y}$. That is, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Cok}(l(u_X)) & \xrightarrow{\theta} & \mathrm{L}_T(T_{B_X}) \\ {\scriptstyle h_{\mathrm{Cok}}}\downarrow & & \downarrow{\scriptstyle \mathrm{L}_T(t)} \\ \mathrm{Cok}(l(u_Y)) & \underset{\pi_{l_Y}}{\longleftarrow} & L_T(A_Y) \end{array}$$

Note that $\mathrm{L}_T(T_{B_X})$ is a projective–injective object in $\mathcal{RC}(S)$, so $h_{\mathrm{Cok}} = 0$ in $\underline{\mathcal{RC}}(S)$.

4.1.7 The functor $\mathbf{S}_T : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$

We will show that the functor $\mathbf{S}_T$ factors through $\underline{\mathcal{RC}}(R)$.

To see this, it is enough to show that $\mathbf{S}_T(X)$ is a projective object in $\mathcal{RC}(S)$ whenever X is a projective object in $\mathcal{RC}(R)$.

Without loss of generality, we assume that $X = \{X_i\}$ is an indecomposable projective object in $\mathcal{RC}(R)$. Thus, X is of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_R(DR, I) \overset{1}{\rightsquigarrow} I \rightsquigarrow 0 \rightsquigarrow \cdots,$$

where I is indecomposable injective and is on the $(k-1)$th position, for some k [18, 2.2 Lemma].

Note that $X_k = \mathrm{Hom}_R(DR, I) \in \mathrm{add}_{\mathrm{mod}R}R \subseteq \mathcal{B}$ and $X_{k-1} = I \in \mathrm{add}_{\mathrm{mod}R}DR \subseteq \mathcal{A}$, so, following 4.1.4, we can choose A_X to be of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow T_k \rightsquigarrow I \rightsquigarrow 0 \rightsquigarrow \cdots,$$

where $A_{X_k} = T_k \in \mathrm{add}_{\mathrm{mod}R}T$. And we have that the homomorphism $u_X : X \to A_X$ is of the form

$$\begin{array}{lccccccccccccc} X: & \cdots & \rightsquigarrow & 0 & \rightsquigarrow & 0 & \rightsquigarrow & \mathrm{Hom}_R(DR, I) & \rightsquigarrow & I & \rightsquigarrow & 0 & \rightsquigarrow & \cdots \\ \downarrow u_X & & & & & & & \downarrow u_k & & \downarrow 1 & & & & \\ A_X: & \cdots & \rightsquigarrow & 0 & \rightsquigarrow & 0 & \rightsquigarrow & T_k & \rightsquigarrow & I & \rightsquigarrow & 0 & \rightsquigarrow & \cdots \end{array}$$

Then, from the structure of $l(u_X)$, we can see that $l(u_X)$ is of the form

$$\begin{array}{lccccccccc} X \hat{\otimes} DT: & 0 & \rightsquigarrow & 0 & \rightsquigarrow & \mathrm{Hom}_R(DR, I) \otimes_R DT & \rightsquigarrow & I \otimes_R DT & \rightsquigarrow & 0 \\ \downarrow l(u_X) & & & \downarrow & & \downarrow (-\theta_{l_k}, u_k \otimes_R DT) & & \downarrow 1 & & \\ \mathrm{L}_T(A_X): & 0 & \rightsquigarrow & \mathrm{Hom}_R(T, T_k) & \overset{(0,\delta_{L_{k+1}})}{\rightsquigarrow} & \mathrm{Hom}_R(T, I) \oplus T_k \otimes_R DT & \rightsquigarrow & I \otimes_R DT & \rightsquigarrow & 0, \end{array}$$

where θ_{l_k} is defined as in 4.1.3 and $\delta_{L_{k+1}}$ is defined as in 4.1.1, respectively. One checks that both homomorphisms θ_{l_k} and $\delta_{L_{k+1}}$ are, in fact, isomorphisms. So we obtain that

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$S_T(X) = \mathrm{Coker}(l(u_X))$ is of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_R(T, T_k) \overset{\delta'_{k+1}}{\rightsquigarrow} T_k \otimes_R DT \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow \cdots,$$

where δ'_{k+1} is the induced isomorphism: $\mathrm{Hom}_R(T, T_k) \otimes_S DS \to T_k \otimes_R DT$. Since $T_k \otimes_R DT \in \mathrm{add}_{\mathrm{mod}S}(T \otimes_R DT) = \mathrm{add}_{\mathrm{mod}S} DS$, we see that $T_k \otimes_R DT$ is an injective S-module and that $\mathbf{S}_T(X)$ is a projective object in $\mathcal{RC}(S)$.

It follows that the functor $\mathbf{S}_T$ factors through $\underline{\mathcal{RC}}(R)$. We still denote by $\mathbf{S}_T$ the induced functor from $\underline{\mathcal{RC}}(R)$ to $\underline{\mathcal{RC}}(S)$.

4.2 From $\underline{\mathcal{RC}}(S)$ to $\underline{\mathcal{RC}}(R)$: the functor $\mathbf{Q}_{DT}$

The functor $\mathbf{Q}_{DT}$ is indeed defined in a way dual to the construction of $\mathbf{S}_T$.

4.2.1 The functor $\mathrm{R}_{DT} : \mathcal{RC}^{\mathrm{tr}}(S) \to \mathcal{RC}(R)$

Dually to 4.1.1, for any $X = \{X_i\} \in \mathcal{RC}^{\mathrm{tr}}(S)$, we define $\mathrm{R}_{DT}(X) \in \mathcal{RC}(R)$ as follows:

(r1) the underlying module is $\mathrm{R}_{DT}(X)_i = \mathrm{Hom}_S(DT, X_i) \oplus X_{i+1} \otimes_S T$ and
(r2) the structure map $\delta_i^{\mathrm{H}}(\mathrm{R}_{DT}(X))$ is given by $\begin{pmatrix} 0 & \delta_{R_i}^{\mathrm{H}} \\ 0 & 0 \end{pmatrix}$, where $\delta_{R_i}^{\mathrm{H}}$ is the composition

$$\mathrm{Hom}_S(DT, X_i) \xrightarrow{\mathrm{Hom}_S(DT, \eta_{X_i}^T)} \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, X_i \otimes_S T))$$
$$\simeq \mathrm{Hom}_R(DT \otimes_S T, X_i \otimes_S T) \simeq \mathrm{Hom}_R(DR, X_i \otimes_S T).$$

Equivalently, the structure map $\delta_i^{\otimes}(\mathrm{R}_{DT}(X))$ is given by $\begin{pmatrix} 0 & \delta_{R_i}^{\otimes} \\ 0 & 0 \end{pmatrix}$, where $\delta_{R_i}^{\otimes}$ is the composition

$$\mathrm{Hom}_S(DT, X_i) \otimes_R DR \simeq \mathrm{Hom}_S(DT, X_i) \otimes_R DT \otimes_S T \xrightarrow{\epsilon_{X_i}^{DT}} X_i \otimes_S T.$$

It is easy to see that R_{DT} is a functor.

Remark.

(1) If $X \in \mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_{\mathrm{mod}S} DT)$, that is, $X = \{X_i\}$ with each $X_i \in \mathrm{add}_{\mathrm{mod}S} DT$, then $\mathrm{Hom}_S(DT, X_i) \in \mathrm{add}_{\mathrm{mod}R} R$ and δ_{R_i} defined above is an isomorphism for each i. It follows that $\mathrm{R}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$ in this case.
(2) As a special case, if $T = S$, then we obtain the functor $\mathrm{R}_{DS} : \mathcal{RC}^{\mathrm{tr}}(S) \to \mathcal{RC}(S)$ which sends objects in $\mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_{\mathrm{mod}S} DS)$ to a projective object in $\mathcal{RC}(S)$.

4.2.2 The functor $\hat{\mathrm{H}}\mathrm{om}(DT, -) : \mathcal{RC}(S) \to \mathcal{RC}(R)$

Let $Y = \{Y_i, \delta_i^{\mathrm{H}}(Y)\} \in \mathcal{RC}(S)$. We define $\hat{\mathrm{H}}\mathrm{om}(DT, Y) \in \mathcal{RC}(R)$ by setting

(h1) the underlying module is $\hat{\mathrm{H}}\mathrm{om}(DT, Y)_i = \mathrm{Hom}_S(DT, Y_i)$ and
(h2) the structure map $\delta_i^{\mathrm{H}}(\hat{\mathrm{H}}\mathrm{om}(DT, Y))$ is given by the composition

$$\mathrm{Hom}_S(DT, Y_i) \xrightarrow{\mathrm{Hom}_S(DT, \delta_i^{\mathrm{H}}(Y))} \mathrm{Hom}_S(DT, \mathrm{Hom}_S(DS, Y_{i-1}))$$
$$\simeq \mathrm{Hom}_R(DR, \mathrm{Hom}_S(DT, Y_{i-1})).$$

Then from the functor property of $\mathrm{Hom}_S(DT, -)$, one can see that $\hat{\mathrm{H}}\mathrm{om}(DT, -)$ is a functor from $\mathcal{RC}(S)$ to $\mathcal{RC}(R)$.

4.2.3 The homomorphism r_Y^X

Dually to the homomorphism l_Y^X, for any $X \in \mathcal{RC}^{\mathrm{tr}}(S)$ and $Y \in \mathcal{RC}(S)$, we have a canonical homomorphism

$$r_Y^X : \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(S)}(X, Y) \to \mathrm{Hom}_{\mathcal{RC}(R)}(\mathrm{R}_{DT}(X), \hat{\mathrm{H}}\mathrm{om}(DT, Y)),$$

which is functorial in both variables, defined by

$$r_Y^X : u = \{u_i\} \longmapsto f = \{f_i\} \quad \text{with } f_i = \begin{pmatrix} \mathrm{Hom}_S(DT, u_i) \\ -\zeta_{r_i} \end{pmatrix},$$

where $\zeta_{r_i} : X_{i+1} \otimes_S T \to \mathrm{Hom}_S(DT, Y_i)$ equals to

$$(u_{i+1} \otimes_S T) \cdot (\delta_{i+1}^{\mathrm{H}}(Y) \otimes_S T) \cdot \epsilon^T_{\mathrm{Hom}_S(DT, Y_i)},$$

that is, the composition

$$\begin{aligned} X_{i+1} \otimes_S T &\overset{u_{i+1} \otimes_S T}{\longrightarrow} Y_{i+1} \otimes_S T \overset{\delta^H_{i+1}(Y) \otimes_S T}{\longrightarrow} \mathrm{Hom}_S(DS, Y_i) \otimes_S T \\ &\simeq \mathrm{Hom}_S(T \otimes_R DT, Y_i) \otimes_S T \\ &\simeq \mathrm{Hom}_R(T, \mathrm{Hom}_S(DT, Y_i)) \otimes_S T \\ &\overset{\epsilon^T_{\mathrm{Hom}_S(DT, Y_i)}}{\longrightarrow} \mathrm{Hom}_S(DT, Y_i). \end{aligned}$$

Remark. Using the fact that ${}_S DS_S \simeq {}_S T \otimes_R DT_S$ and the adjoint isomorphism

$$\mathbf{\Gamma}^{DT} : \mathrm{Hom}_S(X_{i+1} \otimes_S DS, Y_i) \simeq \mathrm{Hom}_S(X_{i+1} \otimes_S T, \mathrm{Hom}_S(DT, Y_i)),$$

one can easily check that ζ_{r_i} is the image of the natural homomorphism $(u_{i+1} \otimes_S DS) \cdot \delta^{\otimes}_{i+1}(Y)$ under Γ^{DT}, that is, $\zeta_{r_i} = \mathbf{\Gamma}^{DT}((u_{i+1} \otimes_S DS) \cdot \delta^{\otimes}_{i+1}(Y))$.

In the following, we simply write r instead of r_Y^X.

4.2.4 An epimorphism $v_Y : G_Y \to Y$ in $\mathcal{RC}^{\mathrm{tr}}(\mathcal{S})$ with $G_Y \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{G})$, for $Y \in \mathcal{RC}(S)$

Since $(\mathcal{G}, \mathcal{K})$ is a complete hereditary cotorsion pair in mod$S$, it follows that, for any $Y = \{Y_i\} \in \mathcal{RC}(S)$, there is an exact sequence $0 \to K_Y \overset{k_Y}{\longrightarrow} G_Y \overset{v_Y}{\longrightarrow} Y \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(\mathcal{S})$ with $K_Y \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{K})$ and $G_Y \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{G})$.

Moreover, for any $h \in \mathrm{Hom}_{\mathcal{RC}(S)}(X, Y)$, there is an induced commutative diagram as follows since $\mathrm{Ext}^1_S(\mathcal{G}, \mathcal{K}) = 0$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_X & \overset{k_X}{\longrightarrow} & G_X & \overset{v_X}{\longrightarrow} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_K} & & \downarrow{\scriptstyle h_G} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & K_Y & \overset{k_Y}{\longrightarrow} & G_Y & \overset{v_Y}{\longrightarrow} & Y & \longrightarrow & 0. \end{array}$$

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4.2.5 The kernel $\mathrm{Ker}(r(v_Y))$

Applying the functor $\hat{\mathrm{H}}\mathrm{om}(DT, -)$ to the bottom exact sequence in the above diagram, we obtain an induced exact sequence

$$0 \to \hat{\mathrm{H}}\mathrm{om}(DT, K_Y) \to \hat{\mathrm{H}}\mathrm{om}(DT, G_Y) \to \hat{\mathrm{H}}\mathrm{om}(DT, Y) \to 0$$

since $(K_Y)_i \in \mathcal{K} \subseteq \mathrm{KerExt}^1_S(DT, -)$ for each i. Thus, after applying the homomorphism r in 4.2.3 to the homomorphism v_Y in 4.2.4, we obtain the following exact sequence in $\mathcal{RC}(R)$:

$$0 \to \mathrm{Ker}(r(v_Y)) \xrightarrow{\lambda_{r_Y}} \mathrm{R}_{DT}(G_Y) \xrightarrow{r(v_Y)} \hat{\mathrm{H}}\mathrm{om}(DT, Y) \to 0.$$

Moreover, for any $h \in \mathrm{Hom}_{\mathcal{RC}(S)}(X, Y)$, by applying the homomorphism r to the right part of the commutative diagram in 4.2.4, we obtain the following commutative diagram in $\mathcal{RC}(R)$, for some h_{Ker}:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(r(v_X)) & \xrightarrow{\lambda_{r_X}} & \mathrm{R}_{DT}(G_X) & \xrightarrow{r(v_X)} & \hat{\mathrm{H}}\mathrm{om}(DT, X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h_{\mathrm{Ker}}} & & \downarrow{\scriptstyle \mathrm{R}_{DT}(h_G)} & & \downarrow{\scriptstyle \hat{\mathrm{H}}\mathrm{om}(DT,h)} & & \\
0 & \longrightarrow & \mathrm{Ker}(r(v_Y)) & \xrightarrow{\lambda_{r_Y}} & \mathrm{R}_{DT}(G_Y) & \xrightarrow{r(v_Y)} & \hat{\mathrm{H}}\mathrm{om}(DT, Y) & \longrightarrow & 0.
\end{array}$$

4.2.6 The assignment $\mathbf{Q}_{DT} : \mathcal{RC}(S) \to \underline{\mathcal{RC}}(R)$ given by $Y \longmapsto \mathrm{Ker}(r(v_Y))$ is a functor

By 4.2.5, it is sufficient to prove that $\mathbf{Q}_{DT}(h) := h_{\mathrm{Ker}} = 0$ in $\underline{\mathcal{RC}}(R)$ provided $h = 0$. This is also divided into two steps.

Step 1: Consider each piece in the commutative diagram in 4.2.4. If $h = \{h_i\} = 0$, then $h_i = 0$ for each i. Thus, we have that $(h_G)_i(v_Y)_i = 0$ and, consequently, $(h_G)_i = g_i(k_Y)_i$ for some $g_i : (G_X)_i \to (K_Y)_i$. Since $(K_Y)_i \in \mathcal{K} \subseteq {}_{DT}\mathcal{X}$ for all i, there are exact sequences $0 \to (K'_Y)_i \to DT_{(K_Y)_i} \xrightarrow{b_i} (K_Y)_i \to 0$ with $DT_{(K_Y)_i} \in \mathrm{add}_{\mathrm{mod}S} DT$ and $(K'_Y)_i \in \mathcal{K} \subseteq \mathrm{KerExt}^1_S(\mathcal{G}, -)$. It follows that there exists $t_i \in \mathrm{Hom}_R((G_X)_i, DT_{(K_Y)_i})$ such that $g_i = t_i b_i$. Altogether, we obtain the following commutative diagram:

$$\begin{array}{ccc}
(G_X)_i & \xrightarrow{t_i} & DT_{(K_Y)_i} \\
\downarrow{\scriptstyle (h_G)_i} & \searrow^{g_i} & \downarrow{\scriptstyle b_i} \\
(G_Y)_i & \xleftarrow[(k_Y)_i]{} & (K_Y)_i
\end{array}$$

It follows that there is a commutative diagram in $\mathcal{RC}^{\mathrm{tr}}(S)$,

$$\begin{array}{ccc}
G_X & \xrightarrow{t} & DT_{K_Y} \\
\downarrow{\scriptstyle h_G} & \searrow^{g} & \downarrow{\scriptstyle b} \\
G_Y & \xleftarrow[k_Y]{} & K_Y,
\end{array}$$

where $DT_{K_Y} := \{DT_{(K_Y)_i}\}$.

Set $\beta := bk_Y$. Then $\mathrm{R}_{DT}(h_G) = \mathrm{R}_{DT}(t\beta) = \mathrm{R}_{DT}(t)\mathrm{R}_{DT}(\beta)$.

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Step 2: Consider the commutative diagram in 4.2.5. Since $(k_Y)_i(v_Y)_i = 0$ in 4.2.4, we see that $\mathrm{R}_{DT}(\beta)r(v_Y) = 0$ by the definitions. Hence, there is some $\theta \in \mathrm{Hom}_{\mathcal{RC}(R)}(\mathrm{R}_{DT}(DT_{K_Y}), \mathrm{Ker}(r(v_Y)))$ such that $\mathrm{R}_{DT}(\beta) = \theta\lambda_{r_Y}$. Consequently, we have that $\mathrm{R}_{DT}(h_G) = \mathrm{R}_{DT}(t)\mathrm{R}_{DT}(\beta) = \mathrm{R}_{DT}(t)\theta\lambda_{r_Y}$. Now we obtain that $h_{\mathrm{Ker}}\lambda_{r_Y} = \lambda_{r_X}\mathrm{R}_{DT}(h_G) = \lambda_{r_X}\mathrm{R}_{DT}(t)\theta\lambda_{r_Y}$. Since λ_{r_Y} is monomorphic, we get that $h_{\mathrm{Ker}} = \lambda_{r_X}\mathrm{R}_{DT}(t)\theta$, that is, the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Ker}(r(v_X)) & \xrightarrow{\lambda_{r_X}} & \mathrm{R}_{DT}(G_X) \\ {\scriptstyle h_{\mathrm{Ker}}}\downarrow & & \downarrow{\scriptstyle \mathrm{R}_{DT}(t)} \\ \mathrm{Ker}(r(v_Y)) & \xleftarrow[\theta]{} & \mathrm{R}_{DT}(DT_{K_Y}) \end{array}$$

Note that $\mathrm{R}_{DT}(DT_{K_Y})$ is a projective–injective object in $\mathcal{RC}(R)$, so $h_{\mathrm{Ker}} = 0$ in $\underline{\mathcal{RC}}(R)$.

4.2.7 The functor $\mathbf{Q}_{DT} : \underline{\mathcal{RC}}(S) \to \underline{\mathcal{RC}}(R)$

We will show that the functor $\mathbf{Q}_{DT}$ factors through $\underline{\mathcal{RC}}(S)$.

To see this, it is enough to show that $\mathbf{Q}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$, whenever X is a projective object in $\mathcal{RC}(S)$.

Without loss of generality, we assume that $X = \{X_i\}$ is an indecomposable projective object in $\mathcal{RC}(S)$. Thus, we have that X has the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow P \overset{1}{\rightsquigarrow} P \otimes_S DS \rightsquigarrow 0 \rightsquigarrow \cdots,$$

where P is indecomposable projective and is on the $(k+1)$th position, for some k; see [18].

Note that $X_{k+1} = P \in \mathrm{add}_{\mathrm{mod}S} S \subseteq \mathcal{G}$ and that $X_k = P \otimes_S DS \in \mathrm{add}_{\mathrm{mod}S} DS \subseteq \mathcal{K}$, so, following 4.2.4, we can choose G_X to be of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow P \rightsquigarrow DT_k \rightsquigarrow 0 \rightsquigarrow \cdots,$$

where $(G_X)_k = DT_k \in \mathrm{add}_{\mathrm{mod}S} DT$. And we have an epimorphism $v_X : G_X \to X$ in $\mathcal{RC}^{\mathrm{tr}}(\mathcal{S})$ which is of the form

$$\begin{array}{lcccccccccccc} G_X: & \cdots & \rightsquigarrow & 0 & \rightsquigarrow & 0 & \rightsquigarrow & P & \rightsquigarrow & DT_k & \rightsquigarrow & 0 & \rightsquigarrow \cdots \\ \downarrow v_X & & & & & & & \downarrow 1 & & \downarrow v_k & & & \\ X: & \cdots & \rightsquigarrow & 0 & \rightsquigarrow & 0 & \rightsquigarrow & P & \rightsquigarrow & P \otimes_S DS & \rightsquigarrow & 0 & \rightsquigarrow \cdots \end{array}$$

Then, from the structure of $r(v_X)$ in 4.2.3, we can see that $r(v_X)$ is of the form

$$\begin{array}{lccccc} \mathrm{R}_{DT}(G_X): & 0 \rightsquigarrow & \mathrm{Hom}_S(DT, P) \rightsquigarrow & \mathrm{Hom}_S(DT, DT_k) \oplus P \otimes_S T \overset{(\delta_{R_k}, 0)^{\mathrm{T}}}{\rightsquigarrow} & DT_k \otimes_S T \rightsquigarrow & 0 \\ \downarrow r(v_X) & & \downarrow 1 & \downarrow (\mathrm{Hom}_S(DT, v_k), \zeta_{r_k})^{\mathrm{T}} & \downarrow & \\ \hat{\mathrm{H}}\mathrm{om}(DT, X): & 0 \rightsquigarrow & \mathrm{Hom}_S(DT, P) \rightsquigarrow & \mathrm{Hom}_S(DT, P \otimes_S DS) \rightsquigarrow & 0 \rightsquigarrow & 0, \end{array}$$

where ζ_{r_k} is defined as in 4.2.3 and δ_{R_k} is defined as in 4.2.1. One checks that both homomorphisms ζ_{r_k} and δ_{R_k} are, in fact, isomorphisms. So we obtain that $\mathbf{Q}_{DT}(X) = \mathrm{Ker}(r(v_X))$ is of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_S(DT, DT_k) \overset{\delta'_k}{\rightsquigarrow} DT_k \otimes_S T \rightsquigarrow 0 \rightsquigarrow \cdots,$$

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where δ'_k is an induced isomorphism: $\operatorname{Hom}_S(DT, DT_k) \to \operatorname{Hom}_R(DR, DT_k \otimes_S T)$. Since $\operatorname{Hom}_S(DT, DT_k) \in \operatorname{add}_{\operatorname{mod}R}(\operatorname{Hom}_S(DT, DT)) = \operatorname{add}_{\operatorname{mod}R} R$, we see that $\operatorname{Hom}_S(DT, DT_k)$ is projective and that $\mathbf{Q}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$.

It follows that the functor $\mathbf{Q}_{DT}$ factors through $\underline{\mathcal{RC}}(S)$. The induced functor from $\underline{\mathcal{RC}}(S)$ to $\underline{\mathcal{RC}}(R)$ is still denoted by $\mathbf{Q}_{DT}$.

4.3 The isomorphism $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$

4.3.1 Computing the composition $\mathbf{Q}_{DT}\mathbf{S}_T$

Take any $X = \{X_i, \delta_i^{\otimes}\} \in \mathcal{RC}(R)$. From the chosen exact sequence $0 \to X \xrightarrow{u_X} A_X \xrightarrow{\pi_X} B_X \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_X \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_X \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$, as in 4.1.4, we obtain an exact sequence

$$0 \to X \hat{\otimes} DT \xrightarrow{l(u_X)} \mathrm{L}_T(A_X) \xrightarrow{s} \mathbf{S}_T(X) \to 0$$

by the construction of the functor $\mathbf{S}_T$ in 4.1.5. Note that, for each i, $\mathbf{S}_T(X)_i$ is given by the pushout diagram

$$\begin{array}{ccc} X_i \otimes_R DT & \xrightarrow{\theta_{l_i}} & \operatorname{Hom}_R(T, (A_X)_{i-1}) \\ {\scriptstyle (u_X)_i \otimes_R DT}\downarrow & & \downarrow{\scriptstyle s_i^1} \\ (A_X)_i \otimes_R DT & \xrightarrow{s_i^2} & \mathbf{S}_T(X)_i. \end{array}$$

Now we take a projective R-module $P_{(A_X)_i}$ such that $P_{(A_X)_i} \xrightarrow{p_i} (A_X)_i \to 0$ is exact. Then we have a pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X}_i & \xrightarrow{(\overline{u}_X)_i} & P_{(A_X)_i} & \longrightarrow & (B_X)_i & \longrightarrow & 0 \\ & & {\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle p_i} & & \| & & \\ 0 & \longrightarrow & X_i & \xrightarrow{(u_X)_i} & (A_X)_i & \longrightarrow & (B_X)_i & \longrightarrow & 0. \end{array}$$

Since $\mathcal{B}$ is closed under kernels of epimorphisms, we see that $\overline{X}_i \in \mathcal{B}$. By applying the functor $- \otimes_R DT$, the diagram above induces the following commutative diagram with exact rows since $\mathcal{B} \subseteq \operatorname{KerTor}_1^R(-, DT)$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X}_i \otimes_R DT & \xrightarrow{(\overline{u}_X)_i \otimes_R DT} & P_{(A_X)_i} \otimes_R DT & \longrightarrow & (B_X)_i \otimes_R DT & \longrightarrow & 0 \\ & & {\scriptstyle q_i \otimes_R DT}\downarrow & & \downarrow{\scriptstyle p_i \otimes_R DT} & & \| & & \\ 0 & \longrightarrow & X_i \otimes_R DT & \xrightarrow{(u_X)_i \otimes_R DT} & (A_X)_i \otimes_R DT & \longrightarrow & (B_X)_i \otimes_R DT & \longrightarrow & 0 \end{array}$$

Now, one can check that the following diagram is commutative with exact rows, for each i, where the lower row is obtained from the first pushout diagram in this section. Here, $t_i^P = (-(q_i \otimes_R DT) \cdot \theta_{l_i}, (\overline{u}_X)_i \otimes_R DT)$, $t_i = (-\theta_{l_i}, (u_X)_i \otimes_R DT)$, $s_i^P = \binom{s_i^1}{(p_i \otimes_R DT) \cdot s_i^2}$

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and $s_i = \left(\begin{smallmatrix} s_i^1 \\ s_i^2 \end{smallmatrix}\right)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{X}_i \otimes_R DT & \xrightarrow{t_i^P} & \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT & \xrightarrow{s_i^P} & \mathbf{S}_T(X)_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q_i \otimes_R DT} & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & p_i \otimes_R DT \end{pmatrix} & & \| & & \\ 0 & \longrightarrow & \mathbf{S}_T(X)_i & \xrightarrow{t_i} & \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus (A_X)_i \otimes_R DT & \xrightarrow{s_i} & \mathbf{S}_T(X)_i & \longrightarrow & 0. \end{array}$$

Denote $\mathfrak{L}_{A_X}^P := \{\mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT\} \in \mathcal{RC}^{\mathrm{tr}}(S)$. Note that $\overline{X}_i \otimes_R DT \in \mathcal{K}$ and $\mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT \in \mathcal{G}$, so we have an exact sequence in $\mathcal{RC}^{\mathrm{tr}}(S)$ from the first row in the above commutative diagram

$$0 \to \overline{X} \otimes_R DT \longrightarrow \mathfrak{L}_{A_X}^P \xrightarrow{s^P} \mathbf{S}_T(X) \to 0$$

with $\overline{X} \otimes_R DT \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{K})$ and $\mathfrak{L}_{A_X}^P \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{G})$, as in 4.2.4. By applying the homomorphism r in 4.2.3 to the homomorphism $s^P : \mathfrak{L}_{A_X}^P \to \mathbf{S}_T(X)$, we have an exact sequence in $\mathcal{RC}(R)$ by the construction of the functor $\mathbf{Q}_{DT}$ in 4.2.5

$$0 \to \mathbf{Q}_{DT}\mathbf{S}_T(X) \xrightarrow{\lambda} \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P) \xrightarrow{r(s^P)} \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X)) \to 0,$$

where $r(s^P)$ is defined as in 4.2.3.

4.3.2 The object $X \oplus \mathrm{L}_R(P_{A_X}^+)$ in $\mathcal{RC}(R)$

Denote $P_{A_X}^+ := \{P_{(A_X)_{i+1}}\}$, then $P_{A_X}^+ \in \mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_{\mathrm{mod}R} R)$. Applying the functor L_R in the remark in 4.1.1, we obtain that $\mathrm{L}_R(P_{A_X}^+)$ is a projective object in $\mathcal{RC}(R)$. Hence, the object $X \oplus \mathrm{L}_R(P_{A_X}^+)$ is isomorphic to X in $\underline{\mathcal{RC}}(R)$.

We will prove that $\mathbf{Q}_{DT}\mathbf{S}_T(X) \simeq X \oplus \mathrm{L}_R(P_{A_X}^+)$ naturally. And then, $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$.

The general strategy is as follows. First, we construct a natural homomorphism $\xi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)$. Second, we show that $\xi \cdot r(s^P) = 0$, that is, the composition of ξ and the homomorphism $r(s^P) : \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P) \to \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X))$ in the exact sequence above is 0. Thus, we obtain a homomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$. Finally, we prove that ϕ is indeed a natural isomorphism.

4.3.3 The homomorphism $\xi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)$

Recall from the construction in 4.3.1 that $X = \{X_i, \delta_i^{\otimes}\}$ and,
$\mathrm{L}_R(P_{A_X}^+) = \{\mathrm{Hom}_R(R, P_{(A_X)_i}) \oplus P_{(A_X)_{i+1}} \otimes_R DR\} = \{P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR\}$, where the structure map $\delta_i^{\otimes}(\mathrm{L}_R(P_{A_X}^+)) : (\mathrm{L}_R(P_{A_X}^+))_i \otimes DR \to (\mathrm{L}_R(P_{A_X}^+))_{i-1}$ is given by $\left(\begin{smallmatrix} 0 & 1_{P_{(A_X)_i} \otimes_R DR} \\ 0 & 0 \end{smallmatrix}\right)$, and that

$$\begin{aligned} \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P) &= \{\mathrm{Hom}_S(DT, (\mathfrak{L}_{A_X}^P)_i) \oplus (\mathfrak{L}_{A_X}^P)_{i+1} \otimes_S T\} \\ &= \{\mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT) \\ &\qquad \oplus (\mathrm{Hom}_R(T, (A_X)_i) \oplus P_{(A_X)_{i+1}} \otimes_R DT) \otimes_S T\}, \end{aligned}$$

where the structure map

$$\delta_i^{\otimes}(\mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)) : (\mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P))_i \otimes_R DR \to (\mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P))_{i-1}$$

is defined in 4.2.1.

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By the definition, we have that $\delta_i^{\otimes}(\mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)) = \begin{pmatrix} 0 & 0 & \gamma_i^{11} & 0 \\ 0 & 0 & 0 & \gamma_i^{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where

$\gamma_i^{11}: \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1})) \otimes_R DR \to \mathrm{Hom}_R(T, (A_X)_{i-1}) \otimes_S T$ is given by the composition

$$\begin{aligned}\gamma_i^{11} &: \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1})) \otimes_R DR \\ &\simeq \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1})) \otimes_R DT \otimes_S T \\ &\xrightarrow{\epsilon^{DT}_{\mathrm{Hom}_R(T,(A_X)_{i-1})} \otimes_S T} \mathrm{Hom}_R(T, (A_X)_{i-1}) \otimes_S T,\end{aligned}$$

and $\gamma_i^{22}: \mathrm{Hom}_S(DT, P_{(A_X)_i} \otimes_R DT) \otimes_R DR \to P_{(A_X)_i} \otimes_R DT \otimes_S T$ is defined similarly as γ_i^{11} by replacing $\mathrm{Hom}_R(T, (A_X)_{i-1})$ with $P_{(A_X)_i} \otimes_R DT$.

Let $\xi = \{\xi_i\}: X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)$ be a homomorphism. We may assume that $\xi_i = (\xi_i^a, \xi_i^b)$, where

$$\xi_i^a: X_i \oplus P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR \to \mathrm{Hom}_S(DT, (\mathfrak{L}_{A_X}^P)_i)$$

and

$$\xi_i^b: X_i \oplus P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR \to (\mathfrak{L}_{A_X}^P)_{i+1} \otimes_S T.$$

4.3.3.1 The homomorphism ξ_i^a in $\mathrm{mod}R$

We set $\xi_i^a = \begin{pmatrix} \xi_i^{a11} & \xi_i^{a12} \\ \xi_i^{a21} & \xi_i^{a22} \\ \xi_i^{a31} & \xi_i^{a32} \end{pmatrix}$:

$$X_i \oplus P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR \longrightarrow \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT).$$

Using the isomorphism ${}_SDS_S \simeq {}_ST \otimes_R DT_S$ and the adjoint isomorphism

$$\mathbf{\Gamma}^{DT}: \mathrm{Hom}_S(- \otimes_R DT, -) \simeq \mathrm{Hom}_R(-, \mathrm{Hom}_S(DT, -)),$$

we define the components of ξ_i^a as follows.

- The morphism $\xi_i^{a11} = \mathbf{\Gamma}^{DT}(\theta_{l_i}): X_i \to \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1}))$, where $\theta_{l_i}: X_i \otimes_R DT \to \mathrm{Hom}_R(T, (A_X)_{i-1})$ is defined in 4.1.3.

In other words, the morphism ξ_i^{a11} is given by the composition:

$$\eta_{X_i}^{DR} \cdot \mathrm{Hom}_R(DR, \delta_i^{\otimes}(X) \cdot \mathrm{Hom}_R(DR, (u_X)_{i-1})) = \delta_i^{\mathrm{H}}(X) \cdot \mathrm{Hom}_R(DR, (u_X)_{i-1})$$

and some natural isomorphisms

$$\begin{aligned}X_i &\xrightarrow{\eta_{X_i}^{DR}} \mathrm{Hom}_R(DR, X_i \otimes_R DR) \xrightarrow{\mathrm{Hom}_R(DR, \delta_i^{\otimes}(X))} \mathrm{Hom}_R(DR, X_{i-1}) \\ &\xrightarrow{\mathrm{Hom}_R(DR, (u_X)_{i-1})} \mathrm{Hom}_R(DR, (A_X)_{i-1}) \simeq \mathrm{Hom}_R(DT \otimes_S T, (A_X)_{i-1}) \\ &\simeq \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1})).\end{aligned}$$

- The morphism

$$\begin{aligned}\xi_i^{a22} &= \mathbf{\Gamma}^{DT}(1_{(P_{(A_X)_i} \otimes_R DT)}) \\ &= \eta^{DT}_{P_{(A_X)_i}}: P_{(A_X)_i} \to \mathrm{Hom}_S(DT, P_{(A_X)_i} \otimes_R DT).\end{aligned}$$

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- The remaining morphisms $\xi_i^{a_{12}}, \xi_i^{a_{21}}, \xi_i^{a_{31}}, \xi_i^{a_{32}}$ are all 0.

So we have that $\xi_i^a = \begin{pmatrix} \xi_i^{a_{11}} & 0 \\ 0 & \xi_i^{a_{22}} \\ 0 & 0 \end{pmatrix}$, where

$$\begin{aligned} \xi_i^{a_{11}} &= \mathbf{\Gamma}^{DT}(\theta_{l_i}), \qquad \text{and} \\ \xi_i^{a_{22}} &= \mathbf{\Gamma}^{DT}(1_{(P_{(A_X)_i} \otimes_R DT)}). \end{aligned}$$

4.3.3.2 The homomorphism ξ_i^b in mod$R$ We set $\xi_i^b = \begin{pmatrix} \xi_i^{b_{11}} & \xi_i^{b_{12}} \\ \xi_i^{b_{21}} & \xi_i^{b_{22}} \\ \xi_i^{b_{31}} & \xi_i^{b_{32}} \end{pmatrix}$:

$$X_i \oplus P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR \to (\mathrm{Hom}_R(T, (A_X)_i) \oplus P_{(A_X)_{i+1}} \otimes_R DT) \otimes_S T,$$

where the components are defined naturally as follows.

- The morphism $\xi_i^{b_{11}} = (u_X)_i \cdot (\epsilon^T_{(A_X)_i})^{-1} : X_i \to \mathrm{Hom}_R(T, (A_X)_i) \otimes_S T$ (note that $\epsilon^T_{(A_X)_i}$ is an isomorphism since $(A_X)_i \in \mathcal{A}$), that is, is given by the composition

$$X_i \xrightarrow{(u_X)_i} (A_X)_i \xrightarrow{(\epsilon^T_{(A_X)_i})^{-1}} \mathrm{Hom}_R(T, (A_X)_i) \otimes_S T.$$

- The morphism $\xi_i^{b_{21}} = p_i \cdot (\epsilon^T_{(A_X)_i})^{-1} : P_{(A_X)_i} \to \mathrm{Hom}_R(T, (A_X)_i) \otimes_S T$, that is, is given by the composition

$$P_{(A_X)_i} \xrightarrow{p_i} (A_X)_i \xrightarrow{(\epsilon^T_{(A_X)_i})^{-1}} \mathrm{Hom}_R(T, (A_X)_i) \otimes_S T.$$

- The morphism $\xi_i^{b_{32}} : P_{(A_X)_{i+1}} \otimes_R DR \to P_{(A_X)_{i+1}} \otimes_R DT \otimes_S T$ is the natural isomorphism given by ${}_R(DT \otimes_S T)_R \simeq {}_R DR_R$.
- The remaining morphisms $\xi_i^{b_{12}}, \xi_i^{b_{22}}, \xi_i^{b_{31}}$ are all 0.

So we have that $\xi_i^b = \begin{pmatrix} \xi_i^{b_{11}} & 0 \\ \xi_i^{b_{21}} & 0 \\ 0 & \xi_i^{b_{32}} \end{pmatrix}$, where $\xi_i^{b_{11}} = (u_X)_i \cdot (\epsilon^T_{(A_X)_i})^{-1}$, $\xi_i^{b_{21}} = p_i \cdot (\epsilon^T_{(A_X)_i})^{-1}$, and $\xi_i^{b_{32}}$ is the natural isomorphism.

4.3.3.3 ξ is a homomorphism in $\mathcal{RC}(R)$ We now show that the above-defined morphism $\xi : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})$ is compatible with structure maps, that is,

$$\xi_i \otimes_R DR \cdot \delta_i^{\otimes}(\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})) = \delta_i^{\otimes}(X \oplus \mathrm{L}_R(P^+_{A_X})) \cdot \xi_{i-1}$$

holds for each i.

Indeed, by the involved definitions, we have that

$$\begin{aligned} \xi_i \otimes_R DR \cdot \delta_i^{\otimes}(\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})) &= \begin{pmatrix} \xi_i^{a_{11}} & 0 & \xi_i^{b_{11}} & 0 \\ 0 & \xi_i^{a_{22}} & \xi_i^{b_{21}} & 0 \\ 0 & 0 & 0 & \xi_i^{b_{32}} \end{pmatrix} \otimes_R DR \cdot \begin{pmatrix} 0 & 0 & \gamma_i^{11} & 0 \\ 0 & 0 & 0 & \gamma_i^{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & \xi_i^{a_{11}} \otimes_R DR \cdot \gamma_i^{11} & 0 \\ 0 & 0 & 0 & \xi_i^{a_{22}} \otimes_R DR \cdot \gamma_i^{22} \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

and that

$$\begin{aligned}\delta_i^{\otimes}(X \oplus \mathrm{L}_R(P^+_{A_X})) \cdot \xi_{i-1} &= \begin{pmatrix} \delta_i^{\otimes}(X) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^{a_{11}}_{i-1} & 0 & \xi^{b_{11}}_{i-1} & 0 \\ 0 & \xi^{a_{22}}_{i-1} & \xi^{b_{21}}_{i-1} & 0 \\ 0 & 0 & 0 & \xi^{b_{32}}_{i-1} \end{pmatrix} \\ &= \begin{pmatrix} \delta_i^{\otimes}(X) \cdot \xi^{a_{11}}_{i-1} & 0 & \delta_i^{\otimes}(X) \cdot \xi^{b_{11}}_{i-1} & 0 \\ 0 & 0 & 0 & \xi^{b_{32}}_{i-1} \\ 0 & 0 & 0 & 0 \end{pmatrix}\end{aligned}$$

- $\delta_i^{\otimes}(X) \cdot \xi^{a_{11}}_{i-1} = 0$. In fact, since $\xi^{a_{11}}_{i-1}$ is the composition of the morphism $\delta^{\mathrm{H}}_{i-1}(X) \cdot \mathrm{Hom}_R(DR, (u_X)_{i-2})$ and some natural isomorphisms by the construction, we see that $\delta_i^{\otimes}(X) \cdot \xi^{a_{11}}_{i-1}$ factors through $\delta_i^{\otimes}(X) \cdot \delta^{\mathrm{H}}_{i-1}(X)$. But the latter is 0 as X is a repe-complex. Thus, $\delta_i^{\otimes}(X) \cdot \xi^{a_{11}}_{i-1} = 0$.
- $\xi^{a_{11}}_i \otimes_R DR \cdot \gamma^{11}_i = \delta_i^{\otimes}(X) \cdot \xi^{b_{11}}_{i-1}$. This follows from the involved definitions and the following commutative diagram:

$$\begin{array}{ccc}
X_i \otimes_R DR & \xrightarrow{\quad 1 \quad} & X_i \otimes_R DR \\
\downarrow{\scriptstyle \delta^{\mathrm{H}}_i(X) \otimes_R DR} & & \downarrow{\scriptstyle \delta^{\otimes}_i(X)} \\
\mathrm{Hom}_R(DR, X_{i-1}) \otimes_R DR & \xrightarrow{\epsilon^{DR}_{X_{i-1}}} & X_{i-1} \\
\downarrow{\scriptstyle \mathrm{Hom}_R(DR, (u_X)_{i-1}) \otimes_R DR} & & \downarrow{\scriptstyle (u_X)_{i-1}} \\
\mathrm{Hom}_R(DR, (A_X)_{i-1}) \otimes_R DR & \xrightarrow{\epsilon^{DR}_{(A_X)_{i-1}}} & (A_X)_{i-1} \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1})) \otimes_R DT \otimes_S T & \xrightarrow{\omega} & \mathrm{Hom}_R(T, (A_X)_{i-1}) \otimes_S T,
\end{array}$$

  where $\omega := \epsilon^{DT}_{\mathrm{Hom}_R(T,(A_X)_{i-1})} \otimes_S T$. The last square is commutative since $\epsilon^{DT \otimes_S T}_M = \epsilon^{DT}_{\mathrm{Hom}_R(T,M)} \otimes_S T \cdot \epsilon^T_M$.
- $\xi^{a_{22}}_i \otimes_R DR \cdot \gamma^{22}_i = \xi^{b_{32}}_{i-1}$. This follows from the involved definitions and the equality $1_{(P_{(A_X)_i} \otimes_R DT)} = \eta^{DT}_{(P_{(A_X)_i})} \otimes_R DT \cdot \epsilon^{DT}_{(P_{(A_X)_i} \otimes_R DT)}$.

Then we can easily conclude that the morphism $\xi : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})$ is, in fact, a homomorphism in $\mathcal{RC}(R)$.

4.3.4 The composition $\xi \cdot r(s^P) = 0$, and so ξ factors through a homomorphism $\phi : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$

4.3.4.1 The analysis of the homomorphism $s : \mathrm{L}_T(A_X) \to \mathbf{S}_T(X)$ in 4.3.1 Recall from 4.3.1 that $s = \{s_i\} : \mathrm{L}_T(A_X) \to \mathbf{S}_T(X)$ is a homomorphism in $\mathcal{RC}(S)$ which is the cokernel of the homomorphism $l(u_X)$. Note that $(\mathrm{L}_T(A_X))_i = \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus (A_X)_i \otimes_R DT$, so we write that $s_i = \left(s^1_i \, s^2_i \right)$ as we have done in the last commutative diagram in 4.3.1.

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The fact that s is a homomorphism in $\mathcal{RC}(S)$ implies that there is the following commutative diagram, for each i:

$$\begin{array}{ccc} [\mathrm{Hom}_R(T,(A_X)_{i-1}))\oplus(A_X)_i)\otimes_R DT]\otimes_S DS & \xrightarrow{\left(s_i^1 s_i^2\right)\otimes_S DS} & \mathbf{S}_T(X)_i\otimes_S DS \\ \Big\downarrow{\scriptstyle \delta_i^{\otimes}(\mathrm{L}_T(A_X))} & & \Big\downarrow{\scriptstyle \delta_i^{\otimes}(\mathbf{S}_T(X))} \\ \mathrm{Hom}_R(T,(A_X)_{i-2}))\oplus(A_X)_{i-1})\otimes_R DT & \xrightarrow{\left(s_{i-1}^1 s_{i-1}^2\right)} & \mathbf{S}_T(X)_{i-1}. \end{array}$$

By the definition of $\delta_i^{\otimes}(\mathrm{L}_T(A_X))$ (see 4.1.1) and the above commutative diagram, we obtain that $(s_i^2\otimes_S DS)\cdot\delta_i^{\otimes}(\mathbf{S}_T(X))=0$ and that $(s_i^1\otimes_S DS)\cdot\delta_i^{\otimes}(\mathbf{S}_T(X))\simeq(\epsilon^T_{A_{X_{i-1}}}\otimes_R DT)\cdot s_{i-1}^2$.

4.3.4.2 The homomorphism $r(s^P):\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})\to\hat{\mathrm{H}}\mathrm{om}(DT,\mathbf{S}_T(X))$ Recall from 4.3.1 that

$$s^P=\{s_i^P\}=\left\{\begin{pmatrix} s_i^1 \\ (p_i\otimes_R DT)\cdot s_i^2\end{pmatrix}\right\}:\mathfrak{L}^P_{A_X}\to\mathbf{S}_T(X).$$

By 4.2.3, we know that

$$\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})=\{\mathrm{Hom}_S(DT,(\mathfrak{L}^P_{A_X})_i)\oplus(\mathfrak{L}^P_{A_X})_{i+1}\otimes_S T\}$$

and that

$$r(s^P)_i=\left(\mathrm{Hom}_S(DT,s_i^P) - \zeta_{r_i}\right),$$

where $\zeta_{r_i}=\mathbf{\Gamma}^{DT}((s^P_{i+1}\otimes_S DS)\cdot\delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$.

For convenience, we set $r(s^P)_i=\left(r_i^1 r_i^2\right)$, where

$$r_i^1=\mathrm{Hom}_S(DT,s_i^P):\mathrm{Hom}_S(DT,(\mathfrak{L}^P_{A_X})_i)\to\mathrm{Hom}_S(DT,\mathbf{S}_T(X)_i)$$

and

$$r_i^2=-\zeta_{r_i}:(\mathfrak{L}^P_{A_X})_{i+1}\otimes_S T\to\mathrm{Hom}_S(DT,\mathbf{S}_T(X)_i).$$

4.3.4.3 Checking $\xi\cdot r(s^P)=0$ To check $\xi\cdot r(s^P)=0$, we need only to check that $\xi_i^a r_i^1+\xi_i^b r_i^2=0$ for each i, since $\xi_i=(\xi_i^a,\xi_i^b)$ and $r(s^P)_i=\left(r_i^1 r_i^2\right)$. Note that $r_i^1=\mathrm{Hom}_S(DT,s_i^P)$ and $r_i^2=-\zeta_{r_i}$, so it is enough to check that $\xi_i^a\cdot\mathrm{Hom}_S(DT,s_i^P)=\xi_i^b\cdot\zeta_{r_i}$.

Since $\xi_i^a=\begin{pmatrix}\xi^a_{11} & 0\\ 0 & \xi^a_{22}\\ 0 & 0\end{pmatrix}$, $\xi_i^b=\begin{pmatrix}\xi^b_{11} & 0\\ \xi^b_{21} & 0\\ 0 & \xi^b_{32}\end{pmatrix}$, $\zeta_{r_i}=\mathbf{\Gamma}^{DT}((s^P_{i+1}\otimes_S DS)\cdot\delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$ and $s_i^P=\begin{pmatrix} s_i^1 \\ (p_i\otimes_R DT)\cdot s_i^2\end{pmatrix}$, we just check the following.

(1) $\xi^a_{11}\cdot\mathrm{Hom}_S(DT,s_i^1)=\xi^b_{11}\cdot\mathbf{\Gamma}^{DT}((s^1_{i+1}\otimes_S DS)\cdot\delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$.
By 4.3.3.1, we have that

$$\xi^a_{11}\cdot\mathrm{Hom}_S(DT,s_i^1)=\mathbf{\Gamma}^{DT}(\theta_{l_i})\cdot\mathrm{Hom}_S(DT,s_i^1)=\mathbf{\Gamma}^{DT}(\theta_{l_i}\cdot s_i^1),$$

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where the later equality uses the naturality of $\mathbf{\Gamma}^{DT}$. On the other hand, by 4.3.3.2 and 4.3.4.1, we obtain that

$$\begin{aligned}&\xi_{11}^{b}\cdot\mathbf{\Gamma}^{DT}((s_{i+1}^{1}\otimes_{S}DS)\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_{T}(X)))\\&\quad=((u_{X})_{i}\cdot(\epsilon_{(A_X)_i}^{T})^{-1})\cdot\mathbf{\Gamma}^{DT}((\epsilon_{(A_X)_i}^{T}\otimes_{R}DT)\cdot s_{i}^{2})\\&\quad=\mathbf{\Gamma}^{DT}((((u_{X})_{i}\cdot(\epsilon_{(A_X)_i}^{T})^{-1})\otimes_{R}DT)\cdot(\epsilon_{(A_X)_i}^{T}\otimes_{R}DT)\cdot s_{i}^{2})\\&\quad=\mathbf{\Gamma}^{DT}(((u_{X})_{i}\otimes_{R}DT)\cdot s_{i}^{2}).\end{aligned}$$

But $((u_X)_i\otimes_R DT)\cdot s_i^2=\theta_{l_i}\cdot s_i^1$ by the pushout diagram on $\mathbf{S}_T(X)_i$ in 4.3.1. Hence, equality (1) holds.

(2) $\xi_{22}^{a}\cdot\mathrm{Hom}_S(DT,(p_i\otimes_R DT)\cdot s_i^2)=\xi_{21}^{b}\cdot\mathbf{\Gamma}^{DT}((s_{i+1}^{1}\otimes_S DS)\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_T(X)))$.
By 4.3.3.1 and the naturality of $\mathbf{\Gamma}^{DT}$,

$$\begin{aligned}&\xi_{22}^{a}\cdot\mathrm{Hom}_S(DT,(p_i\otimes_R DT)\cdot s_i^2)\\&\quad=\mathbf{\Gamma}^{DT}(1_{P_{(A_X)_i}\otimes_R DT})\cdot\mathrm{Hom}_S(DT,(p_i\otimes_R DT)\cdot s_i^2)\\&\quad=\mathbf{\Gamma}^{DT}(1_{P_{(A_X)_i}\otimes_R DT}\cdot((p_i\otimes_R DT)\cdot s_i^2))\\&\quad=\mathbf{\Gamma}^{DT}((p_i\otimes_R DT)\cdot s_i^2).\end{aligned}$$

On the other hand, by 4.3.3.2 and 4.3.4.1 and the naturality of $\mathbf{\Gamma}^{DT}$,

$$\begin{aligned}&\xi_{21}^{b}\cdot\mathbf{\Gamma}^{DT}((s_{i+1}^{1}\otimes_{S}DS)\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_{T}(X)))\\&\quad=(p_i\cdot(\epsilon_{(A_X)_i}^{T})^{-1})\cdot\mathbf{\Gamma}^{DT}((\epsilon_{(A_X)_i}^{T}\otimes_{R}DT)\cdot s_{i}^{2})\\&\quad=\mathbf{\Gamma}^{DT}(((p_i\cdot(\epsilon_{(A_X)_i}^{T})^{-1})\otimes_{R}DT)\cdot(\epsilon_{(A_X)_i}^{T}\otimes_{R}DT)\cdot s_{i}^{2})\\&\quad=\mathbf{\Gamma}^{DT}((p_i\otimes_{R}DT)\cdot s_{i}^{2}).\end{aligned}$$

Hence, equality (2) holds.

(3) $0=\xi_{32}^{b}\cdot\mathbf{\Gamma}^{DT}(((p_{i+1}\otimes_R DT)\cdot s_{i+1}^2)\otimes_S DS\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_T(X)))$.
In fact, the equality holds by observing that

$$\begin{aligned}&\mathbf{\Gamma}^{DT}(((p_{i+1}\otimes_R DT)\cdot s_{i+1}^2)\otimes_S DS\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_T(X)))\\&\quad=\mathbf{\Gamma}^{DT}((p_{i+1}\otimes_R DT\otimes_S DS)\cdot(s_{i+1}^2\otimes_S DS)\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_T(X)))\\&\quad=0\end{aligned}$$

since $(s_{i+1}^2\otimes_S DS)\cdot\delta_{i+1}^{\otimes}(\mathbf{S}_T(X))=0$ by 4.3.4.1.

Altogether, we prove that $\xi\cdot r(s^P)=0$ and, therefore, ξ factors through $\mathbf{Q}_{DT}\mathbf{S}_T(X)=\mathrm{Ker}(r(s^P))$ by a homomorphism

$$\phi: X\oplus \mathrm{L}_R(P_{A_X}^{+})\to\mathbf{Q}_{DT}\mathbf{S}_T(X)$$

in $\mathcal{RC}(R)$, that is, $\xi=\phi\cdot\lambda$.

4.3.5 The induced homomorphism $\phi: X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is an isomorphism

We now prove that the induced homomorphism $\phi: X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is an isomorphism. Clearly, it is equivalent to show that $\phi_i: (X \oplus \mathrm{L}_R(P^+_{A_X}))_i \to \mathbf{Q}_{DT}\mathbf{S}_T(X)_i$ is an isomorphism, for each i.

We will show that there is the following commutative diagram $(*)$ with exact rows, for each i:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{X}_i & \xrightarrow{(-q_i,(\overline{u}_X)_i,0)} & X_i \oplus \mathrm{L}_R(P^+_{A_X})_i & \xrightarrow{\beta_i} & C_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta^{DT}_{\overline{X}_i}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \sigma_i} & & \\
0 & \longrightarrow & \mathrm{Hom}_S(DT, \overline{X}_i \otimes_R DT) & \xrightarrow{a_i} & \mathbf{Q}_{DT}\mathbf{S}_T(X)_i & \xrightarrow{\lambda_i^2} & C_2 & \longrightarrow & 0,
\end{array}$$

where

$$C_1 := (A_X)_i \oplus P_{(A_X)_{i+1}} \otimes_R DR,$$
$$C_2 := \mathrm{Hom}_R(T, (A_X)_i) \otimes_S T \oplus P_{(A_X)_{i+1}} \otimes_R DT \otimes_S T,$$

$\overline{X}_i \in \mathcal{B}$ is obtained in 4.3.1, the morphism a_i is given in 4.3.5.2, the morphism $\beta_i = \begin{pmatrix} (u_X)_i & 0 \\ p_i & 0 \\ 0 & 1 \end{pmatrix}$ and the morphism σ_i is the direct sum of two canonical isomorphisms. Note that $\mathrm{L}_R(P^+_{A_X})_i = P_{(A_X)_i} \oplus P_{(A_X)_{i+1}} \otimes_R DR$.

Then, since $\overline{X}_i \in \mathcal{B}$ implies that $\eta^{DT}_{\overline{X}_i}$ is an isomorphism, we obtain that ϕ_i is also an isomorphism from the above commutative diagram.

4.3.5.1 The upper row in the diagram $(*)$ **is exact** In fact, the pullback of $p_i: P_{(A_X)_i} \to (A_X)_i$ and $(u_X)_i: X_i \to (A_X)_i$ in 4.3.1 gives an exact sequence

$$0 \to \overline{X}_i \xrightarrow{(-q_i,(\overline{u}_X)_i)} X_i \oplus P_{(A_X)_i} \xrightarrow{\begin{pmatrix} (u_X)_i \\ p_i \end{pmatrix}} (A_X)_i \to 0$$

since p_i is surjective. The direct sum of the above exact sequence and the trivial exact sequence

$$0 \to 0 \to P_{(A_X)_{i+1}} \otimes_R DR \xrightarrow{1} P_{(A_X)_{i+1}} \otimes_R DR \to 0$$

gives us the exact sequence in the upper row in the diagram $(*)$.

4.3.5.2 The bottom row in the diagram $(*)$ **is exact** Note that we have the following exact sequence in 4.3.1

$$0 \to \mathbf{Q}_{DT}\mathbf{S}_T(X) \xrightarrow{\lambda} \mathrm{R}_{DT}(\mathfrak{L}^P_{A_X}) \xrightarrow{r(s^P)} \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X)) \to 0$$

and that

$$\begin{aligned}
\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})_i &= \mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i) \oplus (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T \\
&= \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT) \\
&\quad \oplus (\mathrm{Hom}_R(T, (A_X)_i) \oplus P_{(A_X)_{i+1}} \otimes_R DT) \otimes_S T.
\end{aligned}$$

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So we have the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ker}\lambda_i^2 & \xrightarrow{e} & \mathbf{Q}_{DT}\mathbf{S}_T(X)_i & \xrightarrow{\lambda_i^2} & C_3 & \longrightarrow & 0 \\ & & \downarrow q & & \downarrow \lambda_i^1 & & \downarrow -r_i^2 & & \\ 0 & \longrightarrow & \mathrm{Hom}_S(DT, \overline{X}_i \otimes_R DT) & \xrightarrow{b_i} & C_4 & \xrightarrow{r_i^1} & \mathrm{Hom}_S(DT, \mathbf{S}_T(X)_i) & \longrightarrow & 0, \end{array}$$

where

$$C_3 := (\mathrm{Hom}_R(T, (A_X)_i) \oplus P_{(A_X)_{i+1}} \otimes_R DT) \otimes_S T,$$
$$C_4 := \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT),$$

r_i^1, r_i^2 and λ_i^1, λ_i^2 are the components of the homomorphisms $r(s^P)_i$ and λ_i, respectively, and $b_i = \mathrm{Hom}_S(DT, t_i)$ with

$$t_i = (-(q_i \otimes_R DT) \cdot \theta_{l_i}, (\overline{u}_X)_i \otimes_R DT):$$
$$\overline{X}_i \otimes_R DT \to \mathrm{Hom}_R(T, (A_X)_{i-1}) \oplus P_{(A_X)_i} \otimes_R DT$$

is given in 4.3.1.

As the morphism $\lambda_i : \mathbf{Q}_{DT}\mathbf{S}_T(X)_i \to (\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X}))_i$ is injective, we obtain that the morphism q in the left column is an isomorphism. Note that $r_i^1 := \mathrm{Hom}_S(DT, s_i^P)$ is surjective since $\overline{X}_i \otimes_R DT \in \mathcal{K} \subseteq \mathrm{KerExt}_S^{>0}(DT, -)$ by the construction (see 4.3.1), so we can deduce that the upper row in the above diagram is exact. Thus, we get the bottom exact sequence in the diagram $(*)$ by setting $a_i = q^{-1}e$.

4.3.5.3 The diagram $(*)$ is commutative At first, it is easy to see that the right part of the diagram $(*)$ is commutative from the constructions of the morphisms ξ in 4.3.3 and ϕ in 4.3.4, which show that $\beta_i\sigma_i = \xi_i^b = \phi_i\lambda_i^2$.

As for the left part of the diagram $(*)$, we first show the following equality of compositions:

$$(\dagger_1) \qquad \eta_{\overline{X}_i}^{DT} \cdot a_i \cdot \lambda_i^1 = (-q_i, (\overline{u}_X)_i, 0) \cdot \phi_i \cdot \lambda_i^1.$$

Indeed, we have that

$$\begin{aligned} &\eta_{\overline{X}_i}^{DT} \cdot a_i \cdot \lambda_i^1 \\ &\quad = \eta_{\overline{X}_i}^{DT} \cdot b_i \quad \text{(by the commutative diagram in 4.3.5.2)} \\ &\quad = \eta_{\overline{X}_i}^{DT} \cdot \mathrm{Hom}_S(DT, t_i) \quad (\text{since } b_i = \mathrm{Hom}_S(DT, t_i)) \\ &\quad = \mathbf{\Gamma}^{DT}(1_{\overline{X}_i \otimes_R DT}) \cdot \mathrm{Hom}_S(DT, t_i) \\ &\quad = \mathbf{\Gamma}^{DT}(1_{\overline{X}_i \otimes_R DT} \cdot t_i) \quad (\text{by the naturality of } \mathbf{\Gamma}^{DT}) \\ &\quad = \mathbf{\Gamma}^{DT}(t_i) \\ &\quad = \mathbf{\Gamma}^{DT}((-(q_i \otimes_R DT) \cdot \theta_{l_i}, (\overline{u}_X)_i \otimes_R DT)) \end{aligned}$$

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and we also have that

$$\begin{aligned}
&(-q_i, (\overline{u}_X)_i, 0) \cdot \phi_i \cdot \lambda_i^1 \\
&\quad = (-q_i, (\overline{u}_X)_i, 0) \cdot \xi_i^a \quad (\text{since } \xi_i = (\xi_i^a, \xi_i^b) = \phi_i \cdot \lambda_i) \\
&\quad = (-q_i, (\overline{u}_X)_i, 0) \cdot \big(\xi_{11}^a \quad 00 \quad \xi_{22}^a 0 \quad 0\big) \quad (\text{by } 4.3.3.1) \\
&\quad = (-q_i \cdot \xi_{11}^a, (\overline{u}_X)_i \cdot \xi_{22}^a)
\end{aligned}$$

Since $\xi_{11}^a = \mathbf{\Gamma}^{DT}(\theta_{l_i})$ and $\xi_{22}^a = \mathbf{\Gamma}^{DT}(1_{(P_{(A_X)_i} \otimes_R DT)})$ by the construction in 4.3.3.1, we obtain that

$$q_i \cdot \xi_{11}^a = q_i \cdot \mathbf{\Gamma}^{DT}(\theta_{l_i}) = \mathbf{\Gamma}^{DT}(q_i \otimes_R DT \cdot \theta_{l_i})$$

and that

$$\begin{aligned}
(\overline{u}_X)_i \cdot \xi_{22}^a &= (\overline{u}_X)_i \cdot \mathbf{\Gamma}^{DT}(1_{(P_{(A_X)_i} \otimes_R DT)}) \\
&= \mathbf{\Gamma}^{DT}((\overline{u}_X)_i \otimes_R DT \cdot 1_{(P_{(A_X)_i} \otimes_R DT)}) = \mathbf{\Gamma}^{DT}((\overline{u}_X)_i \otimes_R DT).
\end{aligned}$$

Hence, we see that the equality ($\dagger_1$) holds.

Since $a_i \cdot \lambda_i^2 = 0$ and

$$(-q_i, (\overline{u}_X)_i, 0) \cdot \phi_i \cdot \lambda_i^2 = (-q_i, (\overline{u}_X)_i, 0) \cdot \xi_i^b = 0,$$

we also get that

$$(\dagger_2) \qquad \eta_{\overline{X}_i}^{DT} \cdot a_i \cdot \lambda_i^2 = (-q_i, (\overline{u}_X)_i, 0) \cdot \phi_i \cdot \lambda_i^2.$$

Now, from the property of the pullback in 4.3.5.2, we know that the two equalities ($\dagger_1$) and ($\dagger_2$) together imply that

$$\eta_{\overline{X}_i}^{DT} \cdot a_i = (-q_i, (\overline{u}_X)_i, 0) \cdot \phi_i.$$

Thus, the left part of the diagram is also commutative.

4.3.6 The isomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is natural on X

For any $X, Y \in \mathcal{RC}(R)$ and $h \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$, there is an induced morphism $h_A \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(A_X, A_Y)$, following from the construction in 4.1.4. Moreover, following from the construction in 4.3.1 and the definition of $P_{A_X}^+$ in 4.3.2, we see that the morphism h_A induces a morphism $p_{h_A}^+ \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(P_{A_X}^+, P_{A_Y}^+)$. Then, one can prove that

$$\begin{pmatrix} h & 0 \\ 0 & \mathrm{L}_R(p_{h_A}^+) \end{pmatrix} : X \oplus \mathrm{L}_R(P_{A_X}^+) \to Y \oplus \mathrm{L}_R(P_{A_Y}^+)$$

is a morphism in $\mathcal{RC}(R)$.

It is not hard to show that the following diagram is commutative:

$$\begin{array}{ccc}
Y \oplus \mathrm{L}_R(P_{A_X}^+) & \xrightarrow{\phi_X} & \mathbf{Q}_{DT}\mathbf{S}_T(X) \\
{\scriptstyle \big(h \quad 00 \quad \mathrm{L}_R(p_{h_A}^+)\big)} \downarrow & & \downarrow {\scriptstyle \mathbf{Q}_{DT}\mathbf{S}_T(h)} \\
Y \oplus \mathrm{L}_R(P_{A_Y}^+) & \xrightarrow{\phi_Y} & \mathbf{Q}_{DT}\mathbf{S}_T(Y).
\end{array}$$

Thus, the isomorphism ϕ is natural on X. This means that $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$ naturally.

4.4 The isomorphism $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$

Dually to the proof of 4.3, one can show that $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally.

Namely, for an object $Y \in \mathcal{RC}(S)$, one uses that $(\mathcal{G}, \mathcal{K})$ is a complete hereditary cotorsion pair in mod$S$ to obtain exact sequences $0 \to (K_Y)_i \to (G_Y)_i \to Y_i \to 0$, for each i. Then taking an injective S-module $I_{(G_Y)_i}$ and a monomorphism $(G_Y)_i \to I_{(G_Y)_i}$, one can show that there is a natural isomorphism $\mathbf{S}_T\mathbf{Q}_{DT}(Y) \to Y \oplus \mathrm{R}_{DS}(I^-_{G_Y})$, where $I^-_{G_Y} = \{I_{(G_Y)_{i-1}}\}$ and $\mathrm{R}_{DS}(I^-_{G_Y})$ is a projective object in $\mathcal{RC}(S)$ by Remark (2) in 4.2.1. And then one gets that $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally.

4.5 The last proof of Theorem 1

Recall that the natural functor [1] is an automorphism of repetitive categories, where $(X[1])_i = X_{i-1}$ for an object in a repetitive category.

Define $\mathbf{F}_T := [-1]\mathbf{S}_T : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$ and $\mathbf{G}_T := \mathbf{Q}_{DT}[1] : \underline{\mathcal{RC}}(S) \to \underline{\mathcal{RC}}(R)$. Then we have that $\mathbf{F}_T\mathbf{G}_T \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally and that $\mathbf{G}_T\mathbf{F}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$ naturally. So $\mathbf{F}_T$ and $\mathbf{G}_T$ give a repetitive equivalence between R and S.

It is easy to check that $\mathbf{F}_T|_{\mathcal{A}} \simeq \mathrm{Hom}_R(T, -)$ and that $\mathbf{G}_T|_{\mathcal{G}} \simeq - \otimes_S T$ from the definitions of the two functors. Now the proof of the theorem is complete.

4.6 The proof of Proposition 2

Assume that the equivalence is given by the functor $F : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$. By assumptions, F restricts to an equivalence $\mathcal{A} \to \mathcal{G}$. Note that $\mathcal{G}$ is resolving and $S \in \mathcal{G}$. Let $T = F^{-1}(S)$. Then $T \in \mathcal{A}$. By the triangle equivalence, we have that, for any $A \in \mathcal{A}$,

$$\mathrm{Ext}^i_R(T, A) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(R)}(T, \Sigma^i A) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(S, \Sigma^i F(A)) \simeq \mathrm{Ext}^i_S(S, F(A)),$$

where Σ is the translation functor in stable repetitive categories. In particular, we obtain that $\mathrm{Hom}_R(T, A) \simeq \mathrm{Hom}_S(S, F(A)) \simeq F(A)$ and that $\mathrm{Ext}^i_R(T, A) = 0$ for all $i > 0$. It follows that $S \simeq \mathrm{End}(T_R)^{op}$ and that $\mathrm{Ext}^i_R(T, T) = 0$ for all $i > 0$. Note that $\mathcal{A}$ is coresolving and $DR \in \mathcal{A}$, so we also have that $F(DR) \simeq \mathrm{Hom}_R(T, DR) \simeq DT$. Thus, we get that

$$\begin{aligned}\mathrm{Ext}^i_S({}_ST, {}_ST) &\simeq \mathrm{Ext}^i_S(DT, DT) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(DT, \Sigma^i DT)\\ &\simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(F(DR), \Sigma^i F(DR)) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(R)}(DR, \Sigma^i DR) \simeq \mathrm{Ext}^i_R(DR, DR).\end{aligned}$$

It follows that $\mathrm{End}({}_ST)^{op} \simeq R$ and that $\mathrm{Ext}^i_S({}_ST, {}_ST) = 0$ for all $i > 0$. Thus, T is a Wakamatsu-tilting module.

By assumption, $F|_{\mathcal{A}} \simeq \mathrm{Hom}_R(T, -)$ gives the equivalence $\mathcal{A} \to \mathcal{G}$. It follows that $F^{-1}|_{\mathcal{G}} \simeq - \otimes_S T$ by the uniqueness of the adjoint. Note that $\mathrm{Hom}_R(T, -)$ and $- \otimes_S T$ are exact functors on $\mathcal{A}$ and $\mathcal{G}$, respectively, since F is a triangle functor. As $\mathcal{A}$ is coresolving, for any $A \in \mathcal{A}$, the exact sequence $0 \to A \to I \to A' \to 0$ with $I \in \mathrm{inj}R$ is a sequence in $\mathcal{A}$. Applying the exact functor $\mathrm{Hom}_R(T, -)$, we obtain that $\mathrm{Ext}^1_R(T, A) = 0$. It follows that $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(-, \mathcal{A})$, i.e, T is Ext-projective in $\mathcal{A}$. Dually, we have also that $\mathrm{Tor}^S_1(X, T) = 0$ for any $X \in \mathcal{G}$. In particular, for any $A \in \mathcal{A}$, suppose that $A = X \otimes_S T$ for some $X \in \mathcal{G}$ and take an exact sequence $0 \to X' \to P \to X \to 0$ with $P \in \mathrm{proj}S$, then the sequence is in $\mathcal{G}$ since $\mathcal{G}$ is resolving, and, hence, there is an induced exact sequence $0 \to X' \otimes_S T \to P \otimes_S T \to X \otimes_S T \to 0$ since $- \otimes_S T$ is exact in $\mathcal{G}$. The last sequence gives an exact sequence $0 \to A'' \to T_A \to A \to 0$ with $T_A = P \otimes_S T \in \mathrm{add}_{\mathrm{mod}R}T$ and $A'' = X' \otimes_S T \in \mathcal{A}$. It follows that T is an Ext-projective generator in $\mathcal{A}$. Now applying Corollary 3.2.6, we conclude that T is a good Wakamatsu-tilting module.

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§5. Examples

5.1 Tilting modules and cotilting modules

Let R be an Artin algebra. Recall that an R-module $T \in \mathrm{mod}R$ is tilting provided the following three conditions are satisfied:

(1) the projective dimension of T is finite;
(2) $\mathrm{Ext}_R^i(T,T)=0$ for all $i>0$;
(3) there is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some integer n, where each $T_i \in \mathrm{add}_{\mathrm{mod}R}T$.

Dually, an R-module $T \in \mathrm{mod}R$ is cotilting provided the following three conditions are satisfied:

(1) the injective dimension of T is finite;
(2) $\mathrm{Ext}_R^i(T,T)=0$ for all $i>0$;
(3) there is an exact sequence $0 \to T_n \to \cdots \to T_0 \to DR \to 0$ for some integer n, where each $T_i \in \mathrm{add}_{\mathrm{mod}R}T$.

An R-module T is a tilting module if and only if DT is a cotilting left R-module if and only if DT is a cotilting S-module, where $S=\mathrm{End}(T_R)^{op}$. Note also that both tilting modules and cotilting modules are Wakamatsu-tilting modules.

We need the following well-known results on tilting modules and cotilting modules.

Proposition.

(1) *The cotorsion pair* $(\mathrm{KerExt}_R^1(-,{}_T\mathcal{X}),{}_T\mathcal{X})$ *is complete provided that* T *is a tilting module.*
(2) *The cotorsion pair* $(\mathcal{X}_T,\mathrm{KerExt}_R^1(\mathcal{X}_T,-))$ *is complete provided that* T *is a cotilting module.*

Proof. (2) follows from [3, Section 5] and (1) is just the dual of (2). ∎

5.1.1 Tilting modules are good Wakamatsu-tilting

Assume T_R is a tilting module of finite projective dimension. Let $S=\mathrm{End}(T_R)^{op}$. Then ${}_ST_R$ is a good Wakamatsu-tilting module. Hence, there is an equivalence between stable repetitive categories $\underline{\mathcal{RC}}(R)$ and $\underline{\mathcal{RC}}(S)$.

Indeed, if ${}_ST_R$ is a tilting module of finite projective dimension, then T is Wakamatsu-tilting and ${}_RDT_S$ is a cotilting module of finite injective dimension. By Proposition 3.1, we obtain that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the complete hereditary cotorsion pair $(\mathrm{KerExt}_R^1(-,{}_T\mathcal{X}),{}_T\mathcal{X})$ in $\mathrm{mod}R$ and the complete hereditary cotorsion pair $(\mathcal{X}_{DT},\mathrm{KerExt}_S^1(\mathcal{X}_{DT},-))$ in $\mathrm{mod}S$. It follows from the definition that ${}_ST_R$ is a good Wakamatsu-tilting bimodule.

5.1.2 Cotilting modules are good Wakamatsu-tilting

Assume now T_R is a cotilting module of finite injective dimension with $S=\mathrm{End}(T_R)^{op}$. Then ${}_ST_R$ is also a good Wakamatsu-tilting module. Hence, there is an equivalence between stable repetitive categories $\underline{\mathcal{RC}}(R)$ and $\underline{\mathcal{RC}}(S)$.

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Indeed, dually to 5.1.1, if ${}_ST_R$ is a cotilting module of finite injective dimension, then ${}_RDT_S$ is a tilting module of finite projective dimension. By Proposition 3.1 again, we obtain that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the complete hereditary cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ in mod$R$ and the complete hereditary cotorsion pair $(\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$. It follows from the definition that ${}_ST_R$ is also a good Wakamatsu-tilting bimodule.

5.2 Wakamatsu-tilting modules of finite type

5.2.1 We say that a Wakamatsu-tilting R-module T is of finite type provided that either the subcategory $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X})$ or $\mathrm{KerExt}^1_R(\mathcal{X}_T, -)$ is of finite representation type. In particular, if R is an algebra of finite representation type, then each subcategory of mod$R$ is of finite representation type, and, hence, every Wakamatsu-tilting module in mod$R$ is of finite type.

We note that, if T is a Wakamatsu-tilting R-module of finite type with $S = \mathrm{End}(T_R)^{op}$, then DT is a Wakamatsu-tilting S-module of finite type. This follows from the equivalences in Proposition 3.1.

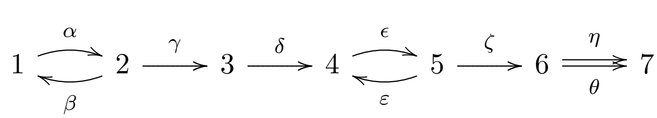
Proposition. A Wakamatsu-tilting module of finite type is always a good Wakamatsu-tilting module. In particular, every Wakamatsu-tilting module over an algebra of finite representation type is good.

Proof. Let T be a Wakamatsu-tilting R-module of finite type with $S = \mathrm{End}(T_R)^{op}$. Assume first that the subcategory $\mathrm{KerExt}^1_R(\mathcal{X}_T, -)$ is of finite representation type. Then the hereditary cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ in mod$R$ is complete. Moreover, by the equivalence in Proposition 3.1(3), the subcategory $\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X})$ is also of finite representation type. Thus, the hereditary cotorsion pair $(\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$ is also complete. It follows that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the complete hereditary cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ in mod$R$ and the complete hereditary cotorsion pair $(\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$. Similarly, in case that $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X})$ is of finite representation type, we have that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the complete hereditary cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ in mod$R$ and the complete hereditary cotorsion pair $(\mathcal{X}_{DT}, \mathrm{KerExt}^1_S(\mathcal{X}_{DT}, -))$ in mod$S$. Altogether, we see that T is a good Wakamatsu-tilting module in either case. □

5.2.2 Two trivial examples of Wakamatsu-tilting modules of finite type over an algebra R are the modules R and DR. In the first case, the subcategory $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}) = \mathrm{proj}R$ is of finite representation type, while the subcategory $\mathrm{KerExt}^1_R(\mathcal{X}_T, -) = \mathrm{inj}R$ is of finite representation type in the second case.

The following is an example of a Wakamatsu-tilting module of finite type over an algebra of infinite representation type.

Example. Let R be the bound path algebra given by the following quiver over a field with the relation given by $\mathrm{rad}^2 R = 0$.



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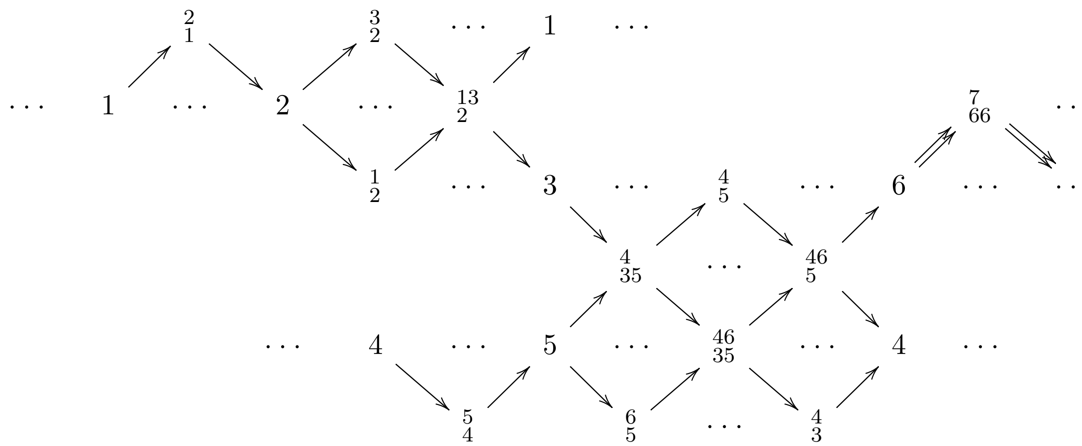
The following is the preprojective part of the AR-quiver of the algebra:



The algebra is of infinite representation type. Over this algebra, we have a Wakamatsu-tilting module of finite type (and, hence, a good Wakamatsu-tilting module)

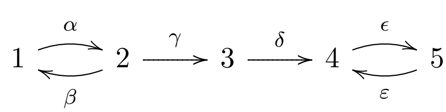
$$T = \begin{matrix} 2 \\ 1 \end{matrix} \oplus \begin{matrix} 1\,3 \\ 2 \end{matrix} \oplus 3 \oplus \begin{matrix} 4 \\ 3\,5 \end{matrix} \oplus \begin{matrix} 5 \\ 4 \end{matrix} \oplus \begin{matrix} 6 \\ 5 \end{matrix} \oplus \begin{matrix} 7 \\ 6\,6 \end{matrix}.$$

Indeed, one can check that the subcategory $\mathrm{KerExt}_R^1(-, {}_T\mathcal{X})$ is of finite representation type, while the subcategory ${}_T\mathcal{X}$ is of infinite representation type.

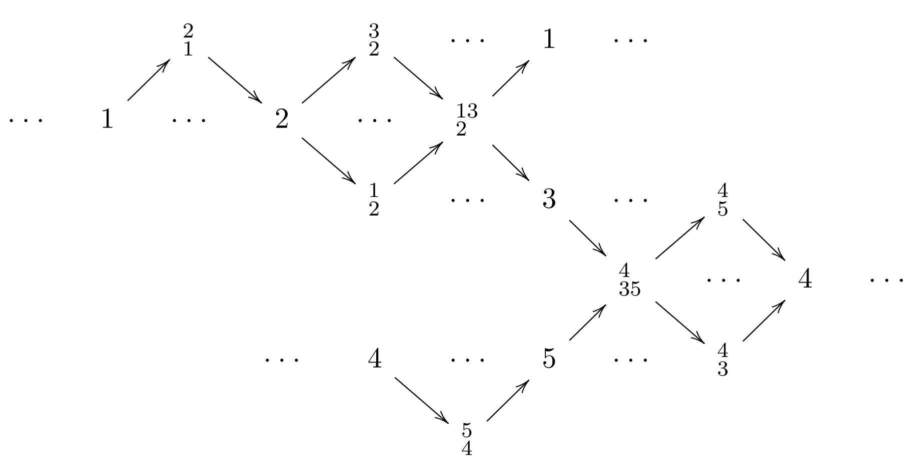
5.3 Repetitive equivalences are not derived equivalences

The following example shows that repetitive equivalences are not derived equivalences.

Example. [26] Let R be the bound path algebra given by the following quiver over a field with the relation given by $\mathrm{rad}^2 R = 0$:



The algebra is of finite representative type and the following is the AR-quiver of the algebra:



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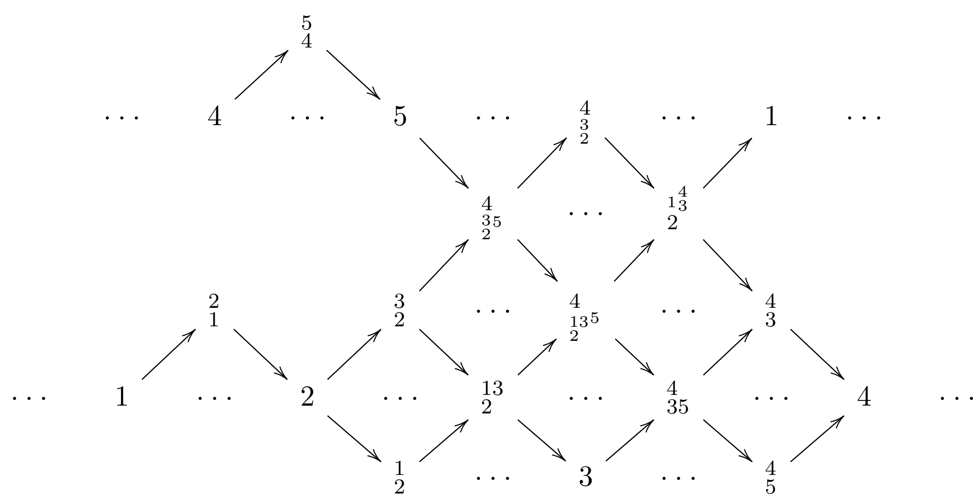
Over this algebra, we have a Wakamatsu-tilting module of finite type (and, hence, a good Wakamatsu-tilting module)

$$T = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,3 \\ 2 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3\,5 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix}.$$

The endomorphism algebra $S := \mathrm{End}(T_R)^{op}$ is the algebra defined by the following quiver over the field with the relation given by $\mathrm{rad}^2 R = 0$ except the path $2 \to 3 \to 4$.

$$1 \rightleftarrows 2 \longrightarrow 3 \longrightarrow 4 \rightleftarrows 5$$

The algebra is also of finite representative type and the AR-quiver of the algebra is as follows:



Then, R is *repetitive equivalent* to S. However, R is *not derived equivalent* to S. Indeed, these two algebras are not even singularity equivalent. Recall that the singularity category of the algebra R, denoted by $D_{sg}(R)$, is the quotient triangulated category of $D^b(\mathrm{mod}R)$ with respect to the full subcategory formed by perfect complexes (a complex in $D^b(\mathrm{mod}R)$ is perfect provided that it is isomorphic to a bounded complex consisting of finitely generated projective modules). Two algebras R and S are *singularity equivalent* if there is a triangle equivalence between $D_{sg}(R)$ and $D_{sg}(S)$. It is obvious that derived equivalences induce singularity equivalences.

In fact, let us consider the simple module $S_4 = 4$ in $\mathrm{mod}R$. It is easy to see that, for any natural number n, the $(2n)$th syzygy $\Omega^{2n}(S_4) = 2^{(n)} \oplus 4$ and the $(2n+1)$th syzygy $\Omega^{2n+1}(S_4) = 1^{(n)} \oplus 3 \oplus 5$. Note that, considered in the singularity category $D_{sg}(R)$, the endomorphism algebra $\mathrm{Hom}_{D_{sg}(R)}(S_4, S_4) = \lim\limits_{n\to\infty} \underline{\mathrm{Hom}}_R(\Omega^n(S_4), \Omega^n(S_4))$, where $\underline{\mathrm{Hom}}_R(-,-)$ is defined in the (projectively) stable module category $\underline{\mathrm{mod}}R$; see [8, Proposition 2.3]. Thus, the endomorphism algebra $\mathrm{Hom}_{D_{sg}(R)}(S_4, S_4)$ is infinite dimensional.

Assume that there is an equivalence between $D_{sg}(R)$ and $D_{sg}(S)$, and denote the image of S_4 to be S_4'. Note that every complex in $D_{sg}(S)$ is isomorphic to a finitely generated S-module (see [8, Lemma 2.2]), so we may assume that S_4' is in $\mathrm{mod}S$. Since every simple module in $\mathrm{mod}S$ satisfies that all its syzygies have composition length 1 (up to projective direct summands), we see that every module in $\mathrm{mod}S$ satisfies that all its syzygies have composition length less than a fixed number (up to projective direct summands). Altogether, we have that the endomorphism algebra $\mathrm{Hom}_{D_{sg}(S)}(S'_4, S'_4) = \lim_{n\to\infty} \underline{\mathrm{Hom}}_S(\Omega^n(S'_4), \Omega^n(S'_4))$ is finite dimensional. However, the singularity equivalence assumption makes $\mathrm{Hom}_{D_{sg}(S)}(S'_4, S'_4) \simeq \mathrm{Hom}_{D_{sg}(R)}(S_4, S_4)$ to be infinite dimensional, which leads to a contradiction. Hence, R and S are not singularity equivalent. In particular, R and S are not derived equivalent.

From the above example, we also see that repetitive equivalences do not imply singularity equivalences. We do not know whether or not singularity equivalences imply repetitive equivalences.

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