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On numbers having finite beta-expansions

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Abstract. Let β be a real number greater than one, and let $\mathbb{Z}_\beta$ be the set of real numbers which have a zero fractional part when expanded in base β. We prove that β is a Pisot number when the set $\mathbb{N}_\beta - \mathbb{N}_\beta - \mathbb{N}_\beta$ is discrete, where $\mathbb{N}_\beta = \mathbb{Z}_\beta \cap [0, \infty[$. We also give partial answers to some related open problems, and in particular, we show that β is a Pisot number when a sum $\mathbb{Z}_\beta + \cdots + \mathbb{Z}_\beta$ is a Meyer set.

1. *Introduction*

Representations of real numbers with an arbitrary real base greater than one, say β, called beta-expansions, were introduced by Rényi [**18**]. They arise from orbits of the transformation $x \mapsto \beta x \pmod 1$ of the unit interval, and have been studied in ergodic theory (see [**6, 12**] and [**16**]). As usual for a real number t we denote by $I(t)$ the largest rational integer not exceeding t, and by $F(t)$ the difference $t - I(t)$. We also denote the ring of rational integers, the field of real numbers and the set of non-negative rational integers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{N}$, respectively. The following definitions can be found in [**12, 16**] and [**18**]. Let x be a positive real number and let $p = p(x) \in \mathbb{Z}$ be such that $\beta^p \leq x < \beta^{p+1}$. Then, the beta-expansion of x in base β, or simply the beta-expansion of x, is the infinite sequence $(\varepsilon_k)_{k\leq p} = (\varepsilon_k(x))_{k\leq p}$ defined by the relations $\varepsilon_p = I(x/\beta^p)$, $r_p = r_p(x) = F(x/\beta^p)$, and $\varepsilon_k = I(\beta r_{k+1})$ and $r_k = r_k(x) = F(\beta r_{k+1})$ for k running through the set $\mathbb{Z} \cap]-\infty, p[$. In this case, we write

$$x \equiv (\varepsilon_k)_{k\leq p}$$

and we have

$$x = \varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0 + \varepsilon_{-1}\beta^{-1} + \varepsilon_{-2}\beta^{-2} + \cdots ,$$

$r_k \in [0, 1[$ and $\varepsilon_k \in [0, \beta[\cap \mathbb{N}$.

If there is $N \in \mathbb{Z}$ such that $\varepsilon_n = 0$ for all $n \leq N - 1$, then we say that x has a finite beta-expansion; in this case we write

$$x \equiv (\varepsilon_k)_{N\leq k\leq p},$$

where N is the greatest rational integer such that $\varepsilon_n = 0$ for all $n \leq N - 1$.

We say that a sequence $(\varepsilon_k)_{k\leq p}$, where $\varepsilon_k \in \mathbb{N}$ and $p \in \mathbb{Z}$, is admissible if it is the beta-expansion of a certain positive number, that is, when there is $x \in]0, \infty[$ such that $x \equiv (\varepsilon_k)_{k\leq p}$. A sequence $(\eta_k)_{k\leq p}$, where $\eta_k \in \mathbb{N}$ and $p \in \mathbb{Z}$, is said to be lexicographically less than a sequence $(\gamma_k)_{k\leq q}$, where $\gamma_k \in \mathbb{N}$ and $q \in \mathbb{Z}$, if there is $l \in \mathbb{N}$ such that $\eta_{p-l} < \gamma_{q-l}$, and $\eta_{p-k} = \gamma_{q-k}$ for all $k \in \{0, \dots, l-1\}$. Let $(\varepsilon^*_k)_{k\leq 0}$ be the sequence of non-negative rational integers defined as follows: $(\varepsilon^*_k)_{k\leq 0}$ is the purely periodical sequence

$$((\beta-1)(\beta-1)(\beta-1)\cdots) = ((\beta-1)^\omega)$$

with period one (and only term $\beta - 1$), when $\beta \in \mathbb{N}$. If $\beta \notin \mathbb{N}$ and $F(\beta) \equiv (\varepsilon_k)_{k\leq n}$, where $\varepsilon_k \neq 0$ for infinitely many k (respectively, and $F(\beta) \equiv (\varepsilon_k)_{N\leq k\leq n}$), then

$$\text{for all } k \leq 0, \quad \varepsilon^*_k = \varepsilon_k$$

(respectively, then $(\varepsilon^*_k)_{k\leq 0}$ is the purely periodical sequence

$$(\varepsilon_0\varepsilon_{-1}\cdots\varepsilon_{N+1}(\varepsilon_N - 1))^\omega$$

with period $1 - N$), where $\varepsilon_0 := I(\beta)$ and $\varepsilon_m := 0$ for all $m \in \{-1, -2, \dots, n+1\}$.

A result of Parry [**16**] says that a sequence $(\varepsilon_k)_{k\leq p}$, where $\varepsilon_k \in \mathbb{N}$ and $p \in \mathbb{Z}$, is admissible, if and only if each sequence of the form $(\varepsilon_k)_{k\leq p_0}$, where $p_0 \in \mathbb{Z} \cap]-\infty, p]$, is lexicographically less than $(\varepsilon^*_k)_{k\leq 0}$. The closure of the set of admissible sequences is called a beta-shift. It is a symbolic dynamical system, that is, a closed shift-invariant subset of $\{0, 1, \dots, I(\beta)\}^{\mathbb{N}}$ (see [**6**] and [**12**]). By analogy with the decimal representation, the beta-expansion of a negative real number x is the sequence $(-\varepsilon_k(-x))_{k\leq p(-x)}$, and by convention $0 \equiv (0)$ (for definitions and results on beta-expansions, see for instance [**15**, Ch. 7]).

The real number x is called a beta-integer if $\varepsilon_j = 0$ for all $j < 0$. Note that beta-integers were introduced in [**4**]. Clearly, a beta-integer has a finite beta-expansion. Let $\mathbb{N}_\beta$ be the set of non-negative beta-integers. Then,

$$\mathbb{N}_\beta = \{\varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0,\ p \in \mathbb{N},\ (\varepsilon_p\cdots\varepsilon_0 000\cdots) \text{ is admissible}\} \cup \{0\},$$

$\mathbb{N}_\beta = \mathbb{N}$ when $\beta \in \mathbb{N}$, and the set $\mathbb{Z}_\beta$ of beta-integers, satisfies

$$\mathbb{Z}_\beta = \mathbb{N}_\beta \cup (-\mathbb{N}_\beta).$$

Consider also the sets

$$A_m = A_m(\beta) = \{\eta_p\beta^p + \eta_{p-1}\beta^{p-1} + \cdots + \eta_0,\ p \in \mathbb{N},\ \eta_i \in \{0, 1, \dots, m\}\}$$

and

$$B_m = B_m(\beta) = A_m - A_m = \{\gamma_p\beta^p + \cdots + \gamma_0,\ p \in \mathbb{N},\ \gamma_i \in \{-m, -m+1, \dots, m\}\},$$

where $m \in \mathbb{Z}^+ := \mathbb{Z} \cap [1, \infty[$. Clearly, $\mathbb{N}_\beta \subset A_{I(\beta)}$ and $\mathbb{Z}_\beta \subset B_{I(\beta)}$. Recall that a Pisot number is a positive algebraic integer whose other conjugates over the field of the rationals $\mathbb{Q}$ are of modulus less than one. By the pigeonhole principle it is easy to see that, when β is a Pisot number, each set B_m is discrete, i.e. B_m has no finite limit point (see also [**7, 9, 19–21**]). Using a result of Frougny [**11**] from automata theory, Bugeaud [**7**] proved that the converse of the last proposition is true, and after this Erdös and Komornik [**9**] showed that the condition '*B_m is discrete, where m is the smallest integer satisfying $m \geq \beta - 1/\beta$*' is sufficient to deduce that β is a Pisot number. Recently [**21**], the present author proved that the implication '*$B_{I(\beta)} - A_{I(\beta)}$ is discrete $\Longrightarrow \beta$ is a Pisot number*' is also true. The question whether Pisot numbers are the only numbers β such that 0 is not a limit point of $B_{I(\beta)}$, remains open. The first aim of this paper is to prove the following.

THEOREM 1. *If $\mathbb{N}_\beta - \mathbb{N}_\beta - \mathbb{N}_\beta$ is discrete, then β is a Pisot number.*

It is worth noting that the above result is an improvement of [**21**, Theorem 1], since the inclusion $A_{I(\beta)} \subset \mathbb{N}_\beta$, is true only when β is a root of a polynomial of the form

$$x^d - a(x^{d-1} + x^{d-2} + \cdots + x) - b,$$

where $d \geq 2$, $a \in \mathbb{N}$, $b \in \mathbb{N}$ and $a \geq b \geq 1$ (see [**10**]); numbers β of this kind are called confluent Pisot numbers (see [**5**]).

The following definitions and results can be found in [**2**] and [**14**]. A subset X of $\mathbb{R}$ is called a Delaunay set if it is relatively dense in $\mathbb{R}$ (i.e. there is $\varepsilon > 0$ such that any closed interval of length ε contains at least one element of X; to be more precise we also say that X is ε-dense), and uniformly discrete (i.e. there is $\varepsilon > 0$ such that the usual distance between two distinct points of X is greater than ε). The Delaunay set X is a Meyer set if the set $X - X = \{x - x', x \in X, x' \in X\}$ is also a Delaunay set. The implication 'β is a Pisot number $\Longrightarrow \mathbb{Z}_\beta$ is a Meyer set', due to authors of [**8**], also appears in [**2**, Proposition 1]. Lagarias [**14**] has proved that a Delaunay set X is a Meyer set if and only if there is a finite subset F of $\mathbb{R}$ such that $X - X \subset X + F$. Using essentially this last result and Theorem 1, we obtain the following.

THEOREM 2. *We have the following.*

(i) *If β is a Pisot number, then each finite sum $\sum_{1 \leq k \leq n} \mathbb{Z}_\beta = \mathbb{Z}_\beta + \cdots + \mathbb{Z}_\beta$, where $n \in \mathbb{Z}^+$, is a Meyer set.*

(ii) *If there is $N \in \mathbb{Z}^+$ such that $\sum_{1 \leq k \leq N} \mathbb{Z}_\beta$ is a Meyer set, then so are all sets $\sum_{1 \leq k \leq n} \mathbb{Z}_\beta$, where $n \in \mathbb{Z}^+$.*

(iii) *If some finite sum $\sum_{1 \leq k \leq N} \mathbb{Z}_\beta$, where $N \in \mathbb{Z}^+$, is a Meyer set, then β is a Pisot number.*

It follows immediately that β is a Pisot number when $\mathbb{Z}_\beta$ is a Meyer set; thus the converse of [**2**, Proposition 1] is true. Note also that if the assertion '*Pisot numbers are the only numbers β such that $\mathbb{Z}_\beta$ is a Delaunay set*' is true, then so is the proposition '*Pisot numbers are the only numbers β such that* 0 *is not a limit point of $B_{I(\beta)}$*' (see the proof of Theorem 2).

Now, let Fin_β be the set of real numbers which have finite beta-expansions: $x \in \mathrm{Fin}_\beta$ if there is $N \in \mathbb{Z}$ such that $\varepsilon_j(x) = 0$ for all $j \leq N$. Then,

$$\mathbb{Z}_\beta \subset \mathrm{Fin}_\beta \subset \left(\mathbb{N}[\beta] + \mathbb{N}\left[\frac{1}{\beta}\right]\right) \cup \left(-\mathbb{N}[\beta] - \mathbb{N}\left[\frac{1}{\beta}\right]\right) \subset \mathbb{Z}[\beta] + \mathbb{Z}\left[\frac{1}{\beta}\right]$$

(if $S \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$, then $S[\alpha]$ is the set of polynomials with coefficients in S, evaluated at α). Clearly, if β is an algebraic integer, then $\mathbb{Z}[\beta] \subset \mathbb{Z}[1/\beta]$ and so $\mathbb{Z}[1/\beta] = \mathbb{Z}[\beta] + \mathbb{Z}[1/\beta]$. Recall also that a Salem number is a real algebraic integer greater than one whose other conjugates over $\mathbb{Q}$ are all of modulus at most one and with a conjugate of modulus one. In [**12**, Lemma 1] it was asserted that β is a Pisot or a Salem number when $\mathbb{N} \subset \mathrm{Fin}_\beta$. In fact, by the same arguments as in the proof of the last mentioned result, we obtain the following.

THEOREM 3. *Assume that there is $N \in \mathbb{N}$ such that $\{I(\beta^n) + 1, n \in \mathbb{N}, n \geq N\} \subset \mathrm{Fin}_\beta$. Then, β is a Pisot number and β has at most one positive conjugate over $\mathbb{Q}$.*

It follows immediately when $\mathrm{Fin}_\beta = \mathbb{Z}[\beta] + \mathbb{Z}[1/\beta]$ that β is a Pisot number (with at most one positive conjugate) and so $\mathrm{Fin}_\beta = \mathbb{Z}[1/\beta]$. It is easy to see when $F(\beta) \in \mathrm{Fin}_\beta$, that β is an algebraic integer with no conjugate over $\mathbb{Q}$ in [0, 1]. In fact, the implication $\mathrm{Fin}_\beta = \mathbb{Z}[\beta] + \mathbb{Z}[1/\beta] \Longrightarrow \beta$ is a Pisot number with no conjugate over $\mathbb{Q}$ in [0, 1], has already been proved in [**12**]. Conversely, Frougny and Solomyak have shown that if the minimal polynomial over $\mathbb{Q}$ of the Pisot number β is of the form $x^d - a_1x^{d-1} - a_2x^{d-2} - \cdots - a_d$, where $a_1 \geq a_2 \geq \cdots \geq a_d \geq 1$, then $\mathrm{Fin}_\beta = \mathbb{Z}[1/\beta]$ (see [**12**]). A complete characterization of Pisot numbers β satisfying $\mathrm{Fin}_\beta = \mathbb{Z}[1/\beta]$ is not known. In [**1**], Akiyama used Pisot units, say also β, with the same property to construct tilings of $\mathbb{R}^{d-1}$, were d is the degree of β over $\mathbb{Q}$; a tiling close to these was obtained by Rauzy [**17**] in connection with substitutative dynamical systems. In his thesis [**13**] Hollander found another class of Pisot numbers β satisfying $\mathrm{Fin}_\beta = \mathbb{Z}[1/\beta]$, and studied the following weak finiteness property: *if* $x \in \mathbb{Z}[1/\beta]$, *then*

$$\exists\, (y_n(x))_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}, \quad y_n \in \mathrm{Fin}_\beta, \quad y_n - x \in \mathrm{Fin}_\beta \quad \text{and} \quad \lim_{n\to\infty} y_n = x. \qquad (W_x)$$

If (W_x) is satisfied for all $x \in \mathbb{Z}[1/\beta]$, then we say that β satisfies (W). Clearly, if $\mathrm{Fin}_\beta = \mathbb{Z}[1/\beta]$, then β satisfies (W) (choose, for instance, $y_n(x) = x$, where $n \in \mathbb{N}$ and $x \in \mathbb{Z}[1/\beta]$). In [**3**], Akyama *et al* found a class of Pisot numbers which satisfy (W), conjectured that this property holds for all Pisot numbers, and proved that if (W) holds for some β, then β is a Pisot or a Salem number. In a similar manner as in the proof of Theorem 3, we show the following result.

THEOREM 4. *If (W_x) holds when x takes infinitely many values of the form $I(\beta^n) + 1$, where $n \in \mathbb{N}$, then β is a Pisot or a Salem number.*

2. *Proofs*

Proof of Theorem 1. The idea of the first part of the present proof is similar to that of [**21**, Theorem 1] with minor modifications. Let $\mathbb{I}_\beta := \mathbb{N}_\beta - \mathbb{N}_\beta$,

$$b \in \mathbb{I}_\beta \cap [1, \infty[$$

and $b \equiv (\varepsilon_k)_{k\leq p}$, where $p \in \mathbb{N}$. Then, the sequence $(\varepsilon_p \cdots \varepsilon_0 000 \cdots)$ is admissible, $\varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0 \equiv (\varepsilon_p\varepsilon_{p-1} \cdots \varepsilon_0)$ and so

$$\varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0 \in \mathbb{N}_\beta.$$

Moreover, the number $b - (\varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0)$ belongs to the finite set

$$F_0 := (\mathbb{I}_\beta - \mathbb{N}_\beta) \cap [0, 1[,$$

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since $\mathbb{I}_\beta - \mathbb{N}_\beta$ is discrete and $b - (\varepsilon_p\beta^p + \varepsilon_{p-1}\beta^{p-1} + \cdots + \varepsilon_0) = r_0(b) \in [0, 1[$. Consider an element

$$d \in \mathbb{I}_\beta - \mathbb{I}_\beta.$$

Then, d can be written

$$d = b - b',$$

where

$$b = a_1 - a_2, \quad b' = a'_1 - a'_2$$

and $\{a_1, a_2, a'_1, a'_2\} \subset \mathbb{N}_\beta$. Let n be a sufficiently large rational integer so that $\beta^n + b \geq 1$, $\beta^n + b' \geq 1$, $\beta^n + a_1 \in \mathbb{N}_\beta$ and $\beta^n + a'_1 \in \mathbb{N}_\beta$. Such an integer n exists because the first inequalities hold trivially when n is large, and the last inequalities follow from the fact that if a sequence, say $(\alpha_p\alpha_{p-1}\alpha_{p-2}\cdots)$, is admissible, then the sequence $(10\cdots 0\alpha_p\alpha_{p-1}\alpha_{p-2}\cdots)$ containing n vanishing terms before the term α_p, where $n \geq -s$ and s is the greatest negative rational integer such that $\varepsilon_s^* \geq 1$, is also admissible. It follows that $\beta^n + b \in \mathbb{I}_\beta$, $\beta^n + b' \in \mathbb{I}_\beta$ and there are $a \in \mathbb{N}_\beta$, $r \in F_0$, $a' \in \mathbb{N}_\beta$ and $r' \in F_0$ satisfying $\beta^n + b = a + r$ and $\beta^n + b' = a' + r'$; thus,

$$d = (a + r) - (a' + r') = (a - a') + (r - r'),$$

and

$$\mathbb{I}_\beta + \mathbb{I}_\beta = \mathbb{I}_\beta - \mathbb{I}_\beta \subset \mathbb{I}_\beta + F,$$

where F is the finite set $F_0 - F_0$. Hence, $\mathbb{I}_\beta + \mathbb{I}_\beta + \mathbb{I}_\beta \subset \mathbb{I}_\beta + \mathbb{I}_\beta + F \subset \mathbb{I}_\beta + F + F$, and by induction we have

$$\sum_{i=1}^{N} \mathbb{I}_\beta \subset \mathbb{I}_\beta + \sum_{i=1}^{N-1} F,$$

where $N \in \mathbb{Z} \cap [2, \infty[$. As $\sum_{i=1}^{N-1} F$ is a finite set, and the sum of a finite set and a discrete set is also a discrete set, by the last inclusion we deduce that each set $\sum_{i=1}^{N} \mathbb{I}_\beta$, where $N \in \mathbb{Z}^+$, is discrete. To complete the proof, we claim that it suffices to show that there exists $M = M(\beta) \in \mathbb{Z}^+$ such that

$$A_{I(\beta)} \subset \sum_{1 \leq i \leq M} \mathbb{N}_\beta.$$

Indeed, in this case we have

$$A_{I(\beta)} - A_{I(\beta)} - A_{I(\beta)} \subset \sum_{1 \leq i \leq M} \mathbb{I}_\beta - \sum_{1 \leq i \leq M} \mathbb{N}_\beta \subset \sum_{1 \leq i \leq 2M} \mathbb{I}_\beta,$$

and the result follows immediately by [**21**, Theorem 1], as $A_{I(\beta)} - A_{I(\beta)} - A_{I(\beta)}$ is a subset of the discrete set $\sum_{1 \leq i \leq 2M} \mathbb{I}_\beta$. Let a be a non-zero element of the set $A_{I(\beta)}$. Then,

$$a = \eta_p\beta^p + \eta_{p-1}\beta^{p-1} + \cdots + \eta_0,$$

where $p \in \mathbb{N}$, $\eta_i \in \{0, 1, \ldots, I(\beta)\}$ and $\eta_p \geq 1$. For each $i \in \{0, 1, \ldots, \min(p, M - 1)\}$, where $M := 1 - s$, set

$$a_i := \sum_{j=0}^{I(p-i)/M} \eta_{i+jM}\beta^{i+jM}.$$

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Then, $a_i \in A_{I(\beta)}$, and each non-zero coefficient, say η_{i+jM}, in the last expression of a_i is followed by at least $(M-1)$ vanishing coefficients. Since the first term of the sequence $(\varepsilon_k^*)_{k\le 0}$, namely ε_0^*, is the greatest rational integer less than β, and this term is followed (in the sequence $(\varepsilon_k^*)_{k\le 0}$) by exactly $M-2$ vanishing terms, the sequence

$$(\eta_{i+MI(p-i)/M}0\cdots 0\eta_{i+MI(p-i)/M-M}\cdots\eta_{i+M}0\cdots 0\eta_i 000\cdots)$$

is admissible,

$$a_i \in \mathbb{N}_\beta$$

and

$$a_i = \sum_{j=0}^{I(p-i)/M} \eta_{i+(I((p-i)/M)-j)M}\beta^{i+(I((p-i)/M)-j)M}.$$

To end the proof it remains to verify that

$$a = \sum_{i=0}^{\min(p,M-1)} a_i,$$

as this equality implies $A_{I(\beta)} \subset \sum_{0\le i\le\min(p,M-1)} \mathbb{N}_\beta \subset \sum_{1\le i\le M} \mathbb{N}_\beta$. If $p \le M-1$, then for each $i \in \{0, 1, \ldots, p\}$ we have $a_i = \eta_i\beta^i$, and the equality is trivial. To prove the result when $p \ge M$, it suffices to show that $\{P_1, P_2, \ldots, P_{M-1}\}$, where

$$P_i = \left\{i, i+M, i+2M, \ldots, i+I\left(\frac{p-i}{M}\right)M\right\}$$

and $i \in \{0, 1, \ldots, M-1\}$, is a partition of the set $\{0, 1, \ldots, p\}$. Clearly, we have $I((p-i)/M) \le (p-i)/M$, $i + I((p-i)/M)M \le p$ and so $P_i \subset \{0, 1, \ldots, p\}$. Moreover, if there are $(i, i') \in \{0, 1, \ldots, M-1\} \times \{0, 1, \ldots, M-1\}$ and $(k, k') \in \mathbb{N} \times \mathbb{N}$ such that $i < i'$ and $i + kM = i' + k'M$, then $(k-k')M > 0$, $k - k' \ge 1$ and so $i' - i \ge M$; the last inequality leads to a contradiction since $i' - i \le i' \le M-1$. □

Proof of Theorem 2. From the definition of the beta-expansion of a real number in an arbitrary base β, we see that the set $\mathbb{Z}_\beta$ is 1-dense; thus, each finite sum $\sum_{1\le i\le n} \mathbb{Z}_\beta$, where n is a positive rational integer, is relatively dense. By the inclusion

$$\sum_{1\le i\le n} \mathbb{Z}_\beta \subset B_{n[\beta]},$$

we have that the set $\sum_{1\le i\le n} \mathbb{Z}_\beta$ is discrete when β is a Pisot number (recall that each set B_m is discrete when β is a Pisot number). It follows, in particular, that 0 is not a limit point of $\sum_{1\le i\le 2n} \mathbb{Z}_\beta$, and so by the equalities $\mathbb{Z}_\beta = -\mathbb{Z}_\beta$ and

$$\sum_{1\le i\le n} \mathbb{Z}_\beta - \sum_{1\le i\le n} \mathbb{Z}_\beta = \sum_{1\le i\le 2n} \mathbb{Z}_\beta,$$

we have that the set $\sum_{1\le i\le n} \mathbb{Z}_\beta$ is uniformly discrete; thus, $\sum_{1\le i\le n} \mathbb{Z}_\beta$ is a Delaunay set and Theorem 2(i) is true, since $\sum_{1\le i\le 2n} \mathbb{Z}_\beta$ is also a Delaunay set.

Now, assume that $\sum_{1\le i\le N} \mathbb{Z}_\beta$ is a Meyer set for some $N \in \mathbb{Z}^+$. It is clear that Theorem 2(ii) is a corollary of Theorems 1 and 2(i), when $N \ge 2$, as $\mathbb{N}_\beta - \mathbb{N}_\beta - \mathbb{N}_\beta \subset \mathbb{Z}_\beta + \mathbb{Z}_\beta + \mathbb{Z}_\beta$ and $\mathbb{Z}_\beta + \mathbb{Z}_\beta + \mathbb{Z}_\beta$ is contained in a (uniformly) discrete set (by the same arguments it suffices to show that $\mathbb{Z}_\beta + \mathbb{Z}_\beta + \mathbb{Z}_\beta$ is discrete when $N = 1$). In fact, using the above-mentioned result of Lagarias, we can easily show Theorem 2(ii) for any $N \geq 1$. Indeed, when X is a Meyer set and $X - X \subset X + F$, where F is a finite set, note that $X - X$ is a Delaunay set,

$$(X - X) - (X - X) \subset X + F - (X + F) = (X - X) + (F - F)$$

and so $X - X$ is a Meyer set, as $F - F$ is a finite subset of $\mathbb{R}$. It follows that $\sum_{1 \leq i \leq N} \mathbb{Z}_\beta - \sum_{1 \leq i \leq N} \mathbb{Z}_\beta = \sum_{1 \leq i \leq 2N} \mathbb{Z}_\beta$ is a Meyer set, and by induction we have that each set of the form

$$\sum_{1 \leq i \leq 2^n N} \mathbb{Z}_\beta,$$

where $n \in \mathbb{N}$, is also a Meyer set. Moreover, since any finite sum $\sum_{1 \leq i \leq m} \mathbb{Z}_\beta$, where $m \in \mathbb{Z}^+$, is relatively dense and is contained in a uniformly discrete set (for instance we have $\sum_{1 \leq i \leq m} \mathbb{Z}_\beta \subset \sum_{1 \leq i \leq 2^m N} \mathbb{Z}_\beta$), we deduce that $\sum_{1 \leq i \leq m} \mathbb{Z}_\beta$ is a Delaunay set; thus $\sum_{1 \leq i \leq 2m} \mathbb{Z}_\beta$ is a Delaunay set and so $\sum_{1 \leq i \leq m} \mathbb{Z}_\beta$ is a Meyer set. Finally, note that Theorem 2(ii) together with Theorem 1 yield immediately Theorem 2(iii). □

Proof of Theorem 3. Clearly, β is a Pisot number when it is a rational integer. Assume that $\beta \notin \mathbb{N}$. Let p be a sufficiently large rational integer so that

$$\beta^p + 1 < \beta^{p+1}$$

and

$$I(\beta^p) < \beta^p.$$

Such an integer p exists, since the first inequality holds trivially when p is large, and the second inequality follows from the fact that if there is $n \in \mathbb{N}$ such that if $I(\beta^n) = \beta^n$ and $I(\beta^{n+1}) = \beta^{n+1}$, then β is an algebraic integer, $\beta = \beta^{n+1}/\beta^n = (I(\beta^{n+1})/I(\beta^n)) \in \mathbb{Q}$, and so $\beta \in \mathbb{N}$. Let

$$x := I(\beta^p) + 1.$$

Then,

$$\beta^p < x < \beta^p + 1 < \beta^{p+1},$$
$$1 < \frac{x}{\beta^p} < 1 + \frac{1}{\beta^p} < 2,$$

$\varepsilon_p(x) = 1$ and $r_p(x) < 1/\beta^p$; thus,

$$x \equiv (10 \cdots 0\varepsilon_{-1} \cdots),$$

where $\varepsilon_{-1} \geq 1$, and

$$x \notin \mathbb{N}_\beta.$$

Since $x \in \mathrm{Fin}_\beta$, there is a positive rational integer m such that

$$x \equiv (10 \cdots 0\varepsilon_{-1} \cdots \varepsilon_{-m});$$

in particular, we have

$$x = \beta^p + \varepsilon_{-1}\beta^{-1} + \varepsilon_{-2}\beta^{-2} + \cdots + \varepsilon_{-m}\beta^{-m}.$$

It follows immediately from the last equality that β is an algebraic integer, and if γ is a conjugate of β over $\mathbb{Q}$, then

$$x = \gamma^p + \varepsilon_{-1}\gamma^{-1} + \varepsilon_{-2}\gamma^{-2} + \cdots + \varepsilon_{-m}\gamma^{-m}$$

and so

$$\beta^p - \gamma^p = \sum_{k=1}^{m} \varepsilon_{-k}(\gamma^{-k} - \beta^{-k}).$$

Hence, for $|\gamma| > 1$ we see that

$$\frac{|\beta^p - \gamma^p|}{I(\beta)} \leq \sum_{k=1}^{m}(|\gamma^{-k}| + \beta^{-k}) < \frac{1}{|\gamma| - 1} + \frac{1}{\beta - 1}$$

and so $\gamma = \beta$, since otherwise $\limsup_p |\gamma^p - \beta^p| = \infty$ (indeed, without loss of generality set $\gamma = |\gamma| e^{i\pi t}$, where $t \in [0, 1]$ and $i^2 = -1$. Clearly, we have $\lim_{n\longrightarrow\infty} |\gamma^n - \beta^n| = \infty$ when $t = 0$ and $\gamma \neq \beta$. If $t = a/b$, where $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}^+$, then $|\gamma^{2nb+1} - \beta^{2nb+1}| \geq |\gamma|^{2nb+1} |\sin(\pi t)|$ for all $n \in \mathbb{N}$, and so $\lim_{n\longrightarrow\infty} |\gamma^{2nb+1} - \beta^{2nb+1}| = \infty$, except when $t = 1$; for $t = 1$ we have $\lim_{n\longrightarrow\infty} |\gamma^{2n+1} - \beta^{2n+1}| = \infty$. Finally, if $t \notin \mathbb{Q}$, then $(F(tn))_{n\in\mathbb{N}}$ is dense mod(1) and so there is a sequence $(n_k)_{k\in\mathbb{N}}$ of positive rational integers such that $\lim_{k\longrightarrow\infty} F(tn_k) = 1/2$; thus $\lim_{k\longrightarrow\infty} |\gamma^{2n_k} - \beta^{2n_k}| = \lim_{k\longrightarrow\infty}(|\gamma|^{2n_k} + \beta^{2n_k}) = \infty$). Note also, when $|\gamma| = 1$, that $1/\gamma$ is a conjugate of β ($1/\gamma$ is the complex conjugate of γ) and so is $1/\beta$; thus,

$$x = \beta^{-p} + \varepsilon_{-1}\beta + \cdots + \varepsilon_{-m}\beta^m,$$

and so

$$m \leq p,$$

since we have $\varepsilon_{-m} \geq 1$ and $x < \beta^{p+1}$. It follows by the inequalities

$$x \leq |\gamma^p + \varepsilon_{-1}\gamma^{-1} + \cdots + \varepsilon_{-m}\gamma^{-m}| \leq 1 + \varepsilon_{-1} + \cdots + \varepsilon_{-m} \leq I(\beta)(m+1),$$

that $x \leq I(\beta)(p+1)$, and this last relation leads to a contradiction when p is large (recall that $x > \beta^p$). Hence, $|\gamma| < 1$ and β is a Pisot number. Now, suppose that γ is a positive real number. Then, the inequality $\gamma\beta \geq 1$ (if $\gamma\beta < 1$, then the norm of β over $\mathbb{Q}$ will be 0) together with the relation $x = \gamma^p + \varepsilon_{-1}\gamma^{-1} + \cdots + \varepsilon_{-m}\gamma^{-m}$ yield

$$x < 1 + \varepsilon_{-1}\beta + \cdots + \varepsilon_{-m}\beta^m \leq I(\beta)\frac{\beta^{m+1} - 1}{\beta - 1};$$

thus, m is large when p is so. It follows when β has a real conjugate, say η, satisfying

$$\gamma \leq \eta < 1$$

that

$$\frac{1}{\gamma^m} - \frac{1}{\eta^m} \leq \sum_{i=1}^{m} \varepsilon_{-i}(\gamma^{-i} - \eta^{-i}) = \eta^p - \gamma^p < 1$$

and so $\gamma = \eta$, since $\lim_{m\to\infty}(1/\gamma^m - 1/\eta^m) = \infty$ when $\gamma \neq \eta$. Consequently, the Pisot number β can not have more than one positive conjugate. Finally, note that the inclusion $\mathbb{N}[\beta] \subset \mathrm{Fin}_\beta$ has been proved in [**12**] when β is a quadratic Pisot number with a positive conjugate. □

Proof of Theorem 4. In a similar manner as in the beginning of the proof of Theorem 3, suppose that $\beta \notin \mathbb{N}$ and let

$$x := I(\beta^p) + 1,$$

where $p \in \mathbb{N}$ is sufficiently large so that $\beta^p + 1 < \beta^{p+1}$ and $I(\beta^p) < \beta^p$. Let δ be a positive real number satisfying

$$x + \delta < \beta^p + 1.$$

From the hypothesis, there are y and $z \in \mathrm{Fin}_\beta$, such that

$$y = x + z$$

and

$$z < \delta.$$

Hence,

$$\beta^p < y < \beta^p + 1 < \beta^{p+1},$$

$1 < (y/\beta^p) < 1 + (1/\beta^p) < 2$, $\varepsilon_p(y) = 1$ and $r_p(y) < (1/\beta^p)$; thus, there is a positive rational integer m such that

$$y \equiv (10 \cdots 0\varepsilon_{-1} \cdots \varepsilon_{-m}),$$

where $\varepsilon_{-m} \geq 1$, and so $y = \beta^p + \varepsilon_{-1}\beta^{-1} + \varepsilon_{-2}\beta^{-2} + \cdots + \varepsilon_{-m}\beta^{-m}$. Let

$$z \equiv (\eta_{-K}\eta_{-K-1} \cdots \eta_{-K-t}),$$

where $K \in \mathbb{N}$ and $t \in \mathbb{N}$. Then,

$$\begin{aligned} x &= \beta^p + \varepsilon_{-1}\beta^{-1} + \cdots + \varepsilon_{-m}\beta^{-m} \\ &\quad - (\eta_{-K}\beta^{-K} + \eta_{-K-1}\beta^{-K-1} + \cdots + \eta_{-K-t}\beta^{-K-t}), \\ x\beta^u &= \beta^{u+p} + \varepsilon_{-1}\beta^{u-1} + \cdots + \varepsilon_{-m}\beta^{u-m} \\ &\quad - (\eta_{-K}\beta^{u-K} + \eta_{-K-1}\beta^{u-K-1} + \cdots + \eta_{-K-t}\beta^{u-K-t}), \end{aligned}$$

where $u = \max(m, K + t)$, and so β is an algebraic integer. Let γ be a conjugate of β over $\mathbb{Q}$ with modulus greater than one. Then,

$$\begin{aligned} x &= \gamma^p + \varepsilon_{-1}\gamma^{-1} + \cdots + \varepsilon_{-m}\gamma^{-m} \\ &\quad - (\eta_{-K}\gamma^{-K} + \eta_{-K-1}\gamma^{-K-1} + \cdots + \eta_{-K-t}\gamma^{-K-t}), \end{aligned}$$

and so

$$\beta^p - \gamma^p = \sum_{i=1}^{u} s_{-i}(\gamma^{-i} - \beta^{-i}),$$

where $s_i \in \{-I(\beta), \ldots, 0, \ldots, I(\beta)\}$ for each $i \in \{0, 1, \ldots, u\}$. Hence,

$$\frac{|\beta^p - \gamma^p|}{I(\beta)} \leq \sum_{i=1}^{u}(|\gamma^{-i}| + \beta^{-i}) < \frac{1}{|\gamma| - 1} + \frac{1}{\beta - 1}$$

and similarly as in the end of the proof of Theorem 3, the last relation leads to a contradiction when $\beta \neq \gamma$; thus, β is a Pisot or a Salem number. □

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