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ON THE CONSTRUCTION OF HÖLDER AND PROXIMAL SUBDERIVATIVES

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ABSTRACT. We construct Lipschitz functions such that for all $s > 0$ they are s-Hölder, and so proximally, subdifferentiable only on dyadic rationals and nowhere else. As applications we construct Lipschitz functions with prescribed Hölder and approximate subderivatives.

1. **Introduction.** Let f be an extended real-valued lower semicontinuous function defined on an open set $U \subset \mathbb{R}$ and $x \in U$. We assume throughout that $s > 0$.

DEFINITION 1. $\xi \in \mathbb{R}$ is called an *s-Hölder* subgradient of f at x if $f(x)$ is finite and for some $\sigma > 0$ and $\delta > 0$ one has

$$f(y) \geq f(x) + \xi(y-x) - \sigma|y-x|^{1+s} \quad \text{when } |y-x| < \delta.$$

We write $\xi \in \partial_{hs} f(x)$. When $s = 1$ such a subdifferential is called a *proximal subdifferential*, denoted by $\partial_p f$.

DEFINITION 2. f is *s-Hölder smooth* at x if there exists $c > 0$, $\delta > 0$, and $\xi \in \mathbb{R}$ such that

$$|f(y) - f(x) - \xi(y-x)| \leq c|y-x|^{1+s} \quad \text{whenever } |y-x| < \delta.$$

When $s = 1$ we say that f is *Lipschitz smooth* at x. More generally, we are considering derivatives and subdifferentials with *power modulus of smoothness*, [2].

In [2] Borwein and Preiss show, *inter alia*, that $\{x \mid \partial_{hs} f(x) \neq \emptyset \text{ and } x \in U\}$ is dense in U. In [5] Clarke, Ledyaev and Wolenski construct a C^1 function, f, on $\mathbb{R}$ such that both $\partial_p f$ and $\partial_p(-f)$ are nonempty only on a set that is small in the sense of both measure and category. In [1] Benoist shows that for every countable dense set D in $\mathbb{R}$ there exist infinitely many (uncountably many as may be seen from his proof) Lipschitz functions f, differing by more than a constant, such that $\partial_p f(x) = (-1, 1)$ if $x \in D$ and $\partial_p f(x) = \emptyset$ if $x \notin D$. Benoist's proof is lengthy. A slight modification of Benoist's proof, allows us to see that for each countable dense set D in $\mathbb{R}$ there exist uncountably many different Lipschitz functions, f, such that for every $s > 0$ we have $\partial_{hs} f(x) = (-1, 1)$ if $x \in D$ and $\partial_{hs} f(x) = \emptyset$ if $x \notin D$. Moreover, we have shown in [4] that

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PROPOSITION 1. *Assume S_1 and S_2 are two arbitrary countable dense sets in $\mathbb{R}$ with $S_1 \cap S_2 = \emptyset$. Then there exist two countable sets $D_1 \subset S_2$ and $D_2 \subset S_1$ with D_1 and D_2 dense in $\mathbb{R}$ such that there exists a Lipschitz function $f : \mathbb{R} \longmapsto \mathbb{R}$ having the following properties: for every $s > 0$*
(i) $\partial_{hs} f(x) = (-1, 1)$ if $x \in D_2$ and $\partial_{hs} f(x) = \emptyset$ if $x \in \mathbb{R} \setminus D_2$.
(ii) $\partial_{hs}(-f)(x) = (-1, 1)$ if $x \in D_1$ and $\partial_{hs}(-f)(x) = \emptyset$ if $x \in \mathbb{R} \setminus D_1$.

Our goal here is to construct Lipschitz functions, f, whose s-Hölder subdifferential is nonempty only on dyadic rationals and nowhere else. Needless to say, one may deduce this from Benoist's result but the construction method given herein is more explicit and much simpler, and has certain other virtues.

2. **Main Result.**

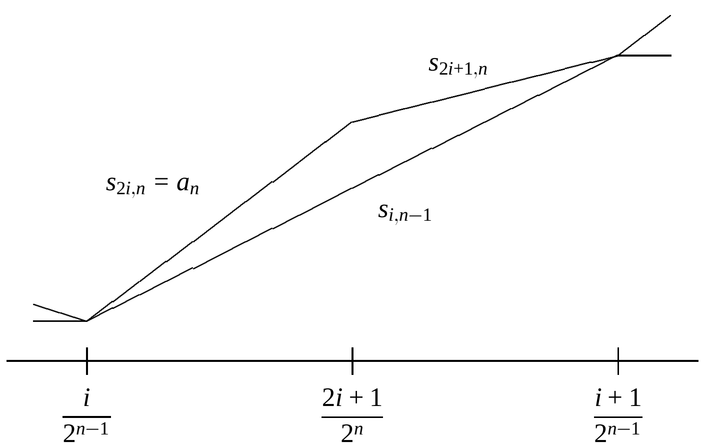
PROPOSITION 2. *For every sequence (a_n) satisfying:*
(0) $0 < a_1 < a_2 < \cdots < 1$, $a_n \longrightarrow 1$,
(1) $(2^n)^s(1 - a_n) \longrightarrow \infty$ *for all* $s > 0$,
there exists a 1-Lipschitz function $f : [0, 1] \longrightarrow \mathbb{R}$ such that $f(0) = 0$ and $f(1/2) = \frac{a_1}{2}$, for all $s > 0$ we have $\partial_{hs} f(x) = (-1, 1)$ when $x \in (0, 1)$ is a dyadic rational, and $\partial_{hs} f(x) = \emptyset$ when $x \in (0, 1)$ is not a dyadic rational.

PROOF. As in [1], f will be the limit of a sequence of functions f_n which are affine on the intervals $[i/2^n, (i+1)/2^n]$ for $i = 0, 1, \ldots, 2^n - 1$. Denote the slope of f_n on this interval by $s_{i,n}$.

Start with $f_0 \equiv 0$. Now assume that f_{n-1} is already defined. Then set $f_n(0) := 0$ and

$$\begin{aligned} s_{2i,n} &:= a_n, \quad s_{2i+1,n} := 2s_{i,n-1} - a_n, \quad \text{if } s_{i,n-1} \geq 0, \\ s_{2i,n} &:= 2s_{i,n-1} + a_n, \quad s_{2i+1,n} := -a_n, \quad \text{if } s_{i,n-1} \leq 0. \end{aligned}$$

In this way, f_n is defined and Lipschitz on the whole interval $[0, 1]$ and satisfies $f_n(2i/2^n) = f_{n-1}(i/2^{n-1})$ for $i = 0, \ldots, 2^{n-1}$.



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CLAIM 1. $s_{i,n} \in [-a_n, a_n]$ for all $i = 0, 1, \dots, 2^n - 1$, $n \in \mathbb{N}$.

PROOF. The claim is true for f_0, f_1, and if it is true for $n-1$, then it is also true for n: If $s_{i,n-1} \geq 0$, then $s_{2i,n} = a_n$ and $s_{2i+1,n} = 2s_{i,n-1} - a_n \leq 2a_{n-1} - a_n \leq 2a_n - a_n$ and $s_{2i+1,n} \geq 0 - a_n$, and similarly for $s_{i,n-1} \leq 0$.

This proves in particular that: $f_n \geq f_{n-1}$ on $[0, 1]$ for all n. In order to see what f_n looks like, we take $a_n := 1 - (15/16)^n$. After 9 iterations, Maple gives figure 1:

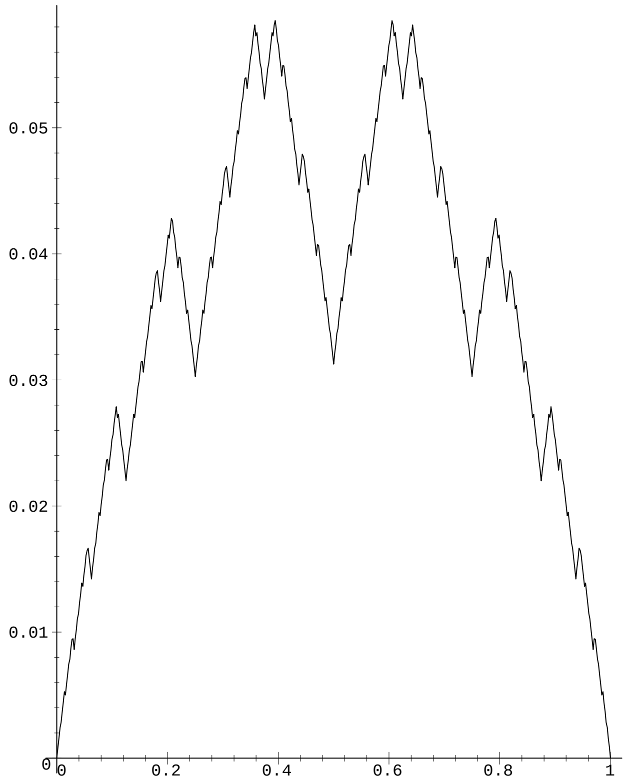

FIGURE 1: f_9

CLAIM 2. The f_n are uniformly convergent to a Lipschitz function f.

PROOF. For all x, we have $0 \leq f_n(x) - f_{n-1}(x) \leq a_n \cdot 1/2^n \leq 1/2^n$, which proves that the f_n are convergent in the uniform norm towards some f. Since $|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq 2/2^n + a_n \cdot |x - y| \leq 2/2^n + |x - y|$ for all n, f is a Lipschitz function.

CLAIM 3. If $x \in (0, 1)$ is a dyadic rational, then $\partial_{hs} f(x) = (-1, 1)$.

PROOF. Assume $x = i/2^n$.

CLAIM 3A. There exists a $k_0 \in \mathbb{N}$ such that for all $k > k_0$:

$$\left.\begin{array}{ll} f(y) \geq f(x) + a_k \cdot (y - x) & \text{for all } y \in [x, x + 1/2^k] \quad \text{and} \\ f(y) \geq f(x) - a_k \cdot (y - x) & \text{for all } y \in [x - 1/2^k, x], \end{array}\right\} \tag{1}$$

with equality for $y = x \pm 1/2^k$.

PROOF. For symmetry reasons, it is enough to prove the first inequality. Since for all $k \geq n$, we have $f \geq f_k \geq f_n$ and $f(x) = f_k(x) = f_n(x)$ and $f(x \pm 1/2^k) = f_k(x \pm 1/2^k)$, it is sufficient to show the existence of a $k_0 \geq n$ such that the first inequality holds for all $k > k_0$ with f replaced by f_k. For this it is enough to find a k_0 such that the slope of f_{k_0} to the right of x, namely $s_{i2^{k_0-n},k_0}$, is positive. The assertion then follows, because then $s_{i2^{k-n},k} = a_k$ for all $k > k_0$. Assume therefore that $s_{i,n}$ is negative. Then $s_{2i,n+1} = 2s_{i,n} + a_{n+1}$, and if that is still negative, then $s_{4i,n+2} = 2s_{2i,n+1} + a_{n+2}$, and so on. Also, because of Claim 1, $s_{2i,n+1} = 2s_{i,n} + a_{n+1} \geq s_{i,n} - a_n + a_{n+1} > s_{i,n}$. This implies $s_{4i,n+2} - s_{2i,n+1} = s_{2i,n+1} + a_{n+2} > s_{i,n} + a_{n+1} = s_{2i,n+1} - s_{i,n} > 0$. Therefore, in each step one increases the previous slope by a positive, increasing amount. After finitely many steps the slope will then itself become positive.

CLAIM 3B. $\partial_{hs} f(x) \subseteq (-1, 1)$.

PROOF. Assume $\xi \in \partial_{hs} f(x)$. That means that there exists a $\sigma > 0$ such that for k big enough, $f(x+1/2^k) \geq f(x)+\xi \cdot 1/2^k - \sigma \cdot (1/2^k)^{1+s}$. We can assume $k > k_0$, such that, by Claim 3A, $f(x+1/2^k) = f(x)+a_k \cdot 1/2^k$. This implies $a_k \cdot 1/2^k \geq \xi \cdot 1/2^k - \sigma \cdot (1/2^k)^{1+s}$ and therefore $a_k \geq \xi - \sigma \cdot (1/2^k)^s$. Letting k tend to infinity, we get $\xi \leq 1$. Moreover, $\xi = 1$ is impossible because of $(2^k)^s \cdot (1 - a_k) \longrightarrow \infty$. In the same way we prove $\xi > -1$.

CLAIM 3C. $(-1, 1) \subseteq \partial_{hs} f(x)$.

PROOF. The first inequality in (1) implies $f(y) \geq f(x)+\xi\cdot(y-x)$ for all $y \in [x, x+1/2^k]$ and all $\xi \leq a_k$, the second inequality implies $f(y) \geq f(x)+\xi\cdot(y-x)$ for all $y \in [x-1/2^k, x]$ and all $\xi \geq -a_k$. Joining the two intervals, we get $f(y) \geq f(x) + \xi \cdot (y - x)$ for all $y \in [x - 1/2^k, x + 1/2^k]$ and all $\xi \in [-a_k, a_k]$. Taking k large enough, we find any $\xi \in (-1, 1)$ in such an interval. This proves Claim 3.

CLAIM 4. If x is not a dyadic rational, then $\partial_{hs} f(x) = \emptyset$.

PROOF. Assume that $\xi \in \partial_{hs} f(x)$; we will show that this leads to a contradiction. If x is not a dyadic rational, then for every n, x lies in a uniquely determined interval of the form $[i/2^n, (i + 1)/2^n]$. Denote by p_n the slope of f_n in this interval.

CLAIM 4A. $p_n \longrightarrow \xi$ for $n \longrightarrow \infty$, in fact $|\xi - p_n| \leq \sigma/(2^n)^s$.

PROOF. Set $y_1 := i/2^n$ and $y_2 := (i + 1)/2^n$. Then

$$f(y_1) \geq f(x) + \xi \cdot (y_1 - x) - \sigma \cdot |y_1 - x|^{1+s} \quad \text{and}$$
$$f(y_2) \geq f(x) + \xi \cdot (y_2 - x) - \sigma \cdot |y_2 - x|^{1+s},$$

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if n is large enough. Since $f_n(x) = f(y_1) + p_n \cdot (x - y_1)$ and $f_n(x) = f(y_2) - p_n \cdot (y_2 - x)$ and $f(x) \geq f_n(x)$, it follows that

$$f(y_1) \geq f(y_1) + p_n \cdot (x - y_1) + \xi \cdot (y_1 - x) - \sigma \cdot |y_1 - x|^{1+s} \quad \text{and}$$
$$f(y_2) \geq f(y_2) - p_n \cdot (y_2 - x) + \xi \cdot (y_2 - x) - \sigma \cdot |y_2 - x|^{1+s}.$$

These are equivalent to $p_n - \xi \leq \sigma \cdot |x - y_1|^s$ and $\xi - p_n \leq \sigma \cdot |y_2 - x|^s$, which implies the claim since $|y_{1,2} - x| \leq 1/2^n$.

CLAIM 4B. $\xi = 1$ or $\xi = -1$.

PROOF. Because of $p_n \in [-1, 1]$, the only other possibility is $|\xi| < 1$. This is only possible if the case $p_n = \pm a_n$ does not occur after an initial phase. That means that for n large enough,

$$p_n = 2p_{n-1} - a_n \quad \text{if } p_{n-1} \geq 0 \text{ and}$$
$$p_n = 2p_{n-1} + a_n \quad \text{if } p_{n-1} \leq 0.$$

But as we saw in the proof of Claim 3A, each of these two cases can happen only finitely many times in a row, after which time p_n changes its sign. Therefore the p_n must converge to 0. But this is also impossible, because if we choose n large enough so that $0 \leq p_{n-1} \leq \varepsilon$ and $1 - a_n \leq \varepsilon$, then $p_n = 2p_{n-1} - a_n \leq -1 + 3\varepsilon$, a contradiction. Similarly for $p_n = 2p_{n-1} + a_n$.

[Note that the arguments of Claim 4A and Claim 4B imply that $f'(x) = -1$ or 1 for all $x \in (0, 1)$ except for a Lebesgue null set.]

CLAIM 4C. $\xi = \pm 1$ is impossible.

PROOF. Assume $\xi = 1$. Claim 4A now says that $1 - p_n \leq \sigma/(2^n)^s$. Since $p_n \leq a_n$, we also have $1 - a_n \leq 1 - p_n$. Therefore we get a contradiction to $(2^n)^s(1 - a_n) \longrightarrow \infty$. Similarly for $\xi = -1$.

All of this proves Claim 4. ■

THEOREM 1. *There exist uncountably many different Lipschitz functions* $f : \mathbb{R} \longmapsto \mathbb{R}$ *with* $f(0) = 0$ *such that for all* $s > 0$ *one has* $\partial_{hs} f(x) = (-1, 1)$ *if* x *is a dyadic number and* $\partial_{hs} f(x) = \emptyset$ *otherwise.*

PROOF. We extend f in Proposition 2 to all of $\mathbb{R}$. Since $f(0) = f(1) = 0$, we may extend f periodically as a Lipschitz function. By the same arguments as in Claim 3 we have $\partial_{hs} f(x) = (-1, 1)$ if x is an integer. In particular, we have $f(0) = 0$ and $f(1/2) = f_1(1/2) = a_1/2$. Changing a_1, we obtain uncountably many functions, different at 1/2 and annulling 0, such that they share the same s-Hölder subdifferential for *all* $s > 0$. ■

COROLLARY 1. *There exist uncountably many distinct nonnegative Lipschitz functions on* $\mathbb{R}$ *of compact support such that for all* $s > 0$ *the functions share the same* s*-Hölder subdifferential. Moreover such a function,* b*, only countably has* $\partial_{hs} b(x) \not\subset \{0\}$.

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PROOF. Define $b:\mathbb{R} \longmapsto \mathbb{R}$ by $b(x) := f(x)$ for $x \in [0,1]$ and 0 otherwise. Then b is a *nonnegative bump* and for all $s > 0$

$$\partial_{hs} b(x) = \begin{cases} (-1,1) & \text{if } x \in (0,1) \text{ is dyadic,} \\ \emptyset & \text{if } x \in (0,1) \text{ is not dyadic,} \\ [0,1) \text{ if } x = 0, \text{ and } (-1,0] & \text{if } x = 1, \\ \{0\} & \text{otherwise.} \end{cases}$$

Theorem 1 assures us that we may choose uncountably many such f's. ■

REMARK 1. (i) Note that if f is s-Hölder smooth at x then f is differentiable at x and $\partial_{hs} f(x)$ is a singleton. All the Lipschitz functions in Theorem 1 are nowhere s-Hölder smooth for every $s > 0$, and therefore are nowhere Lipschitz smooth. Moreover, from Claim 3A we see that all the Lipschitz functions in Theorem 1 achieve (strict) local minima at dyadic rationals and nowhere else.

(ii) Dyadic translation and dilation each produce countably many different Lipschitz functions sharing the same Hölder subdifferentials. For any Lipschitz function f given by Theorem 1, we define $f_n:\mathbb{R} \longmapsto \mathbb{R}$ by $f_n(x) := \frac{1}{2^n} f(2^n x)$ with $n \in \mathbb{N}$. Then $\partial_{hs} f_n = \partial_{hs} f$, and $f_n(0) = 0 = f(0)$. By the construction $f(x) > 0$ if $0 < x < 1$. Now $f_n(\frac{1}{2}) = \frac{1}{2^n} f(2^{n-1}) = 0 \neq \frac{a_1}{2} = f(\frac{1}{2})$. If $n, m \in \mathbb{N}$ and $n \neq m$, we have $f_m \neq f_n + \text{Constant}$, because the periods of f_n and f_m are different.

We may also define $f_b:\mathbb{R} \longmapsto \mathbb{R}$ by $f_b(x) := f(x+b) - f(b)$ for any dyadic number b. Then $\partial_{hs} f_b = \partial_{hs} f$, and $f_b(0) = f(0)$. For infinitely many dyadic number b, $f_b(\frac{1}{2}) \neq f(\frac{1}{2})$. If not, we have $f_b(\frac{1}{2}) = f(\frac{1}{2})$ for all dyadic rationals except for a finite number of them. By density we have $f(\frac{1}{2} + x) = f(x) + f(\frac{1}{2})$ for all $x \in \mathbb{R}$. In particular, when $x = \frac{1}{2}$ we have $0 = f(1) = 2f(\frac{1}{2}) = a_1 > 0$. This is a contradiction.

(iii) Note that the proximal normal cone and approximate normal cone for a locally Lipschitz function f are:

$$N^p_{\operatorname{epi} f}\big(x, f(x)\big) := \big\{t(\xi, -1) : \xi \in \partial_p f(x), t > 0\big\} \cup \big\{(0,0)\big\},$$
$$N^a_{\operatorname{epi} f}\big(x, f(x)\big) := \big\{t(\xi, -1) : \xi \in \partial_a f(x), t > 0\big\} \cup \big\{(0,0)\big\}.$$

We thus see that the Lipschitz functions in Theorem 1 have the following properties:

$$(2) \quad N^p_{\operatorname{epi} f}\big(x, f(x)\big) = \begin{cases} \big\{t(\xi,-1) : -1 < \xi < 1, t > 0\big\} \cup \big\{(0,0)\big\} & \text{if } x \text{ is dyadic,} \\ \big\{(0,0)\big\} & \text{otherwise.} \end{cases}$$

$$N^a_{\operatorname{epi} f}\big(x, f(x)\big) = \big\{t(\xi,-1) : -1 \leq \xi \leq 1, t > 0\big\} \cup \big\{(0,0)\big\} \quad \text{for every } x \in \mathbb{R}.$$

It is well known that the set of points in the boundary of $\operatorname{epi} f$ for which $N^p_{\operatorname{epi} f}\big(x, f(x)\big) \neq \{(0,0)\}$ is dense in $\operatorname{bdry}(\operatorname{epi} f)$. (2) shows that $\big\{\big(x, f(x)\big) : N^p_{\operatorname{epi} f}\big(x, f(x)\big) \neq \{(0,0)\}\big\}$ may be countable. Indeed by appropriately choosing $\{a_n\}$ we may ensure that both x and $f(x)$ are dyadic rationals, and so the proximal normal cone is non-trivial only at a subset of the dyadic rationals in the plane. For $s \geq \varepsilon > 0$ an appropriate sequence is given by $a_n := 1 - (1 - 2^{-k})^n$ for k a sufficiently large integer.

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(iv) If f is differentiable at $x \in U$, then $\partial_{hs} f(x) \subset \{f'(x)\}$. Therefore $\partial_{hs} f$ is singleton or empty almost everywhere, and there is no Lipschitz function f such that $\partial_{hs} f(x) = (-1, 1)$ for every $x \in \mathbb{R}$.

3**. Further applications.** Theorem 1 allows us to construct Lipschitz functions with prescribed Hölder subdifferentials. In the sequel we assume $0 < s \leq 1$. Given a set $D \subset \mathbb{R}$ and a function f defined on $\mathbb{R}$, by f^{-1} we denote the inverse function of f, and we define $f^{-1}(D) := \{x \mid f(x) \in D\}$.

DEFINITION 3. Assume $h: U \longmapsto \mathbb{R}$. h is said to be $C^{1,s}$ on U provided that h is differentiable on U, with h' being locally s-Hölder continuous on U, *i.e.*, for each $x \in U$ there exists $K > 0$ such that $|h'(y) - h'(z)| \leq K|y - z|^s$ whenever y and z are near x.

An application of the mean value theorem shows that each $C^{1,s}$ function defined on U is everywhere s-Hölder smooth.

LEMMA 1. *A vector v is an s-Hölder subgradient of f at x if and only if on some neighbourhood of x there is a $C^{1,s}$ function $h \leq f$ with $h(x) = f(x)$, $h'(x) = v$.*

PROOF. Suppose there exists an h being $C^{1,s}$ with $v = h'(x)$, $h(x) = f(x)$, and $f \geq h$ on a neighbourhood of x. Choose $\delta > 0$ such that h' is s-Hölder continuous with Hölder constant K and exists by the mean value theorem an η between x and y with

$$\begin{aligned} h(y) &= \big[h(y) - h(x) - h'(x)(y - x)\big] + h(x) + h'(x)(y - x) \\ &= \big(h'(\eta) - h'(x)\big)(y - x) + h(x) + h'(x)(y - x) \\ &\geq -K|y - x|^{1+s} + h(x) + h'(x)(y - x). \end{aligned}$$

Since $f(y) \geq h(y) \geq f(x) + h'(x)(y - x) - K|y - x|^{1+s}$ for $y \in (x - \delta, x + \delta)$, we have $v \in \partial_{hs} f(x)$. Conversely, from Definition 1 we need to verify that $h: \mathbb{R} \longmapsto \mathbb{R}$ defined by $h(y) := f(x) + \xi(y - x) - \sigma|y - x|^{1+s}$ is $C^{1,s}$ with respect to y.

CLAIM 1. If $0 < s \leq 1$, then $(1 + t)^s \leq 1 + t^s$ for $0 \leq t < +\infty$.

PROOF. Let $\phi(t) := 1 + t^s - (1 + t)^s$. The claim follows from: $\phi'(t) = s\big(t^{s-1} - (1 + t)^{s-1}\big) \geq 0$ and $\phi(0) = 0$.

CLAIM 2. $h(t) := f(x) + \xi \cdot t - \sigma \cdot |t|^{1+s}$ is differentiable and h' is s-Hölder continuous.

PROOF.
$$h'(t) = \begin{cases} \xi - \sigma(1 + s)t^s & \text{if } t > 0 \\ \xi + \sigma(1 + s)(-t)^s & \text{if } t < 0 \\ \xi & \text{if } t = 0. \end{cases}$$

(i) If $t_1 \geq t_2 > 0$, by Claim 1 we have

$$t_1^s - t_2^s = t_2^s\big[(t_1/t_2)^s - 1\big] \leq t_2^s(t_1/t_2 - 1)^s = (t_1 - t_2)^s.$$

The case that $0 > t_1 \geq t_2$ is similar.

(ii) If either t_1 or t_2 is 0, then $t_1^s - t_2^s \leq |t_1 - t_2|^s$ is clearly true.

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(iii) If $t_1 > 0 > t_2$, then

$$t_1^s + (-t_2)^s \le |t_1 - t_2|^s + |t_1 - t_2|^s = 2|t_1 - t_2|^s.$$

Hence $|h'(t_1) - h'(t_2)| \le 2(1+s)\sigma|t_1 - t_2|^s$ for t_1 and t_2 near 0. ∎

In $\mathbb{R}$ we know that $\xi \in \partial^- f(x)$, the *Dini subdifferential*, if and only if there is another locally Lipschitz function h such that (i) $f(y) \ge h(y)$ for all y near x, $h(x) = f(x)$, and (ii) $h'(x) = \xi$ and h' is continuous at x. Comparing this with Lemma 1 we see that the more restrictive the subdifferential the more restrictive the derivative of h.

LEMMA 2. *(i) Let g and g^{-1} both be $C^{1,s}$. Then*

$$\partial_{hs} f \circ g(x) = g'(x)\partial_{hs} f(z) \quad \text{with } z = g(x).$$

In particular $\partial_{hs} f \circ g(x) = \emptyset$ if and only if $\partial_{hs} f(z) = \emptyset$ where $z = g(x)$.
(ii) Let g be $C^{1,s}$ and f be locally Lipschitz. If $g'(x) = 0$, then $\partial_{hs} f \circ g(x) = \{0\}$.

PROOF. (i) Suppose for a $C^{1,s}$ function h we have $f(y) \ge h(y)$ for y near $g(x)$ and $f\big(g(x)\big) = h\big(g(x)\big)$. Then $f\big(g(y)\big) \ge h\big(g(y)\big)$ for y near x. Since the map $y \longrightarrow h'\big(g(y)\big)g'(y)$ is s-Hölder continuous around x, Lemma 1 shows that $h'\big(g(x)\big)g'(x) \in \partial_{hs} f \circ g(x)$, thus $g'(x)\partial_{hs} f(z) \subset \partial_{hs} f \circ g(x)$ with $z = g(x)$. Conversely let $\xi \in \partial_{hs} f \circ g(x)$. Then there exists h being $C^{1,s}$ such that $f\big(g(y)\big) \ge h(y)$ for y near x. We have $f(y) \ge h\big(g^{-1}(y)\big)$ for y near $g(x)$. Since the map $y \longrightarrow h'\big(g^{-1}(y)\big)(g^{-1})'(y)$ is s-Hölder continuous around $g(x)$, Lemma 1 shows $(h \circ g^{-1})'\big(g(x)\big) = h'(x)/g'(x) \in \partial_{hs} f(z)$ with $z = g(x)$, so we have $h'(x) \in g'(x)\partial_{hs} f(z)$ with $z = g(x)$. That is $\partial_{hs} f \circ g(x) \subset g'(x)\partial_{hs} f(z)$ where $z = g(x)$.

(ii) Assume the Lipschitz constant of f around $g(x)$ is K and the s-Hölder constant of g' around x is $\hat{K}$. By the mean value theorem there exists $\eta \in [x, x+h]$ with

$$|f \circ g(x+h) - f \circ g(x)| \le K|g(x+h) - g(x)| = K|g'(\eta) \cdot h - g'(x) \cdot h| \le K\hat{K}h^{1+s}.$$

This means $0 \in \partial_{hs} f \circ g(x)$. On the other hand,

$$\left|\frac{f \circ g(x+h) - f \circ g(x)}{h}\right| \le K\left|\frac{g(x+h) - g(x)}{h}\right|,$$

we have $(f \circ g)'(x) = 0$. Hence $\partial_{hs} f \circ g(x) = \{0\}$. ∎

It is well known that for a real-valued function g with domain an open interval, $\triangle$, the inverse g^{-1} exists if and only if g is strictly monotone on $\triangle$. If we assume that g is C^1 and $g'(x) \ne 0$ then g^{-1} exists locally around x. In Lemma 2(i) we only need to assume that g^{-1} exists locally.

LEMMA 3. *Let U be an open subset of $\mathbb{R}$. Suppose that $f: U \longmapsto (-\infty, +\infty]$ is lower semicontinuous, and $x \in U$. Suppose further that g is s-Hölder smooth at x. Then $f + g$ is s-Hölder subdifferentiable at x if and only if f is s-Hölder subdifferentiable at x. Furthermore*

$$\partial_{hs}(f+g)(x) = \partial_{hs} f(x) + g'(x).$$

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PROOF. It suffices to show $\partial_{hs}(f+g)(x) \subset \partial_{hs} f(x) + \partial_{hs} g(x)$. Let $\xi \in \partial_{hs}(f+g)(x)$. By assumption there exist $\sigma_1, \sigma_2, \delta > 0$ such that

$$-\sigma_1 |y-x|^{1+s} \le g(y) - g(x) - g'(x)(y-x) \le \sigma_1 |y-x|^{1+s},$$
$$f(y) + g(y) - f(x) - g(x) + \sigma_2 |y-x|^{1+s} \ge \xi(y-x),$$

whenever $|y-x| < \delta$. Then

$$\begin{aligned} &f(y) - f(x) + (\sigma_1 + \sigma_2)|y-x|^{1+s} \\ &\quad \ge f(y) - f(x) + \sigma_2 |y-x|^{1+s} + g(y) - g(x) - g'(x)(y-x) \\ &\quad \ge \big(\xi - g'(x)\big)(y-x). \end{aligned}$$

Note that if $\partial_{hs} f(x) \ne \emptyset$, then $\partial_{hs}(f+g)(x) \ne \emptyset$, conversely if $\partial_{hs}(f+g)(x) \ne \emptyset$ then $f = (f+g) - g$ shows $\partial_{hs} f(x) \ne \emptyset$. ∎

We may now formulate our main application:

THEOREM 2. *Assume $f, g\colon \mathbb{R} \mapsto \mathbb{R}$ are locally s-Hölder continuous functions with $f \ge g$ and $f \not\equiv g$. Define $F, G\colon \mathbb{R} \mapsto \mathbb{R}$ by*

$$F(x) := \int_0^x f(s)\,ds \quad \textit{and} \quad G(x) := \int_0^x g(s)\,ds.$$

Set $D_1 := \{x \mid x \in \mathbb{R} \text{ is dyadic}\}$. Then there exist uncountably many locally Lipschitz functions h, differing by more than a constant, on $\mathbb{R}$ with

$$\partial_{hs} h(x) = \begin{cases} \big(g(x), f(x)\big) & \text{if } x \in \{y \mid g(y) < f(y)\} \cap (F-G)^{-1}(D_1), \\ \emptyset & \text{if } x \in \{y \mid g(y) < f(y)\} \setminus (F-G)^{-1}(D_1), \\ \{g(x)\} & \text{if } f(x) = g(x). \end{cases}$$

PROOF. By Theorem 1 we may choose a Lipschitz H on $\mathbb{R}$ with

$$\partial_{hs} H(x) = \begin{cases} (0,1) & \text{if } x \in D_1, \\ \emptyset & \text{otherwise.} \end{cases} \tag{3}$$

Since $f \not\equiv g$, for some x_0 we have $f(x_0) > g(x_0)$. There exists $\delta > 0$ such that $f > g$ on $[x_0 - \delta, x_0 + \delta]$. Since H is periodic, by translation and dilation (see Remark 1(ii)) we may assume at least one period of H is a subset of $[(F-G)(x_0 - \delta), (F-G)(x_0 + \delta)]$. Now $(F-G)' = f - g$ implies that $F - G$ is $C^{1,s}$. When $f(x) \ne g(x)$ we have $(F-G)' \ne 0$ around x, and the inverse function theorem in [7] shows that $F - G$ is locally invertible around x and the local inverse is $C^{1,s}$ around $(F-G)(x)$. By Lemma 2 we have

$$\partial_{hs} H \circ (F-G)(x) = \begin{cases} \big(0, f(x) - g(x)\big) & \text{if } F(x) - G(x) \in D_1 \text{ and } f(x) > g(x), \\ \emptyset & \text{if } F(x) - G(x) \notin D_1 \text{ and } f(x) > g(x), \\ \{0\} & \text{if } f(x) = g(x). \end{cases}$$

Since G is $C^{1,s}$, it follows from Lemma 3 that

$$\partial_{hs}\big(G + H \circ (F-G)\big)(x) = \begin{cases} \big(g(x), f(x)\big) & \text{if } F(x) - G(x) \in D_1 \text{ and } f(x) > g(x), \\ \emptyset & \text{if } F(x) - G(x) \notin D_1 \text{ and } f(x) > g(x), \\ \{g(x)\} & \text{if } f(x) = g(x). \end{cases}$$

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Denote $h := G + H \circ (F - G)$. Therefore,

$$(4) \qquad \partial_{hs} h(x) = \begin{cases} \big(g(x), f(x)\big) & \text{if } x \in \{y \mid f(y) > g(y)\} \cap (F-G)^{-1}(D_1), \\ \emptyset & \text{if } x \in \{y \mid f(y) > g(y)\} \setminus (F-G)^{-1}(D_1), \\ \{g(x)\} & \text{if } f(x) = g(x). \end{cases}$$

By Theorem 1 plus translation and dilation we can choose uncountably many Lipschitz functions H_1, different from H by more than a constant, which satisfy (3) and have at least one period as a subset of $[(F-G)(x_0-\delta), (F-G)(x_0+\delta)]$. Then $h_1 \colon \mathbb{R} \mapsto \mathbb{R}$ defined by $h_1(x) := \big(G + H_1 \circ (F-G)\big)(x)$ also satisfies (4). Now there exist $y_1, y_2 \in [(F-G)(x_0-\delta), (F-G)(x_0+\delta)]$ such that $H(y_1) - H_1(y_1) \neq H(y_2) - H_1(y_2)$. Write $y_i = (F-G)(x_i)$. Then $(h-h_1)(x_1) = (H-H_1) \circ (F-G)(x_1) = (H-H_1)(y_1) \neq (H-H_1)(y_2) = (H-H_1) \circ (F-G)(x_2) = (h-h_1)(x_2)$. ■

REMARK 2. Assume that $f, g \colon \mathbb{R} \mapsto \mathbb{R}$ are locally s-Hölder continuous with $f \not\equiv g$. With the same notations Theorem 2 becomes:

$$\partial_{hs} h(x) = \begin{cases} \big(\min\{g(x), f(x)\}, \max\{g(x), f(x)\}\big) & \text{if } x \in \{y \mid g(y) \neq f(y)\} \cap (F-G)^{-1}(D_1), \\ \emptyset & \text{if } x \in \{y \mid g(y) \neq f(y)\} \setminus (F-G)^{-1}(D_1), \\ \{g(x)\} & \text{if } f(x) = g(x). \end{cases}$$

As an example, let $f, g \colon \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) := \sin^2(x/2)$ and $g(x) := 1/2$. Then there exist uncountably many Lipschitz functions, h, on $\mathbb{R}$ differing by more than constants such that $\partial_p h(x) =$

$$\begin{cases} \big(\min\{\sin^2(x/2), 1/2\}, \max\{\sin^2(x/2), 1/2\}\big) & \text{if } x \in \mathbb{R} \setminus A \text{ with } \sin x \text{ dyadic}, \\ \emptyset & \text{if } x \in \mathbb{R} \setminus A \text{ with } \sin x \text{ non-dyadic}, \\ \{1/2\} & \text{if } x \in A, \end{cases}$$

where $A := \{k\pi + \pi/2 \mid k \text{ is any integer}\}$.

Note that, in general, if $f > g$ on $\mathbb{R}$, the strict monotonicity of $F - G$ shows:

$$D := \big\{x \mid \big(F(x) - G(x)\big) \in D_1\big\} \text{ is countable,}$$

and we have the following Corollary:

COROLLARY 2. *Suppose f and g are locally s-Hölder continuous functions with $f(x) > g(x)$ for all $x \in \mathbb{R}$. Then there exists a countable dense set $D \subset \mathbb{R}$ such that there exist uncountably many locally Lipschitz functions h, differing by more than a constant, on $\mathbb{R}$ with*

$$\partial_{hs} h(x) = \begin{cases} \big(g(x), f(x)\big) & \text{if } x \in D, \\ \emptyset & \text{otherwise.} \end{cases}$$

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Recall the *approximate subdifferential* [9, 8] and the *symmetric subdifferential* [10] are given by:

$$\partial_a h(x) := \Big\{ \lim_{i\to\infty} \xi_i : \xi_i \in \partial_p h(x_i), x_i \longrightarrow x, h(x_i) \longrightarrow h(x) \Big\},$$
$$\partial_s h(x) := \partial_a h(x) \cup \Big(-\partial_a(-h)(x) \Big).$$

If h is Lipschitz near x, we have the *Clarke subdifferential*: $\partial_c h(x) = \operatorname{co} \partial_a h(x)$ where "co" stands for the convex hull. Borwein and Fitzpatrick [3] have shown that in one dimension the symmetric subdifferential and the Clarke subdifferential are the same. The fact that, when $s = 1$, Theorem 2 holds for the proximal subdifferentials shows:

COROLLARY 3. *Suppose f and g are locally Lipschitz functions on $\mathbb{R}$ with $f \not\equiv g$. Then there exist uncountably many locally Lipschitz functions, differing by more than a constant, on $\mathbb{R}$ such that they share the same approximate subdifferential everywhere. For each such Lipschitz function h one has:*

$$\partial_c h(x) = \partial_s h(x) = \partial_a h(x) = \operatorname{co}\Big\{ g(x), f(x) \Big\} \quad \textit{for every } x \in \mathbb{R}.$$

REFERENCES

1. J. Benoist, *Intégration du sous-différentiel proximal: un contre exemple*. Canad. J. Math. (2) **50**(1998), 242–265.
2. J. M. Borwein and D. Preiss, *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*. Trans. Amer. Math. Soc. **303**(1987), 517–527.
3. J. M. Borwein and S. Fitzpatrick, *Characterization of Clarke subgradients among one-dimensional multifunctions*. In: Proc. of the Optimization Miniconference II (Eds. B. M. Glover and V. Jeyakumar), 1995, 61–73.
4. J. M. Borwein and X. Wang, *Lipschitz functions with prescribed Hölder subderivatives*. In preparation.
5. F. H. Clarke, Yu. S. Ledyaev and P. R. Wolenski, *Proximal analysis and minimization principles*. J. Math. Anal. Appl. **196**(1995), 722–735.
6. F. H. Clarke, R. J. Stern and P. R. Wolenski, *Subgradient criteria for monotonicity, the Lipschitz condition, and convexity*. Canad. J. Math. (6) **45**(1993), 1167–1183.
7. W. Fleming, *Functions of several variables*. Springer-Verlag, 1977.
8. A. D. Ioffe, *Approximate subdifferentials and applications I: The finite dimensional theory*. Trans. Amer. Math. Soc. **281**(1984), 390–416.
9. B. Mordukhovich, *Maximum principle in problems of time optimal control with nonsmooth constraints*. J. Appl. Math. Mech. **40**(1976), 960–969.
10. ———, *Metric approximations and necessary optimality conditions for general classes of nonsmooth extremal problems*. Soviet Math. Dokl. **22**(1980), 526–530.
11. R. T. Rockafellar and R. J-B. Wets, *Variational analysis.* Springer-Verlag, Berlin, 1998.

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