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Control of flexible manipulators: A survey

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SUMMARY

A survey of the field of control for flexible multi-link robots is presented. This research area has drawn great attention during the last two decades, and seems to be somewhat less "attractive" now, due to the many satisfactory results already obtained, but also because of the complex nature of the remaining open problems. Thus it seems that the time has come to try to deliver a sort of "state of the art" on this subject, although an exhaustive one is out of scope here, because of the great amount of publications. Instead, we survey the most salient progresses – in our opinion – approximately during the last decade, that are representative of the essential different ideas in the field. We proceed along with the exposition of material coming from about 119 included references. We do not pretend to deeply present each of the methods quoted hereafter; however, our goal is to briefly introduce most of the existing methods and to refer the interested reader to more detailed presentations for each scheme. To begin with, a now well-established classification of the flexible arms control goals is given. It is followed by a presentation of different control strategies, indicating in each case whether the approach deals with the one-link case, which can be successfully treated via linear models, or with the multi-link case which necessitates nonlinear, more complex, models. Some possible issues for future research are given in conclusion.

KEYWORDS: Flexible manipulators; Control; Survey.

1. INTRODUCTION

The question of modeling and control of multi-link flexible manipulators has received a thorough attention during the past two decades, as can be seen from previous survey papers.[1–3] Because of high performance requirements in robotics (high speed operation, better accuracy), and space applications (construction of large space structures by using lightweight space robot manipulator‡) consideration of structural flexibility in robots arms is a real challenge. Unfortunately, taking into account the flexibility of the arm leads to the appearance of oscillations at the tips of the links during the motion. These oscillations make the control problems of such systems really difficult. A neat presentation of the main issues in control of flexible link manipulators can be found, e.g. in reference [1]. Actually, the control problems of a flexible robot arm can be divided into four principal objectives which are, of increasing difficulty:

- End-effector position regulation.
- Rest to rest end-effector motion in fixed time.
- Trajectory tracking in the joint space (tracking of a desired angular trajectory).
- Trajectory tracking in the operational space (tracking of a desired end-effector trajectory).

To address these objectives, many techniques coming from control theory have been used and adapted to flexible robots: Input/output linearization via static state feedback,[5–10] proportional-derivative regulator,[11–15] adaptive control,[16–19] neural networks,[20] lead-lag controller,[21,22] output redefinition,[23–31] singular perturbations,[32] sliding mode control,[33] stable inversion in the frequency domain,[34–38] stable inversion in the time domain,[40–44] algebraic schemes,[45,46] poles placement,[47–50] optimal trajectories planning,[51–55] optimal and robust control,[13,56–61] input shaping,[52,62–68] boundary control,[69] input-state linearization via dynamical state space feedback,[70,71] control synthesis for a family of flexible manipulators,[60] manipulators coupling,[72] mechanical wave approach,[73,74] optimal arms design.[75,76] Hereafter are detailed and analyzed most of these results. In Section II, some basic concepts, that are standard for flexible arms modeling, are introduced, mainly for fixing notations. Section III is dedicated to a classified description of the flexible arms control goals, together with a list of works in each category. Then, in Section IV are detailed most of the works frequently quoted in the literature. It is worth noting that, in each case, it is indicated whether the result deals with the control of the one-link arm case, or with the multi-link flexible arms. This separation is mainly due to the difference between the robot models used in each case. For the one-link case, linear models have shown to be sufficient to solve all the above control problems. The multi-link case, instead, necessitates more precise and complex nonlinear models, leading to the use of nonlinear control schemes. Section V concludes this work with a presentation of some control objectives that are, to our knowledge, still open for investigations, as well as further reading references.

II. DYNAMIC MODELING

Different models have been used, to analyze and control flexible multi-body systems: the basic spring-mass discrete models,[52,63,73] linear Euler-Bernoulli PDE,[12,69] generalized

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‡ e.g. the SSRMS, Space Station Remote Manipulator System, a six-link flexible manipulator characterized by 17, 6 m length, used by the Canadian space agency for assembly tasks in space environment.[4]

Newton-Euler algorithms,[77,78] Lagrangian equations, associated to a Rayleigh-Ritz elastic field decomposition method, finite element decomposition, e.g.[34,35,79] or modal decomposition, which is actually used for most of the works that are surveyed here. Hence, it is useful to recall now the general lagrangian model form, based on a modal elastic decomposition for multi-link flexible arms undgrgoing *small elastic displacements*. For model derivation details and model properties, the reader can look at references [80], [81] and references therein. Also, for large *elastic* displacements models, some references are [82], [83], [84], [85].

Consider a multi-body system composed of n flexible links $S_1, \dots, S_n$ loaded by a rigid body S_{n+1}. The base S_0 is supposed to be rigid as well as the n joints. A reference frame F_i, $i \in \{1, \dots, n+1\}$ rigidly attached to each body, to describe its position in the motion space. The elastic deformation in S_i, $i \in \{1, \dots, n\}$ is represented as:

$$\omega^i(x, t) = \phi^i(x) Q^i(t) \quad (1)$$

with, ϕ^i the row vectors of shape functions (for detailed discussion about shape functions choice see [86]), Q^i the column vector of elastic coordinates. Lagrangian formulation can be used to determine the final equations of motion, which are of the following general form:[80]

$$M(q)\ddot{q} + h_c(q, \dot{q}) + Kq + D\dot{q} + g(q) = Wu \quad (2)$$

where, $q = (\theta_1, \dots \theta_n, q_{11}, \dots, q_{1m_1}, \dots, q_{n1}, \dots q_{nm_n})^T$ is the vector of generalized coordinates, $u = (u_1, \dots, u_n)^T$ the vector of joint torques, M the positive definite symmetric inertia matrix, h_c the vector of Coriolis and centrifugal forces, K the stiffness matrix, D the damping matrix, $g(q)$ gravity effects term and W the input weighting matrix.

When *clamped shape* functions are chosen, W assumes the form $[I_{n\times n}, 0_{n\times n_e}]^T$ (where $n_e = m_1 + \dots + m_n$ and I is the identity matrix*). The mass matrix M, the stiffness matrix K, the damping matrix D, the Coriolis and centrifugal vector h_c and the gravity terms $g(q)$ can be partitioned consistently with the definition of q as:

$$M = \begin{pmatrix} M_{rr} & M_{re} \\ M_{re}^T & M_{ee} \end{pmatrix} \quad K = \begin{pmatrix} 0 & 0 \\ 0 & K_{ee} \end{pmatrix}$$
$$D = \begin{pmatrix} 0 & 0 \\ 0 & D_{ee} \end{pmatrix} \quad h_c = (h_r, h_e)^T, \; g = (g_r, g_e)^T \quad (3)$$

then, the dynamics can be rewritten separately for the rigid and flexible parts as follows:

$$M_{rr}\ddot{\Theta} + M_{re}\ddot{Q} + h_r(\Theta, Q, \dot{\Theta}, \dot{Q}) + g_r(q) = I_{n\times n} \,.\, u \quad (4)$$

$$M_{re}^T\ddot{\Theta} + M_{ee}\ddot{Q} + h_e(\Theta, Q, \dot{\Theta}, \dot{Q}) + K_{ee}Q + D_{ee}\dot{q} + g_e(q) = 0 \quad (5)$$

where:

$$\Theta = (\theta_1, \dots \theta_n)^T, \; Q = (q_{11}, \dots, q_{1m_1}, \dots, q_{n1}, \dots, q_{nm_n})^T$$

Also when considering the non-collocated end-effector angular position output:

$$Y = [y_1, \dots, y_n]^T = [\theta_1, \dots, \theta_n]^T + \left[atan\left(\frac{\omega_1(L_1, t)}{L_1}\right), \dots, atan\left(\frac{\omega_n(L_n, t)}{L_n}\right) \right]^T \quad (6)$$

which writes for the case of small elastic displacements as:

$$Y = \Theta + \Gamma Q \quad (7)$$

* In the following, $A_{n\times m}$ denotes a matrix with n lines, m columns.

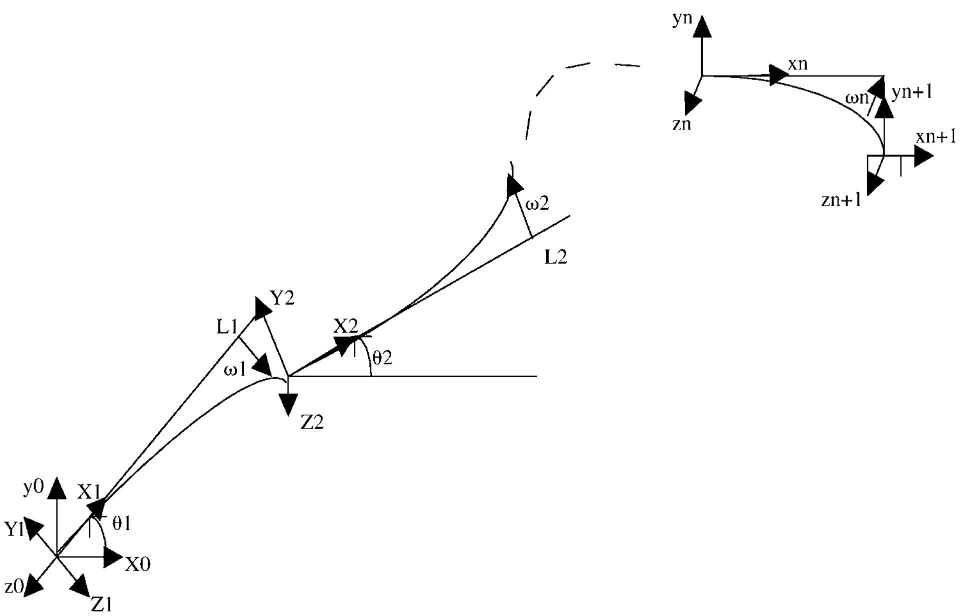

Fig. 1. A multi-link flexible manipulator.

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with

$$\Gamma=\begin{bmatrix} \frac{\phi_{11}}{L_1}\cdots\frac{\phi_{1m_1}}{L_1} & 0\ldots & \ldots 0 \\ 0\ldots 0 & \frac{\phi_{21}}{L_2}\cdots\frac{\phi_{2m_2}}{L_2} & 0\ldots 0 \\ \ddots & \ddots & \ddots \\ 0\ldots & \ldots 0 & \frac{\phi_{n1}}{L_n}\cdots\frac{\phi_{nm_n}}{L_n} \end{bmatrix}$$

the system dynamic writes:

$$M_{rr}\ddot{Y}+(M_{re}-M_{rr}\Gamma)\ddot{Q}+h_r(Y, Q, \dot{Y}, \dot{Q})=u \quad (8)$$

$$M_{re}^T\ddot{Y}+(M_{ee}-M_{re}^T\Gamma)\ddot{Q}+h_e(Y, Q, \dot{Y}, \dot{Q})+K_{ee}Q=0 \quad (9)$$

In the following section, the main control objectives for flexible link robots are summarized with relevant works listed in each case.

III. CONTROL OBJECTIVES FOR THE FLEXIBLE LINK ROBOTS

In the control of flexible link arms, four control objectives are usually distinguished.[1] It can be seen that even for the simple end-effector regulation problem, research has been conducted still recently. For the last and most difficult problem of end-effector tracking in the operational space, frequency domain as well as time domain techniques are still investigated.

(a) End effector regulation problem: In this case, the goal is to achieve a point to point motion of the arm end-effector, in an optimal time motion with respect to the minimization of the end-effector residual tip oscillations. It is the subject of the following works: [11–16, 19, 20, 24–26, 32–33, 45–49. 58–60, 64, 69, 72, 73, 87–93].

(b) End effector to rest motion in a desired fixed time: This problem includes the previous one, since the end-effector positioning is requested to be achieved in any *given desired fixed time*, together with the complete elimination of the tip oscillations at the final motion time (i.e. rest to rest end-effector motion). It has been investigated, e.g. in references [43], [46], [50], [52], [53], [54], [55], [63], [70], [71], [94], [95].

(c) Joint-trajectory tracking: The joint variables must track a desired angular trajectories, and at the same time the end-effector must reach the final desired position with minimal residual oscillations. Control strategies for this objective have been obtained in references [5], [7], [8], [10], [17], [18], [51], [56], [61], [96], [97].

(d) End-effector trajectory tracking: As previously mentioned, this problem is the most difficult one, due to the non-minimum phase nature of the system dynamics when dealing with the non-collocated end-effector position output. The end-effector must track the desired cartesian operational trajectories and must reach the final equilibrium point with minimal oscillations. As a consequence, many works have been dedicated to solve the above control problem, as in references [3], [6], [9], [22], [23], [27], [28], [29], [30], [34], [35], [36], [37], [38], [42], [44], [98], [99], [100], [101], [102], [103].

In the following section, a classification of the related methods is given, with respect to the control methods used in each case. For each method, the corresponding papers are quoted before a summarized presentation of the control method together with the associated results.

IV. CONTROL SCHEMES APPLIED TO THE FLEXIBLE LINKS ROBOTS

(e) Proportional-Derivative (PD) control:[11–15,91] Two forms have been used. The joint collocated PD regulator and the non-collocated PD, which associates tip position and joint velocity feedback. In reference [12], the non-collocated feedback has been applied to the Euler-Bernoulli PDE one-link flexible arm model. The authors proved that this control leads to asymptotic stability of the end-effector equilibrium point, hence solving the regulation problem. A similar result has been underlined in references [13], [14], where the advantages of collocated and non-collocated PD laws have been studied on a one-link flexible testbed. Also the passivity property of a simple collocated joint PD controller has been studied in reference [91], and successfully applied to the end-effector regulation for a one-link flexible arm linear lagrangian model *without taking into account the elastic damping terms* (i.e. the $D\dot{q}$ term in Equation (2)). In a more general setting, the authors of reference [15] have proved, that under some nonlinear lagrangian dynamic model conditions, the collocated joint PD regulator achieves end-effector regulation for multi-link flexible robot, also without taking into account the elastic damping terms.

(f) Input-Output linearization via static state feedback: the "computed torque":[5–10,90,104,105] The so called "computed torque" method is based on input-output linearization via static state feedback.[106] This scheme implies the appearance of non-observable dynamics, called "internal dynamics". For the flexible link robot, if the chosen output is the joint collocated position, the associated internal dynamics are the elastic dynamics (Equation (5), where the desired joint coordinates trajectories Θ_d, $\dot{\Theta}_d$ and $\ddot{\Theta}_d$ are substituted for Θ, $\dot{\Theta}$ and $\ddot{\Theta}$). In this case, the equilibrium point of the associated internal dynamics is locally asymptotically stable ([5], for a one-link case study, based on a nonlinear lagrangian model), thus the solution of the internal dynamics equations is bounded[107] (p. 219, lemma 5.4). Otherwise, if the chosen output is the non-collocated end-effector position (7), the stability of the internal dynamics equilibrium point is no longer assured (reference [5], for the one-link case). Then, the internal dynamics solutions could make a finite time escape phenomenon appear, yielding unbounded joint control torques. For this reason, the computed torque scheme has been used mainly for the joint trajectory tracking problem. This approach permits to successfully track the desired angular trajectories, by adding a simple joint position and velocity tracking error dynamics to the linearizing control torque. However, damping of the tip oscillations is not directly accounted for. To overcome this problem, one solution consists of adding linear elastic velocity terms to the previous control law. This method is known as *active damping*.[10] This way, the end-effector

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oscillations are effectively damped, but the price to pay is the loss of the precise joint trajectory tracking. Also, the application of the open-loop form of the computed torque scheme, necessitates measurement or estimation of the elastic coordinates. Measurement of the position coordinates could be done by strain gauges and the velocities could be estimated using classical observers algorithms. However, the elastic measurements (or the non-collocated measurements in general), could lead to the spillover problem.[108] Furthermore, the estimation algorithms are time consuming. To avoid these problems, one solution is to compute the elastic coordinates "off-line", by integrating the elastic dynamics (Equation (5)) associated to the desired joint trajectories. The obtained elastic trajectories as well as the desired joint trajectories are then substituted into Equation (4), yielding "off-line" control torques. These torques are then added to joint/elastic coordinates error dynamics to construct the final control law, called *open-loop* computed torque. The computed torque has been also used to solve the end-effector tracking problem in reference [6], where the output regulation theory has been used to compute a bounded open-loop control torque (see reference [109] for an exposition of the output regulation theory for nonlinear systems in a general setting). This open-loop control has been completed with linear joint/elastic error dynamics feedback. Good end-effector trajectory tracking has been obtained on a simulation model of a one-link flexible arm. However, the computed controls present discontinuities at the initial motion time, which is a well known inconvenient of the output regulation approach.[109]

(g) Adaptive control:[16–19] In reference [17], experiments are conducted on a two-link flexible arm. The control scheme used is an LQR regulator, computed on a linearization of the robot nonlinear lagrangian dynamic. The regulator is based on joint coordinates, elastic coordinates and error dynamics vector states feedback, including parametric model uncertainties and unstructured uncertainties. The authors propose an algorithm to estimate this error vector during the robot motion. This is done via a "Strong Tracking Filter-STF", which is fed in with the control torque and the end-effector position. The scheme has been applied to solve the regulation problem as well as the joint trajectories tracking. The experiments show the robustness of this scheme with respect to the end-effector load mass variations, however, tip residual oscillations are still present at the end of the desired motion.

(h) Neural network based control:[20] A neural network algorithm is added to classical control laws, to enhance the controller robustness with respect to model uncertainties. In [20], a three stages controller has been proposed. The first term is based on a nonlinear decoupling law, the second is due to an optimal control approach and the final term is based on a neural network algorithm. Simulation tests, have been conducted on a five-link flexible robot, within a modal decomposition modeling approach. This controller has shown robust performances with respect to model uncertainties, when treating the end-effector regulation problem.

(i) Lead-Lag control:[21,22] In reference [21], the authors treat the problem of the end-effector trajectory tracking for a one-link flexible arm. The dynamical model takes into account the joint friction phenomenon. The proposed controller is based on two control loops. The first one, corresponds to a lead-lag circuit applied to the linear motor identified transfer function. The second loop uses a PID regulator added to an open-loop term, computed through the minimization of the end-effector angular position integral square error. This controller has been tested on a one-link testbed. It appears that the obtained results are close to the numerical ones obtained via the computed torque scheme in reference [5], i.e. satisfactory end-effector trajectory tracking but with important residual tip-oscillations. Thus, the complexity of the two loops based regulator, which necessitates the tuning of many parameters, is not really justified. However, the paper remains interesting, due to the numerous implementation practical details presented therein. In reference [22], two regulators are proposed for the end-effector trajectory tracking of a one-link flexible robot. The arm is modelled with an identified discrete transfer function. This non-minimum phase transfer B, relates the end-effector linear position to the joint torque. The control strategy is based on a feedforward plus a feedback terms. The authors use a classical linear *full state; i.e. joint and elastic measurements* static feedback. The paper novelty concerns the feedforward terms. Two feedforward laws are proposed, based on a stable approximation, via lead-lag transfers, of the non-minimum phase transfer inverse. The first approach called "Extended bandwidth zero phase error tracking control–EBZPETC" goes as follows: the non-minimum phase transfer is decomposed as $B(Z^{-1}) = B^u \,.\, B^a$, where B^u, B^a are, respectively, the unstable and the stable parts of B. The unstable inverse $1/B^u(Z^{-1})$ is then approximated with the stable transfer $B^u(Z)/[B^u(1)]^2 \cdot [\sum_{i=0}^{n-1} (1-G(z))^i]$, $G(Z) = B^u(Z^{-1})B^u(Z)/[B^u(1)]^2$ where $[\sum_{i=0}^{i=n-1} (1-G(z))^i] \rightarrow 1/G(Z)$ for $n \rightarrow +\infty$. The second stable inverse approximation H is obtained by minimizing the term:

$$\sum_{w \in [0\ 2\pi]} W(\omega)^2 \left| e^{-j\omega\tau} B^{-1}(\omega) - H(e^{-j\omega}) \right|^2$$

with respect to a_i, b_i variables, such that:

$$H(Z^{-1}) = \frac{\sum_{i=0}^{m-1} b_i \,.\, Z^{-i}}{\sum_{i=0}^{n-1} a_i \,.\, Z^{-i}}$$

Where $W(\omega)$ is a positive loading function, and τ is a pure delay, insuring the stability of the transfer H. The two controllers are tested on a one-link flexible arm. When applied to a smooth ninth degree polynomial trajectory, the first law insures the end-effector tracking with 1 mm maximal tracking error, the second yields 18 mm tracking error. Furthermore, the residual tip oscillations are small. The results are then satisfactory, although many parameters (e.g. the loading function W, the lead-lad circuits degrees n, m) have to be tuned "off line".

(j) Output redefinition:[3,23–31] As underlined before, the end-effector trajectory tracking problem, is critical due to

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the non-minimum phase nature of the system dynamic when considering this non-collocated output. The idea of "output redefinition" schemes is to search for a "new" non-collocated output, such that the associated dynamics remains minimum phase. In reference [25], the authors prove that for a one-link flexible robot of length L, the "new" output $y_t = \theta L - \omega(L, t)$, where θ is the joint angle and $\omega(L, t)$ the tip elastic deflection, yields a minimum phase motor torque to y_t transfer, if *the hub motor inertia is important*. Furthermore, under those conditions, the obtained transfer is passive, permitting to stabilize the output, with any passive regulator, e.g. a simple collocated PD regulator.[26] This idea is extended to the multi-link case in reference [28], where a multi-link robot (including flexible and rigid links) is modelled with a lagrangian scheme associated to a modal elastic decomposition. The non-minimum phase output is the tip linear* velocities vector:

$$\dot{\rho}_t = J_\theta(\theta, q_e)\dot{\theta} + J_e(\theta, q_e)\dot{q}_e$$

where θ, $\dot{\theta}$ represent the joint positions vector and the joint velocities vector respectively, q_e denotes the elastic modal coordinates and J_θ, J_e are the rigid and the elastic jacobian matrices. The proposed *redefined* output writes:

$$\dot{\rho}_{t\mu} = \mu\dot{\rho}_t + (1-\mu)J_\theta(\theta, q_e)\dot{\theta}, \ \mu \in \Re$$

The initial non-minimum phase output is recovered for $\mu = 1$. Firstly, the model is linearized at the equilibrium point $(\theta, q_e) = (\theta_d, 0)$. The authors prove that the transfer matrix between $\dot{\rho}_{t\mu}$ and the input $\tilde{u} = J_\theta^{-T} u$, where u denotes the joint torques vector, is positive-real (PR) matrix, provided that:

(i) $\mu \leq 1$,
(ii) the robot has no redundancy, i.e. $\theta \in \Re^6$ (reference [28, p. 275]) and
(iii) the arm load mass is much more important than the robot intrinsic mass.

Under these three conditions, and applying the passivity theorem, the new end-effector position output can be regulated using any strictly passive regulator on a feedback loop in reference [28, p. 277]. Then, this result is extended to the nonlinear case, using a more general notion that the PR transfer; i.e. the notion of nonlinear passive systems. The author prove that the dynamic relating the couple (input $\tilde{u}$, output $\dot{\rho}_{t\mu}$) is passive under the above three hypotheses, yielding the same output stability result. Some numerical tests are conducted on a "Space shuttle manipulator" model: point to point desired motions are fixed for the end-effector, and the associated elastic trajectories are computed via two different models. The first, more accurate one, includes the "dynamics stiffening" phenomenon.† The second is the simplified model, used for the conducted theoretical analysis. This tests permits to validate the *numerous model approximations*. However, in this paper, no simulation is done to validate the control law objectives. For simulations tests on a six-link "Space Shuttle Remote Manipulator – SSRM", we refer the reader to reference [29]. The load mass considered is about 15,000 kg, the links masses vary from 8 kg to 138 kg (see reference [30], for a detailed SSRM robot description). The μ factor is fixed to 0.99 in reference [28, p. 226]. The desired operational trajectories, present a maximal end-effector displacement amplitude varying between 5.30 cm and 16.2 cm. The authors compare the tracking performances of two algorithms. The first is the classical law presented above, the second is an adaptive version of this law (with respect to the load mass changes). The classical controller leads to a maximal tracking error of about 9 cm and the adaptive form yields 2 cm maximal tracking error. Experimental validations, are presented in reference [27], where this approach is implemented on a three-link arm, with two flexible forearm links. The conditions quoted above, are of course verified. Since the load mass/inertia: 8, 66 kg, 480.36 g.m^2 are higher then the links masse/inertia: first link 0.19 kg, 9.79 g.m^2, the second link 2.10 kg, 14.93 g.m^2 and the third link 0.16 kg, 5.69 g.m^2. Also the maximal links length is 39.2 cm. For these experiments the μ factor is fixed to 0.92. The tracking results for operational five degree polynomial trajectories are satisfactory.

(k) Singular perturbations:[32] The main idea (see also reference [1]) is the separation between two dynamics, the first one called "the slow dynamics" characterize the "rigid joint motion". The second one called "fast dynamics" are associated to the "elastic links" motion. The control law uses two terms:

(i) The first term, based on the "computed torque" method, is applied to the "slow dynamics", and permits to treat the joint trajectory tracking problem.
(ii) The second term, based on elastic state feedback, stabilizes the "fast" elastic dynamics.

This law applied in reference [32] to a one-link flexible arm, yields good results for the regulation as well as the joint trajectory tracking objectives.

(l) Sliding mode control:[33] This scheme is based on an augmented state feedback. The additional state variables, called "surface variables", writes as a function of the rigid/elastic states tracking errors. The tuning of the state feedback gains, insures asymptotic stability for the error dynamics equilibrium point. In [33], this scheme has been compared to a classical LQ regulator on a one-link testbed. Good results have been obtained for the end-effector regulation problem, even when considering load mass variations.

(m) Stable inversion in the frequency domain:[34–39,79,101,110,111] In reference [34], a finite element approach is used to drive a linear transfer G, relating the joint torque input U and the linear end-effector acceleration $\ddot{V}_t$:

$$\ddot{V}_t = G \,.\, U$$

for a desired output trajectory υ_{td}. The Fourier transform of the associated open-loop control law is obtained, via the transfer inversion:

$$U_d(f) = \ddot{V}_{td} \,.\, G^{-1}$$

* By opposition to angular.
† See reference [84] for more details about these notions.

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then, a bounded *non-causal*, closed-loop control is computed through an inverse Fourier transform (using a "Fast Fourier Transform – FFT" algorithm). The relationship between the non causality and the bounded nature of the obtained open-loop control, taking into account the non-minimum phase nature of the direct transfer G is presented in reference [40], where this stable inversion scheme has been reformulated in the time domain. The bounded open-loop control torque has been computed using:

$$u(t)=\int_{t_0}^{t_f} g(t-\tau)\upsilon_t^{(2)}(\tau)d\tau \qquad (10)$$

where $[t_0, t_f]$ represents the tip acceleration bounded time support and $g(t)$ is the *non-causal system impulse response*:

$$TF(g(t))=\int_{-\infty}^{+\infty} g(t) \, . \, e^{(-2\pi ft)}dt=G(f) \qquad (11)$$

This shows that the non-causal stable inversion, obtained initially in reference [34], is due to the use of non-causal Fourier transforms, which has been clearly proved in reference [41] (this paper is analyzed hereafter, in the "time domain stable inversion" section). In reference [38], the previous frequency stable inversion approach, has been generalized to the nonlinear model of a one-link flexible arm, and successfully applied to an experimental testbed. In reference [35], the scheme is extended to the planar multi-link case, using an iterative algorithm and simulation tests conducted on a two-link flexible robot yield good tracking results. An experimental validation, has been presented in reference [79], where the authors study a possible extension of this method to manipulators with closed-loop configurations. A theoretical analysis is made in reference [111], where the existence of solutions for the iterative algorithm proposed in reference [35], is proved under the following conditions:

(i) Hyperbolicity of the equilibrium point associated to the inversed dynamics* (p. 79).
(ii) The desired tips motions present desired velocities and accelerations with bounded time supports (p. 77).
(iii) Small joint "velocities/accelerations" vectors norms† (p. 88). In all these works, the bounded open-loop control has been completed with simple collocated joint positions and velocities feedback, to achieve good end-effector tracking, under the model conditions quoted above. Stability analysis for tracking error dynamics equilibrium point is also effected in reference [111], based on Lyapunov theory and valid only for "small elastic displacements".

(n) Stable inversion in the time domain:[40–44] In reference [41], the end-effector trajectory tracking for a one-link flexible robot is addressed. A linear lagrangian model is considered:

$$M\ddot{q}+D_e\dot{q}+Kq=B \, . \, u$$

where $q=(q_r, q_e)^T$, q_r is the end-effector position (non-collocated output), q_e the elastic position coordinates vector and u the control torque. This model is then rewritten under the *inverse* state space form:

$$\begin{cases} \dot{X}=A \, . \, X+B \, . \, \upsilon \\ u=C \, . \, X+D \, . \, \upsilon \end{cases}$$

with $X=(q_e, \dot{q}_e)^T$ et $\upsilon=(\dot{q}_r, \ddot{q}_r)^T$ and where A, B, C, D writes as a function of M, D_e, K. Hence, under the previous system *equilibrium point hyperbolicity* hypothesis, it can be rewritten as:

$$\begin{pmatrix} \dot{\tilde{x}}_c \\ \dot{\tilde{x}}_{nc} \end{pmatrix}=\begin{pmatrix} A_c & 0 \\ 0 & A_{nc} \end{pmatrix} \cdot \begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{nc} \end{pmatrix}+\begin{pmatrix} B_c \\ B_{nc} \end{pmatrix} \cdot \upsilon \qquad (12)$$

$$\begin{pmatrix} u_c \\ u_{nc} \end{pmatrix}=\begin{pmatrix} C_c \\ C_{nc} \end{pmatrix} \cdot \begin{pmatrix} \tilde{x}_c \\ \tilde{x}_{nc} \end{pmatrix}+\frac{1}{2} \cdot \begin{pmatrix} D \\ D \end{pmatrix} \cdot \upsilon \qquad (13)$$

with $X=T \, . \, \tilde{X}$, $\tilde{X}=(\tilde{x}_c, \tilde{x}_{nc})^T$, T is the transformation matrix (i.e. constructed with the generalized eigenvectors of A) and indices stand: "c" for "causal" and "nc" for "noncausal". This bloc-diagonalization leads to a decomposition of the whole dynamics into two subsystems; the causal one, associated to the stable system eigenvalues and the *non-causal* subsystem, associated to the unstable eigenvalues. Then, to obtain a bounded control torque associated to a desired end-effector trajectory, the causal subsystem, should be integrated forward in time from the initial desired motion time. Instead, the non-causal part, have to be integrated backward in time, from the final desired time motion. This yields a bounded non-causal open-loop control torque. Also, it is underlined in p. 198, that, in order to obtain a bounded time support, the backward integration should be truncated, and the authors propose to stop the backward integration when the control is being under a threshold fixed to 0.5% of the maximal torque value. In this paper, the authors prove the equivalence between their scheme and the one initially introduced in references [34, 40]. This non-causal open-loop law is completed with a simple collocated joint PD regulator, and leads to good numerical and experimental results. A novel causal stable inversion approach has been introduced in [43]. It is applicable to *hyperbolic and non-hyperbolic systems* and proceeds as follows: the linear tip position output versus joint torque input transfer is rewritten under a linear differential equation of the form:

$$b_m u^{(m)}(t)+\ldots+b_0 u(t)=a_n y_{tip}^{(n)}+\ldots+a_0 y_{tip}(t) \qquad (14)$$

where a_i, $b_j \in \Re$, $i=1 \ldots n$, $j=1 \ldots m$. Due to the linear nature of equation (14), its solution is composed of two terms (the transient and steady-state solutions of the equation). Since the system has a non-minimum phase characteristics, the transient solution of (14), contains divergent terms (corresponding to the effect of unstable zeros of the system). Hence, the output trajectory is planned in order that it cancels these undesirable terms, which can be done using a polynomial form for the output trajectory,

* The linearized system around the equilibrium point, has no purely imaginary poles.
† The Euclidian vector norm is used in this paper.

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depending on the number of the output initial and final constraints as well as the number of the unstable zeros associated to (14). It is worth noting that this is done by solving a simple linear system in the coefficients a_i, leading to a closed-form expression for the planned output y_d, as well as to a causal open-loop control trajectory that will force the system output to track exactly the planned trajectory. Eventually, this causal open-loop torque is added to a joint collocated feedback. This control has led to good numerical results.[43] Good experimental end-effector trajectory tracking results, has been obtained as well, and are detailed in reference [112], where the scheme is presented in a more general setting and is compared to other non-causal stable inversion techniques.

The previous methods concern linear models of a one-link robot. We focus now on nonlinear stable inversion time domain methods. In reference [42], the authors treat the end-effector trajectory tracking for a planar two-link flexible robot. The scheme applied has been introduced in reference [113]. The robot is modelled using a lagrangian formulation, leading a nonlinear model of the form (4), (5), with $\Theta \in \Re^2$, $Q \in \Re^4$. The controlled operational output is the end-effector non-collocated position (7), for $n=2$. The associated internal dynamics (9) are linearized along the elastic coordinates trajectories, and are rewritten under the time varying linear form:

$$\dot{\eta} = A(t)\,.\,\eta + B(t),\; A(t) = \begin{bmatrix} 0 & I \\ -H_1^{-1}H_3 & -H_1^{-1}H_2 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ H_1^{-1}H_4 \end{bmatrix} \tag{15}$$

where $\eta = (q^T, \dot{q}^T)^T$ and $A(t)$, $B(t)$ write as functions of the model matrices M_{ee}, M_{re}, h_e, K_{ee}, D_{ee} and their derivatives along the present elastic trajectories. Following reference [113], under the equilibrium point *hyperbolicity* hypothesis, a bounded solution of this linear time varying internal dynamics can be obtained if equation (15) is associated to boundary conditions satisfying:

$$\eta(t_0) \in E^u,\; \eta(t_f) \in E^s \tag{16}$$

where E^s, E^u are respectively the stable and the unstable vector subspace* of system (15). The linear system (15), associated to (16) is rewritten under two decoupled time varying subsystems, where the first one is associated to an initial condition and the other is associated to a final condition. These subsystems are integrated iteratively, at each iteration the initial system (15) is evaluated along the new elastic trajectories solutions. The iterations are stopped when the difference between two successive solutions are under a fixed threshold. The final bounded elastic solutions are then used together with equations (4) and (5) to compute a bounded openloop nominal torque. The convergence of this scheme is insured under two conditions:

* See reference [114] for more details concerning these concepts.

(i) The elastic damping terms must be taken into account in the dynamical modeling, to insure a hyperbolic equilibrium point for the systems (15) (p. 318).

(ii) The desired end-effector motion trajectories are *very smooth*, i.e. they are constructed such that, they present an initial *status quo* interval and a small motion amplitude (such that the linearized model remains valid). The bounded nominal torque added to a collocated joint PD regulator yields good simulation tip tracking performances. In reference [44], a simple idea has been introduced to solve the end-effector trajectory tracking for multilink planar flexible manipulators. Desired trajectories are fixed for the tips angular motions vector Y (6), which leads, for small elastic displacements, to the internal dynamics (9). As said above, in this case, the integration of this internal dynamics (9) from initial conditions, yields radially unbounded solutions. To avoid this finite time escape phenomenon, the authors propose to reformulate this integration problem, as the two point boundary value problem:

$$\begin{aligned} & M_{re}^T \ddot{Y}_d + (M_{ee} - M_{re}^T \Gamma)\ddot{Q} + h_e(Y_d, Q, \dot{Y}_d, \dot{Q}) + K_{ee}Q = 0 \\ & Q(t_0) = 0 \\ & Q(t_f) = 0 \end{aligned} \tag{17}$$

Then, the associated solution remains bounded over all the integration interval; i.e. the time motion interval $[t_0, t_f]$. This method works equally for hyperbolic as well as non-hyperbolic dynamics.[103] However, it can lead to small initial/final control discontinuities, that increase with the desired motion speed; i.e. when $t_f - t_0$ decreases. We refer the reader to reference [115] for a more theoretical analysis of this approach.

(o) Algebraic control:[45,46] In reference [45] the Euler-Bernoulli EDP of a one-link flexible robot is considered. Using the so called "Mikusiński" operators (i.e. one generalization of Laplace calculus), all the PDE variables, once the PDE transformed into an operational ODE, are written as a function of a given base of the ring of operators $\mathbb{R}[d/dt]$ (this is connected to the so called "flatness theory" in the special case of ODE's dynamics). Using this new system of equations, it is straightforward to plan a joint trajectory, connecting two desired rest robot positions, in the same manner the control torque is directly obtained through this parametric model form. The main practical diffficulty is when inverting the obtained control law to write it in the time domain. Good end-effector regulation results has been obtained on a one-link testbed. Another parametric form has been used initially in reference [46]. This work, considers a linear lagrangian model of a one-link arm, with two elastic modes:

$$\mathcal{D}_1 \begin{cases} M_{11}\ddot{\theta} + M_{12}\ddot{q}_1 + M_{13}\ddot{q}_2 - u \\ M_{21}\ddot{\theta} + M_{22}\ddot{q}_1 + K_1 q_1 \\ M_{31}\ddot{\theta} + M_{33}\ddot{q}_2 + K_2 q_2 \end{cases} \tag{18}$$

where θ, q_1, q_2, u are respectively, the joint angle, the elastic modes and the joint torque. The theory of parametrization of linear differential operators is used to transform this model

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into a parametric form (which reaches the well known "controllable from"); where all the system variables are written as a function of a new parametric function and its time derivatives:

$$\mathcal{D}_0 \begin{cases} \theta(t)=A_1\lambda(t)+A_2\lambda^{(2)}(t)+A_6\lambda^{(4)}(t) \\ q_1(t)=A_3\lambda^{(2)}(t)+A_7\lambda^{(4)}(t) \\ q_2(t)=A_4\lambda^{(2)}(t)+A_8\lambda^{(4)}(t) \\ u(t)=A_5\lambda^{(2)}(t)+A_9\lambda^{(4)}(t)+A_{10}\lambda^{(6)}(t) \end{cases} \quad (19)$$

where the A_i, $i=1\ldots10$, write as a function of the dynamical model constants. This form permits then to resolve easily the rest to rest motion in fixed time [46] as well as the end-effector trajectory tracking problems.[85] The obtained controls are written in *closed-form expressions*, also for the end-effector trajectory tracking problem, the control torque obtained is a *causal law and no hyperbolicity condition is needed*. Nice/experimental results have been obtained; i.e. rest to rest motion/end-effector trajectory tracking, without any residual tip oscillations (refer to reference [115] for more calculus details on general models of n elastic modes, as well as experimental results).

(p) Poles placement:[47–50] The poles placement technique has been used mainly to solve the end-effector regulation problem. In [48], the authors treat the regulation problem for a one-link robot. The joint frictions have been taken into account in the proposed lagrangian model. A two stages controller is proposed. The first loop is based on a pole placement scheme, the second loop represents a simple integral action. This controller is validated on a one-link flexible testbed, and leads to good regulation results.

(q) Optimal trajectory planning:[51–55] In reference [54], the rest to rest motion in fixed time is studied for multi-link flexible arms. The authors propose a planning of the joint trajectories, in such a way as to minimize the elastic tip deflections at the end of the desired point to point motion. These joint trajectories are planned, in order to minimize the energetic cost:

$$\frac{1}{2}\,\dot{q}_f^T \,.\, M_{ee} \,.\, \dot{q}_f + \frac{1}{2}\, q_f^T \,.\, K \,.\, q_f$$

where q_f, $\dot{q}_f$ denote the elastic coordinates at the desired final time motion t_f. M_{ee}, K are respectively the elastic mass matrix and the arm stiffness matrix (see equations (4), (5)). Two algorithms are proposed to compute the optimal trajectories. Next, a simple open-loop computed torque law is calculated, along those optimal trajectories (i.e. using equation (4)). The presented simulation results on a two-link flexible arm show that the point to point motion is achieved in the desired fixed time, with damped residual tip oscillations. See also reference [53], for a simple joint trajectory planning for a linear one-link dynamical model, associated to a high gain joint PD regulator, to solve the rest to rest end-effector motion in a desired fixed time. In reference [51], a novel approach has been used to solve the joint trajectory tracking for planar multi-link flexible robots. The method is mainly based on feedforward torque computations. For a desired joint vector trajectory, the associated elastic trajectories $Q_d(t)$ are obtained through backward integration of the internal elastic dynamics (5), starting this integration from the desired rest elastic position:

$$Q_d(t_f)=0_{n_e\times 1},\ \dot{Q}_d(t_f)=0_{n_e\times 1}$$

For slow joint motions, the obtained elastic positions/velocities trajectories present a small initial error at the desired initial time motion to. However, this error is easily compensated by adding to the obtained feedforward (substituting (Θ_d, Q_d into (4)) a simple joint error tracking PD feedback. The whole controller drive the system dynamics trajectories along those desired joint/elastic trajectories, reaching at the final instant the desired rest elastic position. Nevertheless, for fast desired joint motions, the backward integration of the elastic dynamics, yields more important initial elastic positions/velocities values. The simple collocated joint PD is no longer able to compensate the associate initial error. To overcome this limitation, without changing the simple collocated feedback, the authors propose to plan non-causal joint trajectories, in such a way as to pre-load the robot elastic coordinates near the initial values obtained through the backward integration. This way, the first non-causal control part drives the system dynamics along the planned joint trajectories (these trajectories start from the system desired initial position and reach this same position) and the associated elastic non-causal trajectories, to decrease the elastic error at the instant t_0, permitting then to the second causal control part (feedforward plus a simple joint error dynamics PD regulator) to drive the system dynamics along the desired joint/elastic causal trajectories, reaching the final rest position. This scheme has led to good experimental results on a two-link flexible arm. The same idea has been used to solve the rest to rest motion in a desired fixed time for planar multi-link robots.[55] For this problem, no desired joint trajectories are specified between the two rest end-effector positions. Thus, the authors shape this causal joint trajectories in such a way as to minimize the initial elastic error, due to the backward integration and no non-causal action is longer needed.

(r) Optimal and robust control:[13,56–61] In reference [61], the joint trajectory tracking objective is treated on a three-links manipulator with a flexible forearm. The-authors use an input/ouput linearizing law added to an LQR regulator. Good simulation results are obtained. However, the LQR regulator is based on a full state feedback, which is to be avoided in a practical setting for flexible structure due partly to the well known spillover problem.[108] In reference [13], the LQG scheme is compared to simple collocated/non-collocated PD regulator, to treat the end-effector regulation problem for a one-link arm. We quote also,[59] where the H_∞ method is used to insure robust (with respect to the arm elastic damping coefficients) tip regulation performances, for a one-link robot. See also references [58] and [60] for other results concerning the H_∞ regulator.

(s) Mechanical wave approach:[73,74] In reference [73], a lumped-parameter mass-spring model is considered. A novel regulation approach based on wave propagation and absorbtion techniques is presented. The main idea is to

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inject into the system the "adequate" waves to move the load to the desired point and to be able to absorb all the system reflected waves, avoiding in this way any residual elastic oscillations.

(t) Input shaping:[52,62–68]* In reference [63], a method based on the input shaping control is presented on spring-mass oscillating systems. The method called *ramped sinusoids forcing function* amounts to a decomposition of the input signal on a set of sine, cosine and ramp functions. The coordinates of the decomposition are then computed via minimization of the controller energy during the motion and the elastic acceleration at the end of the desired motion. In reference [52], the oscillating system of springmass is also considered to present an optimal shaped trajectory approach. The desired trajectories are shaped in order to minimize elastic velocity, acceleration and jerk. In reference [65], it is shown that an open-loop strategy based on shaped-input filters is a simple method to reduce the end-effector vibrations and ensure good performance even when the actual system frequencies are imprecisely known. In reference [64], the authors treat the regulation problem, for the class of multi-link manipulator with one flexible link, using dominant elastic vibration frequencies filters added to simple rigid controllers. For the same manipulator class, see also references [66] and [67] for a vibration suppression scheme, based onn a closed-loop input shaping technique. Experimental results on a two-link flexible arm testbed are reported in reference [68], where the rest to rest problem is treated. The desired joint point to point motions are firstly shaped via linear filters (computed using the input shaping techniques), the obtained "shaped" joint motions are then used to compute a feedforward control torque (obtained via a classical "rigid" computed torque technique), finally, this torque law is added to a feedback joint friction compensator (based on a sliding mode controller). This "three stage" controller leads to satisfactory results for the rest to rest motion problem.

(u) Boundary control:[69] In reference [69], a one-link flexible arm is controlled through a two acting point controller. The first controller is a joint collocated PD, the second controller part acts directly at the tip level (force controller). The theoretical analysis of an Euler-Bernoulli PDE model proves the asymptotic regulation property for the end-effector.

(v) Exact linearization via dynamical state feedback:[70,71] Reference [70] addresses the rest to rest motion in fixed time for a linear model of a one-flexible link. The idea is based on the design of an auxiliary output such that the associated transfer function has no zeros. This method yields the same parametric form as the one obtained in reference [46], through a different approach. This scheme has been extended in reference [71] to the particular nonlinear case of a two-link robot with a flexible forearm, when considering one elastic vibration mode to model the arm flexibility. The authors searched for a set of two outputs with respect to which the system has no zero dynamics, leading then to a closed-form solution for the rest to rest motion problem.

(w) Control synthesis for a family of flexible manipulators:[60] In reference [60] an H_∞ regulator is computed for a "non-dimensioned" model, representing a whole arm family. The tuned regulator is then available for the whole arm family. This approach, permits to "safely" design/test controllers on small size robots before a real implementation into a real arm, belonging to the same arm family of the prototype robot.

(x) Manipulators coupling:[72] In reference [72] a small flexible manipulator called "mini-manipulator" is superposed, at the tip level, to a flexible arm called "macro-manipulator". The idea is to be able to "easily" control the mini-manipulator, due to its small dimensions, using this way the precision and the speed of this small arm, and taking in the same time, advantage from the operational accessible space offered by the well dimensioned macro-manipulator. However, the authors show that some coupling problems appear when using PD regulators, decreasing the end-effector positioning precision.

(y) Control-structure design methodology/optimal arms design:[75,76] In reference [76] is presented an experimental validation of an optimization-based integrated controls-structures design methodology. The idea is to design *at the same time* an optimal controller (e.g. LQG and dissipative regulators), together with an optimal system (tuning the system mechanical parameters). Other works, search for the optimal flexible structure (e.g. presenting the highest first modal frequency), to facilitate the control design parts.[75]

V. CONCLUSION AND FURTHER READING

This paper has presented a survey study of control methods for flexible-link manipulators. The task is not easy, due to the numerous good results obtained in this research field. Nevertheless, we have tried to present the essential ideas of the research field. About 119 works have been indexed, specifying in each case, the nature of controlled robots. Important differences (concerning the control methods) may appear when dealing with the one-link case, which can be precisely modelled with linear dynamics and the multi-link case, which needs more precise nonlinear modeling. As a result of this bibliographical study, we can conclude that, in our opinion, the control objectives have been mostly reached for the planar case, when considering small elastic displacements. The more general results dealing with the multi-link case, have been obtained using the feedforward based techniques, e.g. open loop input shaping for the regulation problem and the feedforward computed torques for the tracking problems. This is partly due to the complexity of the nonlinear multi-link models, since it is difficult even if not impossible to apply directly some "theoretical" closed-loop control strategies, which need closed-form manipulations of these complex system dynamics. Instead, feedforward laws could be computed and generalized easily for the multi-link general case. However,

* Refer also to the web page (http://www.me.gatech.edu/input-shaping/) for other publications concerning this method.

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a principal shortcoming of these feedforward based controllers concerns the robustness aspect. For this reason, many works have been dedicated to the extension of the available feedforward schemes to more robust laws, either by adding to these controls robust feedback regulators or by proposing adaptive forms of the feedforward algorithms. We have also noted that many of the quoted works deal with planar motion or 3D small motions. In fact, in the case of large 3D motions, controllability problems could appear,[116,117] increasing the control design difficulty. Furthermore, to our knowledge, all the presently available works treat the case of small elastic deflections. In the presence of large elastic displacements, the available models are much more complex,[84] and the control methods should take into account this new difficulty. Moreover, it appears to us that the case of manipulators with flexible "closed loop" arm configurations, has not been deeply studied,[118] and should be more investigated. Eventually, we refer to reference [119], for a survey study of controlling mechanical systems with backlash phenomenon, which can hold for robots with flexible links.

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