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Parametrizations of sub-attractors in hyperbolic balance laws

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We investigate the properties of the global attractor of hyperbolic balance laws on the circle, given by

$$u_t + f(u)_x = g(u).$$

The new tool of sub-attractors is introduced. They contain all solutions on the global attractor up to a given number of zeros. The paper proves finite dimensionality of all sub-attractors, provides a full parametrization of all sub-attractors and derives a system of ordinary differential equations for the embedding parameters that describe the full partial differential equation dynamics on the sub-attractor.

1. Introduction

Existence of global attractors has been proven for many partial differential equations. However, in most cases little is known about their exceeding existence and bounds on the dimension of the attractor. Exceptions to this rule are hyperbolic balance laws with dissipative source term:

$$u_t(x,t) + [f(u(x,t))]_x = g(u(x,t)). \tag{H}$$

Despite the fact that the global attractor of (H) is infinite dimensional, a lot is known about the structure of the attractor and the connecting properties of rotating waves.

We consider (H) for $x \in S^1$ with $S^1 := \mathbb{R}/(2\pi\mathbb{Z})$, which is equivalent to imposing periodic boundary conditions on a domain of length 2π. By a scaling argument, all results remain true for the situation of periodic boundary conditions in a domain of size L for any bounded and fixed $L \in \mathbb{R}$. u is a function mapping $S^1 \times \mathbb{R} \to \mathbb{R}$. The nonlinearities f, g map $\mathbb{R} \to \mathbb{R}$. Furthermore, we require the following hypotheses:

(H1) f is C^2 and strictly convex (there exists $\gamma \in \mathbb{R}$ such that $f'' > \gamma > 0$);

(H2) g is C^1 and dissipative, i.e. there exists a constant $M > 0$ such that

$$ug(u) < M \tag{1.1}$$

for all $|u| > M$;

(H3) g has finitely many zeros at $u_1 < u_2 < \cdots < u_n$; all zeros are simple; (H2) implies that n is odd.

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Hypotheses (H1)–(H3) guarantee the existence of a global attractor (see the next section). One of the remaining questions regarding this attractor is its dynamic description. This paper closes this gap for all solutions on the attractor with arbitrary but finite zero set. For our description we introduce sub-attractors for hyperbolic balance laws which will turn out to be of finite dimension. This approach allows us to overcome several difficulties arising from the infinite dimensionality of the full global attractor and from solutions with an infinite (countable or uncountable) zero set.

In addition, the sub-attractors show some striking similarities to the analogously defined sub-attractors of the parabolically regularized version of (H),

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u),$$

for small viscosity ε. This relation is explored thoroughly in [2].

The paper is organized as follows. The next section reviews what is known about global attractors and the so-called connection problem. It provides the necessary background on hyperbolic balance laws for this paper. In § 3 the notion of sub-attractors is introduced. In § 4 we then formulate and prove the main result of the paper: a parametrization of all sub-attractors, their finite dimensionality and their dynamics. A section with examples follows. The paper concludes in § 6 with a brief discussion of the results.

2. Global attractors and the connection problem

We shall present the tools and methods used in the proofs of the paper and then discuss key results concerning global attractors of scalar hyperbolic balance laws.

The initial-value problem (Cauchy problem) of (H) can be solved by the method of characteristics. The classical solution $u(x,t)$ to an initial condition $u(x,0) =: u_0(x)$ is given by

$$u(\chi(t), t) := \underline{v}(t),$$

where $\underline{v}$, χ are curves that solve the following ordinary differential equation:

$$\left.\begin{aligned} \chi'(t) &= f'(\underline{v}), \\ \underline{v}'(t) &= g(\underline{v}), \\ \chi(0) &= x_0, \\ \underline{v}(0) &= u_0(x_0), \end{aligned}\right\} \tag{2.1}$$

for all $x_0 \in S^1$. Classical solutions in general only exist for finite time. To overcome this difficulty we work with weak solutions of (H).

In the weak framework, solutions are in general not unique. To overcome this obstacle an additional entropy condition can be imposed that singles out a unique weak solution. This idea derives from the physical entropy in thermodynamics. Entropy conditions for hyperbolic balance laws were first considered by Volpert [12] and Kruzhkov [8].

We follow their approach and define an *entropy* or *admissible solution* of the hyperbolic balance law (H) in the following way.

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Definition 2.1. We call $u \in \mathrm{BV}([0,\infty) \times S^1, \mathbb{R})$ an entropy or admissible solution of (H) to the initial condition $u_0(x)$ if the following hold:

(i) $u(x,0) = u_0(x)$;

(ii) it solves (H) in the weak sense,

$$\int_{S^1 \times \mathbb{R}^+} [u\varphi_t + f(u)\varphi_x - g(u)\varphi] \,\mathrm{d}x\mathrm{d}t = 0 \tag{2.2}$$

for all $\varphi \in C_0^1(S^1 \times \mathbb{R}^+, \mathbb{R})$;

(iii) the entropy condition

$$u(x+,t) \leqslant u(x-,t) \tag{2.3}$$

holds for all $t > 0$.

Here $u(x+,t)$ defines the right-hand limit and $u(x-,t)$ the left-hand limit of u in x at time t and $\mathrm{BV}([0,\infty) \times S^1, \mathbb{R})$ denotes the space of functions with bounded variation mapping from $[0,\infty) \times S^1$ to $\mathbb{R}$.

Volpert [12] and later, and for more general initial conditions (L^∞), Kruzhkov [8] were able to prove the following result on the existence of solutions.

Proposition 2.2. *If (H1) holds, the Cauchy problem of (H) possesses a unique entropy solution u with the property $u\colon (0,\infty) \to L^1$ is continuous in time and $u(\cdot,t) \in \mathrm{BV}(S^1)$ for all times $t > 0$.*

Equation (H), together with (2.3), therefore defines a semiflow on $\mathrm{BV}(S^1, \mathbb{R})$. We denote that semiflow by

$$\begin{aligned} \Phi\colon\ \mathrm{BV} \times \mathbb{R}^+ &\to \mathrm{BV}, \\ u_0, t &\mapsto \Phi(u_0,t) := u(\cdot,t), \end{aligned}$$

where $u(\cdot,t)$ is the unique entropy solution to the initial condition u_0 at time t.

In order to work in the weak framework, Dafermos [1] generalized the concept of characteristics.

Definition 2.3. A Lipschitz curve $x = \chi(t)$ defined on the interval $[a,b] \subset \mathbb{R}$ is called a generalized characteristic associated with the solution u of (H) if it satisfies the inequality

$$\dot{\chi} \in [f'(u(\chi+,t)), f'(u(\chi-,t))] \tag{2.4}$$

for almost all $t \in [a,b]$.

Generalized characteristics coincide with classical characteristics $\chi(t)$ defined in (2.1), wherever the solution is differentiable. Filippov was able to show in [6] that there is at least one forward and one backward characteristic through any point $(x,t) \in S^1 \times \mathbb{R}^+$.

Equation 2.4 suggests that there is a lot of freedom in computing forward characteristics. That this is in fact not the case is shown by a proposition found in [6].

Proposition 2.4. *Let $\chi\colon [a,b] \to \mathbb{R}$ be a generalized characteristic. Then the following holds for almost all $t \in [a,b]$:*

$$\dot{\chi}(t) = \begin{cases} f'(u(\chi(t)\pm,t)) & \text{if } u(\chi(t)-,t) = u(\chi(t)+,t), \\[2ex] \dfrac{f(u(\chi(t)+,t)) - f(u(\chi(t)-,t))}{u(\chi(t)+,t) - u(\chi(t)-,t)} & \text{if } u(\chi(t)-,t) > u(\chi(t)+,t). \end{cases}$$

Hence, $\dot{\chi}(t)$ is uniquely defined even at the shock positions. If the solution $u(x,t)$ possesses a shock at position x_0, then the shock speed is given by the Rankine–Hugoniot condition for shock speeds:

$$c_{\text{shock}} = \frac{f(u(x_0+)) - f(u(x_0-))}{u(x_0+) - u(x_0-)}. \tag{2.5}$$

To distinguish between generalized characteristics and the characteristics of classical solutions, the notion of genuine characteristics is important.

Definition 2.5. A characteristic on the interval $[a,b]$ is called genuine if

$$u(\chi(t)-,t) = u(\chi(t)+,t) \quad \text{for almost all } t \in [a,b].$$

The set of backward characteristics through a point $(\bar{x},\bar{t})$ spans a funnel between the *minimal backward characteristic* $\chi^-(t;\bar{x},\bar{t})$ and the *maximal backward characteristic* $\chi^+(t;\bar{x},\bar{t})$.

The additional properties of characteristics that are of importance for us are summarized in the next propositions. For proofs we refer the reader to [1].

Proposition 2.6. *Let $(\bar{x},\bar{t}) \in S^1 \times \mathbb{R}$ be arbitrary. Then the minimal backward characteristic $\chi^-(t;\bar{x},\bar{t})$ and the maximal backward characteristic $\chi^+(t;\bar{x},\bar{t})$ are genuine.*

Proposition 2.7. *Genuine characteristics intersect only at their end points; backward characteristics do not intersect in particular.*

We direct our attention to the existence of global attractors for (H). Fan and Hale [5] have settled the existence question for hyperbolic balance laws, as follows.

Proposition 2.8. *Assume (H1)–(H3) hold. Then*

$$\mathcal{A} := \{u_0 \in \mathrm{BV}(S^1,\mathbb{R})\colon \Phi(u_0,t) \text{ exists for all } t \in \mathbb{R} \text{ and is bounded}\} \tag{2.6}$$

is the global attractor of (H) in $L^p(S^1)$, for any $p \in [1,\infty]$, i.e. it is invariant and attracts bounded sets in $L^p(S^1)$.

This settles the existence of $\mathcal{A}$. We turn to the structure of the global attractor. Several authors have proved Poincaré–Bendixson-type results for the scalar balance laws (see, for example, [4, 9, 11]).

Proposition 2.9. *For $t \to \infty$, any solution of (H) either tends to a homogeneous solution $u \equiv u_i$ for some $i \in \{1,\dots,n\}$ or it converges to a rotating wave solution*

$$u(x,t) = v(x - ct),$$

where the wave speed c can only take the values $c = f'(u_{2i})$ for $i \in \{1,\dots,\frac{1}{2}(n-1)\}$.

For global solutions a theorem similar to proposition 2.9 holds true in backward time. This leads to a description of the global attractor $\mathcal{A}$ as the unification of the homogeneous steady states, the frozen and rotating waves and heteroclinic connections between all these objects. A rotating wave is a solution of (H) of the form

$$u(x,t) = v(x - ct)$$

for a profile $v\colon S^1 \to \mathbb{R}$; c denotes the wave speed. If $c = 0$, the wave is called frozen. For the definition of heteroclinic connections we define by $\mathcal{E}$ the set of homogeneous equilibria of (H), $\mathcal{F}$ the set of frozen waves of (H) and $\mathcal{R}$ the set of rotating waves of (H).

A heteroclinic connection is a solution $u(x,t)$ of (H) that has the property that

$$\left.\begin{aligned} \lim_{t\to+\infty} u(x,t) \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}, \\ \lim_{t\to-\infty} u(x,t) \in \mathcal{E} \cup \mathcal{F} \cup \mathcal{R}. \end{aligned}\right\} \tag{2.7}$$

If we denote the set of heteroclinic connections by $\mathcal{H}$, then the global attractor $\mathcal{A}$ of (H) can be described as

$$\mathcal{A} = \mathcal{E} \cup \mathcal{F} \cup \mathcal{R} \cup \mathcal{H}. \tag{2.8}$$

Fan and Hale showed in [5, theorem 3.7] that if two rotating waves are connected by a heteroclinic orbit, then the waves must have the same velocity. Moreover, if a heteroclinic orbit connects a homogeneous equilibrium $u \equiv u_j$ and a rotating wave with speed $f'(u_{2i})$, then $|j - 2i| = 1$. It is a consequence of [5, proposition 1.5] that all global solutions $u(x,t)$ satisfy

$$u_{2i-1} \leqslant u(x,t) \leqslant u_{2i+1}$$

for some $i \in \{1, \dots, \frac{1}{2}(n-1)\}$. This implies that the homogeneous solutions $u \equiv u_{2i\pm1}$ divide the global attractor into separate pieces, connected only at the homogeneous solutions. Hence, we can treat all these pieces separately and restrict our analysis to the case where g possesses only three zeros. Without loss of generality we can rewrite assumption (H3) as

(H3′) g has three simple zeros at $u_- < u_0 < u_+$ and $u_0 = 0$.

In our case this implies

$$c = f'(0)$$

due to proposition 2.9. The hyperbolic balance law (H) is homogeneous in x and we can perform a coordinate transformation

$$x \mapsto x - f'(0)t,$$

which automatically freezes all rotating waves. Hence, we can assume without loss of generality that

(H4) $f'(0) = 0$.

This assumption fixes our coordinate system where all rotating waves have wave speed $c = 0$; hence, we have $\mathcal{R} = \emptyset$.

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In [10], Sinestrari proved that for any possible wave speed $c = f'(u_0)$ and for any closed set $Z \subset S^1$ there exists a unique rotating wave u_Z with the property

$$Z = \{y \in S^1 \colon u_Z(y) = 0\}.$$

The uniqueness automatically proves that these are all waves and hence $\mathcal{F}$ is fully described. For the connection question we introduce the map $\mathcal{Z}(\cdot)$ that assigns each function $u \colon S^1 \to \mathbb{R}$ its zero set:

$$\mathcal{Z}(u(\cdot)) := \{x \in S^1;\ u(x) = 0\}. \tag{2.9}$$

In addition, we define the zero-number

$$z(u) := \sharp \mathcal{Z}(u);$$

if $\mathcal{Z}(u)$ is uncountable we define $z(u) := \infty$.

Härterich [7] was able to prove the following three theorems which settle the connection question.

Theorem 2.10 (Härterich [7, theorem A]). *For any rotating wave $u_{-\infty}$ there exist heteroclinic orbits which connect $u_{-\infty}$ to the homogeneous states $u \equiv u_-$ and $u \equiv u_+$.*

Theorem 2.11 (Härterich [7, theorem B]). *For any rotating wave $u_{+\infty}$ there exist (several) heteroclinic orbits that connect the spatially homogeneous solution $u \equiv 0$ to $u_{+\infty}$.*

Theorem 2.12 (Härterich [7, theorem C]). *Suppose that for two rotating waves, $u_{-\infty}$ and $u_{+\infty}$, the condition $\mathcal{Z}(u_\infty) \subset \mathcal{Z}(u_{-\infty})$ holds. Then there is a heteroclinic solution that approaches $u_{\pm\infty}$ as the time t tends to $\pm\infty$.*

3. Sub-attractors $\mathcal{A}_n$ of order n

One of the main obstacles in the description of the global attractor of (H) is the huge number of stationary solutions due to Sinestrari's [10] result. This results in an infinite dimensionality of the attractor. To overcome this obstacle, we introduce the notion of sub-attractors in this section. The underlying idea is to only consider solutions with bounded zero number and to define the sub-attractors in such a way that they remain invariant as sets under the semiflow of the equation. This allows us to get rid of all solutions with an infinite or uncountable zero set.

Definition 3.1. Let $n = 2\alpha$ for $\alpha \in \mathbb{N}$. Then we define:

(i) $\mathcal{E}_n := \{u \equiv u_+,\ u \equiv u_-\}$;

(ii) $\mathcal{F}_n := \{u \in \mathcal{F};\ z(u) \leqslant \alpha\}$;

(iii) $\mathcal{H}_n := \{u \in \mathcal{H};\ \lim_{t\to\pm\infty} \in \mathcal{E}_n \cup \mathcal{F}_n\}$.

Then we define the sub-attractor of order n of the hyperbolic balance law (H) by

$$\mathcal{A}_n := \mathcal{E}_n \cup \mathcal{F}_n \cup \mathcal{H}_n. \tag{3.1}$$

We first prove the following lemma.

LEMMA 3.2.

(i) *Let* $\mathcal{A}_n$ *and* $\mathcal{A}_m$ *be defined as above for some* $m, n \in \mathbb{N}$. *Then*

$$\mathcal{A}_n \subset \mathcal{A}_m \Longleftrightarrow n < m.$$

(ii) *We have the following alternative description for* $\mathcal{A}_n$*:*

$$\mathcal{A}_n = \overline{\{W^u(\mathcal{F}_n)\}}.$$

(iii) $\mathcal{A}_n$ *is invariant under the semiflow* Φ *generated by (H).*

Proof. Part (i) is obvious by the definition of $\mathcal{A}_n$. Part (ii) follows directly through

$$\begin{aligned}\mathcal{A}_n &= \bigcup_{\beta=1}^{\alpha} \{W^u(u^0);\ u^0 \in \mathcal{F},\ z(u^0) = \beta\} \cup \mathcal{F}_n \cup \mathcal{E}_n \\ &= \bigcup_{\beta=1}^{\alpha} \overline{\{W^u(u^0);\ u^0 \in \mathcal{F},\ z(u^0) = \beta\}} \\ &= \overline{\{W^u(\mathcal{F}_n)\}}. \end{aligned} \tag{3.2}$$

Part (iii) is a direct consequence of the invariance of $\mathcal{E}_n$ and $\mathcal{F}_n$. □

At first glance it seems strange to denote the sub-attractors by $\mathcal{A}_n$ and not $\mathcal{A}_\alpha$. However, one of the results in the following section will be $\dim \mathcal{A}_n = n$, which justifies the notation.

4. Parametrizations for $\mathcal{A}_n$

We now turn to the question of parametrizing the sub-attractors $\mathcal{A}_n$. We follow an idea introduced by Härterich. In [7, § 4] Härterich presents an example of one heteroclinic connection between two defined states for Burgers's equation ($f(u) = \frac{1}{2}u^2$). The key idea is that the connection consists of stationary profiles that are separated by shocks. The solutions on the connections only change if the connections are at all close to the shocks. This idea guides the path towards the parametrization of the sub-attractors by the position of the stationary profiles on the one hand and the positions of the shocks on the other. A key step in the proof is to show that this approach covers all heteroclinic connections.

We begin with the definition of the stationary profiles. Let $\phi(x)$ be the unique solution of the following equation:

$$v_x = \frac{g(v)}{f'(v)}, \qquad v(0) = 0. \tag{4.1}$$

Then $\phi(x)$ exists for all $x \in \mathbb{R}$ and

$$\lim_{x\to-\infty} \phi(x) = u_-, \qquad \lim_{x\to\infty} \phi(x) = u_+.$$

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Let $n = 2\alpha$ be given for some $\alpha \in \mathbb{N}$. Then we choose a sequence of α zeros $0 \leqslant x_1 < x_2 < \cdots < x_\alpha < 2\pi$. Due to [10], there exists a unique frozen wave $v(x)$ with

$$\mathcal{Z}(v) = \{x_1, \ldots, x_\alpha\}.$$

Without loss of generality, we assume that $x_1 = 0$. All other cases can be generated by a shift.

Note that for every solution of (H) it is true that between two zeros there must be a shock and between two shocks with sign-changing left- and right-hand states there must be a zero. This is even true in the case where f depends explicitly on x [3]. It is, in particular, true for v_α. Hence, there is a unique sequence of shocks $\hat{y}_1, \ldots, \hat{y}_\alpha$ with

$$0 = x_1 < \hat{y}_1 < x_2 < \hat{y}_2 < \cdots < \hat{y}_{\alpha-1} < x_\alpha < \hat{y}_\alpha < 2\pi$$

such that v is given by

$$v = \begin{cases} \phi(x - x_i) & \text{for } x \in [x_i, \hat{y}_i], \\ \phi(x - x_{i+1}) & \text{for } x \in [\hat{y}_i, x_{i+1}]. \end{cases} \tag{4.2}$$

For convenience let us define

$$\{\boldsymbol{x}_\alpha\} := \{x_1, \ldots, x_\alpha\}$$

and denote the unique frozen wave with zero set $\boldsymbol{x}_\alpha$ by $v_{\boldsymbol{x}_\alpha}$.

We now define the solution $u_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}}$, with α shocks located between the zeros $\{x_1, \ldots, x_\alpha\}$, that consists piecewise of shifted copies of $\phi(x)$. In general, $u_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}}$ is not stationary.

Let $0 \leqslant x_1 \leqslant y_1 < x_2 \leqslant \cdots < x_\alpha \leqslant y_\alpha < 2\pi$. Then we define

$$u_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for } x \in [x_i, y_i], \\ \phi(x - x_{i+1}) & \text{for } x \in (y_i, x_{i+1}], \end{cases} \tag{4.3}$$

for $i = 1, \ldots, \alpha$.

Finally, let us define the general solution $\tilde{u}_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}}$, with α or fewer shocks, that consists piecewise of shifted copies of $\phi(x)$, where all shocks have sign-changing left- and right-hand states.

Let $0 \leqslant \tilde{y}_1 \leqslant \tilde{y}_2 \leqslant \cdots \leqslant \tilde{y}_\alpha < 2\pi$. Then, if $\tilde{y}_i < \tilde{y}_{i+1}$, we define

$$\tilde{u}_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for } x \in [x_i, \tilde{y}_i], \\ \phi(x - x_{i+1}) & \text{for } x \in (\tilde{y}_i, x_{i+1}], \end{cases} \tag{4.4}$$

and if $\tilde{y}_i = \tilde{y}_{i+1} = \cdots = \tilde{y}_{i+m}$,

$$\tilde{u}_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}} = \begin{cases} \phi(x - x_i) & \text{for } x \in [x_i, \tilde{y}_i], \\ \phi(x - x_{i+m+1}) & \text{for } x \in (\tilde{y}_{i+m}, x_{i+m+1}]. \end{cases} \tag{4.5}$$

Then the two sets $A_{\{\boldsymbol{x}_\alpha\}}$ and $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ of all these solutions with fixed $\{x_1, \ldots, x_\alpha\} = \{\boldsymbol{x}_\alpha\}$ are given by

$$A_{\{\boldsymbol{x}_\alpha\}} := \{u_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}};\ 0 \leqslant x_1 \leqslant y_1 < x_2 \leqslant \cdots < x_\alpha \leqslant y_\alpha < 2\pi\} \tag{4.6}$$

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and

$$\tilde{A}_{\{\boldsymbol{x}_\alpha\}} := \{\tilde{u}_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}};\ 0 \leqslant y_1 \leqslant \cdots \leqslant y_\alpha < 2\pi\}. \tag{4.7}$$

Then we have the following lemma.

LEMMA 4.1. *Let $\boldsymbol{x}_\alpha$ and $\boldsymbol{x}_\beta$ be given with $\boldsymbol{x}_\beta \subset \boldsymbol{x}_\alpha$. Then we have*

(i) $v_{\boldsymbol{x}_\alpha} \in A_{\{\boldsymbol{x}_\alpha\}} \subset \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$;

(ii) $\tilde{A}_{\{\boldsymbol{x}_\beta\}} \subset \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$;

(iii) *there exists no $u \in \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ with more than α shocks.*

Proof. We only prove (iii). We first argue the case for two zeros: assume that the solution has a zero located at $x_1 = 0$ and another zero at x_2. We explicitly construct the set of all admissible solutions $u(x)$ that consist piecewise of shifted copies of $\phi(x - x_i - 2\pi k_j)$ for some $k_j \in \mathbb{Z}$ and $i \in \{1, 2\}$; with the additional property that $u(x_1 = 0) = 0$ and show that, in fact, $j = 2$ necessarily.

Let us denote all shock positions by $0 < y_1 < \cdots < y_j \leqslant 2\pi$. Due to the fact that between zeros there has to be shock, we obtain $j \geqslant 2$. Let us define the sequence of stationary profiles:

$$\ldots,\ \phi(x+2\pi),\ \phi(x+x_2),\ \phi(x),\ \phi(x-x_2),\ \phi(x-2\pi),\ \phi(x-x_2-2\pi),\ \ldots. \tag{4.8}$$

Because $u(0) = 0$, we start at $x = 0$ with $u(x) = \phi(x)$ locally. At each of the shocks y_i the solution jumps one profile to the right in the above sequence due to the entropy condition (2.3). However, we have to end with $u(x) = \phi(x - 2\pi)$ locally at x close to 2π. Hence, $j \leqslant 2$ and therefore $j = 2$. □

We now state the main theorem.

THEOREM 4.2. *Let $n = 2\alpha$ and $\alpha \in \mathbb{N}$. Then the following are true.*

(a) *The local unstable manifold $W^u_{\mathrm{loc}}(v_{\{\boldsymbol{x}_\alpha\}})$ of $v_{\{\boldsymbol{x}_\alpha\}}$ is given by $A_{\{\boldsymbol{x}_\alpha\}}$ defined in (4.6):*

$$W^u_{\mathrm{loc}}(v_{\{\boldsymbol{x}_\alpha\}}) = A_{\{\boldsymbol{x}_\alpha\}}, \tag{4.9}$$

where $v_{\{\boldsymbol{x}_\alpha\}}$ is the unique frozen wave of (H) with zeros at $x_1, \ldots, x_\alpha$.

(b) *The global unstable manifold $W^u(v_{\{\boldsymbol{x}_\alpha\}})$ of $v_{\{\boldsymbol{x}_\alpha\}}$ is then given by*

$$W^u(v_{\{\boldsymbol{x}_\alpha\}}) = \{\Phi(u, t); u \in A_{\{\boldsymbol{x}_\alpha\}},\ t \in \mathbb{R}^+\}, \tag{4.10}$$

where Φ denotes the semiflow in $\mathrm{BV}(S^1, \mathbb{R})$ generated by equation (H).

(c) *The dynamics on $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ defined in equation (4.7) can be described by the following equation for the shock parameters y_j:*

$$\dot{y}_j(t) = \frac{f(\phi(y_j - x_j)) - f(\phi(y_j - x_{j+1}))}{\phi(y_j - x_j) - \phi(y_j - x_{j+1})}. \tag{4.11}$$

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(d) *The dimension of the sub-attractors $\mathcal{A}_n$ of order n is given by*

$$\dim \mathcal{A}_n = n.$$

(e) *Let v_1 be a frozen wave of equation (H) with*

$$z(v_1) = 1.$$

Then there exist unique heteroclinic connections $\tilde{u}(\cdot,t)$ and $\hat{u}(\cdot,t)$ with

$$\lim_{t\to-\infty} \tilde{u}(\cdot,t) = \lim_{t\to-\infty} \hat{u}(\cdot,t) = v_1,$$
$$\lim_{t\to\infty} \tilde{u}(\cdot,t) \equiv u_+,$$
$$\lim_{t\to\infty} \hat{u}(\cdot,t) \equiv u_-.$$

(f) *Let $0 \leqslant x_1 < x_2 < \cdots < x_\alpha < 2\pi$ and let v_1 and v_2 be frozen waves of equation (H) with the property*

$$\mathcal{Z}(v_1) = \{x_1,\dots,x_\alpha\} \quad \textit{and} \quad \mathcal{Z}(v_2) = \{x_{k_1},\dots,x_{k_\beta}\}$$

with $k_{i+1} - k_i \in \{0,1\}$ for all $1 \leqslant i \leqslant \beta - 1$, where we have set $\beta + 1 = \alpha$. Then there exists, up to shifts in time, a unique heteroclinic connection $u(x,t)$ with the property

$$\lim_{t\to-\infty} u(\cdot,t) = v_1(\cdot) \quad \textit{and} \quad \lim_{t\to\infty} u(\cdot,t) = v_2(\cdot).$$

The proof of the theorem will use the overflowing invariance of the sets $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ and $A_{\{\boldsymbol{x}_\alpha\}}$, which we define, state and prove first.

Definition 4.3. Let $A \subset X$ be a subset of the phase space X of a semiflow $\varPhi$. A is called overflowing invariant if all $u \in A$ have the following property: either $\varPhi(u,t) \in A$ for all $t \in \mathbb{R}$ or there exists $t_0 \in \mathbb{R}$ such that $\varPhi(u,t) \in A$ for $t < t_0$ and $\varPhi(u,t) \notin A$ for $t > t_0$.

Lemma 4.4. *Let $\{\boldsymbol{x}_\alpha\} := \{x_1,\dots,x_\alpha\}$ with $0 \leqslant x_1 < \cdots < x_\alpha < 2\pi$ be given and $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$, $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ be defined as above.*

(i) *The set $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ is overflowing invariant under the semiflow of (H).*

(ii) *The set $A_{\{\boldsymbol{x}_\alpha\}}$ is overflowing invariant under the semiflow of (H).*

Proof. Let $u(x,0) \in \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ such that $u(x,0) = u_{\{\boldsymbol{x}_\alpha,\boldsymbol{y}_\alpha\}}$ with $y_1 > 0$ and $y_\alpha < 2\pi$.

We first show local invariance: local forward invariance of $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ follows from the fact that the profiles ϕ that define $u_{\{\boldsymbol{x}_\alpha,\boldsymbol{y}_\alpha\}}$ are stationary. Hence, $u(x,t)$ is stationary except near the points y_j, and so we only have to prove invariance locally at the shock points. We investigate only the shock located at y_1; the argument works equivalently for any other shock.

Let therefore $u(x,0)$ be given by

$$u(x,0) = \begin{cases} \phi(x) & \text{for } y_1 - \delta x \leqslant y_1, \\ \phi(x - x_2) & \text{for } y_1 + \delta x > y_1, \end{cases} \tag{4.12}$$

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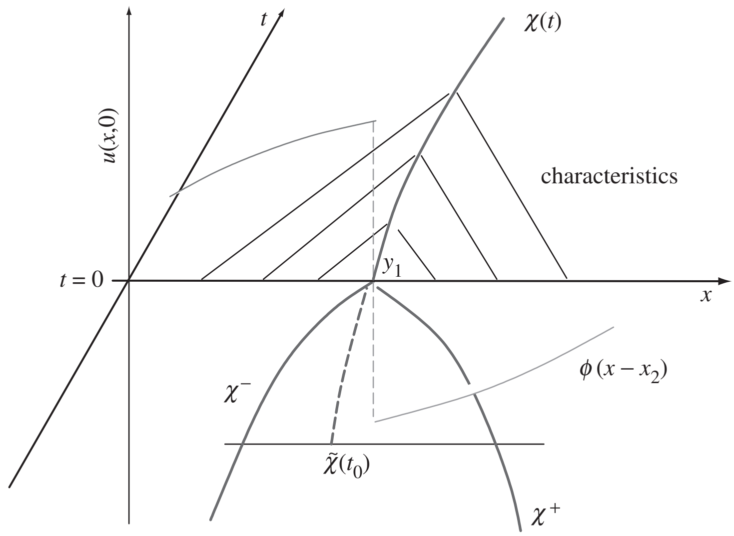

Figure 1. Illustration of the minimal and maximal characteristics emanating in a shock located at y_1. The shape of the solution at $t = 0$ is indicated in light grey. Note that χ^+ and χ^- intersect transversely in $(x, t) = (y_1, 0)$.

for some $\delta > 0$. At y_1 there is a unique forward characteristic $\chi(t)$ on which the shock evolves. The other characteristics in a neighbourhood of y_1 necessarily point towards $\chi(t)$ for $t > 0$. Hence, for $x \notin [y_1 - \delta, y_1 + \delta]$ the solution $u(x, t)$ is stationary and given by $\phi(x)$ for $x \leqslant \chi(t)$ and by $\phi(x - x_2)$ for $x > \chi(t)$. See figure 1 for an illustration.

$\chi(t)$ is uniquely determined by the differential equation

$$\left.\begin{aligned} \dot{\chi}(t) &= \frac{f(\phi(\chi(t))) - f(\phi((\chi(t) - x_2)))}{\phi(\chi(t)) - \phi((\chi(t) - x_2))}, \\ \chi(0) &= y_1. \end{aligned}\right\} \tag{4.13}$$

The slope of $\chi(t)$ is bounded from above and hence, if t is sufficiently small, we have obtained local forward invariance of the shock.

For the backward invariance we observe that for $t < 0$ a minimal characteristic $\chi^-(t)$ and a maximal backward characteristic $\chi^+(t)$ emanate from y_1. For the area between χ^- and χ^+ there are in principle many possibilities to define the solution such that we obtain $u(x, t)$ for $t \geqslant 0$ (there is no backward uniqueness!). For backward invariance it is, however, enough if we can find one $u(x, t) \in \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ for $t < 0$.

Now let $t_0 < 0$ be sufficiently small. Then we define

$$\tilde{u}(x, t_0) := \begin{cases} \phi(x) & \text{for } x \in [\chi^-(t_0), \tilde{\chi}(t_0)], \\ \phi(x - x_2) & \text{for } x \in (\tilde{\chi}(t_0), \chi^+(t_0)], \end{cases}$$

for some $\tilde{\chi}(t_0) \in [\chi^-(t_0), \chi^+(t_0)]$. Local backward invariance follows if we can prove that there is one $\tilde{\chi}(t_0)$ such that if we solve (4.13) with initial condition $\tilde{\chi}(t_0)$ we obtain

$$\tilde{\chi}(0) = \chi(0) = y_1.$$

If we assume $\tilde{\chi}(t_0) = \chi^-(t_0)$, then monotonicity of ϕ and convexity of f imply $\tilde{\chi}(0) < y_1$; if we assume on the other hand $\tilde{\chi}(t_0) = \chi^+(t_0)$, then the same argument yields $\tilde{\chi}(0) > y_1$. The intermediate value theorem yields the existence of a $\tilde{y} \in (\chi^-, \chi^+)$ such that $\tilde{\chi}(t)$ with $\tilde{\chi}(t_0) := \tilde{y}$ has the desired property. Due to the convexity of f and the monotonicity of ϕ the $\tilde{y}$ is even unique. Hence, backward invariance follows.

Although $\tilde{y}$ is unique, the backward solution is not unique in $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ in general, due to the possibility of shock splittings in the backward time direction. However, if we assume that no shock splitting occurs, we even obtain uniqueness of the backward solution in $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$.

Now we turn to the overflowing property. Note that for $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ overflowing now means that if a solution $u \in \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ leaves $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ at time $t = \tilde{t}$, then either $y_1 = x_1$ or $y_\alpha = 2\pi$ in $u_{\{\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha\}} := \Phi(u, \tilde{t})$.

We assume $u(x, 0) \in \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ with $y_1 = 0$. Then the forward characteristic $\chi(t)$ in $x_1 = y_1 = 0$ is given by

$$\chi(t) = \frac{-f(\phi(y_\alpha - 2\pi))}{-\phi(y_\alpha - 2\pi)} < 0$$

for $t \in [0, \delta_1)$, δ_1 positive and small and $\chi(0) = 2\pi$. Thus, after identification of 0 and 2π, we obtain that the solution is locally given by

$$\begin{array}{ll} \phi(x - x_2) & \text{for } 0 < x < y_2, \\ \phi(x - x_2 - 2\pi) & \text{for } \chi(t) < x < 2\pi, \\ \phi(x - 2\pi) & \text{for } y_\alpha < x < \chi(t). \end{array}$$

This proves the overflowing property of $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$, because the above solution is not in $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$ due to the fact that there are only $\alpha - 1$ zeros but α shocks, one of which has the same sign at the left and right states. This proves (i).

Due to the fact that $A_{\{\boldsymbol{x}_\alpha\}} \subset \tilde{A}_{\{\boldsymbol{x}_\alpha\}}$, we conclude invariance of $A_{\{\boldsymbol{x}_\alpha\}}$ by virtue of the same construction. The overflowing property works just as for $\tilde{A}_{\{\boldsymbol{x}_\alpha\}}$; here the boundary is given by the condition $y_j = x_j$ or $y_j = x_{j+1}$ for some $j \in \{1, \ldots, \alpha\}$. □

Corollary 4.5. *For every* $u(x, 0) \in A_{\{\boldsymbol{x}_\alpha\}}$ *there is a unique backward orbit in* $A_{\{\boldsymbol{x}_\alpha\}}$.

Proof. From the proof of the previous lemma we deduce that it is sufficient to show that shocks in u cannot split in backward time. By construction, any solution in $A_{\{\boldsymbol{x}_\alpha\}}$ has exactly α zeros and α shocks because, due to lemma 4.1(iii), shock splitting cannot occur. □

Proof of theorem 4.2. We have already proven part (c). Equation (4.13) yields exactly (4.11) if we replace $\chi(t)\pm$ by the y_j. Hence, we can integrate solutions along the (invariant) manifold $A_{\{\boldsymbol{x}_\alpha\}}$ by using (4.11) for every y_j $(1 \leqslant j \leqslant n)$. Note that y_j and y_{j+1} can meet. Thus, the y_j are Lipschitz only in t and not in C^1.

For (a) we prove that all solutions $u(\cdot, 0) \in A_{\{\boldsymbol{x}_\alpha\}}$ converge in backward time to $v_{\{\boldsymbol{x}_\alpha\}}$. This shows that

$$A_{\{\boldsymbol{x}_\alpha\}} \subset W^u(v_{\{\boldsymbol{x}_\alpha\}}). \tag{4.14}$$

Then we show maximality of $A_{\{\boldsymbol{x}_\alpha\}}$ by proving that all solutions $u(\cdot,t)$ converging to $v_{\{\boldsymbol{x}_\alpha\}}$ in backward time are contained in $A_{\{\boldsymbol{x}_\alpha\}}$ for sufficiently small $t<0$ which proves

$$W^u_{\mathrm{loc}}(v_{\{\boldsymbol{x}_\alpha\}}) \subset A_{\{\boldsymbol{x}_\alpha\}}. \tag{4.15}$$

This yields (a) for appropriately chosen local neighbourhood in $W^u(v_{\{\boldsymbol{x}_\alpha\}})$.

The first part is a consequence of lemma 4.4 and the convexity of f: let $u(\cdot,0) \in A_{\{\boldsymbol{x}_\alpha\}}$. Because of the overflowing invariance and backward uniqueness (corollary 4.5), we conclude that

$$u(\cdot,t) \in A_{\{\boldsymbol{x}_\alpha\}}$$

for all $t<0$. In addition,

$$\lim_{t\to-\infty} u(\cdot,t) \in \mathcal{F} \cup \mathcal{E}$$

due to proposition 2.9 $v_{\{\boldsymbol{x}_\alpha\}}$ is the only frozen wave in $A_{\{\boldsymbol{x}_\alpha\}}$, and hence

$$A_{\{\boldsymbol{x}_\alpha\}} \cap \mathcal{E} \cup \mathcal{F} = \{v_{\{\boldsymbol{x}_\alpha\}}\}.$$

This yields (4.14).

For the other direction we argue indirectly. Assume there exists $\tilde{u}(x,t)$ with

$$\lim_{t\to-\infty} \tilde{u}(x,t) = v_{\{\boldsymbol{x}_\alpha\}} \quad \text{and} \quad \tilde{u}(x,t) \notin A_{\{\boldsymbol{x}_\alpha\}} \quad \text{for all } t<0.$$

Then for sufficiently small $\tilde{t}<0$ there exists $\tilde{x}\in S^1$ such that for all $1 \leqslant j \leqslant \alpha+1$

$$\tilde{u}(\tilde{x},\tilde{t}) \neq \phi(\tilde{x}-x_j), \tag{4.16}$$

where we have set $x_{\alpha+1} = 2\pi$.

Due to the fact that $\tilde{u}$ connects to $v_{\{\boldsymbol{x}_\alpha\}}$ we can always choose $(\tilde{x},\tilde{t})$ such that $\tilde{u}(\tilde{x},\tilde{t})$ is smaller than the maximum and larger than the minimum of the stationary solution with one zero.

We now construct a contradiction by proving that $\lim_{t\to-\infty}\tilde{u}(\cdot,t)$ has a zero x_s not coinciding with one of the $x_1,\dots,x_\alpha$ and therefore $\tilde{u}$ cannot converge to $v_{\{\boldsymbol{x}_\alpha\}}$ in backward time.

We use a stationary solution u_s coinciding with $\tilde{u}(\cdot,0)$ at $\tilde{x}$ to calculate explicitly the backward characteristic of $\tilde{u}$ emanating from $(\tilde{x},\tilde{t})$.

From (4.16) we deduce that there is a stationary solution $u_s \in \mathcal{F}$ with the following properties:

$$u_s(\tilde{x}) = \tilde{u}(\tilde{x},\tilde{t}), \qquad \mathcal{Z}(u_s) = \{x_s\},$$

where $x_s \notin \{x_1,\dots,x_\alpha\}$.

We investigate the (genuine!) backward characteristic $(\chi(t),\underline{v}(t))$ with

$$\begin{aligned}\chi(\tilde{t}) &= \tilde{x},\\ \underline{v}(\tilde{t}) &= u_s(\tilde{x},\tilde{t}) = \tilde{u}(\tilde{x},\tilde{t}).\end{aligned}$$

Because u_s is stationary, the characteristic has the property that

$$\lim_{t\to-\infty}\chi(t) = x_s$$

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and

$$\lim_{t\to-\infty} \underline{v}(t) = 0.$$

From this we deduce

$$(\chi(t), \underline{v}(t)) \Longrightarrow \lim_{t\to-\infty} u_s(\chi(t), t) = u_s(x_s, \cdot) = 0;$$

this implies

$$\lim_{t\to-\infty} u(x_s, t) = \lim_{t\to-\infty} u_s(x_s, t) = 0.$$

Hence,

$$\lim_{t\to-\infty} \tilde{u}(\chi(t), t) = v_{\{x_\alpha\}}(x_s) = 0.$$

This contradicts $x_s \notin \{x_1, \dots, x_\alpha\}$, and maximality of $A_{\{\boldsymbol{x}_\alpha\}}$ is proved.

Part (b) follows from the fact that due to unique forward solvability we obtain the global unstable manifold by using the semiflow to forward-solve the local unstable manifold. $A_{\{\boldsymbol{x}_\alpha\}} \subset \mathcal{A}$ ensures boundedness of the forward iteration; hence, (4.10) follows.

For part (d) we use the fact that

$$\dim(W^u_{\text{loc}}(v_{\{\boldsymbol{x}_\alpha\}})) = \dim(W^u(v_{\{\boldsymbol{x}_\alpha\}})), \tag{4.17}$$

which is true due to the forward uniqueness of solutions.

The sub-attractor of order $n = 2$ consists by definition of all frozen waves with one zero and heteroclinic connections from these waves to $u_\pm$. In other words,

$$\mathcal{A}_2 = W^u(\mathcal{F}_2) \cup \mathcal{E}_2.$$

For fixed x_1 we have

$$\dim(W^u(v_{\{x_1\}})) = \dim(A_{\{x_1\}}) = 1.$$

From the uniqueness of frozen waves with given $x_1 \in S^1$ we deduce that

$$\dim \mathcal{A}_2 = 2.$$

For $n = 2\alpha > 2$ we use

$$\mathcal{A}_n = \{W^u(u); u \in \mathcal{F}_n\} \cup \mathcal{E}_n. \tag{4.18}$$

First, we prove

$$\dim\{W^u(u); u \in \mathcal{F}, z(u) = \alpha\} = 2\alpha = n.$$

For each fixed set of zeros $\{0 \leqslant x_1 < \cdots < x_\alpha < 2\pi\}$ we have by part (a) that

$$\dim(W^u_{\text{loc}}(v_{\{\boldsymbol{x}_\alpha\}})) = \dim(A_{\{\boldsymbol{x}_\alpha\}}) = \alpha.$$

Moreover, all frozen waves v with zero-number $z(v) \leqslant \alpha$ can be parametrized by $(x_1, \dots, x_\alpha) \in (S^1)^\alpha = \mathbb{T}^\alpha$; hence,

$$\dim \mathcal{F}_n = \dim \mathbb{T}^\alpha = \alpha.$$

Putting everything together, by using (4.18) we obtain

$$\dim \mathcal{A}_n = \dim W^u_{\text{loc}}(\{\mathcal{F}_n\}) = \dim W^u_{\text{loc}}(v_{\{\boldsymbol{x}_\alpha\}}) + \dim \mathbb{T}^\alpha = \alpha + \alpha = n.$$

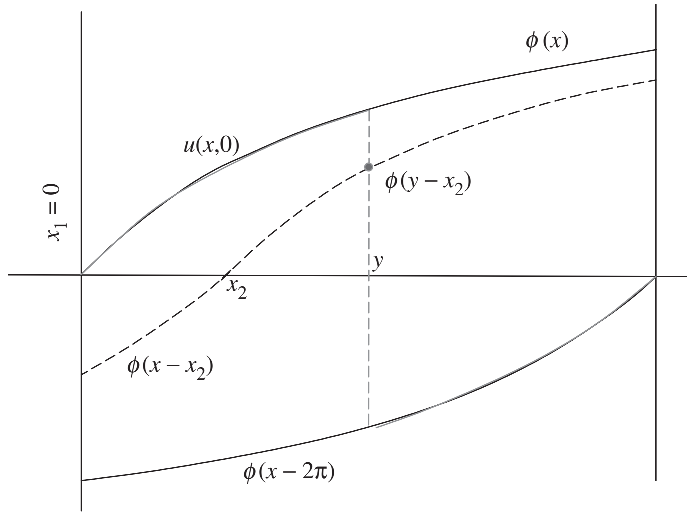

Figure 2. Unique shock-splitting of one shock in backward time in $A_{\{x_1,x_2\}}$.

For (e) we count dimensions to obtain uniqueness. For $\alpha = 1$ the unstable manifold of v_1 is one dimensional; thus, the connection must be unique.

For (f) we argue in the following way: the condition $k_{i+1} - k_i \in \{0, 1\}$ implies that at most every second zero can vanish; hence, we can reduce the proof to the situation where

$$\mathcal{Z}(v_1) = \{0, x_2\} \quad \text{and} \quad \mathcal{Z}(v_2) = \{0\}.$$

Let us denote the unique shock position of v_2 by y and the two unique shock positions of v_1 by y_1 and y_2.

It is a consequence of (c) that in the class of solutions $A_{\{x_1,x_2\}}$ all stationary shocks are unstable. In order to obtain the solution v_2 with only one shock, the two shocks emanating from y_1 and y_2 consequently have to meet at position y in such a way that the resulting shock is stationary.

We define $t = 0$ as the time at which the two shocks collide. Thus, the question of uniqueness of heteroclinic connections reduces to the question of uniqueness of shock collisions in $A_{\{x_1,x_2\}}$, or, in the negative time direction, the question of uniqueness of the splitting of shocks at a given position.

Let $u(x, t)$ be the solution where two shocks meet at time $t = 0$ at position $x = y$. Then the lower state of the left shock and the upper state of the right shock have to coincide. By construction of $\tilde{A}_{\{x_1,x_2\}}$, this state is given by $\phi(y - x_2)$:

$$\lim_{x \searrow y} \lim_{t \nearrow 0} u(x,t) = \lim_{x \nearrow y} \lim_{t \nearrow 0} u(x,t) \stackrel{!}{=} \phi(y - x_2).$$

See figure 2 for an illustration.

Hence, uniqueness of the splitting follows by uniqueness of backward solutions in the case of $u \in A_{\{x_1,x_2\}}$ with two shocks proved in corollary 4.5. This proves (e), and the theorem is proven. □

Note that for the situation in theorem 4.2(e) we can explicitly parametrize the whole heteroclinic connection from v_1 to $u_\pm$. The stationary solution v_1 with

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$\mathcal{Z}(v_1) = \{x_1\}$ has one unique shock at position y_1. Then, using (b) and (c) of theorem 4.2, we can parametrize the whole connection manifold $W^u(v_1)$ as follows: for any $k \in \mathbb{Z}$ and any $y_1 \in [2k\pi, 2(k+1)\pi)$ we define

$$u^*_{\{x_1,y_1\}}(x) := \begin{cases} \phi(x - x_1 + 2k\pi) & \text{for } 0 \leqslant x \leqslant y_1 - 2k\pi, \\ \phi(x - x_1 + 2(k-1)\pi) & \text{for } y_1 - 2k\pi < x < 2\pi. \end{cases} \tag{4.19}$$

Then $W^u(v_1)$ is given by

$$W^u(v_1) := \{u^*_{\{x_1,y_1\}} \in \mathrm{BV}(s^1, \mathbb{R}); y_1 \in \mathbb{R}\}. \tag{4.20}$$

The next section will present a geometric representation of $\mathcal{A}_2 = \overline{W^u(\mathcal{F}_2)}$.

Corollary 4.6. *Again let $\alpha \in \mathbb{N}$ and $n = 2\alpha$. Then the set of heteroclinic connections between two frozen waves with zero-number $z \leqslant \alpha$ is completely contained in*

$$\tilde{\boldsymbol{A}}_n := \{\tilde{A}_{\{\boldsymbol{x}_\alpha\}}; \boldsymbol{x}_\alpha \in \mathbb{T}^\alpha \text{ and } 0 \leqslant x_1 < \cdots < x_\alpha < 2\pi\}. \tag{4.21}$$

Proof. Let v_1, v_2 be two frozen waves with

$$\mathcal{Z}(v_1) = \{x_1, \ldots, x_\beta\},$$
$$\mathcal{Z}(v_2) \subset \mathcal{Z}(v_1),$$

for some given $0 \leqslant x_1 < \cdots x_\beta < 2\pi$ and $\beta \leqslant \alpha$. Let $u(x,t)$ denote a heteroclinic connection between v_1 and v_2. Then

$$u(\cdot, t) \in A_{\{\boldsymbol{x}_\beta\}} \subset \tilde{A}_{\{\boldsymbol{x}_\beta\}} \subset \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for some $x_{\beta+1}, \ldots, x_\alpha$ and t sufficiently small.

Now assume $u(\cdot, \tilde{t}) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$ for some $\tilde{t} \in \mathbb{R}$. Then we conclude

$$u(\cdot, t) \notin \tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$$

for all $t > \tilde{t}$ due to the overflowing property of $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$.

This contradicts

$$\lim_{t\to\infty} u(\cdot, t) = v_2$$

because

$$v_2 \in \mathring{\tilde{A}}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}.$$

Here $\mathring{\tilde{A}}$ denotes the interior of $\tilde{A}$ in the topology of the manifold $\tilde{A}_{\{x_1,\ldots,x_\beta,\ldots,x_\alpha\}}$. □

In the next section a geometric representation of the sub-attractors of order $n = 2$ and $n = 4$ will be given.

5. Examples

5.1. The sub-attractor $\mathcal{A}_2$

According to the definition the sub-attractor $\mathcal{A}_2$ consists of all frozen waves with zero-number $z = 1$, the two stable homogeneous equilibria $u \equiv u_\pm$ and all heteroclinic connections between these objects. The frozen waves can be represented as the S^1 symmetry.

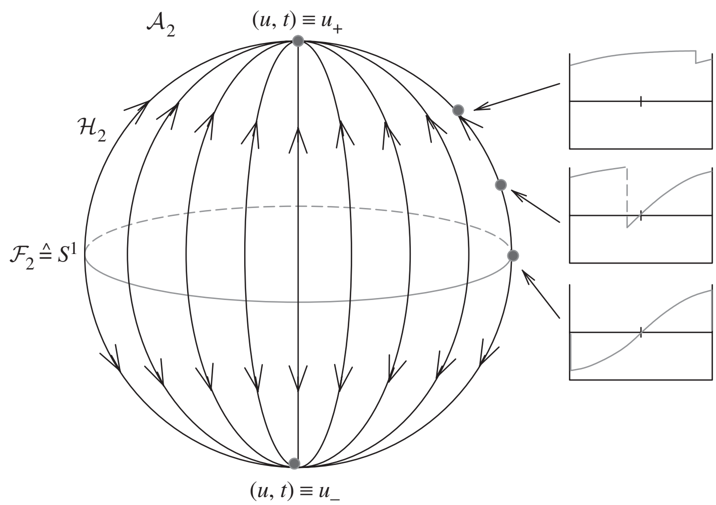

Figure 3. Geometric representation of the sub-attractor $\mathcal{A}_2$.

Due to theorem 2.10 all frozen waves are connected to $u(x) \equiv u_\pm$. Theorem 2.10 states that these are all heteroclinic connections in $\mathcal{A}_2$ and theorem 4.2(e) yields uniqueness of these heteroclinics. Equation (4.19) provides, together with (4.20), an explicit parametrization of these connections. Hence, we can define an explicit embedding

$$\begin{aligned} \Sigma_2 \colon S^1 \times \mathbb{R} &\to \mathrm{BV}(S^1, \mathbb{R}), \\ (x_1, y_1) &\mapsto \Sigma_2(x_1, y_1) := u_{\{x_1, y_1\}}, \end{aligned}$$

where $u^*_{\{x_1,y_1\}}$ is defined in (4.19). The flow on graph(Σ_2) can be computed explicitly and is given by (4.11).

By a stereographic projection $\mathcal{S}$, we can map graph(Σ_2) onto the surface of a ball, thus obtaining a representation of $\mathcal{A}_2$ as an S^2, as shown in figure 3.

The three diagrams on the right in figure 3 show schematically how the shape of these solutions evolves on the S^2 along a heteroclinic connection.

5.2. The sub-attractor $\mathcal{A}_4$

Following the definition of $\mathcal{A}_4 := \mathcal{E}_4 \cup \mathcal{F}_4 \cup \mathcal{H}_4$, we shall first classify all homogeneous equilibria and frozen waves. Due to Sinestrari the frozen waves can be uniquely parametrized by the position of their zeros x_1, x_2 and hence form a two-torus:

$$\mathcal{F}_4 = \mathbb{T}^2 := S^1 \times S^1,$$

and again $\mathcal{E}_4 = \{u_-, u_+\}$.

Each element of this torus has a heteroclinic connection to the homogeneous equilibria $u \equiv u_\pm$. This can be depicted by a spindle with a quadratic horizontal section and $u_\pm$ located at the top and bottom (see figure 4(a)). The heteroclinic connections are drawn with arrows. The edges of the quadratic horizontal section have to be identified in order to obtain the torus. The sub-attractor $\mathcal{A}_2$ is contained in this picture as well and is depicted by the thick lines. Figure 3 is obtained after

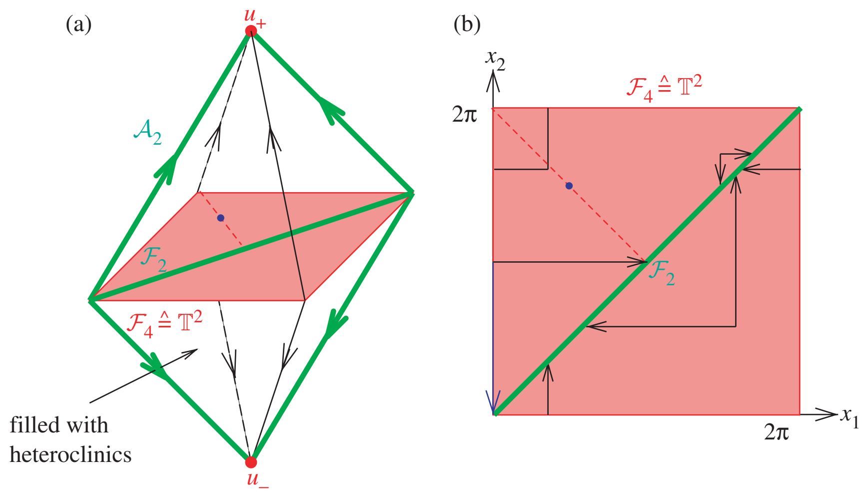

Figure 4. Heteroclinic connections in $\mathcal{A}_4$ with targets $u \equiv u_\pm$.

identification of the two opposite corners on the torus $\mathcal{F}_4$. The spindle is completely filled with heteroclinics starting in $\mathcal{F}_4$ and ending at $u \equiv u_-$ or $u \equiv u_+$, respectively.

The more interesting part of $\mathcal{A}_4$ is the part of the attractor that consists of all heteroclinic connections between $\mathcal{F}_4$ and $\mathcal{F}_2$. Theorem 2.12 yields that every frozen wave $\tilde{u}$ with zero-number $z(\tilde{u}) = 2$ is connected to two waves $\tilde{u}_a, \tilde{u}_b$ with zero-numbers $z(\tilde{u}_{a,b}) = 1$, theorem 4.2(f) yields uniqueness of these connections.

Hence, every point on the torus of frozen waves $\mathcal{F}_4 \setminus \mathcal{F}_2$ has two heteroclinic connections to two points on the diagonal curve on that torus representing $\mathcal{F}_2$. This is shown in figure 4(b), where we have parametrized the torus by the zeros (x_1, x_2) given as the horizontal and vertical axes. Some heteroclinics are shown as arrows for illustration. The lines are vertical if the zero x_1 persists, and horizontal if the zero x_2 persists. Two heteroclinics emerge from every point.

Equations (4.4) and (4.5) provide an explicit parametrization of these connections.

To show the complete connection picture, it is convenient to use another representation that divides out the S^1 symmetry. This representation is shown in figure 5.

To understand the figure it is best to start with the vertical line. This line represents $\mathcal{F}_4/S^1$: the manifold that contains all frozen waves with zero-number $z = 2$ after having divided out the S^1 symmetry. The centre point on this line is the π-periodic frozen wave with equidistant zeros.

The coordinates on the vertical manifold are given by the distance between the two zeros x_1 and x_2. On the bottom the distance is zero, in the middle (at the dot) it is π and then it goes to zero again towards the top. x_1 and x_2 change in such a way that the two shocks always remain in the same position (for Burgers's equation this means due to symmetries that $\frac{1}{2}(x_1 + x_2) = \pi$). Three of the solution profiles in figure 5 show how the solutions evolve along the vertical manifold. This manifold is also included in parts (a) and (b) of figure 4 as a dashed line with a dot on the torus $\mathbb{T}^2$.

Each of the frozen waves has two connections to frozen waves with $z = 1$, one connection where the zero at x_1 persists and one where the one at x_2 persists. These

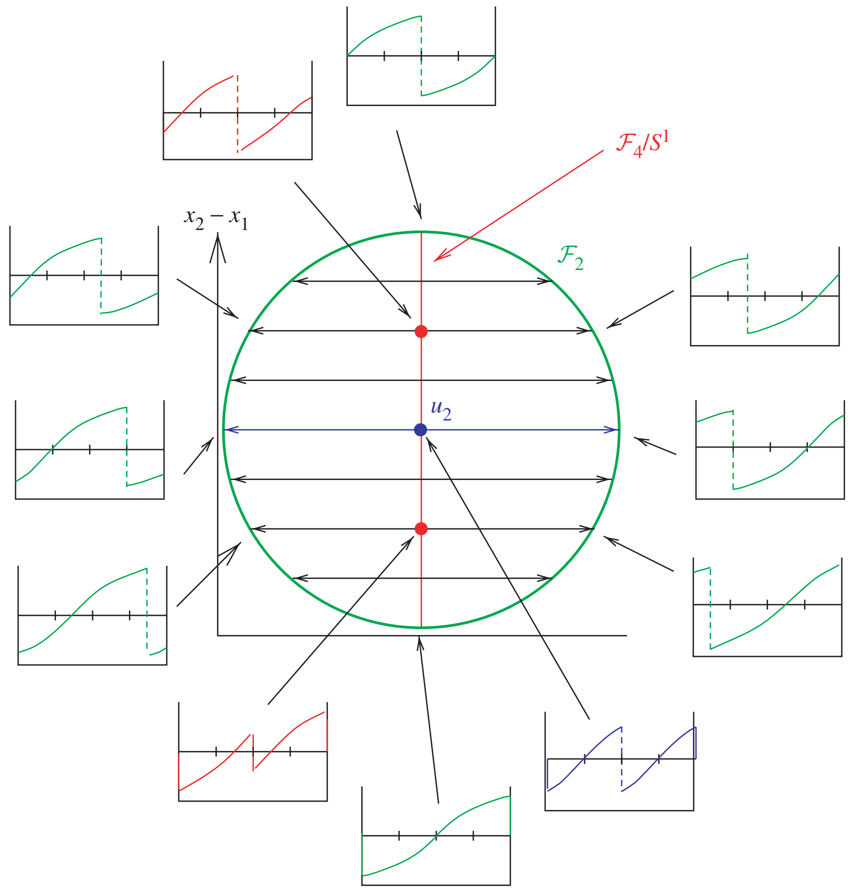

Figure 5. Heteroclinic connections in $\mathcal{A}_4$ from frozen waves with zero-number $z = 2$ to waves with zero-number $z = 1$. The S^1 symmetry is divided out.

are represented by the black arrows connecting to the circle representing $\mathcal{F}_2$ (equal to frozen waves with one zero). To the left x_1 persists and to the right x_2 persist, this induces coordinates on the circle of frozen waves with zero-number $z = 1$. The eight remaining solution profiles in figure 5 indicate how solutions evolve along the circle. A clockwise rotation along the S^1 in the figure corresponds to a shift of the solution to the right.

Now we are ready to include in the figure the S^1 symmetry that was divided out before. To do this we just have to rotate the whole figure along a circle in the transverse direction attached to the dot representing the wave with two equidistant zeros. We obtain a filled torus where we have a figure similar to that in figure 5 in every slice.

Inside the torus, the vertical line and the heteroclinic connections rotate once around the centre point with higher symmetry and therefore form a spiral. Figure 6 shows a geometric representation of this. We have plotted half of the torus. The thick half circular line corresponds to the frozen waves in $\mathcal{A}_4$ with higher symmetry (equidistant zeros). The heteroclinics are shown only in the beginning and the end.

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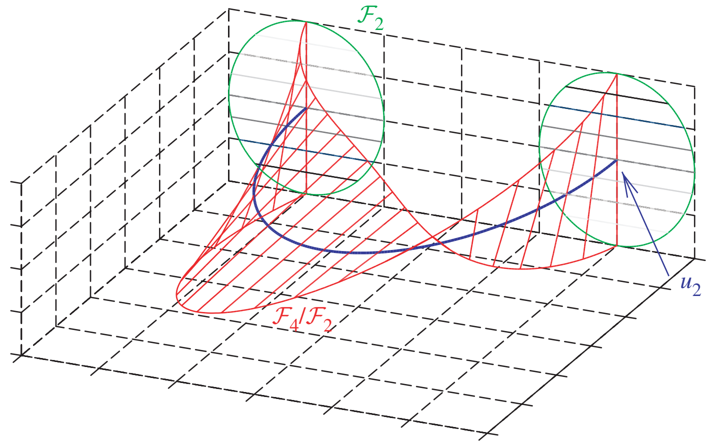

Figure 6. Torus representing $W^u(\mathcal{F}_4)^{\mathrm{T}}W^s(\mathcal{F}_2)$.

They rotate with the vertical manifold. There is a colour gradient (light grey to dark grey) included to illustrate the rotation of the heteroclinics. Note that the there is no rotation on the torus's surface.

To obtain the full picture we have to identify all points on the surface of the torus with the S^1 labelled with $\mathcal{F}_2$, and hence retract the torus surface to the S^1 without rotating it!

6. Conclusions and discussion

Building on earlier results of Fan and Hale [5], Sinestrari [10] and Härterich [7] and others, this paper closes on of the last remaining gaps in the full dynamic description of the global attractor of hyperbolic balance laws. The introduction of finite-dimensional sub-attractors in § 3 allowed us to overcome difficulties coming from the infinite-dimensional nature of the global attractor.

Theorem 4.2 and corollary 4.6 provide explicit parametrizations of all finite-dimensional sub-attractors of the global attractor and allow a geometric interpretation of the results as given in the examples section.

A remaining question concerns the uniqueness of heteroclinic connections in situations where the assumption in theorem 4.2(f) is violated. It is unclear whether convexity of f and monotonicity of the profiles ϕ is enough to guarantee uniqueness of heteroclinic connections in case more than two shocks meet to form a stationary shock.

In addition, the question remains how to describe the remaining part of the global attractor. We believe that, in principle, heteroclinic connections emanating from waves with infinite zero set can be treated analogously. A uniform explicit parametrization covering the whole attractor seems to be difficult, due to the infinite-dimensional nature of the global attractor.

Moreover, we believe that the introduced sub-attractors are a suitable tool to investigate the relation between global solutions of the hyperbolic balance law with global solutions of its parabolically regularized version the viscous balance laws given by

$$u_t + f(u)_x = \varepsilon u_{xx} + g(u). \tag{6.1}$$

This relation is explored in [2]. Due to the non-persistence result in [2], the relation of global solutions of the hyperbolic and parabolic equation is more complicated than one might expect. However, the sub-attractors help facilitating the description of that relation.

Finally, the explicit results on the structure of the connections between waves with finite zero number in this paper open an alternative door for the description of heteroclinic connections in the parabolically regularized (6.1) other than by proving invariant manifold results by spectral methods, which, despite serious efforts by many people in recent decades, is still an unsolved problem.

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