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ADMISSIBLE SOLUTIONS FOR DIRAC EQUATIONS WITH SINGULAR AND NON-MONOTONE NONLINEARITY

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Abstract By making use of Merle's general shooting method we investigate Dirac equations of the form

$$\begin{aligned} u' + \frac{2u}{r} &= v(F(v^2 - u^2) - (M - \omega)),\\ v' &= u(F(v^2 - u^2) - (M + \omega)). \end{aligned}$$

Here it is possible that $F(0) = -\infty$ and that $F(s)$ defined on $(0, +\infty)$ is not monotonously non-decreasing. Our results cover some known ones as a special case.

Keywords: admissible solutions; Dirac equations; singular and non-monotone nonlinearity; general shooting methods

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1. Introduction

We investigate the existence of stationary states for nonlinear Dirac equations of the form

$$\mathrm{i}\sum_{\mu=0}^{3} \gamma^\mu \partial_\mu \psi - M\psi + F(\bar{\psi}\psi)\psi = 0 \tag{1.1}$$

where $\psi\colon \mathbb{R}^4 \mapsto \mathbb{C}^4$, $\partial_\mu = \partial/\partial x_\mu$, M is a positive constant, $\bar{\psi}\psi = (\gamma^0\psi, \psi)$, $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{C}^4$ and γ^μ are complex-valued 4×4 matrices given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \quad \text{for } \mu = 1, 2, 3$$

and

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Solution ψ of the form $\psi(t, x) = \mathrm{e}^{-\mathrm{i}\omega t}\varphi(x)$ will be called a stationary state of (1.1), where $x_0 = t$, $x = (x_1, x_2, x_3)$ and $\omega > 0$ is a constant. This kind of solution was first investigated by Soler [**11**]. The equation for $\varphi\colon \mathbb{R}^3 \mapsto \mathbb{C}^4$ is

$$\mathrm{i}\sum_{\mu=0}^{3}\gamma^\mu\partial_\mu\psi - M\varphi + \omega\gamma^0\varphi + F(\bar{\varphi}\varphi)\varphi = 0 \tag{1.2}$$

As in [**11**, **12**] we seek solutions that are separable in spherical coordinates of the form

$$\varphi(x) = \begin{pmatrix} v(r) & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ \mathrm{i}u(r) & \begin{pmatrix} \cos\theta \\ \sin\theta \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix} \end{pmatrix}. \tag{1.3}$$

Here $r = |x|$ and (θ, ϕ) are the angular parameters. Equation (1.2) then becomes a non-autonomous planar dynamical system in the r variable, which is

$$u' + \frac{2u}{r} = v(F(v^2 - u^2) - (M - \omega)), \tag{1.4}$$

$$v' = u(F(v^2 - u^2) - (M + \omega)). \tag{1.5}$$

In the following we assume that $0 < M < \omega$. We investigate the so-called admissible positive solutions defined first by Balabane *et al*. in [**2**] as follows.

Definition 1.1. (u, v) is an admissible positive solution of system (1.4), (1.5) if there exists a real positive number R such that the following hold:

(i) $u, v \in C^1([0, R))$;

(ii) $0 < u(r) < v(r)$ for all $r \in (0, R)$;

(iii) $u(0) = 0$;

(iv) $v(r) \to 0$ as $r \to R$;

(v) (u, v) solves system (1.4), (1.5) on $(0, R)$.

It is easy to see that the number R is unique and it is called the radius of the solution (u, v). Let

$$H(u, v) = \frac{1}{2}\biggl(\int_0^{v^2-u^2} F(\sigma)\,\mathrm{d}\sigma - M(v^2 - u^2) + \omega(v^2 + u^2)\biggr)$$

and let Γ^0 be the connected component of $\{(u, v) \mid H(u, v) = 0,\ v \geqslant 0\}$ containing $(0, 0)$. The main result of this paper is the following theorem.

Theorem 1.2. *Assume that*

(F1) $F\colon (0, +\infty) \to (-\infty, 0)$ *is differentiable, and* F *is less than* $M - \omega$ *and is integrable near the origin* 0,

(F2) Γ^0 *is bounded,*

(F3) *if* $(0, v) \in \Gamma^0$, *then* $F(v^2) \neq M - \omega$.

Then system (1.4), (1.5) has an admissible solution.

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By verifying these assumptions one by one we have the following corollaries.

Corollary 1.3. *Assume (F1) and (F3) hold. And instead of (F2) we assume that*

(F2′) $H(0,v) \geqslant 0$ *if* $v \geqslant v_0$ *for some* $v_0 \in (0,+\infty)$.

Then (1.4), (1.5) has an admissible solution.

Corollary 1.4 (Balabane *et al.* [2, Theorem 1.2]). *Assume that* $F\colon (0,\infty) \to (-\infty,0)$ *is differentiable and* F *is integrable near the origin; also assume that* F *is non-decreasing and* $F(a^2) - (M-\omega) = 0$, $F'(a^2) > 0$ *for some* $a \in (0,+\infty)$.

Then (1.4), (1.5) has an admissible positive solution.

Equations (1.1) and (1.2), or system (1.4), (1.5), make sense in physics (see [**1–3**, **9**] and the references therein). The papers [**3**, **9**] were concerned with the case $F(0) = 0$, $M - \omega > 0$. Precisely, Cazenave and Vazquez [**3**] assumed further that $F \in C^1(\mathbb{R},\mathbb{R})$ is increasing, $F(x) > M+\omega$ for large x and $F'(a) > 0$ if $F(a^2) = M-\omega$ and $a > 0$. And by making use of shooting methods they showed that the system (1.4), (1.5) has a solution (u,v) on $[0,+\infty)$ satisfying

$$u(0) = 0, \qquad v(0) > 0, \qquad u(r) \leqslant v(r) \leqslant C\mathrm{e}^{-\delta r}, \quad r \in [0,+\infty),$$

where c and δ are positive constants. Merle [**9**] improved this result. He assumed further that F is bounded either from above or from below, and F also satisfies (F2) and (F3) as in Theorem 1.2. By making use of a more general shooting method he obtained a solution similar to the one in [**3**]. In [**2**], Balabane *et al.* investigated a different case: $F(0) = -\infty$, $M - \omega < 0$. They proved Corollary 1.4. Our Theorem 1.2 improves this result. In the proof we will use a general shooting method introduced by Merle in [**9**] as well as some results in [**2**]. In particular, we will use some properties of solutions of the following Hamiltonian system:

$$u' = v(F(v^2 - u^2) - (M-\omega)), \tag{1.6}$$

$$v' = u(F(v^2 - u^2) - (M+\omega)). \tag{1.7}$$

In particular, we use the property that the energy of its solution does not change. It was pointed out in [**3**] that seeking solutions of (1.1) other than of the form (1.3) by variational methods is interesting. Such work has been done by Esteban and Séré and others [**5**, **6**, **10**]. A good survey for recent results concerning Dirac equation has been given in [**4**, **7**].

In the next section we will do some preparation and in the last section we will prove Theorem 1.2.

2. Preliminary results

In this section we prepare for the proof of Theorem 1.2.

As in [**2**, **9**], let $F^0 = \{(u,v);\ \nabla H(u,v) = 0\}$, where

$$\nabla H(u,v) := \left(\frac{\partial}{\partial u}H(u,v), \frac{\partial}{\partial v}H(u,v)\right)$$

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expresses the gradient of H, $H_\tau(u,v) = H(u,v) - v^2/\tau$ and G_τ is the connected component of $\{(u,v) \mid H_\tau(u,v) = 0,\ v \geqslant 0\}$ containing $(0,0)$. For every $a > 0$, we denote by Γ_a^0 the connected component of $\{(u,v) \mid H(u,v) = H(0,a)\} \cap \{(u,v) \mid v > |u|\}$ containing $(0,a)$.

Lemma 2.1.

(i) *$\Gamma^0 \cap \{u = 0\} = \{(0,0),(0,a_0)\}$ for some $a_0 > 0$. Γ^0 divides $\mathbb{R}^2$ into two connected components. Let $\mathrm{Int}(\Gamma^0)$ and $\mathrm{Ext}(\Gamma^0)$ denote the bounded and unbounded components, respectively. Then, for any $(u,v) \in \mathrm{Int}(\Gamma^0)$, $H(u,v) < 0$.*

(ii) *$(u,v) \in F^0$ if and only if $u = 0$ and $F(v^2) = M - \omega$.*

(iii) *If (F3) is assumed, then $F(a_0^2) > M - \omega$ and $\mathrm{dist}(\Gamma^0, F^0) > 0$.*

(iv) *Assume $a > a_0$ and, for every $x \in [a_0,a]$, $F(x^2) - (M-\omega) > 0$. Then, for every $(u,v) \in \Gamma_a^0$, we have $|u| < v, v \in ((H(0,a)/\omega)^{1/2}, a]$ and $(u,v) \in \mathrm{Out}(\Gamma^0)$.*

(v) *For $1/\omega < \tau < 2/(\omega - m)$, G_τ is unbounded. And $\mathrm{Int}(\Gamma^0) \subset \{(u,v) \mid G_\tau(u,v) < 0\} \subset \{(u,v) \mid v > |u|\}$.*

Proof. (i) By definition, for every small $v > 0$,

$$H(0,v) = \frac{1}{2}\int_0^{v^2} (F(\sigma) - (M-\omega))\,\mathrm{d}\sigma < 0.$$

Because Γ^0 is bounded, there exists $v \in (0,+\infty)$ such that $H(0,v) = 0$. Then $a_0 := \inf\{v > 0 \mid H(0,v) = 0\}$ is a positive real number. For every $v \in (0,a_0)$, by definition we have that $H(0,v) < 0$, that $H(v,v) = \omega v^2 > 0$ and that $H(u,v)$ is increasing with respect to u^2; also, there exists uniquely $u(v) \in (0,v)$ such that $H(u(v),v) = 0$. By symmetry of H, $H(-u(v),v) = 0$ for every $v \in (0,a_0)$. Define $u(0) = 0 = u(a_0)$. Then $\Gamma^0 = \{(\pm u(v),v) \mid v \in [0,a_0]\}$ is bounded. And $\mathrm{Int}(\Gamma^0) = \{(u,v) \mid u \in (-u(v),+u(v)), v \in (0,a_0)\}$. By definition of $u(v)$, for every $(u,v) \in \mathrm{Int}(\Gamma^0)$, $H(u,v) < 0$.

(ii) This can be calculated directly.

(iii) By definition, $G(0,v) < 0$ for every $v \in (0,a_0)$, and $G(0,a_0) = 0$. So

$$a_0(F(a_0^2) - (M-\omega)) = \frac{\mathrm{d}}{\mathrm{d}v}G(0,v)\bigg|_{v=a_0} \geqslant 0.$$

Then $F(a_0^2) > M - \omega$ follows from (F3). Moreover, both Γ^0 and F^0 are compact. This and the fact that $\Gamma^0 \cap F^0$ is empty means $\mathrm{dist}(\Gamma^0, F^0) > 0$.

(iv) A simple calculation shows that

$$\frac{\partial}{\partial v^2}H(0,v) = F(v^2) - (M-\omega) > 0$$

for $v \in [a_0,a]$ and $H(v,v) = v^2\omega$. So $H(0,v) < H(0,a)$ for $v \in [0,a)$ and $H(0,v) < H(0,a) < H(v,v)$ for $v \in ((H(0,a)/\omega)^{1/2},a)$. Because $H(u,v)$ is increasing with respect to u^2, again there exists uniquely $u(v)$ satisfying $H(u_1(v),v) = H(0,a)$. If $a_0 \leqslant (H(0,a)/\omega)^{1/2}$, we have $(\pm u_1(v),v) \in \mathrm{Ext}(\Gamma_0)$. If $a_0 > (H(0,a)/\omega)^{1/2}$, then we have $u_1(v) > u(v), (\pm u_1(v),v) \in \mathrm{Ext}(\Gamma_0)$ for $v \in ((H(0,a)/\omega)^{1/2},a_0)$.

(v) For any prescribed v satisfying $1/\omega < \tau < 2/(\omega - m)$, $G_\tau(0, v) < 0$, $G_\tau(v, v) > 0$ if $v > 0$. For every $v > 0$ fixed, by the fact that

$$\frac{\mathrm{d}}{\mathrm{d}u} G_\tau(u, v) = \frac{\mathrm{d}}{\mathrm{d}u} G(u, v) > 0 \quad \text{when } u > 0,$$

there exists a unique $u = u_1(v) \in (0, v)$ such that $G_\tau(v, u_1(v)) = 0$. So $G_\tau = \{(v, \pm u_1(v)) \mid v \in [0, +\infty)\}$ with supplementary definition $u_1(0) = 0$ is unbounded. And compared with $u(v)$ defined in (i), this $u_1(v)$ obviously satisfies $u_1(v) > u(v)$. □

Lemma 2.2. *Assume $(u^0, v^0) \in \operatorname{Int}(\{(u, v) \mid v > |u|\})$ and let $\gamma = (u, v)$ be the solution of (1.6), (1.7) with the initial data (u^0, v^0), defined on maximum interval of existence (R_-, R_+). Let $[\gamma] := \{\gamma(t) \mid t \in (R_-, R_+)\}$.*

(i) *If $(u^0, v^0) \in \Gamma^0$, then $(u(R_-), v(R_-)) = (u(R_+), v(R_+)) = (0, 0)$.*

(ii) *If $(u^0, v^0) \in \Gamma^0_{a_1}$, where $a_1 > a_0$ and, for every $a \in [a_0, a_1]$, $F(a^2) - (M - \omega) \neq 0$, then $(u(R_-), v(R_-))$ and $(u(R_+), v(R_+))$ are different and are in the set $\{(u, v) \in \mathbb{R} \times (0, +\infty) \mid v = |u|\}$.*

Proof. We prove only (i). Because the system (1.6), (1.7) is a Hamiltonian one, the energy of a solution does not change: $H(u(t), v(t)) = H(u(0), v(0)) = H(u^0, v^0)$. That is, $(u(t), v(t)) \in \Gamma \setminus \{(0, 0)\}$ for every $t \in (R_-, R_+)$. If $u^0 > 0$, from (1.6) we have $v'(t) < 0$ and $(u(t), v(t))$ goes around Γ^0 anticlockwise towards $(0, 0)$ as $t \to R_+$. So $(u(R_+), v(R_+))$ does exist and is the unique ω-limit point of $(u(t), v(t))$. □

Remark 2.3. From our definitions and Lemma 2.2 (i) we have $[\gamma] = \Gamma^0_{a_1}$ if $(u^0, v^0) \in \Gamma^0_{a_1}$ and $\Gamma^0_{a_1} = \Gamma^0$.

For every $x > 0$ there exists a unique solution (u_x, v_x) of (1.4), (1.5) satisfying $(u_x(0), v_x(0)) = (0, x)$. Let $R_x > 0$ be such that $[0, R_x)$ is the largest existence interval. Then this solution belongs to $C^1([0, R_x), \mathbb{R}^2)$ and $v_x^2(r) - u_x^2(r) > 0$ for $r \in [0, R_x)$. By standard arguments, as in [**8**], we have the following lemma.

Lemma 2.4.

(i) *For any $x > 0$ we have either $R_x = \infty$ or $R_x < \infty$, and when $r \to R_x$ either $v_x^2(r) - u_x^2(r) \to 0$ or $|v_x(r)| + |u_x(r)| \to \infty$.*

(ii) *Let $x_n > 0$, $x > 0$, $x_n \to x$. Then for every $r \in (0, R_x)$ we have $R_{x_n} > r$ for n large, and $(u_{x_n}, v_{x_n}) \to (u_x, v_x)$ in $C^1([0, r], \mathbb{R}^2)$.*

Lemma 2.5. *Let $(u^0, v^0) \in \mathbb{R}^2$ and let (u_n^0, v_n^0) and r_n be such that $(u_n^0, v_n^0) \to (u^0, v^0)$ and $r_n \to +\infty$. We denote the solution of (1.6), (1.7) with initial data (u^0, v^0) by (u, v) and the solution of*

$$\begin{aligned} u_n' + \frac{2u_n}{r_n + r} &= v_n(F(v_n^2 - u_n^2) - (M - \omega)), \\ v_n' &= u_n(F(v_n^2 - u_n^2) - (M + \omega)), \\ (u_n(0), v_n(0)) &= (u_n^0, v_n^0) \end{aligned}$$

by (u_n, v_n).

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Let (S_n, R_n) and (S, R) be maximal existence intervals of (u_n, v_n) and (u, v), respectively. Then for every compact interval of (S, R):

(i) *(S_n, R_n) covers this interval when n is large enough;*

(ii) *(u_n, v_n) converges to (u, v) uniformly on this interval.*

Proof. This is similar to [**3**, Lemma 2.5]. □

Lemma 2.6. *For every $x > 0$, for all $r \in (0, R_x)$ we have*

$$\frac{\mathrm{d}}{\mathrm{d}r} H(u_x(r), v_x(r)) = \frac{2}{r} u_x^2(r)(F(v_x^2(r) - u_x^2(r)) - (m + \omega)) < 0.$$

3. Proof of the main result

In this section we prove Theorem 1.2. For every $x > 0$, let

$$\rho'_x = \sup\{\rho > 0 \mid u_x(r) > 0 \text{ for } r \in (0, \rho)\}.$$

Recall that Γ^0 divides $\mathbb{R}^2$ into two connected components and $\mathrm{Int}(\Gamma^0)$ and $\mathrm{Ext}(\Gamma^0)$ denote the bounded and unbounded components respectively. Define

$$I = \{x > a_0 \mid \exists \rho_x \in (0, \rho'_x) \colon \forall r \in (\rho_a, \rho'_a), (u_a(r), v_a(r)) \in \mathrm{Int}(\Gamma^0)\}. \tag{3.1}$$

Note that from Lemmas 2.1 (i) and 2.6, we can define I equivalently by

$$I = \{x > a_0 \mid \exists \rho_x \in (0, \rho'_x) \colon (u_a(\rho_x), v_a(\rho_x)) \in \mathrm{Int}(\Gamma^0)\}.$$

Lemma 3.1.

(i) *Let $\delta \in (0, \frac{1}{2}(m + \omega))$ be such that*

$$\alpha := \exp\left\{\frac{-4\omega}{\omega - \delta}\right\}\left(1 + \frac{\delta}{\omega}\right) - \frac{\delta}{\omega} > 0.$$

Then for every $x > 0$, $r \in [0, \min(\rho'_x, 1/(\omega - \delta)))$,

$$v_x^2(r) - u_x^2(r) \geqslant \alpha x^2.$$

(ii) *For every $x \in I$, if $\rho'_x > 1/(\omega - \delta)$, then $H_{1/(\omega-\delta)}(u_x(r), v_x(r)) \leqslant 0$ for every $r \in (1/(\omega - \delta), \rho'_x)$.*

Proof. (i) This can be seen directly from [**2**, Proposition 2.7] and its proof.

(ii) For $r \in (1/(\omega - \delta), \rho'_x)$, from Lemma 2.1 (i) and $u_x(r) - v_x(r) < 0$, we have

$$\frac{u_x(r)}{r} - (\omega - \delta) v_x(r) < 0.$$

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So

$$\frac{\mathrm{d}}{\mathrm{d}r}H_{1/(\omega-\delta)}(u_x(r),v_x(r)) = 2u_x(r)(F(v_x^2(r))-u_x^2(r))-(m+\omega)\left(\frac{u_x(r)}{r}-(\omega-\delta)v_x(r)\right) > 0.$$

Then $H_{1/(\omega-\delta)}(u_x(r),v_x(r))$ is increasing with respect to $t\in(1/(\omega-\delta),\rho'_x)$. Therefore, the result follows from the fact that $(u_x(r),v_x(r))\in\mathrm{Int}(\Gamma^0)$ and hence

$$H_{1/(\omega-\delta)}(u_x(r),v_x(r)) < H(u_x(r),v_x(r)) < 0$$

when $r\in(\rho_x,\rho'_x)$ in view of Lemma 2.1 (i) and the definitions of I and H_τ. □

Let

$$\Delta := \{(u,v)\in[0,+\infty)\times[0,+\infty) \mid H_{1/(\omega-\delta)}(u,v)\leqslant 0 \text{ or } v^2-u^2\geqslant \alpha a_0^2\}. \tag{3.2}$$

It is easy to check that $(\Delta-(0,0))\subset\{(u,v)\in\mathbb{R}^2 \mid |u|<v\}$. Moreover, from Lemma 3.1, $(u_x(r),v_x(r))\in\Delta$ for every $x\in I$, $r\in[0,\rho'_x)$. This will play a crucial role in the proof of Theorem 1.2.

Lemma 3.2. *I is not empty. And I is open.*

Proof. First, we prove that there exists $\epsilon>0$ such that $(a_0,a_0+\epsilon)\subset I$. In fact, from (1.4) we have $3u'_{a_0}(0)=a_0(F(a_0^2)-(M-\omega))>0$, and hence $u_{a_0}(r)>0$ for small $r>0$. Then, from Lemma 2.6, $H(u_{a_0}(r),v_{a_0}(r))<H(0,a_0)=0$ and $(u_{a_0}(r),v_{a_0}(r))\in\mathrm{Int}(\Gamma^0)$ for small r and hence for all $r\in(0,\rho'_{a_0})$. Fix some point $r_0\in(0,\rho'_{a_0})$. From Lemma 2.4 (ii), as $x\to a_0$ we have $\rho'_x>r_0$ and $(u_x(r_0),v_x(r_0))\to(u_{a_0}(r_0),v_{a_0}(r_0))$. Because $\mathrm{Int}(\Gamma^0)$ is open and $(u_{a_0}(r_0),v_{a_0}(r_0))$ is its element, there exists $\epsilon>0$ such that, for every $x\in(a_0,a_0+\epsilon)$, $(u_x(r_0),v_x(r_0))\in\mathrm{Int}(\Gamma^0)$. Again from Lemma 2.6, $H(u_x(r),v_x(r))<H(u_x(r_0),v_x(r_0))<0$ and $(u_x(r),v_x(r))\in\mathrm{Int}(\Gamma^0)$ for every $r\in(r_0,\rho'_x)$. This means that $x\in I$.

Second we prove I is open. Suppose $a\in I$. Then there exists $r_0\in(0,\rho'_a)$ such that $(u_a(r_0),v_x(r_0))\in\mathrm{Int}(\Gamma^0)$. As above, when x is near a, $(u_x(r_0),v_x(r_0))\in\mathrm{Int}(\Gamma^0)$ and $x\in I$. □

Lemma 3.3. *Assume that $F(v^2)-(M-\omega)>0$ for every $v\geqslant a_0$. Then I is bounded.*

Proof. We argue by contradiction. Assume that there exists a sequence $x_n\in I$ and $x_n\to+\infty$. Here and in what follows we denote u_{x_n}, v_{x_n} by u_n, v_n, respectively. By definition there exist some $a>a_0$ and $a-a_0$ small enough such that $(u_{x_n}(r_n),v_{x_n}(r_n))\in\Gamma_a^0\cap\Delta$ for some $r_n\in(0,\rho_{x_n})$, and $0<H(0,a)<H(u_{x_n}(r),v_{x_n}(r))$ for every $r\in[0,r_n)$.

Because Γ_a^0 is bounded, $(u_{x_n}(r_n),v_{x_n}(r_n))$ is also bounded. But, from Lemma 3.1, $u_{x_n}^2(r)+v_{x_n}^2(r)\geqslant\alpha x_n\to+\infty$ as $n\to\infty$ for every $r\in[0,1/(\omega-\delta))$. This means that $r_n>1/(\omega-\delta)$. We divide the following proof into two steps.

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Step 1. We claim that $r_n \to +\infty$ as $n \to +\infty$. In fact, for $r \in (1/(\omega-\delta), r_n)$, we have

$$u'_{x_n}(r) + \frac{2}{r}u_{x_n}(r) = v_{x_n}(r)(F(v^2_{x_n}(r) - u^2_{x_n}(r)) - (M-\omega)), \tag{3.3}$$

$$v'_{x_n}(r) = u_{x_n}(r)(F(v^2_{x_n}(r) - u^2_{x_n}(r)) - (M+\omega)). \tag{3.4}$$

Note that, for every $(u,v) \in \Delta \cap \{(u,v) \mid H(u,v) \geqslant H(0,a)\}$, we have $v^2 - u^2 \geqslant c > 0$ for some constant c. Thus, for every $r \in (1/(\omega-\delta), r_n)$, $1/r$ and $F(v^2_{x_n}(r) - u^2_{x_n}(r))$ are bounded. From (3.3), (3.4), there exists $k > 0$ such that $(u_{x_n}(r) + v_{x_n}(r))' + k(u_{x_n}(r) + v_{x_n}(r)) \geqslant 0$. We then have

$$(u_{x_n}(r_n) + v_{x_n}(r_n))\mathrm{e}^{kr_n} \geqslant \left(v_{x_n}\left(\frac{1}{\omega-\delta}\right) + u_{x_n}\left(\frac{1}{\omega-\delta}\right)\right)\exp\left\{\frac{k}{\omega-\delta}\right\}.$$

Thus, $r_n \to +\infty$ as $n \to +\infty$.

Step 2. We now obtain a contradiction by using the conservative system and the fact that Γ^0_a is a connected component of an equipotential contour and that some parts of it are not in Δ.

Since $\Gamma^0_a \cap \Delta$ is compact and $(u_{x_n}(r_n), v_{x_n}(r_n))$ are in this set, we assume $(u^0, v^0) \in \Gamma^0_a \cap \Delta$ is its limit. Let (u,v) be the unique solution with $(u(0), v(0)) = (u^0, v^0)$. Then $(u(t), v(t))$ covers the whole Γ^0_a. So there exists $t_0 < 0$ such that $(u(t_0), v(t_0))$ is not in Δ. From Lemma 2.5, for large n, $(u_{x_n}(r_n + t_0), v_{x_n}(r_n + t_0))$ is not in Δ either. This is a contradiction. □

Lemma 3.4. *Assume that $F^0 \cap \mathrm{Ext}(\Gamma^0)$ is not empty. Let b be the infimum of that set. Then there exists $\epsilon > 0$ such that $(b-\epsilon, b) \cap I$ is empty.*

Proof. Assume that $x_n \in I$, $x_n < b$ and $x_n \to b$. Then for the $\epsilon > 0$ as in the proof of Lemma 3.2, there exist $r_n \in (0, \rho'_{x_n})$ such that $(u_n(r_n), v_n(r_n)) \in \Gamma^0_{a_0+(\epsilon/2)} \cap \Delta$. So $\mathrm{dist}((u_n(r_n), v_n(r_n)), (0,b)) \geqslant \frac{1}{2}\epsilon$. From Lemma 2.4 (ii) it follows that $r_n \to +\infty$. Assume $(u_n(r_n), v_n(r_n)) \to (u^0, v^0)$ and let $(u(t), v(t))$ be the solution of (1.6) again with initial value (u^0, v^0). This will lead to a contradiction in a way similar to Lemma 3.3. □

Proof of Theorem 1.2. Let $c = \sup I$ if $F^0 \cap \mathrm{Ext}(\Gamma^0)$ is empty, and let $c = \sup I \cap (a_0, b)$ if $F^0 \cap \mathrm{Ext}(\Gamma^0)$ is not empty and b is defined as in Lemma 3.4. We prove that (u_c, v_c) is an admissible solution of (1.4), (1.5).

Because of the openness of I and the fact that c belongs to the boundary of I, we have that $u_c(r) > 0$ for $r \in (0, R_c)$. To finish the proof of Theorem 1.2, we need to prove only that $(u_c(r), v_c(r)) \to (0,0)$ as $r \to R_c$. From (1.5) $v_c(r)$ is decreasing on $(0, R_c)$. Because we also have $0 < u_c(r) < v_c(r)$ for $r \in [0, R_c)$, it is sufficient to prove that $\mathrm{dist}([\gamma_c], (0,0)) = 0$, where $\gamma = (u_c, v_c)$.

Suppose that $\mathrm{dist}((u_c(r), v_c(r)), (0,0)) \geqslant \alpha_1 > 0$ for every $r \in (0, R_c)$. Then from Lemma 2.4 (i) we have $R_c = +\infty$, and there exists $r_n \to +\infty$ such that $(u_c(r_n), v_c(r_n)) \to (u^0, v^0)$. Here (u^0, v^0) is also an element of Δ and $\mathrm{dist}((u^0, v^0), (0,0)) \geqslant \alpha_1 > 0$. From Lemma 2.6, $0 < H(u_c(t), v_c(r)) < H(0,c)$ for every $r \in (0, R_c)$. So we have $0 \leqslant H(u^0, v^0) < H(0,c)$. This means $(u^0, v^0) \in \Gamma^0_a$ for some $a \in (a_0, c)$ or $(u^0, v^0) \in \Gamma^0$.

By definition, there exists a sequence $x_n \to c$ as $n \to +\infty$ with $x_n \in I$ and $(u_n(r_n), v_n(r_n)) \to (u^0, v^0)$. We have, for all $r \in (0, r_n)$,

$$u_n^2(r) + v_n^2(r) \geqslant \alpha_1^2, \tag{3.5}$$

$$(u_n(r), v_n(r)) \in \Delta. \tag{3.6}$$

Let (u, v) be the solution of (1.6), (1.7) with initial value (u^0, v^0). There are two cases, both of which lead to contradictions.

Case 1 ($(u^0, v^0) \in \Gamma^0$). From Lemma 2.2 (i), $(u(R_-), v(R_-)) = (0, 0)$. So there exists some $t_0 < 0$ such that $u^2(r_0) + v^2(r_0) < \alpha_1^2$. From Lemma 2.5, for large n,

$$u_n^2(r_n + r_0) + v_n^2(r_n + r_0) < \alpha_1^2.$$

This contradicts (3.5).

Case 2 ($(u^0, v^0) \in \mathrm{Ext}(\Gamma^0)$). From Lemma 2.2 (ii), $|u(R_-)| = v(R_-) \neq 0$. So there exists some $t_0 < 0$ such that $(u(r_0), v(r_0))$ is not in Δ. From Lemma 2.5 again, for large n, $(u_n(r_n + r_0), v_n(r_n + r_0))$ is not in Δ either. This contradicts (3.6). The proof is finally complete. □

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References

1. M. Balabane, T. Cazenave, A. Douady and F. Merle, Existence of excited states for a nonlinear Dirac field, *Commun. Math. Phys.* **119** (1988), 153–176.
2. M. Balabane, T. Cazenave and L. Vazquez, Existence of standing waves for Dirac fields with singular nonlinearities, *Commun. Math. Phys.* **133** (1990), 53–74.
3. T. Cazenave and L. Vazquez, Existence of localized solutions for a classical nonlinear Dirac field, *Commun. Math. Phys.* **103** (1986), 35–47.
4. M. J. Esteban, M. Lewin and E. Séré, Variational methods in relativistic quantum mechanics, *Bull. Am. Math. Soc.* **45** (2008), 535–593.
5. M. J. Esteban and E. Séré, Stationary states of the nonlinear Dirac equation: a variational approach, *Commun. Math. Phys.* **171** (1995), 323–350.
6. M. J. Esteban and E. Séré, Solutions of the Dirac–Fock equations for atoms and molecules, *Commun. Math. Phys.* **203** (1999), 499–530.
7. M. J. Esteban and E. Séré, An overview on linear and nonlinear Dirac equations, *Discrete Contin. Dynam. Syst.* **8** (2002), 381–397.
8. P. Hartman, *Ordinary differential equations* (Birkhäuser, Boston, MA, 1982).
9. F. Merle, Existence of stationary states for nonlinear Dirac equations, *J. Diff. Eqns* **74** (1988), 50–68.
10. E. Paturel, A new variational principle for a nonlinear Dirac equation on the Schwarzschild metric, *Commun. Math. Phys.* **213** (2000), 249–266.
11. M. Soler, Classical stable nonlinear spinor field with positive rest energy. *Phys. Rev.* D **1** (1970), 2766–2769.
12. M. Wakano, Intensely localized solutions of the classical Dirac–Maxwell field equations, *Prog. Theor. Phys.* **35** (1966), 1117–1141.