Canad. J. Math. 2025, pp. 1–24
http://dx.doi.org/10.4153/S0008414X24001196


# Diffusive convective elliptic problem in variable exponent space and measure data

Safimba Soma, Ibrahime Konaté, and Adama Kaboré

*Abstract.* In this article, we study a class of convective diffusive elliptic problem with Dirichlet boundary condition and measure data in variable exponent spaces. We begin by introducing an approximate problem via a truncation approach and Yosida's regularization. Then, we apply the technique of maximal monotone operators in Banach spaces to obtain a sequence of approximate solutions. Finally, we pass to the limit and prove that this sequence of solutions converges to at least one weak or entropy solution of the original problem. Furthermore, under some additional assumptions on the convective diffusive term, we prove the uniqueness of the entropy solution.

## 1 Introduction

Solving partial differential equations and variational problems combined with assumptions of $p(x)$-growth has undergone significant evolution, both through the theoretical development of mathematics in Sobolev spaces with variable exponent and through their accuracy applications in modeling various real-word phenomena. Indeed, fluids that change their chemical properties when subjected to an electric field can be efficiently modeled in Sobolev spaces with variable exponents [1, 10, 24]. A Leray–Lions type operator with $p(.)$-growth also appears in biology, as it was discovered that blood exhibits electrorheological fluid properties. In [8], Chen *et al.* demonstrated the importance of such equations in image processing. For example, such operator can be used to search for a perfect image from a noisy one.

The aim of this article is to study the existence and uniqueness of solution for the following nonlinear elliptic problem.

$$(1.1)\qquad (\mathcal{P})\begin{cases} -\text{div}\, a(x,\nabla u) + \beta(u) + \text{div}\phi(u) \ni \mu \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \ \partial\Omega, \end{cases}$$

where $\Omega$ is an open bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $\beta$ is a maximal monotone graph with bounded domain on $\mathbb{R}$ (i.e., $\overline{\text{dom}(\beta)} = [m, M] \subset \mathbb{R}$) such that $0 \in \beta(0)$ and $\mu$ is a Radon diffuse measure.

In the literature, there are numerous works related to the problem $(\mathcal{P})$, but it's important to emphasize that none of these studies have addressed the problem $(\mathcal{P})$ under measure data and with $\text{div}\phi \neq 0$ simultaneously. Going into detail, when $\beta$ is

Received by the editors August 30, 2024; revised December 4, 2024; accepted December 16, 2024.
AMS Subject Classification: **36J60, 35J65, 35J20, 35J25.**
Keywords: Sobolev spaces, variable exponent, entropy solution, maximal monotone graph, Radon measure.

assumed to be a continuous and nondecreasing function with $div\phi = 0$, the authors in [6] proved the existence and uniqueness of entropy solution to the problem $(\mathcal{P})$ when the right-hand side datum belongs to $L^1$. For other works in the same direction, we refer to [2, 4, 5, 9, 11]. In the context of classical Sobolev space with constant exponent, Soma *et al.* [16] analyzed the existence and uniqueness of solution of problem $(\mathcal{P})$ when the convective diffusive term $\phi$ is null (see also [3]). Furthermore, they also obtained in [22] the existence and uniqueness of the entropy solution in the framework of variable exponent spaces and measure data. In the case of the right-hand side being in $L^1$, Wittbold and Zimmermann [26] used the bi-monotone technique and the comparison principle to prove the existence and uniqueness of the renormalized solution to the problem $(\mathcal{P})$.

The aim of this article is to extend the main results of [26] to the framework of measure data on the right-hand side. However, due to the lack of regularity in the measure data, we cannot use the same method, therefore, we must proceed differently. To achieve our goal, we first construct an approximate problem $(\mathcal{P}_\varepsilon)$ through approximation by truncation and Yosida regularization. Then, using the technique of maximal monotone operators in Banach spaces, we ensure the existence of a sequence of solutions to the problem $(\mathcal{P}_\varepsilon)$. We conclude by proving that this sequence of solutions converges to the solution of the problem $(\mathcal{P})$.

The remaining part of this article is organized as follows: In Section 2, we introduce some preliminary results that can be useful throughout the article. In Section 3, we present the necessary assumptions on the data of the problem and also we provide the main results. In Section 4, we prove the existence of at least one weak and/or entropy solution. In Section 5, we explore the question of uniqueness of the solution.

## 2 Preliminaires

Let $\Omega$ be a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. In this entire article, $p(.) : \overline{\Omega} \longrightarrow \mathbb{R}^+$ is a continuous function satisfying

(2.1) $$1 < p^- := \min_{x\in\overline{\Omega}} p(x) \leq p^+ := \max_{x\in\overline{\Omega}} p(x) < \infty.$$

We define the set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, the variable exponent Lebesgue space is defined by

$$L^{p(.)}(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u|^{p(x)} dx < \infty \right\}.$$

If the exponent is bounded, i.e., $p^+ < \infty$, then the expression

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

defines a norm in $L^{p(.)}(\Omega)$ called the Luxemburg norm. Then $(L^{p(.)}(\Omega), \|u\|_{p(.)})$ is a separable Banach space. Moreover, if $1 < p^- \le p^+ < \infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ in $\Omega$.

The $p(.)$-modular of the $L^{p(.)}(\Omega)$ space is the mapping $\rho_{p(.)} : L^{p(.)}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)} dx.$$

For any $u \in L^{p(.)}(\Omega)$, the following inequality (see [12, 13]) will be used later.

$$\min\left\{\|u\|_{p(.)}^{p^-}; \|u\|_{p(.)}^{p^+}\right\} \le \rho_{p(.)}(u) \le \max\left\{\|u\|_{p(.)}^{p^-}; \|u\|_{p(.)}^{p^+}\right\}. \tag{2.2}$$

For any $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$, we have the Hölder type inequality (see [19]).

$$\left|\int_\Omega uv dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right) \|u\|_{p(.)}\|v\|_{q(.)}. \tag{2.3}$$

If $\Omega$ is bounded and $p,\ q \in C_+(\overline{\Omega})$ such that $p(x) \le q(x)$ for any $x \in \Omega$, then the embedding $L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous (see [19, Theorem 2.8]).

***Proposition 2.1*** ([19]) *For $u_n, u \in L^{p(x)}(\Omega)$ and $p_+ < \infty$, the following assertions hold true.*

(i) $\|u\|_{p(.)} < 1$ *(resp,* $= 1, 1$*) if and only if* $\rho_{p(.)}(u) < 1$ *(resp,* $= 1, 1$*);*

(ii) $\|u\|_{p(.)} > 1$ *imply* $\|u\|_{p(.)}^{p_-} \le \rho_{p(.)}(u) \le \|u\|_{p(.)}^{p_+}$, *and* $\|u\|_{p(.)} < 1$ *imply* $\|u\|_{p(.)}^{p_+} \le \rho_{p(.)}(u) \le \|u\|_{p(.)}^{p_-}$;

(iii) $\|u_n\|_{p(.)} \to 0$ *if and only if* $\rho_{p(.)}(u_n) \to 0$, *and* $\|u_n\|_{p(.)} \to \infty$ *if and only* $\rho_{p(.)}(u_n) \to \infty$.

Now, we define the variable exponent Sobolev space as follows

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

with the norm

$$||u||_{1,p(.)} = \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

For a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we introduce the following notation

$$\rho_{1,p(.)}(u) := \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

We denote by $W_0^{1,p(.)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$.

The Sobolev exponent is defined as $p^*(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ and $p^*(x) = \infty$ if $p(x) \ge N$.

***Proposition 2.2*** (see [25, 27]) *For $u \in W^{1,p(.)}(\Omega)$, the following properties hold*

(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p^-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p^+}$;

(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p^+} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p^-}$;

(iii) $\|u\|_{1,p(.)} < 1$ *(respectively,* $= 1$, $1$*)* $\Longleftrightarrow \rho_{1,p(.)}(u) < 1$ *(respectively,* $= 1$, $1$*).*

***Theorem 2.3*** ([13, 14])

(i) *Assuming $1 < p_- \le p^+ < \infty$, the space $W^{1,p(.)}(\Omega)$ is a separable and reflexive Banach space.*

(ii) *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$, then the embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.*

(iii) *Poincaré inequality: there exists a constant $C > 0$, such that*

$$\|u\|_{p(.)} \le C\|\nabla u\|_{p(.)}, \ \forall u \in W_0^{1,p(.)}(\Omega).$$

(iv) *Sobolev–Poincaré inequality: there exists a constant $C > 0$, such that*

$$\|u\|_{p^*(.)} \le C\|\nabla u\|_{p(.)}, \ \forall u \in W_0^{1,p(.)}(\Omega).$$

***Remark 2.4*** By (iii) of Theorem 2.3, we deduce that $\|\nabla u\|_{p(.)}$ and $\|u\|_{1,p(.)}$ are equivalent norms in $W_0^{1,p(.)}(\Omega)$.

We denote by $\mathcal{L}^N$ the $N$-dimensional Lebesgue measure of $\mathbb{R}^N$ and by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures in $\Omega$, equipped with its standard norm $\|.\|_{\mathcal{M}_b(\Omega)}$. Note that, if $\mu$ belongs to $\mathcal{M}_b(\Omega)$, then $|\mu|(\Omega)$(the total variation of $\mu$) is a bounded positive measure on $\Omega$.

Given $\mu \in \mathcal{M}_b(\Omega)$, we say that $\mu$ is diffuse with respect to the capacity $W_0^{1,p(.)}(\Omega)$ ($p(.)$-capacity for short) if $\mu(A) = 0$, for every set $A$ such that $Cap_{p(.)}(A, \Omega) = 0$.

For every $A \subset \Omega$, we denote

$$S_{p(.)}(A) = \{u \in W_0^{1,p(.)}(\Omega) \cap C_0(\Omega) : u = 1 \text{ on } A, u \ge 0 \text{ on } \Omega\}.$$

The $p(.)$-capacity of every subset $A$ with respect to $\Omega$ is defined by

$$Cap_{p(.)}(A, \Omega) = \inf_{u \in S_{p(.)}(A)} \left\{ \int_\Omega |\nabla u|^{p(x)} dx \right\}.$$

In the case $S_{p(.)}(A) = \emptyset$, we set $Cap_{p(.)}(A, \Omega) = +\infty$.

The set of bounded Radon diffuse measure in the variable exponent setting is denoted by $\mathcal{M}_b^{p(.)}(\Omega)$. We will use the following decomposition result of bounded Radon diffuse measure due to Nyanquini *et al.* (see [22]).

***Theorem 2.5*** *Let $p(.) : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous function and $\mu \in \mathcal{M}_b(\Omega)$. Then $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$ if and only if $\mu \in L^1(\Omega) + W^{-1,p'(.)}(\Omega)$.*

***Lemma 2.6*** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ ($N \ge 1$). If $u \in W_0^{1,p(x)}(\Omega)$, then*

$$\int_\Omega div(u) dx = 0.$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

If $\gamma$ is a maximal monotone operator defined on $\mathbb{R}$, by $\gamma_0$ we denote the main section of $\gamma$; i.e.,

$$\gamma_0(s) = \begin{cases} \text{minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \varnothing \\ +\infty \text{ if } [s, +\infty) \cap D(\gamma) = \varnothing \\ -\infty \text{if } (-\infty, s] \cap D(\gamma) = \varnothing. \end{cases}$$

We also recall an important result on convergence (see [22]).

***Lemma 2.7*** *Let $(\beta_n)_{n\geq 1}$ be a sequence of maximal monotone graphs such that $\beta_n \to \beta$ in the sense of the graph (for $(x, y) \in \beta$, there exists $(x_n, y_n) \in \beta_n$ such that $x_n \to x$ and $y_n \to y$). We consider two sequences $(z_n)_{n\geq 1} \subset L^1(\Omega)$ and $(w_n)_{n\geq 1} \subset L^1(\Omega)$.*

*We suppose that: $\forall n \geq 1, w_n \in \beta_n(z_n)$, $(w_n)_{n\geq 1}$ is bounded in $L^1(\Omega)$ and $z_n \to z$ in $L^1(\Omega)$. Then,*

$$z \in dom(\beta).$$

Throughout the article, we use the truncation function $T_k$, $(k > 0)$ defined by

$$T_k(s) = \max\{-k, \min\{k; s\}\}. \tag{2.4}$$

It is obvious that $\lim\limits_{k\to\infty} T_k(s) = s$ and $|T_k(s)| = \min\{|s|; k\}$.

We define $\mathcal{T}_0^{1,p(.)}(\Omega)$ as the set of the measurable function $u : \Omega \longrightarrow \mathbb{R}$ such that $T_k(u) \in W_0^{1,p(.)}(\Omega)$.

We denote by

$$H_\varepsilon = \min\left(\frac{s^+}{\varepsilon}; 1\right) \text{ and } sign_0^+(s) = \begin{cases} 1 \text{ if } s > 0, \\ 0 \text{ if } s \leq 0, \end{cases}$$

Remark that as $\varepsilon$ goes to 0, $H_\varepsilon(s)$ goes to $sign_0^+(s)$.

To outline our definition of solution and the principal results, we set

$$\text{int}(\text{dom}(\beta)) = (m, M) \text{ with } -\infty < m \leq 0 \leq M < +\infty.$$

For any $r \in \mathbb{R}$ and any measurable function $u$ on $\Omega$, $[u = r]$, $[u \leq r]$ and $[u \geq r]$ denote the set $\{x \in \Omega : u(x) = r\}$, $\{x \in \Omega : u(x) \leq r\}$, $\{x \in \Omega : u(x) \geq r\}$, respectively.

# 3 Assumptions and main results

## 3.1 Assumptions

We study the problem $(\mathcal{P})$ under the following assumptions on the data.

Let $\Omega$ be a bounded open domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary domain $\partial\Omega$.

We assume that $p(.)$ verifies (2.1) and $a : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ denotes a Carathéodory function satisfying the following conditions.

$(H_1)$ there exists a positive constant $C_1$ such that

$$|a(x, \xi)| \leq C_1\left(j(x) + |\xi|^{p(x)-1}\right), \tag{3.1}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where $j$ is a non-negative function in $L^{p'(.)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$;

$(H_2)$ for all $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$ and for every $x \in \Omega$,

$$(a(x,\xi) - a(x,\eta)).(\xi - \eta) > 0, \tag{3.2}$$

$(H_3)$ there exists a positive constant $C_2$ such that

$$a(x,\xi).\xi \geq C_2|\xi|^{p(x)}, \tag{3.3}$$

for $\xi \in \mathbb{R}^N$ and almost every $x \in \Omega$.

$(H_4)$ $\overline{\text{dom}(\beta)} = [m, M] \subset \mathbb{R}$ where $-\infty < m \leq 0 \leq M < +\infty$.

$(H_5)$ $\phi : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a continuous function with $\phi(0) = 0$ and there exists a constant $C_3 > 0$ such that

$$\forall s \in \mathbb{R}, \ |\phi(s)| \leq C_3|s|^{p(x)-1}. \tag{3.4}$$

## 3.2 Notions of solutions and main results

***Definition 3.1*** Let $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$. We say that a couple $(u,b) \in W_0^{1,p(.)}(\Omega) \times L^1(\Omega)$ is a weak solution of problem $(\mathcal{P})$ if there exists $\nu \in \mathcal{M}_b^{p(.)}(\Omega)$ satisfying $\nu \perp \mathcal{L}^N$ and

$$\begin{cases} u \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \ b \in \beta(u)\mathcal{L}^N - a.e. \text{ in } \Omega, \\ \nu^+ \text{ is concentrated on } \ [u = M], \\ \nu^- \text{ is concentrated on } \ [u = m], \end{cases} \tag{3.5}$$

such that

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx + \int_\Omega b\varphi dx + \int_\Omega \varphi d\nu - \int_\Omega \phi(u).\nabla\varphi dx = \int_\Omega \varphi d\mu, \tag{3.6}$$

for any $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

Moreover,

$$\lim_{n\to+\infty} \int_{\{n<|u|<n+1\}} |\nabla u|^{p(x)} dx = 0. \tag{3.7}$$

***Definition 3.2*** Let $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$. An entropy solution of problem $(\mathcal{P})$ is a couple $(u,b) \in W_0^{1,p(.)}(\Omega) \times L^1(\Omega)$ such that (3.5) holds and

$$\begin{aligned} \int_\Omega a(x,\nabla u).\nabla T_k(u-\varphi)dx + \int_\Omega bT_k(u-\varphi)dx - \int_\Omega \phi(u).\nabla T_k(u-\varphi)dx \\ \leq \int_\Omega T_k(u-\varphi)d\mu, \end{aligned} \tag{3.8}$$

where $k > 0$ and $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \in dom\beta$.

***Theorem 3.3*** *Assuming $(H_1) - (H_5)$ and $\mu \in \mathcal{M}_b^{p(.)}(\Omega)$. Then, the problem $(\mathcal{P})$ admits at least one renormalized solution in the sense of Definition 3.1.*

The connection between our notion of weak solution and the entropy solution is formulated as follows.

***Theorem 3.4*** *A solution of problem* $(\mathcal{P})$ *in the sense of Definition 3.1 is also an entropy solution.*

**Proof** Let $(u, b)$ be a weak solution of $(\mathcal{P})$ and $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ such that $\varphi \in dom(\beta)$.

For any $k > 0$, taking $T_k(u-\varphi)$ as a test function in (3.6) one obtains

$$\int_\Omega a(x,\nabla u).\nabla T_k(u-\varphi)dx - \int_\Omega \phi(u).\nabla T_k(u-\varphi)dx + \int_\Omega bT_k(u-\varphi)dx + \int_\Omega T_k(u-\varphi)d\nu = \int_\Omega T_k(u-\varphi)d\mu. \tag{3.9}$$

Neglecting the positive term $\int_\Omega T_k(u-\varphi)d\nu$ (see [18]), we obtain (3.8). ∎

## 4 Existence of solution for a regular right hand side data

In this section we study the following problem

$$(P^\phi_{g,\gamma})\begin{cases} -\text{div}\, a(x,\nabla u) + g(u) + \text{div}\phi(u) \ = \gamma \ \text{ in } \ \Omega \\ u = 0 \text{on } \ \partial\Omega, \end{cases} \tag{4.1}$$

where $g$ is a continuous and nondecreasing function such that $g(0) = 0$ and $\gamma \in L^\infty(\Omega)$.

***Theorem 4.1*** *Under assumptions* $(H_1)-(H_3)$*, the problem* $(P^\phi_{g,\gamma})$ *admits at least one weak solution in the following sense:*

$u \in W_0^{1,p(.)} \cap L^\infty(\Omega)$, $g(u) \in L^\infty(\Omega)$ *and*

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx + \int_\Omega g(u)\varphi dx - \int_\Omega \phi(u).\nabla\varphi dx = \int_\Omega \varphi\gamma dx. \tag{4.2}$$

**Proof** For any $k > 0$, let us consider the following problem

$$(P^\phi_{T_k(g),\gamma})\begin{cases} -\text{div}\, a(x,\nabla u_k) + T_k(g(u_k)) + \text{div}\phi(u_k) \ = \gamma \ \text{ in } \ \Omega \\ u_k = 0 \text{on } \ \partial\Omega, \end{cases}$$

***Theorem 4.2*** *Under assumptions* $(H_1)-(H_3)$*, the problem* $(P^\phi_{T_k(g),\gamma})$ *admits at least one weak solution in the following sense:*

$u \in W_0^{1,p(.)}(\Omega)$ *and*

$$\int_\Omega a(x,\nabla u_k).\nabla\varphi dx + \int_\Omega T_k(g(u_k))\varphi dx - \int_\Omega \phi(u_k).\nabla\varphi dx = \int_\Omega \varphi\gamma dx. \tag{4.3}$$

*for any* $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

*Moreover*

$$\forall k > \|\gamma\|_\infty,\ |g(u_k)| \leq \|\gamma\|_\infty \text{ a.e. in } \Omega. \tag{4.4}$$

∎

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

**Proof** We define the operators $A_1, A_2$ and $A := A_1 + A_2$, acting from $W_0^{1,p(.)}(\Omega)$ into its dual $W^{-1,p'(.)}(\Omega)$ as follows

$$\langle A_1 u, \varphi\rangle = \int_\Omega \Big(a(x,\nabla u) - \phi(u)\Big).\nabla\varphi dx, \ \forall\, u,\varphi \in W_0^{1,p(.)}(\Omega)$$

and

$$\langle A_2 u, \varphi\rangle = \int_\Omega T_k(g(u))\varphi dx, \ \forall\, u,\varphi \in W_0^{1,p(.)}(\Omega).$$

We have

$$\begin{aligned}\left|\int_\Omega T_k(g(u))\varphi dx\right| &\leq \int_\Omega |T_k(g(u))||\varphi| dx\\ &\leq k\int_\Omega |\varphi| dx\\ &\leq k\left(\frac{1}{p_-}+\frac{1}{p'_-}\right)(\text{meas}(\Omega)+1)^{\frac{1}{p'_-}}\|\varphi\|_{p(.)}\\ &\leq C_4\|\varphi\|_{1,p(.)}.\end{aligned}$$

According to $(H_1)$, one has

$$\begin{aligned}\left|\int_\Omega a(x,\nabla u).\nabla\varphi dx\right| &\leq \int_\Omega |a(x,\nabla u)||\nabla\varphi| dx\\ &\leq C_1\int_\Omega j(x)|\nabla\varphi| dx + C_1\int_\Omega |\nabla u|^{p(x)-1}|\nabla\varphi| dx\\ &\leq C_1\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\left(\|j\|_{p'(.)} + \||\nabla u|\|_{p(.)}^{p(x)-1}\right)\|\nabla\varphi\|_{p(.)}\\ &\leq C_5\|\varphi\|_{1,p(.)}.\end{aligned}$$

By using the growth condition $(H_5)$ on $\phi$, one obtains

$$\begin{aligned}\left|\int_\Omega \phi(u).\nabla\varphi dx\right| &\leq \int_\Omega |\phi(u)||\nabla\varphi| dx\\ &\leq C_3\int_\Omega |u|^{p(x)-1}|\nabla\varphi| dx\\ &\leq C_3\left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)\||u|^{p(x)-1}\|_{p'(.)}\|\nabla\varphi\|_{p(.)}\\ &\leq C_6\|\varphi\|_{1,p(.)}.\end{aligned}$$

**Claim 1:** the operator $A$ is bounded.

Indeed, for any $u, \varphi \in W_0^{1,p(.)}(\Omega)$, one has

$$\Big|\langle Au, \varphi\rangle\Big| \leq C_7\|\varphi\|_{1,p(.)}.$$

Therefore, $A$ is bounded.

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

**Claim 2**: $A$ is coercive.

Indeed, since the divergence theorem implies $\int_\Omega \phi(u).\nabla u dx = 0$, and $\int_\Omega T_k(g(u))u dx \geq 0$, thanks to Proposition 2.1 and the Poincaré-type inequality, one obtains

$$\begin{aligned}\langle Au, u\rangle &= \int_\Omega a(x,\nabla u).\nabla u dx + \int_\Omega T_k(g(u))u dx - \int_\Omega \phi(u).\nabla u dx\\ &\geq \int_\Omega a(x,\nabla u).\nabla u dx\\ &\geq C_1 \int_\Omega |\nabla u|^{p(x)} dx\\ &\geq C_1 \|\nabla u\|^{\alpha}_{p(.)}\\ &\geq C_8 \|u\|^{\alpha}_{1,p(.)}\end{aligned}$$

where

$$\alpha = \begin{cases} p_+ \text{ if } \|\nabla u\|_{p(.)} \leq 1,\\ p_- \text{ if } \|\nabla u\|_{p(.)} > 1.\end{cases}$$

Thus, we obtain

$$\frac{\langle Au, u\rangle}{\|u\|_{1,p(x)}} \longrightarrow \infty \text{ as } \|u\|_{1,p(.)} \longrightarrow \infty.$$

**Claim 3**: $A_1$ is of type $(M)$.

Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(.)}(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u \text{ in } W_0^{1,p(.)}(\Omega) \text{ as } n \to \infty,\\ A_1 u_n \rightharpoonup \chi \text{ in } W^{-1,p'(.)}(\Omega) \text{ as } n \to \infty,\\ \lim_{n\to\infty} \sup\langle A_1 u_n, u_n\rangle \leq \langle \chi, u\rangle.\end{cases} \tag{4.5}$$

Let us set $h_n(x) = b(x, u_n, \nabla u_n)$ where $b(x,s,\varphi) = a(x,\varphi) - \phi(s)$, $\forall (x,s,\varphi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Then, one has

$$|h_n(x)| = |a(x,\nabla u_n) - \phi(u_n)| \leq C_9(j(x) + |u_n|^{p(x)-1} + |\nabla u_n|^{p(x)-1}), \tag{4.6}$$

where $C_9 = \max\{C_1, C_3\}$.

We aim to show that

$$\langle A_1 u_n, u_n\rangle \longrightarrow \langle \chi, u\rangle \text{ as } n \longrightarrow \infty, \text{ where } \chi = A_1 u.$$

Due to the compact embedding $W_0^{1,p(.)}(\Omega) \hookrightarrow\hookrightarrow L^{p(.)}(\Omega)$, one has $u_n \to u$ in $L^{p(.)}(\Omega)$ as $n \to \infty$ (up to a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$). Since $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,p(.)}(\Omega)$, one can deduce from (4.6) that $(h_n)_{n\in\mathbb{N}}$ is bounded in $(L^{p'(.)}(\Omega))^N$. Therefore, there exists a function $h \in (L^{p'(.)}(\Omega))^N$ such that

$$h_n \rightharpoonup h \text{ in } (L^{p'(.)}(\Omega))^N \text{ as } n \to \infty. \tag{4.7}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

For all $\varphi \in W_0^{1,p(.)}(\Omega)$, one has

$$\langle \chi, \varphi \rangle = \lim_{n\to\infty} \langle A_1 u_n, \varphi \rangle = \lim_{n\to\infty} \int_\Omega h_n(x).\nabla\varphi dx = \int_\Omega h(x).\nabla\varphi dx = \langle A_1 u, \varphi \rangle. \tag{4.8}$$

This implies that $\chi = A_1 u$.

Applying (4.5), one obtains

$$\limsup_{n\to+\infty} \langle A_1 u_n, u_n \rangle \leq \int_\Omega h(x).\nabla u dx. \tag{4.9}$$

Using $(H_2)$, for any $\varphi \in (L^{p(.)}(\Omega))^N$, one has

$$\int_\Omega (b(x, u_n, \nabla u_n) - b(x, u_n, \varphi)).(\nabla u_n - \varphi) dx$$
$$= \int_\Omega (a(x, \nabla u_n) - a(x, \varphi)).(\nabla u_n - \varphi) dx \geq 0.$$

This is equivalent to

$$\int_\Omega h_n.\nabla u_n dx - \int_\Omega h_n.\varphi dx - \int_\Omega b(x, u_n, \varphi).(\nabla u_n - \varphi) dx \geq 0. \tag{4.10}$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(.)}(\Omega)$, then up to a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$ one has $u_n \to u$ in $L^{p(.)}(\Omega)$, $u_n \to u$ a.e in $\Omega$ as $n \to \infty$, and $|u_n| \leq v \in L^{p(.)}(\Omega)$.

Since the function $b(x, s, \varphi)$ is continue with respect to $s$, on has

$$b(x, u_n, \varphi) \longrightarrow b(x, u, \varphi) \text{ a.e in } \Omega.$$

On the other hand, one has

$$|b(x, u_n, \varphi)| \leq C_9 \Big( j(x) + |v|^{p(x)-1} + |\varphi|^{p(x)-1} \Big) \in (L^{p'(.)}(\Omega))^N.$$

Then, using Lebesgue dominated convergence theorem, one obtains

$$b(x, u_n, \varphi) \longrightarrow b(x, u, \varphi) \text{ a.e in } (L^{p'(.)}(\Omega))^N.$$

Therefore, we have

$$\lim_{n\to\infty} \int_\Omega b(x, u_n, \varphi).(\nabla u_n - \varphi) dx = \int_\Omega b(x, u, \varphi).(\nabla u - \varphi) dx$$

and

$$\lim_{n\to 0} \int_\Omega h_n.\varphi dx = \int_\Omega h.\varphi dx.$$

Passing to the limit as $n \to \infty$ in (4.10) and using (4.9), we obtain

$$\int_\Omega \Big( h - b(x, u, \varphi) \Big).(\nabla u - \varphi) dx \geq 0. \tag{4.11}$$

By considering $\tilde{\varphi} \in (\mathcal{D}(\Omega))^N$ and replacing in (4.11) $\varphi$ by $\nabla u + t\tilde{\varphi}$, $t \in \mathbb{R}$, one obtains

$$(-t) \int_\Omega \Big( h - b(x, u, \nabla u + t\tilde{\varphi}) \Big).\tilde{\varphi} dx \geq 0. \tag{4.12}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

Dividing the above inequality by $t > 0$ and by $t < 0$, then letting $t$ go to 0, one can deduce from Lebesgue's dominated convergence theorem that

$$\int_\Omega \Big(h - b(x,u,\nabla u)\Big).\tilde{\varphi} dx = 0,$$

this implies that $h = b(x,u,\nabla u)$. Hence $A_1 u = \chi$.

**Claim 4**: $A_2$ is monotone and weakly continue.

Since for any $k > 0$, the function $T_k(g)$ is non-decreasing and satisfies $T_k(g(0)) = 0$, one has

$$\langle A_2 u - A_2 \varphi, u - \varphi \rangle \ = \int_\Omega \Big(T_k(g(u)) - T_k(g(\varphi))\Big)(u-\varphi)dx \geq 0.$$

Hence, $A_2$ is monotone.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,p(.)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p(.)}(\Omega)$ as $n \to \infty$. Then, for all $\varphi \in W_0^{1,p(.)}(\Omega)$, one has

$$\langle A_2 u_n - A_2 u, \varphi \rangle \ = \int_\Omega \Big(T_k(g(u_n)) - T_k(g(u))\Big)\varphi dx.$$

Since $u_n \rightharpoonup u$ in $W_0^{1,p(.)}(\Omega)$, up to a subsequence still denoted $(u_n)_{n\in\mathbb{N}}$, one has $u_n \to u$ in $L^{p(.)}(\Omega)$, $u_n \to u$ a.e in $\Omega$ as $n \to \infty$, and $|u_n| \leq \nu \in L^{p(.)}(\Omega)$.

By the continuity of the function $T_k(g)$, it follows that

$$\Big(T_k(g(u_n)) - T_k(g(u))\Big)\varphi \to 0 \text{ a.e in } \Omega \text{ as } n \to \infty.$$

Moreover,

$$\left|\Big(T_k(g(u_n)) - T_k(g(u))\Big)\varphi\right| \leq 2k|\varphi| \in L^1(\Omega).$$

Leveraging the Lebesgue dominated convergence theorem, one arrives at

$$\int_\Omega \Big(T_k(g(u_n)) - T_k(g(u))\Big)\varphi dx = 0.$$

Therefore, $A_2 u_n \rightharpoonup A_2 u$ as $n \to \infty$.

Since $A$ is the sum of an operator of type $(M)$ and a monotone, weakly continuous operator, $A$ is of type $(M)$. Adding the fact that $A$ is bounded and coercive, we conclude that $A$ is surjective.

Therefore, for any $L \in W^{-1,p'(.)}(\Omega)$, there exists at least one solution $u \in W_0^{1,p(.)}(\Omega)$ such that $A(u_k) = L$.

Setting $L(\varphi) = \int_\Omega \gamma\varphi dx$, we conclude that the problem $(P^{\phi}_{T_k(g),\gamma})$ admits at least one solution.

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

To complete the proof of Theorem 4.2, it remains to prove (4.4). To this end, we take $H_\varepsilon(u_k - R)$ as a test function in (4.2), where $\varepsilon > 0$ and $R > 0$ is a real to be specified later. One obtains

$$\int_\Omega a(x, \nabla u_k).\nabla H_\varepsilon(u_k - R)dx + \int_\Omega T_k(g(u))H_\varepsilon(u_k - R)dx - \int_\Omega \phi(u_k).\nabla H_\varepsilon(u_k - R)dx = \int_\Omega \gamma H_\varepsilon(u_k - R)dx. \tag{4.13}$$

For the first term of (4.13), one has

$$\int_\Omega a(x, \nabla u_k).\nabla H_\varepsilon(u_k - R)dx = \frac{1}{\varepsilon}\int_{\{|u_k - R| < \varepsilon\}} a(x, \nabla u_k).\nabla u_k \geq 0.$$

By setting $\psi_\varepsilon(u_k) = \int_0^{u_k} \phi(s)\chi_{\{0 \leq |u_k - R| < \varepsilon\}}(s)ds$, one obtains

$$\begin{aligned}\int_\Omega \phi(u_k).\nabla H_\varepsilon(u_k - R)dx &= \int_\Omega \frac{1}{\varepsilon}\phi(u_k).\nabla u_k \chi_{\{0 \leq |u_k - R| < \varepsilon\}}dx \\ &= \int_\Omega \nabla.\left(\int_0^{u_k} \phi(s)\chi_{\{0 \leq |u_k - R| < \varepsilon\}}(s)ds\right)dx \\ &= \int_{\partial\Omega} \psi_\varepsilon(u_k).\nu d\sigma = 0 \quad (\text{as } u_k = 0 \text{ on } \partial\Omega).\end{aligned}$$

Consequently, (4.13) becomes

$$\int_\Omega T_k(g(u))H_\varepsilon(u_k - R)dx \leq \int_\Omega \gamma H_\varepsilon(u_k - R)dx. \tag{4.14}$$

Using the inequality above, one can deduce (4.4) (see [15, 17, 22] for the details). ∎

Setting $k = k_0 = \|\gamma\|_\infty + 1$, Theorem 4.1 is a consequence of Theorem 4.2.

# 5 Proof of Theorem 3.3

This section is devoted to the proof of Theorem 3.3.

## 5.1 Approximate problem

For every $\varepsilon > 0$, we consider the Yosida regularization $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ of $\beta$ (see [7]), given by

$$\beta_\varepsilon = \frac{1}{\varepsilon}(I - (I + \varepsilon\beta)^{-1}).$$

We emphasize that the function $\beta_\varepsilon$ is both non-decreasing and Lipschitz-continuous

Since $\mu$ belongs to $\mathcal{M}_b^{p(\cdot)}(\Omega)$, so, by Theorem 2.5, it can be decomposed as $\mu = f - \operatorname{div}(F)$, where $f \in L^1(\Omega)$ and $F \in (L^{p'(\cdot)}(\Omega))^N$.

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

By introducing the function $f_\varepsilon(x) = T_{\frac{1}{\varepsilon}}(f(x))$ for $a.e.x \in \Omega$, the regularized form of the measure $\mu$ is given by

$$\mu_\varepsilon = f_\varepsilon - \nabla.\, F \text{ for any } \varepsilon > 0.$$

Therefore, one has $\mu_\varepsilon \in \mathcal{M}_b^{p(.)}(\Omega),\ \mu_\varepsilon \rightharpoonup \mu$ and $\mu_\varepsilon \in L^\infty(\Omega)$.

Then, we consider the following approximating scheme problem.

$$(5.1)\qquad Pb(\beta_\varepsilon, \phi)(\mu_\varepsilon)\begin{cases} \beta_\varepsilon(u_\varepsilon) - \operatorname{div} a(x, \nabla u_\varepsilon) + \operatorname{div}\phi(u_\varepsilon) = \mu_\varepsilon \text{ in } \ \Omega \\ u_\varepsilon = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

***Theorem 5.1*** *Let $(H_1) - (H_3)$ hold true. Then, the problem $Pb(\beta_\varepsilon, \phi)(\mu_\varepsilon)$ admits at least one weak solution $u_\varepsilon$ in the sense that $u_\varepsilon \in W_0^{1,p(.)}(\Omega)$, $\beta_\varepsilon(u_\varepsilon) \in L^1(\Omega)$ and $\forall \varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,*

$$(5.2)\qquad \int_\Omega a(x, \nabla u_\varepsilon).\nabla\varphi dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)\varphi dx - \int_\Omega \phi(u_\varepsilon).\nabla\varphi dx = \int_\Omega \varphi d\mu_\varepsilon.$$

**Proof** We just need to set $g = \beta_\varepsilon$ and $\gamma = \mu_\varepsilon$ in Theorem 4.1. ■

## 5.2 A priori estimates

Now, we derive a priori estimates for the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ which will enable us to obtain the necessary convergence results.

***Proposition 5.2*** *Let $k > 0$ and $u_\varepsilon$ be a solution to the problem $Pb(\beta_\varepsilon, \phi)(\mu_\varepsilon)$. Then,*

(i) *there exist a constant $C_{10} > 0$ such that*

$$(5.3)\qquad \int_{\{|u_\varepsilon|\le k\}} |\nabla u_\varepsilon|^{p(x)} dx \le C_{10},$$

(ii) *the sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$,*

(iii) *the sequence $(\beta_\varepsilon(T_k(u_\varepsilon)))_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$.*

**Proof** Taking $\varphi = T_k(u_\varepsilon)$ as a test function in (5.2) we obtain

$$(5.4)\qquad \begin{aligned} \int_\Omega a(x, \nabla u_\varepsilon).\nabla T_k(u_\varepsilon)dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)T_k(u_\varepsilon)dx - \int_\Omega \phi(u_\varepsilon).\nabla T_k(u_\varepsilon)dx \\ = \int_\Omega f_\varepsilon T_k(u_\varepsilon)dx + \int_\Omega F.\nabla T_k(u_\varepsilon)dx. \end{aligned}$$

The third term of (5.4) is zero. Indeed, we have

$$(5.5)\qquad \begin{aligned} \int_\Omega \phi(u_\varepsilon).\nabla T_k(u_\varepsilon)dx &= \int_\Omega \phi(T_k(u_\varepsilon)).\nabla T_k(u_\varepsilon)dx \\ &= \int_\Omega \nabla\left(\int_0^{T_k(u_\varepsilon)} \phi(s)ds\right)dx = 0, \end{aligned}$$

The remainder of the proof follows in the same manner as [23] (see also [18]). ■

***Proposition 5.3*** ([23]) *Let $u_\varepsilon$ be a weak solution of $Pb(\beta_\varepsilon, \phi)(\mu_\varepsilon)$ and let $k > 0$ large enough. Then, we have*

$$(5.6)\qquad meas\{|u_\varepsilon| > k\} \le \frac{C(\mu, \Omega)}{\min\{\beta_\varepsilon(k), |\beta_\varepsilon(-k)|\}}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

*and*

$$\text{(5.7)}\qquad meas\Big\{|\nabla u_\varepsilon| > k\Big\} \le \frac{C_{11}(k+1)}{k^{p^-}} + \frac{C(\mu,\Omega)}{\min\{\beta_\varepsilon(k), |\beta_\varepsilon(-k)|\}},$$

*where $C_{11}$ is a positive constant.*

## 5.3 Convergence results

***Proposition 5.4*** ([23]) *Let $u_\varepsilon$ be a weak solution of $Pb(\beta_\varepsilon,\phi)(\mu_\varepsilon)$. Then, there exists $u \in W_0^{1,p(.)}(\Omega) \subset \mathcal{T}_0^{1,p(.)}(\Omega)$ such that $u \in dom(\beta)$ a.e. in $\Omega$ and*

$$\text{(5.8)}\qquad u_\varepsilon \longrightarrow u \text{ in measure and a.e. in } \Omega \text{ as } \varepsilon \longrightarrow 0.$$

***Lemma 5.5*** *For every function $h \in W^{1,+\infty}(\mathbb{R})$, $h \ge 0$ with $supp(h)$ compact,*

$$\text{(5.9)}\qquad \limsup_{\varepsilon\to 0}\int_\Omega \Big[a(x,\nabla u_\varepsilon) - \phi(u_\varepsilon)\Big].\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\Big]dx \le 0,$$

$$\text{(5.10)}\qquad \limsup_{\delta\to 0}\limsup_{\varepsilon\to 0}\int_{\{\delta<|u_\varepsilon|<\delta+1\}} a(x,\nabla u_\varepsilon).\nabla u_\varepsilon dx \le 0$$

*and*

$$\text{(5.11)}\qquad \limsup_{\varepsilon\to 0}\int_\Omega a(x,\nabla u_\varepsilon).[\nabla T_k(u_\varepsilon) - \nabla T_k(u)]dx \le 0.$$

**Proof** By choosing $h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))$ as a test function in (5.2), one obtains

(5.12)
$$\begin{cases} \int_\Omega \Big[a(x,\nabla u_\varepsilon) - \phi(u_\varepsilon)\Big].\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\Big]dx \\ \quad + \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))dx \\ = \int_\Omega f_\varepsilon h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))dx + \int_\Omega F_\varepsilon.\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))\Big]dx. \end{cases}$$

• Let us start by proving (5.9). The following inequality holds

$$\text{(5.13)}\qquad \limsup_{\varepsilon\to 0}\int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))dx \ge 0.$$

Indeed, for any $r > 0$ sufficiently small we set

$$u_r = (u \wedge (M-r)) \vee (m+r).$$

According to [23], for any $k > 0$, $T_k(u_r) \in W_0^{1,p(.)}(\Omega)$, one has

$$\int_\Omega h(u_\varepsilon)(\beta_\varepsilon(u_\varepsilon) - \beta_\varepsilon(u_r))(T_k(u_\varepsilon) - T_k(u_r))dx \ge 0,$$

and

$$\begin{aligned} \int_\Omega \beta_\varepsilon(u_\varepsilon)h(u_\varepsilon)(T_k(u_\varepsilon) - T_k(u))dx &\ge \int_\Omega h(u_\varepsilon)\beta_\varepsilon(u_r)(T_k(u_\varepsilon) - T_k(u_r))dx \\ &\quad + \int_\Omega h(u_\varepsilon)\beta_\varepsilon(u_\varepsilon)(T_k(u_r) - T_k(u))dx \\ &=: I_{\varepsilon,r} + J_{\varepsilon,r}. \end{aligned}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

Having in mind that $m + r \leq u_r \leq M - r$, one has (see [23])

$$\limsup_{\varepsilon\to 0} I_{\varepsilon,r} = \int_\Omega h(u)\beta_0(u_r)(T_k(u) - T_k(u_r))dx \geq 0.$$

We treat the term $J_{\varepsilon,r}$ as follows

$$J_{\varepsilon,r} := \int_\Omega h(u_\varepsilon)\beta_\varepsilon(u_\varepsilon)(T_k(u_r) - T_k(u))dx := A_{\varepsilon,r} + B_{\varepsilon,r} + C_{\varepsilon,r} + D_{\varepsilon,r},$$

where

$$\begin{aligned}
A_{\varepsilon,r} &:= \int_\Omega h(u_\varepsilon)(T_k(u_r) - T_k(u))d\mu_\varepsilon,\\
B_{\varepsilon,r} &:= -\int_\Omega h(u_\varepsilon)a(x,\nabla u_\varepsilon).\nabla(T_k(u_r) - T_k(u))dx,\\
C_{\varepsilon,r} &:= -\int_\Omega h'(u_\varepsilon)(T_k(u_r) - T_k(u))a(x,\nabla u_\varepsilon).\nabla u_\varepsilon dx,\\
D_{\varepsilon,r} &:= -\int_\Omega \phi(u_\varepsilon).\nabla\Big[h(u_\varepsilon)(T_k(u_r) - T_k(u))\Big]dx.
\end{aligned}$$

According to [23], one has $\lim_{r\to 0} A_{\varepsilon,r} = \lim_{r\to 0} B_{\varepsilon,r} = \lim_{r\to 0} C_{\varepsilon,r} = 0$.

$$\begin{aligned}
D_{\varepsilon,r} &= \int_\Omega \phi(u_\varepsilon).\nabla\Big[h(u_\varepsilon)(T_k(u_r) - T_k(u))\Big]dx\\
&= \int_\Omega \phi(T_l(u_\varepsilon)).\nabla\Big[h(u_\varepsilon)(T_k(u_r) - T_k(u))\Big]dx,
\end{aligned}$$

where $l > 0$ is such that supp$h \subset ]-l, l[$.

Thanks to $(H_5)$, one has

$$|\phi(T_l(u_\varepsilon))| \leq |T_l(u_\varepsilon)|^{p(x)-1} \leq (l+1)^{p(x)-1} \leq (l+1)^{p^+-1}.$$

It follows that $(\phi(T_l(u_\varepsilon)))_\varepsilon$ is uniformly bounded. Adding the fact that $\nabla\Big[h(u_\varepsilon)(T_k(u_r) - T_k(u))\Big] \rightharpoonup 0$ in $(L^{p(\cdot)}(\Omega))^N$ (see [23]) as $r \to 0$, one obtains

$$\lim_{r\to 0} D_{\varepsilon,r} = \int_\Omega \phi(T_l(u_\varepsilon)).\nabla\Big[h(u_\varepsilon)(T_k(u_r) - T_k(u))\Big]dx = 0.$$

From above results, one deduces that $\lim_{r\to 0} J_{\varepsilon,r} = 0$ and (5.13).

Therefore, passing to the limit as $\varepsilon \to 0$ in (5.12), one obtains (5.9).

• Taking $\varphi_\delta(u_\varepsilon) = T_1(u_\varepsilon - T_\delta(u_\varepsilon))$ as test function in (5.2), one obtains

$$\begin{aligned}
&\int_\Omega a(x,\nabla u_\varepsilon).\nabla T_1(u_\varepsilon - T_\delta(u_\varepsilon))dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)T_1(u_\varepsilon - T_\delta(u_\varepsilon))dx\\
&\quad - \int_\Omega \phi(u_\varepsilon).\nabla T_1(u_\varepsilon - T_\delta(u_\varepsilon))dx\\
&= \int_\Omega f_\varepsilon T_1(u_\varepsilon - T_\delta(u_\varepsilon))dx + \int_\Omega F.\nabla T_1(u_\varepsilon - T_\delta(u_\varepsilon))dx. \qquad (5.14)
\end{aligned}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

On the other hand, one has

$$\int_\Omega \phi(u_\varepsilon).\nabla\varphi_\delta(u_\varepsilon)dx = \int_\Omega \nabla\left(\int_0^{\varphi_\delta(u_\varepsilon)} \phi((\varphi_\delta)^{-1}\circ(s))ds\right)dx = 0.$$

The rest of the proof of (5.10) and (5.11) follow the same lines as in [23]. ■

The following results are necessary for the sequel.

***Lemma 5.6*** *Let $u_\varepsilon$ be a weak solution of $Pb(\beta_\varepsilon,\phi)(\mu_\varepsilon)$ and $k>0$. Then*

$$\phi(T_k(u_\varepsilon)) \longrightarrow \phi(T_k(u)) \text{ in } L^{p'(.)}(\Omega) \text{ as } \varepsilon\to 0, \tag{5.15}$$

$$\lim_{\varepsilon\to 0}\int_\Omega \phi(u_\varepsilon).\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Big]dx = 0, \tag{5.16}$$

$$\lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))dx = 0 \tag{5.17}$$

*and*

$$\lim_{\varepsilon\to 0}\int_\Omega F_\varepsilon.\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Big]dx = 0. \tag{5.18}$$

**Proof** • Since $\phi(T_k(u_\varepsilon)) \longrightarrow \phi(T_k(u))$ a.e. in $\Omega$, the growth condition $(H_5)$ implies that

$$|\phi(T_k(u_\varepsilon))| \le C_3|T_k(u_\varepsilon)|^{p(x)-1} \in L^{p'(.)}(\Omega).$$

On the other hand, the sequence $(|T_k(u_\varepsilon)|^{p(x)-1})_{\varepsilon>0}$ is bounded in $L^{p'(.)}(\Omega)$ and $|T_k(u_\varepsilon)|^{p(.)-1} \longrightarrow |T_k(u)|^{p(.)-1}$ in $L^{p'(.)}(\Omega)$ as $\varepsilon\to 0$. Thanks to the generalized Lebesgue convergence theorem, we obtain (5.15).

• For $l>0$ such that supp$h \subset ]-l,l[$, one has
$\int_\Omega \phi(u_\varepsilon).\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Big]dx = \int_\Omega \phi(T_l(u_\varepsilon)).\nabla\Big[h(u_\varepsilon)(T_k(u_\varepsilon)-T_k(u))\Big]dx.$

Using the convergence (5.15), one deduces (5.16).

For the proofs of (5.17) and (5.18), see [23]. ■

***Proposition 5.7*** [23] *Let $u_\varepsilon$ be a weak solution of $Pb(\beta_\varepsilon,\phi)(\mu_\varepsilon)$ with $k>0$. Then, as $\varepsilon\to 0$, we have*

(i) $a(x,\nabla T_k(u_\varepsilon)) \rightharpoonup a(x,\nabla T_k(u))$ *in* $(L^{p'(.)}(\Omega))^N$,
(ii) $\nabla T_k(u_\varepsilon)\to\nabla T_k(u)$ *a.e. in* $\Omega$,
(iii) $a(x,\nabla T_k(u_\varepsilon))\cdot\nabla T_k(u_\varepsilon) \longrightarrow a(x,\nabla T_k(u))\cdot\nabla T_k(u)$ *a.e. in* $\Omega$ *and strongly in* $L^1(\Omega)$,
(iv) $\nabla T_k(u_\varepsilon)\to\nabla T_k(u)$ *in* $(L^{p(.)}(\Omega))^N$.

***Remark 5.8*** Since $T_k$ is continuous, for $k>0$, it follows that $T_k(u_\varepsilon)\to T_k(u)$ a.e. in $\Omega$. Finally, applying Lemma 2.7, we deduce that for all $k>0$, $T_k(u)\in dom(\beta)$ a.e. in $\Omega$. Therefore, since $T_k(u)\in dom(\beta)$, we conclude that $u\in dom(\beta)$ $a.e.$ in $\Omega$, and sine $dom(\beta)$ is bounded, we have $u\in W_0^{1,p(.)}(\Omega)$.

***Lemma 5.9*** [23] *For any $h \in C_c^1(\mathbb{R})$ and $\xi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$,*

$$\nabla[h(u_\varepsilon)\varphi] \longrightarrow \nabla[h(u)\varphi] \text{ strongly in } L^{p(.)}(\Omega) \text{ as } \varepsilon \to 0.$$

## 5.4 Existence of solution

Let us introduce, for any $l_0 > 0$, the function $h_0$ defined by

(i) $h_0 \in C_c^1(\mathbb{R})$, $h_0(r) \geq 0$, for all $r \in \mathbb{R}$,
(ii) $h_0(r) = 1$ if $|r| \leq l_0$ and $h_0(r) = 0$ if $|r| \geq l_0 + 1$.

To demonstrate the Theorem 3.3, one chooses $h_0(u_\varepsilon)\varphi$ as a test function in (5.2) to obtain

$$\int_\Omega a(x, \nabla u_\varepsilon).\nabla(h_0(u_\varepsilon)\varphi dx - \int_\Omega \phi(u_\varepsilon).\nabla[h_0(u_\varepsilon)\varphi]dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)h_0(u_\varepsilon)\varphi dx$$

$$= \int_\Omega f_\varepsilon h_0(u_\varepsilon)\varphi dx + \int_\Omega F.\nabla[h_0(u_\varepsilon)\varphi]dx, \tag{5.19}$$

where $\varphi \in W_0^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

By applying the same arguments as [23], one can express

$$\lim_{\varepsilon \to 0} \int_\Omega a(x, \nabla u_\varepsilon).\nabla(h_0(u_\varepsilon)\varphi dx = \int_\Omega a(x, \nabla u).\nabla(h_0(u)\varphi dx$$

$$= \int_\Omega a(x, \nabla u).\nabla\varphi dx, \tag{5.20}$$

$$\lim_{\varepsilon \to 0} \int_\Omega f_\varepsilon h_0(u_\varepsilon)\varphi dx = \int_\Omega f h_0(u)\varphi dx = \int_\Omega f\varphi dx, \tag{5.21}$$

and

$$\lim_{\varepsilon \to 0} \int_\Omega F.\nabla[h_0(u_\varepsilon)\varphi]dx = \int_\Omega F.\nabla[h_0(u)\varphi]dx = \int_\Omega F.\nabla\varphi dx. \tag{5.22}$$

According to Lemma 5.9 and the convergence (5.15), one has

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega \phi(u_\varepsilon).\nabla[h_0(u_\varepsilon)\varphi]dx &= \int_\Omega \phi(T_{l_0+1}(u_\varepsilon)).\nabla[h_0(u_\varepsilon)\varphi]dx \\ &= \int_\Omega \phi(T_{l_0+1}(u)).\nabla[h_0(u)\varphi]dx \\ &= \int_\Omega \phi(u).\nabla[h_0(u)\varphi]dx \end{aligned}$$

Therefore,

$$\lim_{\varepsilon \to 0} \int_\Omega \phi(u_\varepsilon).\nabla[h_0(u_\varepsilon)\varphi]dx = \int_\Omega \phi(u).\nabla\varphi dx. \tag{5.23}$$

In order to pass to the limit in the sequence $(\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ as $\varepsilon$ goes to 0, we need the following lemmas.

***Lemma 5.10*** [20] *Let $j$ be a lower semi-continuous function on $\mathbb{R}$ with $\overline{dom(j)} = [m, M] \subset \mathbb{R}$, and let $j_\varepsilon$ be a sequence of lower semi-continuous functions such that*

$$j_\varepsilon(t) \geq 0, \forall t \in [m, M], \text{ and } j_\varepsilon \uparrow j \text{ as } \varepsilon \downarrow 0.$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

*Consider two sequences* $(v_\varepsilon)_{\varepsilon>0}$ *and* $(z_\varepsilon)_{\varepsilon>0}$ *of measurable functions on* $\Omega$ *satisfying*

$$\begin{cases} v_\varepsilon \longrightarrow v\mathcal{L}^N \text{ a.e in } \Omega, \ v \in dom(j)\mathcal{L}^N a.e. \text{ in } \Omega, \\ \forall \varepsilon > 0, \ z_\varepsilon \in \partial j_\varepsilon(v_\varepsilon)\mathcal{L}^N \text{ a.e in } \Omega. \end{cases}$$

*Assume that there exists* $z \in \mathcal{M}_b^{p(.)}(\Omega) \cap [(W^{1,p(.)}(\Omega))^* + L^1]$ *such that for all* $\varphi \in C_C^1(\Omega)$, $\varphi \geq 0$,

$$\liminf_{\varepsilon\to 0} \int_\Omega (t - v_\varepsilon)\varphi h_0(v_\varepsilon) z_\varepsilon dx \geq \int_\Omega (t-v)\varphi dz, \ \forall t \in \mathbb{R}. \tag{5.24}$$

*Then,*

$$\begin{cases} z = w\mathcal{L}^N + z_s \text{ with } v \perp \mathcal{L}^N, \ w \in \partial j(v)\mathcal{L}^N \text{ a.e in } \Omega, \ w \in L^1(\Omega), \\ z_s^+ \text{ is concentrated on } [u = M], \ z_s^- \text{ is concentrated on } [u = m]. \end{cases}$$

***Lemma 5.11*** *Let* $l_0 > 0$ *such that* $\mathcal{D}(\beta) = [m, M] \subset [-l_0, l_0]$. *Then, there exists* $\sigma \in \mathcal{M}_b^{p(.)}(\Omega)$ *such that* $h_0(u_\varepsilon)\beta_\varepsilon(u_\varepsilon) \overset{*}{\rightharpoonup} \sigma$, *as* $\varepsilon \to 0$.

**Proof** • According to Proposition 5.2-(ii), for any $k > 0$, the sequence $(h_0(u_\varepsilon)\beta_\varepsilon(u_\varepsilon))_{\varepsilon>0}$ is bounded in $L^1(\Omega)$. Then, there exists $\sigma \in \mathcal{M}_b(\Omega)$ such that $h_0(u_\varepsilon)\beta_\varepsilon(u_\varepsilon) \overset{*}{\rightharpoonup} \sigma$ in $\mathcal{M}_b(\Omega)$ as $\varepsilon \to 0$.

• One can write $\sigma \in \mathcal{M}_b^{p(.)}(\Omega) \cap (W^{-1,p'(.)} + L^1(\Omega))$.

Indeed, for any $\varphi \in \mathcal{D}(\Omega)$, one has

$$\begin{aligned}
\int_\Omega \varphi d\sigma &= \int_\Omega h_0(u)\varphi d\sigma = \lim_{\varepsilon\to 0}\int_\Omega h_0(u_\varepsilon)\varphi\beta_\varepsilon(u_\varepsilon)dx \\
&= -\lim_{\varepsilon\to 0}\int_\Omega \Big[a(x, \nabla T_{l_0+1}(u_\varepsilon)) - \phi(T_{l_0+1}(u_\varepsilon))\Big].\nabla[h_0(u_\varepsilon)\varphi]dx \\
&\quad + \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon h_0(u_\varepsilon)\varphi dx + \lim_{\varepsilon\to 0}\int_\Omega F.\nabla[h_0(u_\varepsilon)\varphi]dx \\
&= -\int_\Omega a(x, \nabla T_{l_0+1}(u)).\nabla[h_0(u)\varphi]dx + \int_\Omega \phi(T_{l_0+1}(u).\nabla[h_0(u)\varphi]dx \\
&\quad + \int_\Omega f h_0(u)\varphi dx + \int_\Omega F.\nabla[h_0(u)\varphi]dx \\
&= -\int_\Omega a(x,\nabla u).\nabla\varphi dx + \int_\Omega \phi(u).\nabla\varphi dx + \int_\Omega f\varphi dx + \int_\Omega F.\nabla\varphi dx.
\end{aligned}$$

Therefore, $\sigma = \text{div}a(x,\nabla u) - \text{div}\phi(u) + \mu$ in $\mathcal{D}'(\Omega)$ and $\sigma \in \mathcal{M}_b^{p(.)}(\Omega) \cap (W^{-1,p'(.)}(\Omega) + L^1(\Omega))$. ■

***Remark 5.12*** The measure $\sigma$ can be written as $\sigma = b\mathcal{L}^N + \nu$ with $\nu \perp \mathcal{L}^N$ such that all the properties of (3.5) hold.

Indeed, for any $\varphi \in C_c^1(\Omega)$, $t \in \mathbb{R}$, one has

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega (t - u_\varepsilon)h_0(u_\varepsilon)\varphi\beta_\varepsilon(u_\varepsilon)dx \\
&= -\lim_{\varepsilon\to 0}\int_\Omega \Big[a(x, \nabla T_{l_0+1}(u_\varepsilon)) - \phi(T_{l_0+1}(u_\varepsilon))\Big].\nabla[(t-u_\varepsilon)h_0(u_\varepsilon)\varphi]dx \\
&\quad + \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon(t-u_\varepsilon)h_0(u_\varepsilon)\varphi dx + \lim_{\varepsilon\to 0}\int_\Omega F.\nabla[(t-u_\varepsilon)h_0(u_\varepsilon)\varphi]dx
\end{aligned}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

$$= -\int_\Omega a(x,\nabla T_{l_0+1}(u)).\nabla[(t-u)h_0(u)\varphi]dx$$
$$+\int_\Omega \phi(T_{l_0+1}(u)).\nabla[(t-u)h_0(u)\varphi]dx$$
$$+\int_\Omega (t-u)fh_0(u)\varphi dx+\int_\Omega F.\nabla[(t-u)h_0(u)\varphi]dx$$
$$= -\int_\Omega a(x,\nabla u).\nabla[(t-u)h_0(u)\varphi]dx+\int_\Omega \phi(u).\nabla[(t-u)h_0(u)\varphi]dx$$
$$+\int_\Omega (t-u)fh_0(u)\varphi dx+\int_\Omega F.\nabla[(t-u)h_0(u)\varphi]dx$$

Setting $v_\varepsilon = u_\varepsilon$ and $z_\varepsilon = \beta_\varepsilon$ in Lemma 5.10, one can deduce (3.5). Since $v = (f-b) - div(a(x,\nabla u) - F)$ in $\mathcal{D}'(\Omega)$, one has also $v \in \mathcal{M}_b^{p(.)}(\Omega))$.

Using the results above, by letting $\varepsilon \to 0$, one obtains

$$\int_\Omega a(x,\nabla u).\nabla\varphi dx - \int_\Omega \phi(u).\nabla\varphi dx + \int_\Omega b\varphi dx + \int_\Omega \varphi dv = \int_\Omega \varphi d\mu. \tag{5.25}$$

Now, we focus on the proof of (3.7) to end the demonstration.

For that, one chooses $T_1(u_\varepsilon - T_n(u_\varepsilon))$ as test function in (5.2) to obtain

$$\int_\Omega a(x,\nabla u_\varepsilon).\nabla T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_\Omega \beta_\varepsilon(u_\varepsilon)T_1(u_\varepsilon - T_n(u_\varepsilon))dx$$
$$+\int_\Omega \phi(u_\varepsilon).\nabla T_1(u_\varepsilon - T_n(u_\varepsilon))dx = \int_\Omega T_1(u_\varepsilon - T_n(u_\varepsilon))d\mu_\varepsilon. \tag{5.26}$$

Observing that $\int_\Omega \beta_\varepsilon(u_\varepsilon)T_1(u_\varepsilon - T_n(u_\varepsilon))dx \geq 0$ and $\nabla T_1(u_\varepsilon - T_k(u_\varepsilon)) = \nabla u_\varepsilon \chi_{\{n<|u_\varepsilon|<n+1\}}$, (5.26) becomes

$$\int_{\{n<|u_\varepsilon|<n+1\}} a(x,\nabla u_\varepsilon).\nabla u_\varepsilon dx + \int_{\{n<|u_\varepsilon|<n+1\}} \phi(u_\varepsilon).\nabla u_\varepsilon dx$$
$$\leq \int_\Omega f_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_{\{n<|u_\varepsilon|<n+1\}} F.\nabla u_\varepsilon dx.$$

Using $(H_3)$, we deduce that

$$C_3 \int_{\{n<|u_\varepsilon|<n+1\}} |\nabla u_\varepsilon|^{p(x)} dx + \int_{\{n<|u_\varepsilon|<n+1\}} \phi(u_\varepsilon).\nabla u_\varepsilon dx$$
$$\leq \int_\Omega f_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_{\{n<|u_\varepsilon|<n+1\}} F.\nabla u_\varepsilon dx. \tag{5.27}$$

Let us consider $\Phi(t) = \int_0^t \phi(\tau)d\tau$. Then $\Phi(T_n(u_\varepsilon)) \in (W_0^{1,p(x)}(\Omega))^N$,

By using Lemma 2.6, one gets

$$\int_{\{n<|u_\varepsilon|<n+1\}} \phi(u_\varepsilon).\nabla u_\varepsilon dx$$
$$= \int_\Omega \phi(T_{n+1}(u_\varepsilon)).\nabla T_{n+1}(u_\varepsilon)dx - \int_\Omega \phi(T_n(u_\varepsilon)).\nabla T_n(u_\varepsilon)dx$$
$$= \int_\Omega \operatorname{div}\Phi((T_{n+1}(u_\varepsilon)))dx - \int_\Omega \operatorname{div}\Phi(T_n(u_\varepsilon))dx = 0. \tag{5.28}$$

Consequently, (5.27) becomes

$$C_3\int_{\{n<|u_\varepsilon|<n+1\}}|\nabla u_\varepsilon|^{p(x)}dx \le \int_\Omega f_\varepsilon T_1(u_\varepsilon - T_n(u_\varepsilon))dx + \int_{\{n<|u_\varepsilon|<n+1\}} F.\nabla u_\varepsilon dx.$$

Arguing similarly as in [23], one obtains the rest of the proof of the condition (3.7).

***Lemma 5.13*** *Suppose that $\phi$ is a Lipschitz function. let $s \in W_0^{1,p(.)}(\Omega)$, $\sigma$ in $\mathcal{M}_b^{p(.)}(\Omega)$ and $\lambda \in \mathbb{R}$ such that*

$$\begin{cases} s \le \lambda \text{ a.e. in } \Omega \text{ (resp. } s \ge \lambda) \\ \sigma = -div\, a(x,\nabla s) + div\, \phi(s) \text{ in } \mathcal{D}'(\Omega). \end{cases} \tag{5.29}$$

*Then,*

$$\int_{[s=\lambda]} \varphi d\sigma \ge 0, \tag{5.30}$$

*(resp.)*

$$\int_{[s=\lambda]} \varphi d\sigma \le 0, \tag{5.31}$$

*for any $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$.*

**Proof** For $n \ge 1$, we consider the function $\theta_n$ defined by

$$\theta_n(r) = \inf\{1, (nr - n\lambda + 1)^+\}, \ \forall r \in \mathbb{R}.$$

Note that $\theta_n(r)$ converges to $\chi_{[\lambda,\infty)}(r)$ for every $r \in \mathbb{R}$, so $\theta_n(s(x))$ converges to $\chi_{[\lambda,\infty)}(s(x))$ at every $x$ where $s(x)$ is defined.

Since $s$ is defined quasi everywhere and $\chi_{[\lambda,\infty)} \circ s = \chi_{\{x\in\Omega: s(x)=\lambda\}}$, then the convergence of $\theta_n(s)$ to $\chi_{[\lambda,\infty)}(s)$ is quasi everywhere.

Therefore, since $\sigma$ is diffuse, then $\theta_n(s)$ converges to $\chi_{\{x\in\Omega:s(x)=\lambda\}}$, $\sigma$-a.e. in $\Omega$.

$\forall \varphi \in C_c^1(\Omega)$ such that $\varphi \ge 0$, one has

$$\begin{aligned} \int_{[s=\lambda]} \varphi d\sigma &= \lim_{n\to+\infty}\int_\Omega \varphi\theta_n(s)d\sigma \\ &= \lim_{n\to+\infty}\int_\Omega a(x,\nabla s).\nabla[\varphi\theta_n(s)]dx + \lim_{n\to+\infty}\int_\Omega \text{div}\, \phi(s)(\varphi\theta_n(s))dx \\ &\ge \int_\Omega \theta_n(s)a(x,\nabla s).\nabla\varphi dx + \lim_{n\to+\infty}\int_\Omega \text{div}\, \phi(s)(\varphi\theta_n(s))dx. \end{aligned}$$

Since $\phi$ is a Lipschitz function, one has

$$\int_\Omega \text{div}\, \phi(s)(\theta_n(s))dx = \int_\Omega (\theta_n(s))\phi'(s)\cdot\nabla s dx.$$

It follows that

$$\begin{aligned} \left|\int_\Omega \text{div}\, \phi(s)(\theta_n(s))dx\right| &= \left|\int_\Omega (\theta_n(s))\phi'(s)\cdot\nabla s dx\right| \\ &\le \|\varphi\|_\infty \int_{\{\lambda-\frac{1}{n}\le s\le\lambda\}} |\phi'(s)|\,|\nabla s|dx \\ &\longrightarrow 0 \text{ as } n \to +\infty. \end{aligned}$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

On the other hand, we have

$$\left|\int_\Omega \theta_n(s)a(x,\nabla s).\nabla\varphi dx\right| \leq \|\nabla\varphi\|_\infty \int_{\{\lambda-\frac{1}{n}\leq s\leq\lambda\}} |a(x,\nabla s)|dx$$
$$\longrightarrow 0 \text{ as } n\to+\infty.$$

Hence, the relation (5.30) holds.

In the case where $s\geq\lambda$, one reasons similarly as above after setting $\tilde{s}=-s$, $\tilde{\lambda}=-\lambda$ and $\tilde{a}(x,z)=a(x,-z)$ to obtain (5.31). ■

***Remark 5.14*** Moreover, if $\phi$ is a Lipschitz function, then a weak solution $u$ of problem $(\mathcal{P})$ satisfies

$$\nu^+\leq \mu_s\lfloor\,[u=M], \tag{5.32}$$

$$\nu^-\leq -\mu_s\lfloor\,[u=m]. \tag{5.33}$$

Indeed, since

$$\nu = \text{div}\, a(x,\nabla u) - \text{div}\,\phi(u) - b\mathcal{L}^N + \mu,$$

one has

$$\mu-\nu-b\mathcal{L}^N = -\text{div}\, a(x,\nabla u) + \text{div}\,\phi(u).$$

According to Lemma 5.13, the proof follows the same approach as in [21, Theorem 1.3]).

***Remark 5.15*** In the case where the right-hand side data is a regular function (for example, an $L^1$-function), one has $\mu_s=0$, so that $\nu^+=\nu^-=0$ and the notion of weak solution in this article coincides with the usual one.

## 6 Uniqueness of solution

The study of the uniqueness of the solution depends on additional conditions on the convection term

***Theorem 6.1*** *Let $\phi$ be a Lipschitz function. If $(u_1,b_1)$ and $(u_2,b_2)$ are two solutions of (4.1), then*

$$\int_\Omega (b_1-b_2)sign_0(u_1-u_2)dx = 0. \tag{6.1}$$

**Proof** By choosing $\varphi=u_2$ and $\varphi=u_1$ as tests functions in (3.8) for $(u_1,b_1)$ and $(u_2,b_2)$, respectively, we obtain

$$\int_\Omega a(x,\nabla u_1).\nabla T_k(u_1-u_2)dx + \int_\Omega b_1T_k(u_1-u_2)dx - \int_\Omega \phi(u_1).\nabla T_k(u_1-u_2)dx$$
$$\leq \int_\Omega T_k(u_1-u_2)d\mu \tag{6.2}$$

and

$$\int_\Omega a(x,\nabla u_2).\nabla T_k(u_2-u_1)dx+\int_\Omega b_2 T_k(u_2-u_1)dx-\int_\Omega \phi(u_2).\nabla T_k(u_1-u_2)dx$$
$$\le \int_\Omega T_k(u_2-u_1)d\mu. \tag{6.3}$$

By adding (6.2) and (6.3), we obtain

$$\int_\Omega (a(x,\nabla u_1)-a(x,\nabla u_2)).\nabla T_k(u_1-u_2)dx+\int_\Omega (b_1-b_2)T_k(u_1-u_2)dx$$
$$-\int_\Omega \Big(\phi(u_1)-\phi(u_2)\Big).\nabla T_k(u_2-u_1)dx\le 0. \tag{6.4}$$

Since $a(x,.)$ is monotone, the first term of (6.4) is non-negative, and we deduce from (6.4) that

$$\int_\Omega (b_1-b_2)T_k(u_1-u_2)dx-\int_\Omega \phi(u).\nabla T_k(u_2-u_1)dx\le 0.$$

Dividing the above inequality by $k>0$, we get

$$\frac{1}{k}\int_\Omega (b_1-b_2)T_k(u_1-u_2)dx-\frac{1}{k}\int_\Omega \Big(\phi(u_1)-\phi(u_2)\Big).\nabla T_k(u_2-u_1)dx\le 0. \tag{6.5}$$

Setting $A_k:=\{0\le |u_1-u_2|\le k\}$, the second term of (6.5) gives

$$\begin{aligned}&\left|-\frac{1}{k}\int_\Omega \Big(\phi(u_1)-\phi(u_2)\Big).\nabla T_k(u_2-u_1)dx\right|\\&=\left|\frac{1}{k}\int_\Omega \Big(\phi(u_1)-\phi(u_2)\Big).\nabla (u_2-u_1)\chi_{A_k}dx\right|\\&\le \frac{1}{k}\int_\Omega \left|\Big(\phi(u_1)-\phi(u_2)\Big).\nabla (u_2-u_1)\chi_{A_k}\right|dx\\&\le \frac{C}{k}\int_\Omega |u_2-u_1|\,|\nabla(u_2-u_1)|\chi_{A_k}dx\\&\le C\int_\Omega |\nabla(u_2-u_1)|\chi_{A_k}dx.\end{aligned}$$

Since

$$|\nabla(u_2-u_1)|\chi_{A_k}\longrightarrow 0 \text{ a.e. in } \Omega \text{ as } k\to 0$$

and

$$\Big||\nabla(u_2-u_1)|\chi_{A_k}\Big|\le |\nabla(u_2-u_1)|\in L^1(\Omega).$$

By Lebesgue's dominated convergence theorem, one obtains

$$\lim_{k\to 0}\int_\Omega |\nabla(u_2-u_1)|\chi_{\{0\le|u_1-u_2|\le k\}}dx=0.$$

https://doi.org/10.4153/S0008414X24001196 Published online by Cambridge University Press

Therefore,

$$\lim_{k\to 0} -\frac{1}{k}\int_{\Omega}\Big(\phi(u_1)-\phi(u_2)\Big).\nabla T_k(u_2-u_1)dx = 0.$$

For the first term of (6.5), we have

$$\frac{1}{k}(b_1-b_2)T_k(u_1-u_2) \to (b_1-b_2)\text{sign}_0(u_1-u_2)\ a.e. \text{ in } \Omega, \text{ as } k\to 0$$

and

$$\left|\frac{1}{k}(b_1-b_2)T_k(u_1-u_2)\right| \le (b_1-b_2) \in L^1(\Omega).$$

Hence,

$$\lim_{k\to 0}\frac{1}{k}\int_{\Omega}(b_1-b_2)T_k(u_1-u_2)dx = 0.$$

By taking the limit as $k \to 0$ in (6.5), we arrive at (6.1). ∎

***Corollary 6.2*** *Let $\phi$ be a Lipschitz function and let $\beta$ be a continuous, increasing function on $\mathbb{R}$. Then $b_1 = b_2$ a.e. in $\Omega$.*

**Proof** Let $\beta$ be a continuous and increasing function on $\mathbb{R}$. One can deduce that

$$(b_1-b_2)\text{sign}_0(u_1-u_2) = |b_1-b_2|.$$

Then, using Theorem 6.1, it follows that

$$\|b_1-b_2\|_{L^1(\Omega)} = 0. \tag{6.6}$$

Hence, $b_1 = b_2$ a.e. in $\Omega$. ∎

## References


[1] S. Antontsev and J. F. Rodrigues, *On stationary thermorheological viscous flows*. Ann. Univ. Ferrara Sez. VII Sci. Mat. **52**(2006), 19–36.

[2] M. B. Benboubker, H. Chrayteh, M. E. Moumni, and H. Hjaji, *Entropy and renormalized solutions for nonlinear elliptic problem involving variable exponent and measure data*. Acta Math. Sin. (Engl. Ser.) **31**(2015), 151–169.

[3] P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre, and J. L. Vázquez, *An $L^1$– theory of existence and uniqueness of solutions of nonlinear elliptic equations*. Ann. Sc. Norm. Sup. Pisa, Cl. Sci. **22**(1995), 241–272.

[4] L. Boccardo and T. Gallouet, *Nonlinear elliptic equations with right hand side measures*. Comm Partial Differ. Equ. **17**(1992), 641–655.

[5] L. Boccardo, T. Gallouet, and L. Orsina, *Existence and uniqueness of entropy solutions for nonlinear elliptic equations with measure data*. Ann. Inst. Henri Poincaré, Anal. Non Linéaire. **13**(1996), 539–551.

[6] B. K. Bonzi and S. Ouaro, *Entropy solutions for a doubly nonlinear elliptic problem with variable exponent*. J. Math. Anal. Appl. **370**(2010), 392–405.

[7] H. Brezis, *Opérateurs maximaux monotones et semigroupes de contraction dans les espaces de Hilbert*, North Holland, Amsterdam, 1973.

[8] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*. SIAM J. Appl. Math. **66**(2006), 1383–1406.

[9] G. Dal Maso, F. Murat, L. Orsina, and A. Prignet, *Renormalized solutions of elliptic equations with general measure data*. Ann. Scuola Norm. Sup. Pisa. Cl. Sci. **28**(1999), 741–808.

[10] L. Diening, *Theoretical and numerical results for electrorheological fluids*. PhD. thesis, University of Frieburg, Germany, 2002.
[11] G. Dolzmann, N. Hungerbehler, and S. Muller, *Non-linear elliptic systems with measure-valued right-hand side*. Math Z. **4**(1997), 545–574.
[12] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\vec{\overline{p}}(.)$-Laplacien equations. Complex variable and elliptic equations*. Int. J. **56**(2011), 623–642.
[13] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$*. J. Math. Anal. Appl. **263**(2001), 424–446.
[14] P. Harjulehto and P. Hästö, *Sobolev inequalities for variable exponents attaining the values 1 and $n$*. Publ. Mat. **52**(2008), 347–363.
[15] I. Ibrango and S. Ouaro, *Entropy solutions for anisotropic nonlinear Dirichlet problems*. Ann. Univ. Craiova. Mat. Inform. **42**(2015), 347–364.
[16] N. Igbida, S. Ouaro, and S. Soma, *Elliptic problem involving diffuse measures data*. J. Differ. Equ. **253**(2012), 3159–3183.
[17] I. Konaté and S. Ouaro, *Good Radon measure for anisotropic problems with variable exponent*. Electron. J. Diff. Equ. **2016**(2016), 1–19.
[18] I. Konaté and S. Ouaro, *Nonlinear multivalued problems with variable exponent and diffuse measure data in anisotropic space*. Gulf J. Math. **6**(2018), 13–30.
[19] O. Kovacik and J. Rakosnik, *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*. Czech. Math. J. **41**(1991), 592–618.
[20] E. Nassouri, S. Ouaro, and T. Urbain, *Nonlinear Neumann problems involving $p(x)$-Laplace operator and measure data*. Discuss. Math. Differ. Incl., Control Optim. **39**(2019), 181–212.
[21] N. Igbida, S. Ouaro, and S. Soma, *Elliptic problem involving diffuse measure data*. J. Differ. Equ. **253**(2012), 3159–3183.
[22] I. Nyanquini, S. Ouaro, and S. Soma, *Entropy solution to nonlinear multivalued elliptic problem with variable exponents and measure data*. Ann. Univ. Craiova Ser. Mat. Inform. **40**(2013), 74–198.
[23] S. Ouaro, A. Ouédraogo, and S. Soma, *Multivalued homogeneous Neumann problem involving diffuse measure data and variable exponent*. Nonlinear Dyn. Syst. Theory **16**(2016), 102–114.
[24] C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen, and B. Dolgin, *Electrorheological fluid based force feedback device*. In: Proceeding 1999 SPIE telemanipulator and telepresence technologies VI conference, 3840, Boston, MA, 1999, pp. 88–99.
[25] L. Wang, Y. Fan, and W. Ge, *Existence and multiplicity of solutions for a Neumann problem involving the $p(.)$ -Laplace operator*. Nonlinear Anal. **71**(2009), 4259–4270.
[26] P. Wittbold and A. Zimmermann, *Existence and uniqueness solutions to nonlinear elliptic equations with variable exponent and $L^1$ -data*. Nonlinear Anal. Theory, Methods Appl. **72**(2010), 2990–3008.
[27] J. Yao, *Solutions for Neumann boundary value problem involving $p()$-Laplace operator*. Nonlinear Anal. **68**(2008), 1271–1283.

*Laboratoire de Mathématiques et d'Informatique (LA.M.I), UFR, Sciences Exactes et Appliquées, Université Joseph KI-ZERBO, 03 BP 7021 03 Ouagadougou, Burkina Faso*
*e-mail*: safimba.soma@ujkz.bf

*Laboratoire de Science et technologie (LaST), UFR, Sciences et Technologies, Université Thomas SANKARA, 12 BP 417 12 Ouagadougou, Burkina Faso*
*e-mail*: ibrakonat@yahoo.fr

*Laboratoire de Mathématiques et d'Informatique (LA.M.I), Institut des Sciences et de Technologie, Ecole Normale Superieure, 01 BP 1757 01 Ouagadougou, Burkina Faso*
*e-mail*: kaboreadama59@yahoo.fr