Power of a point: from Jakob Steiner to modern applications

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1. *A useful 200-year-old idea*

"What should we use?" seems to be the question when one approaches a plane geometry problem. In many ways, Euclidean geometry is a laboratory in the realm of logic, an ideal place where one can see how alternative methods can be employed to solve problems. What detail might represent a hint? And from among many choices, what method could one consider? Does the geometric structure suggest a certain type of approach?

In this article, we focus on one such method. Jakob Steiner's power of a point with respect to a circle was born in a very fertile decade in the history of geometry, the 1820s, a golden era when the curvatures of surfaces were introduced by C. F. Gauss, when J. Bolyai investigated the content that led to his *Tantamen*, and when Brianchon and Poncelet published their joint work, which included the nine-point-circle theorem. It is natural to compare the use of Steiner's method with other pathways available to the geometry student today. We will now take a fresh look more appropriate for our times, at this old idea; this method is not to be forgotten, it is to be used.

Ostermann and Wanner [1] point out that the idea of the power of a point with respect to the circle originates in one of the papers published in the initial issue of Crelle's Journal [2], and that the results were included in a manuscript [3], which unfortunately was only published in 1931, long after Steiner passed away. About the birth of this method, we read in [1, p. 98] this interesting story:

> Jakob Steiner (1796-1863) has one of the most incredible biographies of a great mathematician: he was born in a small Swiss village (Utzenstorf, close to Bern). His father forbade him to read, the village priest refused to let him write — he was not good enough in Catechism. At the age of 18 he entered, as the oldest pupil, the Pestalozzi school at Yverdon, where he began his education eagerly and with great energy. Later in Berlin, he was not allowed to teach higher mathematics at the Werden Gymnasium — he had not understood Hegel's philosophy well enough. So he survived as "Privatlehrer" [private teacher] and contributed five articles (No. 5, 18, 25, 31 and 32) to the first volume of the newly founded *Crelle Journal*.

The idea of the power of a point with respect to the circle is quite useful in advanced Euclidean geometry, the realm of mathematics where most students see it. Here we try to illustrate the concept.

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2. *Opening the toolbox*

Throughout the article we use the notation AB for a line passing through points A and B, and also for the segment joining these points and for the length of the segment. Appropriate comments for distinction between them will be made if necessary. Angles are generally denoted with $\angle PQR$. In the case of an angle in a triangle, the simplified form $\angle A$ may be used.

Jakob Steiner's idea about the power of a point with respect to a circle involves two separate cases. The following theorems encapsulate the most common usage of this concept.

Theorem 1: (Intersecting chords theorem) If two chords of a circle, AB and CD, intersect at a point P inside the circle, then the products of the lengths of the two segments on each chord are equal. Thus, using the notation in Figure 1, we have

$$PA \cdot PB = PC \cdot PD.$$

Proof: From the similarity of triangles ΔPAC and ΔPDB ($\angle PAC = \angle PDB$ and $\angle PCA = \angle PBD$) we get

$$\frac{PA}{PD} = \frac{PC}{PB} \Leftrightarrow PA \cdot PB = PC \cdot PD.$$

The converse of this theorem is also true. Hence, if two line segments AB and CD intersect at a point P such that the relation $PA \cdot PB = PC \cdot PD$ holds true, points A, B, C, D lie on the same circle. The converse can also be proved using the similarity of triangles ΔPAC and ΔPDB ($\angle CPA = \angle BPD$ and $\frac{PA}{PD} = \frac{PC}{PB}$), which implies that $\angle BAC = \angle CBD$, meaning that A, B, C, D lie on the same circle, according to the properties of cyclic quadrilaterals.

Theorem 2: (Tangent-secant theorem) Given a secant line l intersecting a circle at points E and F, and a tangent t intersecting the same circle at point T, if l and t intersect at point P_1 (see Figure 1), then the following equation holds:

$$P_1T^2 = P_1E \cdot P_1F.$$

Proof: Since triangles ΔP_1TE and ΔP_1FT are similar ($\angle P_1TE = \angle P_1FT$ and $\angle TP_1E = \angle FP_1T$), we have

$$\frac{P_1T}{P_1E} = \frac{P_1F}{P_1T} \Leftrightarrow P_1T^2 = P_1E \cdot P_1F.$$

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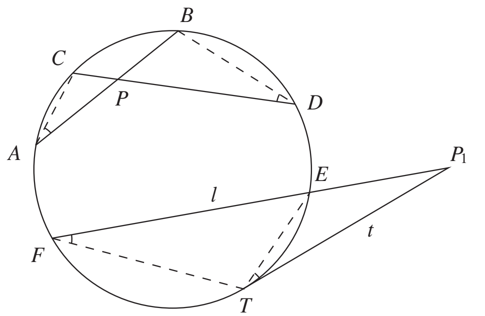

FIGURE 1: Theorem 1 and Theorem 2

The converse of this theorem also holds true.

If line l, cutting a circle at E and F, intersects line t, cutting the circle at T, at P_1 outside of the circle, given that $P_1T^2 = P_1E \cdot P_1F$ holds true, it follows that t is tangent at T_1 to the circle at T. To prove this, consider the tangent t_1 from P_1 to the circle, and join T and T_1 with the centre of the circle, O. By Theorem 2, we have that $P_1T_1^2 = P_1E \cdot P_1F$, which implies $P_1T_1 = P_1T$. It follows that triangles ΔP_1T_1O and P_1TO are congruent, as all corresponding sides have equal length, meaning that $\angle P_1TO = \angle P_1T_1O = 90°$. The last relation implies that t is tangent to the given circle at T.

In order to solve Problem 6 from the next section, we need Pascal's theorem. Here we present this theorem with an elementary proof which only uses Menelaus' theorem and the results discussed above.

For the other elements of fundamental geometry we are using in this paper, we refer the interested reader to [4], a reference which invites further explorations.

Theorem 3: (Pascal's Theorem) Given six points on a circle, A, B, C, D, E, F in some order, if we denote by $P = AB \cap DE$, $Q = BC \cap EF$ and $R = CD \cap FA$, then P, Q, R are collinear.

Proof: We use directed lengths. Denote by $X = AB \cap CD$, $Y = CD \cap EF$ and $Z = EF \cap AB$ as in Figure 2. Now apply Menelaus' theorem three times for triangle ΔXYZ and the lines determined by segments BC, DE and FA:

$$\frac{BX}{BZ} \cdot \frac{RZ}{RY} \cdot \frac{CY}{CX} = -1$$

$$\frac{DY}{DX} \cdot \frac{PX}{PZ} \cdot \frac{EZ}{EY} = -1$$

$$\frac{AX}{AZ} \cdot \frac{FZ}{FY} \cdot \frac{QY}{QX} = -1.$$

Multiplying these relations and rearranging terms we get

$$\frac{AX \cdot BX}{AZ \cdot BZ} \cdot \frac{CY \cdot DY}{CX \cdot DX} \cdot \frac{EZ \cdot FZ}{EY \cdot FY} \cdot \frac{RZ}{RY} \cdot \frac{QY}{QX} \cdot \frac{PX}{PZ} = -1.$$

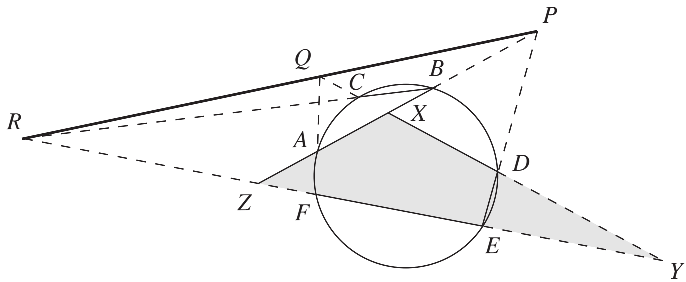

FIGURE 2: Pascal's Theorem

Now, by Theorems 1 and 2, we know that $AX \cdot BX = CX \cdot DX$, $CY \cdot DY = DY \cdot EY$ and $EZ \cdot FZ = AZ \cdot BZ$. On simplifying, (1) becomes

$$\frac{RZ}{RY} \cdot \frac{QY}{QX} \cdot \frac{PX}{PZ} = -1.$$

Now, by the converse of Menelaus' theorem applied to ΔXYZ, P, Q, R lie on the same line, which completes the proof.

3. *Problems and Solutions*

In this section we provide applications of Steiner's method of using the power of a point with respect to a circle.

Problem 1: Let ABC be a non-isosceles triangle. The internal angle bisector of $\angle A$ intersects side BC at D. The circumcircle of ΔADA_1 intersects AB and AC at P and Q respectively, see Figure 3. Prove that $BP = CQ$.

Solution:

The power of point B with respect to the circle is $BP \cdot AB = BD \cdot BA_1$ and the power of C with respect to the same circle yields $CQ \cdot AC = CD \cdot CA_1$. By the angle bisector theorem in ΔABC,

$$\frac{AB}{AC} = \frac{BD}{CD} \Leftrightarrow \frac{BD}{AB} = \frac{CD}{AC}.$$

Since A_1 is the midpoint of BC, $BA_1 = CA_1$, we have

$$BP = \frac{BD \cdot BA_1}{AB} = \frac{CD \cdot CA_1}{AC} = CQ,$$

which is what we wanted to prove.

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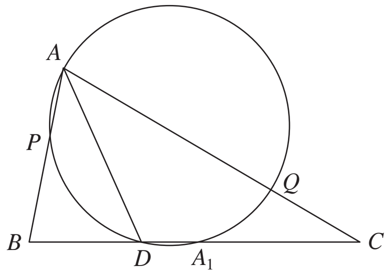

FIGURE 3: Problem 1

Remark: As the statement of the angle bisector theorem in the case of the external bisector is similar to the one for the internal bisector, the solution remains valid if we consider D as the intersection of BC with the external angle bisector of $\angle A$, instead of the internal one.

Our next example is from a 1930 book by Aubert and Papelier, where the problem is presented in a context that suggests an analytic solution. As we will see, a synthetic approach based on the methods presented in this article is more natural than an analytic one.

Problem 2: ([5, p. 39]) Let ΔABC be an acute-angled and let X lie on the interior bisector of angle $\angle A$ and inside the triangle. The circumcircle of ΔAXB intersects AC at D and the circumcircle of ΔAXC intersects AB at E. Prove that $BE = CD$.

Solution: Denote by N the intersection of the circumcircle of ΔABX with line BC and by M the intersection of the circumcircle of ΔAXC with the same line. Consider $Y = AX \cap BC$. Note that M and N do not necessarily lie inside the segment BC: one or both could lay outside the segment. As shown in Figure 4, we treated the situation with M and N inside the segment BC, but all other situations can be solved in a similar way.

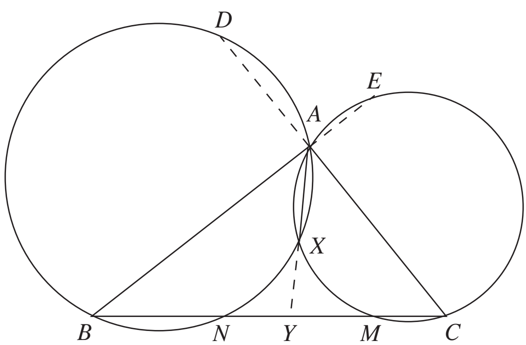

FIGURE 4: Problem 2

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Denote the lengths of the sides of ΔABC as follows: $AB = c$, $AC = b$ and $BC = a$.

By writing the powers of points B, C and Y, respectively, and combining them with the angle bisector theorem, we obtain:

$$BA \cdot BE = BM \cdot BC \Leftrightarrow c \cdot BE = BM \cdot a \Rightarrow BE = BM \cdot \frac{a}{c}$$

$$CN \cdot CB = CA \cdot CD \Leftrightarrow CN \cdot a = b \cdot CD \Rightarrow CD = CN \cdot \frac{a}{b}$$

$$YN \cdot YB = YX \cdot YA = YM \cdot YC \Rightarrow \frac{YB}{YC} = \frac{YM}{YN} = \frac{c}{b}.$$

Hence

$$\frac{BE}{CD} = \frac{BM}{CN} \cdot \frac{b}{c} = \frac{b \cdot (BY + YM)}{c \cdot (CY + YN)} = \frac{b \,.\, BY + b \cdot YM}{c \cdot CY + c \cdot YN} = \frac{c \,.\, CY + c \cdot YN}{c \cdot CY + c \cdot YN} = 1.$$

We conclude that $BE = CD$.

Remark: The result remains true if point X is taken outside or inside the triangle on either the internal or external bisector. The proof works in essentially the same way in each case.

Problem 3: (A. Gîrban) Let $ABCD$ be a trapezium with $BC \,//\, AD$. Let the incircle of ΔABC be ω which is tangential to BC at E. The tangents from D to ω intersect BC at points K and L. If AE intersects the circumcircle of ΔAKL at F, prove that the quadrilateral $ABCF$ is cyclic.

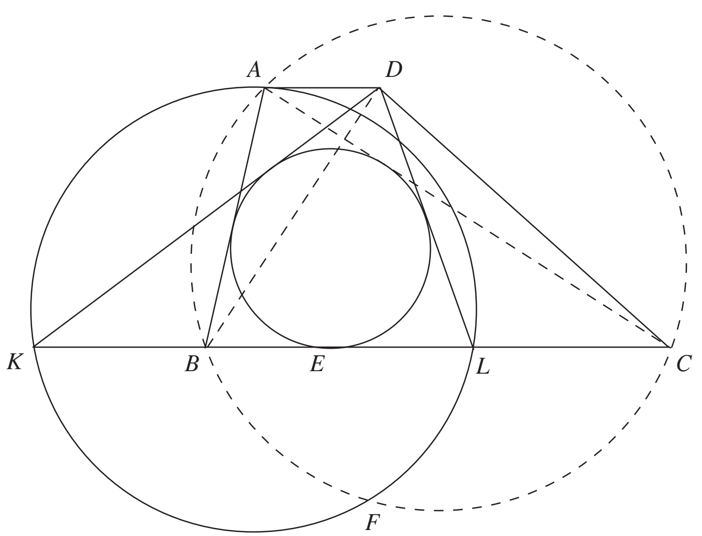

FIGURE 5: Problem 3

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Solution:

By the converse of theorem 1 it is enough to prove that $BE \cdot EC = FE \cdot EA$.

From the same theorem, we know that $FE \cdot EA = KE \cdot EL$, hence the problem is equivalent to showing that $BE \cdot EC = KE \cdot EL$.

The products in the previous relation concern triangles ΔABC and ΔDKL, which have equal heights and share the same incircle. Thus, if we express said products for the triangles in terms of only corresponding heights and the inradius, the products themselves will be equal as the heights and the inradii are the same. Denoting by r the inradius and by h the length of the heights corresponding to sides BC and KL, we express the product (of segments formed by the point of contact with the incircle) using only r and h. We denote the sides of the triangle by a, b, c and the semiperimeter by p.

We derive the following

$$EB \cdot EC = (p-a)(p-b) = \frac{(a+c-b)(a+b-c)}{4} = \frac{a^2-(b-c)^2}{4}$$

$$= \frac{a^2+2bc-2bc\cos A-a^2}{4} = \frac{bc}{2} \cdot (1-\cos A) = bc \cdot \frac{1-\cos A}{2}$$

$$= bc \cdot \sin^2\frac{A}{2} = bc \cdot \frac{4\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}{4\cos^2\frac{1}{2}A} = bc \cdot \frac{\sin^2 A}{4\cos^2\frac{1}{2}A}$$

$$= \frac{bc\sin A}{2} \cdot \frac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{2\cos^2\frac{1}{2}A} = \frac{bc\sin A}{2} \cdot \tan\frac{A}{2}.$$

Now, using different formulas for the area of ΔABC (denoted by S_{ABC}), the previous equation becomes

$$EB \cdot EC = S_{ABC} \cdot \tan\frac{A}{2} = S_{ABC} \cdot \frac{r}{p-a} = r^2 \cdot \frac{p}{p-a}. \tag{2}$$

It remains to express the ratio $\dfrac{p}{p-a}$ only in terms of r and h. Denoting this ratio by k, we get

$$k = \frac{p}{p-a} \Rightarrow \frac{1}{k} = 1 - \frac{a}{p} = 1 - \frac{2r}{h} \Rightarrow k = \frac{h}{h-2r}. \tag{3}$$

From equations (2) and (3), we have

$$EB \cdot EC = \frac{r^2h}{h-2r} = EK \cdot EL$$

and the conclusion follows.

Remark: Another way of proving the invariance of the product $EB \cdot EC$ with respect to r and h uses repeated applications of Pythagoras' theorem and a simple auxiliary construction.

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Problem 4: [6] The triangle ΔABC is acute-angled and T is the midpoint of the arc BC of the circumcircle. The feet of the perpendiculars from A and T to BC are D and A_1 respectively, and the feet of the perpendiculars from B and C to AT are H and L respectively. Prove that:

(a) The quadrilateral DHA_1L is cyclic;

(b) The lines A_1H, DL, DH and A_1L are parallel to AC, BT, TC and AB respectively;

(c) If C_1 is the midpoint of AB, then the circumcentre of quadrilateral DHA_1L lies on the circumcircle of ΔDC_1A_1.

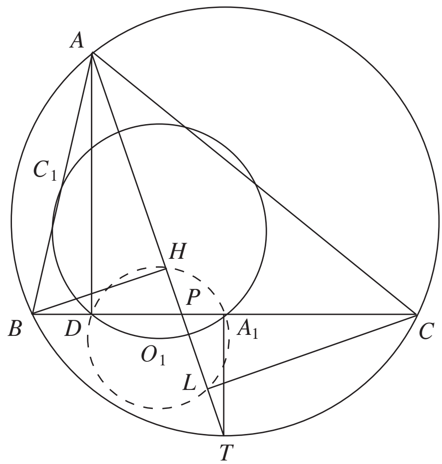

FIGURE 6: Problem 4

For this problem we present two solutions. The method of the second solution is different from the proof published in *Amer. Math. Monthly*, January 2005, pp. 90-91. The whole investigation we pursue here is to illustrate the difference between the *methods* and, ultimately, to test out the effectiveness of each one of them.

First Solution: (a) Let P be the intersection of AT with BC. The quadrilateral $ABDH$ is cyclic (since $\angle BHA = \angle ADB = \frac{1}{2}\pi$). Therefore $\angle HDP = \angle BAH$. Similarly, from the cyclic quadrilateral CA_1LT one gets $\angle A_1CT = \angle PLA_1$, and since $\angle A_1CT = \angle BAT$ it follows that $\angle HDA_1 = \angle HLA_1$, which implies that DHA_1L is cyclic.

(b) The quadrilateral $ADLC$ is cyclic since $\angle ADC = \angle ALC = \frac{1}{2}\pi$, therefore $\angle DCA = \angle DLA = \angle DA_1H$ and thus $\angle DA_1H = \angle DCA$, yielding $HA_1 \,//\, AC$.

To see that $DL \,//\, BT$, we look first at the cyclic quadrilateral BTA_1H. We have $\angle HTB = \angle HA_1B$. Using (a), in the quadrilateral DHA_1L we have $\angle HA_1B = \angle HLD$ which means that $\angle HLD = \angle HTB$, so $DL \,//\, BT$.

Now we prove that $HD /\!/ TC$. From (a), $\angle DHL = \angle DA_1L$. From the cyclic quadrilateral A_1LTC we have $\angle DA_1L = \angle LTC$, so $\angle DHL = \angle DA_1L = \angle LTC$. Therefore $HG /\!/ TC$.

Finally, since $\angle ABC = \angle ATC$, the above arguments also yield $\angle ABC = \angle BA_1L$, so $AB /\!/ A_1L$ as desired.

(c) Let us denote by O_1 the centre of the circumcircle of quadrilateral DHA_1L. We remark first that

$$\angle DLA_I = \angle DLA + \angle ALA_1 = \angle BCA + \angle BAL = \angle C + \tfrac{1}{2}\angle A.$$

In the circle with centre O_1 we have

$$\angle A_1O_1D = 2 \cdot \angle DLA_1 = 2\left(\angle C + \tfrac{1}{2}\angle A\right) = 2 \cdot \angle C + \angle A.$$

But

$$\angle A_1C_1D = \angle\left(DC_1; AC\right) = \pi - \angle C_1DC - \angle C = \angle C_1DB - \angle C = \angle B - \angle C$$

and adding term by term these last two relations we get

$$\angle A_1O_1D + \angle A_1C_1D = \angle A + \angle B + \angle C = \pi,$$

and the conclusion follows.

Remark: The circumcircle of ΔDC_1A_1 in (c) is the nine-point circle of ΔABC.

Second Solution: This approach uses the concept of the power of a point and yields a nice solution for (a) and (b). By applying the power of the point P with respect to the circumcircles of quadrilaterals A_1LTC, $ABDH$ and $ABTC$ we get

$$PA_1 \cdot PC = PT \cdot PL \tag{4}$$

$$PD \cdot PB = PH \cdot PA \tag{5}$$

$$PT \cdot PA = PB \cdot PC.$$

Multiplying these equations, it follows that, after reducing terms, $PA_1 \cdot PD = PL \cdot PH$, so DHA_1L is a cyclic quadrilateral, proving (a).

For (b), we prove that $A_1H /\!/ AC$. All other parallelisms are shown in the same way. For this case, it is enough to show that

$$\frac{PA_1}{PC} = \frac{PH}{PA} \Leftrightarrow PA_1 \cdot PA = PH \cdot PC.$$

This is obtained immediately if we multiply equations (4) and (5) together with $PL \cdot PA = PD \cdot PC$ (which represents the power of the point P with respect to the circumcircle of quadrilateral $ADLC$).

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Problem 5: (A. Gîrban) The incircle of ΔABC touches the sides BC, CA and AB at points D, E and F, respectively. The line EF meets the circumcircle of ΔABC at points X and Y. Prove that the circumcircle of ΔXDY passes through the midpoint A_1 of BC.

Solution:

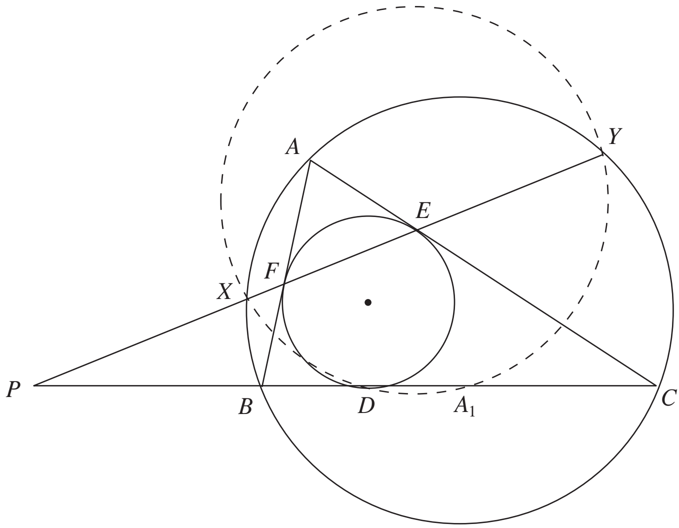

FIGURE 7: Problem 5

Let P be the intersection of the supporting lines of segments BC and XY. For simplicity, we will use the notation a, b, c for the lengths of the sides of $\angle ABC$ and p for its semiperimeter. Also, let $PB = x$.

By the converse of Theorem 1 we need to prove that $PD \cdot PA_1 = PX \cdot PY$.

By using the power of point P, $PX \cdot PY = PB \cdot PC$, it is sufficient to show that $PB \cdot PC = PD \cdot PA_1$. Using the previous notation, this relation becomes

$$x(x + a) = (x + p - b)\left(x + \frac{a}{2}\right) \Leftrightarrow 2x^2 + 2ax = 2x^2 + ax + 2px + ap - 2bx - ab.$$

Rearranging and reducing terms we now need to prove

$$ax - 2px + 2bx = ap - ab \Leftrightarrow x(b - c) = a(p - b) \Leftrightarrow x = \frac{a(p - b)}{b - c}. \quad (6)$$

By using Menelaus' theorem for ΔABC and transversal EFP, we have that (using directed lengths)

$$\frac{p - a}{p - b} \cdot \frac{-x}{x + a} \cdot \frac{p - c}{p - a} = -1 \Leftrightarrow \frac{x}{p - b} \cdot \frac{p - c}{x + a} = 1$$

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Rearranging and reducing terms yields

$$px - cx = px + pa - bx - ba \Leftrightarrow x(b - c) = a(p - b)$$

which is equivalent to (6), and the proof is complete.

Remark: This problem admits an extension. The claim holds true for any three arbitrary cevians AD, BE and CF which meet at point P. As cevians induce a harmonic ratio, denoting by P the intersection of lines EF and BC one finds that $(P, D; B, C) = -1$. Since A_1 is the midpoint of segment BC, using a well-known property, we have $PB \cdot PC = PD \cdot PA_1$. The last relation implies that the points D, A_1, X and Y are concyclic, via the converse of Theorem 2.

Problem 6: (A. Gîrban) In the isosceles trapezium $ABCD$, AD is parallel to BC. The incircle of $\triangle ABC$ touches BC at E. The intersection of AE and BD is M and the intersection of DE and AC is N. The line MN intersects the incircle of $\triangle ABC$ at P and Q. Prove that B, C, P and Q are concyclic.

Solution:

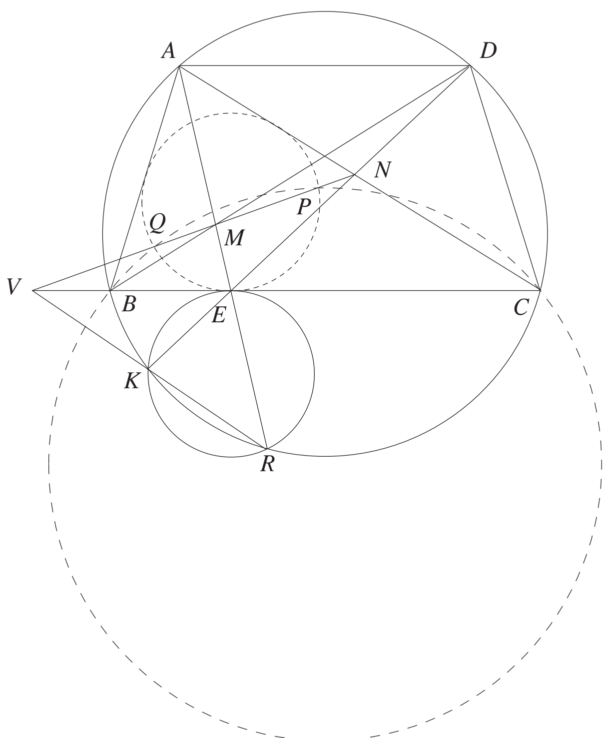

FIGURE 8: Problem 6

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This is a difficult problem, because it is not obvious where to start, and the solution requires an additional construction.

Let $V = PQ \cap BC$ and consider K and R the intersection points of the circumcircle of the trapezium $ABCD$ with lines AE and DE respectively, as in Figure 8.

Since we have six points A, B, C, K, D, R on the same circle, we can apply Pascal's Theorem (Theorem 3) and conclude that points V, M, N are on the same line, because $V = CB \cap RK$, $M = BD \cap RA$ and $N = AC \cap KD$.

We can now prove that the circumcircle of ΔRKE is tangent to the line BC by observing the angle equalities $\angle RKE = \angle RKD = \angle RAD = \angle REC$, which shows that BC is tangent to the circumcircle RKE at point E. Finally, by using the power of a point principle in the form of Theorem 2 applied to point V with respect to the circumcircles of ΔABC, ΔKER and ΔQEP, we find that

$$VB \cdot VC = VK \cdot VR = VE^2 = VQ \cdot VP,$$

which by the converse of Theorem 2 proves that points B, C, P, Q are concyclic.

4. *Conclusions*

The central concept we treated in the previous examples is *cyclicity*. This property is directly related to invariants in the plane under the action of the Möbius group in the plane. Therefore, by studying properties related to cycles we are actually investigating invariant properties, as stated by Felix Klein's 1872 Erlangen Program (for a recent discussion of the relevance of this connection, see [7]). A thorough investigation of cyclicity is by no means an effort focused on particular cases, and a display of methods useful to understand it reaches deep into the very structure of the foundations of geometry. For a further discussion of this idea, see e.g. [8]. As mentioned in the chronological order from Section 1, some of these methods existed a few decades before the Erlangen Program was published. Some configurations are even several hundred years old.

The classical viewpoint on these geometric configurations presented by Coxeter and Greitzer in [4] or Lalescu in [9], is the common ground of many textbooks. The problem-solving viewpoint is represented in the literature in [10], with a thorough discussion of the power of a point with respect to a circle. It is quite interesting that the question of whether there exists a power of the point with respect to an arbitrary planar algebraic curve was raised before World War I in a paper published in *Gazeta matematică* by Sebastian Kaufmann; his story is told in [11].

One of the best presentations available for the classical viewpoint is in Coxeter and Greitzer [4, pp. 27-31]. The problems we included in our present work are different from the approaches in all these other sources, and we were very interested to illustrate how a power of the point, as a problem-solving strategy, is related to other elementary methods available.

At the end of the day, was Jakob Steiner right to advocate for the use of purely geometric methods? It is this thought that we find valuable, and the important take-away of the present method-oriented analysis.

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