J. Plasma Phys. (2021), *vol.* 87, 905870309 © The Author(s), 2021.
Published by Cambridge University Press
doi:10.1017/S0022377821000568

Lower hybrid current drive in a tokamak for correlated passes through resonance

Peter J. Catto †

Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA, USA

(Received 24 February 2021; revised 13 May 2021; accepted 17 May 2021)

Standard quasilinear descriptions are based on the constant magnetic field form of the quasilinear operator so improperly treat the trapped electron modifications associated with tokamak geometry. Moreover, successive poloidal transits of the Landau resonance during lower hybrid current drive in a tokamak are well correlated, and these geometrical details must be properly retained to account for the presence of trapped electrons that do not contribute to the driven current. The recently derived quasilinear operator in tokamak geometry accounts for these features and finds that the quasilinear diffusivity is proportional to a delta function with a transit or bounce averaged argument (rather than a local Landau resonance condition). The new quasilinear operator is combined with the Cordey (*Nucl. Fusion*, vol. 16, 1976, pp. 499–507) eigenfunctions to properly derive a rather simple and compact analytic expression for the trapped electron modifications to the driven lower hybrid current and the efficiency of the current drive.

Key words: plasma confinement, fusion plasma, plasma flows

1. Introduction

The original proposals to use radio frequency waves (rf) (Fisch 1978; Karney & Fisch 1979) and early estimates (Fisch & Boozer 1980; Fisch & Karney 1981) and analytic evaluations (Antonsen & Chu 1982; Cordey, Edlington & Start 1982; Taguchi 1983) relied on the constant magnetic field treatment of quasilinear theory (Kennel & Engelmann 1966). Indeed, all subsequent analytic treatments (Karney & Fisch 1985; Yoshioka & Antonsen 1986; Cohen 1987; Giruzzi 1987; Chiu *et al.* 1989; Ehst & Karney 1991) of the amount of parallel electron current that can be driven with lower hybrid waves in a tokamak continued to employ the Kennel–Engelmann quasilinear (QL) operator. More recent work, starting in the late 1980s, focused on numerical simulations of lower hybrid current drive (LHCD) and is the subject of an extensive review by Bonoli (2014). These treatments work quite well and also lead to sensible estimates of the LHCD efficiency even though tokamak geometry is not properly retained. Recently, a QL description that properly accounts for the correlated nature of successive poloidal passes through the Landau damping resonance in a tokamak has been derived (Catto & Tolman 2021). In this new formulation the wave–particle resonance condition is a transit or bounce averaged

† Email address for correspondence: catto@psfc.mit.edu

resonance condition in velocity space, rather than a resonance at a local poloidal angle in the torus. This improved QL operator is used herein to discover the changes that arise because of this subtle difference, and to demonstrate how only minor changes are required in standard treatments to properly account for toroidal geometry and the presence of trapped electrons.

In the next section, the collision operators employed are given and the adjoint procedure of Antonsen & Chu (1982) slightly extended. Section 3 presents a solution of the adjoint kinetic equation for the electrons by using the Sturm–Liouville eigenfunctions originally introduced by Cordey (1976) – a procedure that avoids the limitations of employing model collision operators that only retain pitch angle scattering collisions (Taguchi 1983) or a Legendre equation approximation to the pitch angle scattering operator when energy diffusion is retained (Cordey *et al.* 1982; Karney & Fisch 1985; Cohen 1987). The final solution is rather simple and compact in form because only a single Cordey eigenfunction is required. It is used in § 4 to evaluate the parallel current that can be driven by lower hybrid waves. The resulting expression properly retains for the first time the electron trapping modifications that reduce the amount of driven current. In § 5 the power needed to drive the current is used to evaluate the current drive efficiency with these modifications due to the presence of trapped electrons as well as poloidal and radial mode structure effects retained. The discussion in § 6 also gives an estimate of the density at which nonlinear effects are expected to enter as well as a summary.

2. Adjoint method and collision operators

Using Gaussian cgs units, and the drift kinetic variables of spatial location $\boldsymbol{r}$, total energy $E = v^2/2 - e\Phi/m_e$, magnetic moment $\mu = v_\perp^2/2B$ and gyrophase φ, such that the velocity is

$$\boldsymbol{v} = \boldsymbol{v}_\perp + v_{||}\boldsymbol{n} = v_\perp[\boldsymbol{e}_1(\boldsymbol{r})\cos\varphi + \boldsymbol{e}_2(\boldsymbol{r})\sin\varphi] + v_{||}\boldsymbol{n}(\boldsymbol{r}), \tag{2.1}$$

with $v_{||}^2 = v^2 - 2\mu B$, the QL equation for the unperturbed electron distribution function f is

$$v_{||}\boldsymbol{n}\cdot\nabla f = C\{f\} + Q\{f\}. \tag{2.2}$$

In the preceding, Q denotes the QL operator, C is the collision operator, the three orthonormal spatial unit vectors are related by $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{n} = \boldsymbol{B}/B$, and the unperturbed magnetic field is

$$\boldsymbol{B} = B\boldsymbol{n} = \nabla\alpha \times \nabla\psi = I(\psi)\nabla\zeta + \nabla\zeta \times \nabla\psi, \tag{2.3}$$

where m_e and e are the mass and magnitude of the charge on an electron, and Φ is the electrostatic potential. The spatial variables employed are the poloidal flux function ψ, and the poloidal, ϑ, and toroidal, ζ, angles, where q is the safety factor and $I = RB_t$, with B_t the toroidal magnetic field and R the major radius. The detailed forms of Q and C will be given as needed.

Linearizing about an electron Maxwellian of density n_e and temperature T_e,

$$f_0 = f_0(\psi, E) = n_e(\psi)\left[\frac{m_e}{2\pi T_e(\psi)}\right]^{3/2} \mathrm{e}^{-m_e v^2/2T_e(\psi)} = n_e(\psi)\left[\frac{m_e}{2\pi T_e(\psi)}\right]^{3/2} \mathrm{e}^{-[m_e E + e\Phi(\psi)]/T_e(\psi)}, \tag{2.4}$$

by writing

$$f = f_0 + f_1 + \cdots, \tag{2.5}$$

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and using

$$C\{f\} = C_{ee}\{f\} + C_{ei}\{f\}, \tag{2.6}$$

with $C\{f_0\} = 0$, and C_{ee} and C_{ei} the electron–electron and electron–ion collision operators, leads to the unperturbed linearized equation to be solved by the adjoint method, namely

$$v_{||}\boldsymbol{n}\cdot\nabla f_1 = C\{f_1\} + Q\{f_0\}. \tag{2.7}$$

The electron–ion collision operator (Hinton & Hazeltine 1976; Helander & Sigmar 2002) is

$$C_{ei}\{f_1\} = \frac{\nu_u}{x^3}L\left\{f_1 - \frac{m_e}{T_e}V_{||}v_{||}f_0\right\} = \frac{\nu_u}{x^3}\left[L\{f_1\} + \frac{m_e}{T_e}V_{||}v_{||}f_0\right], \tag{2.8}$$

with $x = v/v_e$ and $v_e = (2T_e/m_e)^{1/2}$ the electron thermal speed. The Lorentz operator L is self-adjoint and defined here as

$$L\{h\} = \frac{1}{2}\nabla_v\cdot[(v^2\boldsymbol{I} - \boldsymbol{vv})\cdot\nabla_v h] = \frac{2B_0}{B}\xi\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial h}{\partial\lambda}\right), \tag{2.9}$$

with $L\{v_{\|}f_0\} = -v_{\|}f_0$, $\lambda = 2\mu B_0/v^2 = B_0 v_\perp^2/Bv^2$, $\xi = v_{\|}/v$ and B_0 a normalization constant to be made explicit shortly. The parallel ion mean velocity is $V_{\|}$ and the unlike collision frequency is

$$\nu_u \equiv \sqrt{2}\pi Z^2 e^4 n_i \ell n\Lambda/m_e^{1/2}T_e^{3/2} \to 3\sqrt{\pi}\nu_{ei}/4, \tag{2.10}$$

where $\nu_{ei} = 4\sqrt{2\pi}Z^2e^4n_i\ell n\Lambda/3m^{1/2}T^{3/2} = Z\nu_{ee}$ for a quasineutral plasma with $Zn_i = n_e$, ν_{ee} the electron–electron collision frequency, and Z the ion charge number.

The electron–electron collision operator is self-adjoint. As only a lowest-order solution is desired, the standard high speed expansion of the electron–electron collision operator in its self-adjoint form is employed, namely

$$\begin{aligned} C_{ee}\{h\} &= \nu_\ell\left\{\frac{1}{x^3}L\{h\} + \nabla_v\cdot\left[\frac{T_e f_0}{m_e x^3}\nabla_v\left(\frac{h}{f_0}\right)\right]\right\} \\ &= \frac{2B_0\nu_\ell\xi}{Bx^3}\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial h}{\partial\lambda}\right) + \frac{\nu_\ell T_e}{m_e v^2}\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}\left(\frac{h}{f_0}\right)\right], \end{aligned} \tag{2.11}$$

where

$$\nu_\ell \equiv \sqrt{2}\pi Z^4 e^4 n_e \ell n\Lambda/m_e^{1/2}T_e^{3/2} \to 3\sqrt{\pi}\nu_{ee}/4, \tag{2.12}$$

where $\nu_{ee} = 4\sqrt{2\pi}e^4n_e\ell n\Lambda/3m^{1/2}T^{3/2}$ and h is a perturbed distribution function. The preceding like particle collision operator is just the usual non-momentum conserving high speed expansion of the Rosenbluth potentials for collisions with a Maxwellian (Karney & Fisch 1979).

The adjoint method provides an explicit means of evaluating the parallel current provided the adjoint equation is easier to solve than the original kinetic equation. It has been used for LHCD by Antonsen & Chu (1982), Karney & Fisch (1985) and Cohen (1987). It has also been used to calculate the bootstrap current in stellarators (Helander, Geiger & Maaβberg 2011), and in the plateau regime to evaluate the bootstrap current in a tokamak (Pusztai & Catto 2010).

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To account for the non-self-adjointness of the electron–ion collision operator it is necessary to use the modified adjoint equation

$$v_{||}\boldsymbol{n}\cdot\nabla h + C\{h\} = -\left(\frac{B}{I} - \frac{m\nu_u}{Tx^3}V_{||}\right)v_{||}f_0. \tag{2.13}$$

Then, defining the flux surface average of any quantity A as

$$\langle A\rangle \equiv [\oint \mathrm{d}\vartheta A/\boldsymbol{B}\cdot\nabla\vartheta]/[\oint \mathrm{d}\vartheta/\boldsymbol{B}\cdot\nabla\vartheta], \tag{2.14}$$

using

$$\left\langle \int \mathrm{d}^3v f_0^{-1} v_{||}(h\boldsymbol{n}\cdot\nabla f_1 + f_1\boldsymbol{n}\cdot\nabla h)\right\rangle = \left\langle \boldsymbol{B}\cdot\nabla\left(B^{-1}\int \mathrm{d}^3v v_{||}hf_1/f_0\right)\right\rangle = 0, \tag{2.15}$$

and the self-adjointness of the electron–electron collision operator,

$$\left\langle \int \mathrm{d}^3v f_0^{-1}(f_1C_{ee}\{h\} - hC_{ee}\{f_1\})\right\rangle = 0, \tag{2.16}$$

and the Lorentz operator

$$\left\langle \int \mathrm{d}^3v f_0^{-1}x_e^{-3}(f_1L\{h\} - hL\{f_1\})\right\rangle = 0, \tag{2.17}$$

yields the desired adjoint relation

$$\left\langle B\int \mathrm{d}^3v v_{||}f_1\right\rangle = I\left\langle \int \mathrm{d}^3v h\left(f_0^{-1}Q\{f_0\} - \frac{m\nu_u}{Tx_e^3}V_{||}v_{||}\right)\right\rangle. \tag{2.18}$$

Therefore, only the adjoint equation need be solved to evaluate the parallel electron current driven by the lower hybrid waves. It is convenient to define $B_0^2 \equiv \langle B^2\rangle$.

Based on the direction of the poloidal magnetic field, the Ohmic current is in the positive toroidal direction. Consequently, the lower hybrid (LH) parallel electron flow is to be driven in the opposite or negative direction to make $\langle B\int \mathrm{d}^3v v_{\|}f_1\rangle < 0$.

The parallel ion flow term in (2.18) is normally ignored as

$$\frac{m\nu_u\langle V_{||}\int \mathrm{d}^3v x^{-3}hv_{||}\rangle}{T\langle\int \mathrm{d}^3v h f_0^{-1}Q\{f_0\}\rangle} \sim \frac{V_{||}\nu_{ee}f_0}{v_eQ} \sim \left(\frac{m_e}{M}\right)^{1/2}\frac{\rho_{pi}}{a}\frac{\nu_{ee}f_0}{Q\{f_0\}} \ll 1, \tag{2.19}$$

where M is the ion mass, and $V_{\|} \sim v_i\rho_{pi}/a$, with v_i the ion thermal speed, $\rho_{pi}$the poloidal ion gyroradius and a the minor radius. Consequently, $V_{\|}$ can be ignored and only

$$\left\langle B\int \mathrm{d}^3v v_{||}f_1\right\rangle = I\int \mathrm{d}^3v\frac{v_{||}h}{Bf_0}\left\langle \frac{B}{v_{||}}Q\{f_0\}\right\rangle = I\left(\int \mathrm{d}^3v\frac{v_{||}h}{Bf_0}\tau_f\overline{Q\{f_0\}}\right)/(\oint \mathrm{d}\vartheta/\boldsymbol{B}\cdot\nabla\vartheta), \tag{2.20}$$

need be evaluated, where the solution for h is shown in the next section to satisfy $\partial h/\partial\vartheta = 0$ to lowest order. The transit average over a full (f) poloidal circuit is defined using $\mathrm{d}\tau = \mathrm{d}\vartheta/v_{\|}\boldsymbol{n}\cdot\nabla\vartheta$ to be

$$\bar{A} \equiv \oint_f \mathrm{d}\tau A/(\oint_f \mathrm{d}\tau). \tag{2.21}$$

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3. Eigenfunction solution procedure in tokamak geometry

Rewriting the adjoint equation yields a Spitzer & Härm (1953) equation in toroidal geometry,

$$v_{||}\boldsymbol{n}\cdot\nabla h + C_{ee}\{h\} + \nu_u x^{-3}L\{h\} = -I^{-1}Bv_{||}f_0. \tag{3.1}$$

Writing $h = \bar{h} + \tilde{h} + \cdots$ with $\tilde{h} \ll \bar{h}$ then to lowest order

$$v_{||}\boldsymbol{n}\cdot\nabla\bar{h} = 0, \tag{3.2}$$

giving

$$\bar{h} = \bar{h}(\psi, \alpha, E, \mu, \sigma), \tag{3.3}$$

where $\sigma = v_{\|}/|v_{\|}|$ for the passing and $\sigma = 0$ for the trapped. Then transit averaging the next order equation,

$$v_{||}\boldsymbol{n}\cdot\nabla\tilde{h} + C_{ee}\{\bar{h}\} + \nu_u x^{-3}L\{\bar{h}\} = -I^{-1}Bv_{||}f_0, \tag{3.4}$$

leads to

$$\overline{C_{ee}\{\bar{h}\}} + \nu_u x^{-3}\overline{L\{\bar{h}\}} = -I^{-1}\overline{Bv_{||}}f_0. \tag{3.5}$$

Integration over a full bounce for the trapped (t) electrons gives $\overline{Bv_{\|}} = 0$, implying that the trapped response $\bar{h}_t$ vanishes,

$$\bar{h}_t = 0. \tag{3.6}$$

For the passing electrons

$$\overline{Bv_{||}}\oint_f \mathrm{d}\tau = \oint \mathrm{d}\vartheta B^2/\boldsymbol{B}\cdot\nabla\vartheta = \langle B^2\rangle \oint \mathrm{d}\vartheta/\boldsymbol{B}\cdot\nabla\vartheta. \tag{3.7}$$

Using the flux surface average to rewrite the passing (p) adjoint equation leads to

$$I\left\langle \frac{B}{v_{||}}C_{ee}\{\bar{h}_p\}\right\rangle + \frac{I\nu_u}{x_e^3}\left\langle\frac{B}{v_{||}}L\{\bar{h}_p\}\right\rangle = -\langle B^2\rangle f_0, \tag{3.8}$$

where

$$\left\langle\frac{B}{v_{||}}L\{\bar{h}_p\}\right\rangle = \frac{2B_0}{v}\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\bar{h}_p}{\partial\lambda}\right), \tag{3.9}$$

and

$$\left\langle\frac{B}{v_{||}}C_{ee}\{\bar{h}_p\}\right\rangle = \frac{2B_0\nu_\ell}{x^3 v}\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\bar{h}_p}{\partial\lambda}\right) + \frac{\nu_\ell T_e}{m_e v^3}\left\langle\frac{B}{\xi}\right\rangle\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}\left(\frac{\bar{h}_p}{f_0}\right)\right]. \tag{3.10}$$

The last term of (3.10) proportional to $v^{-3}\partial\bar{h}_p/\partial v$ is drag and the remainder is energy scattering.

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In the large aspect ratio limit

$$\langle \xi \rangle = \frac{2\sqrt{2\varepsilon}E(k)}{\pi\sqrt{(1-\varepsilon)k^2+2\varepsilon}}, \tag{3.11}$$

and

$$\frac{1}{B_0}\left\langle \frac{B}{\xi} \right\rangle = -2\frac{\partial\langle\xi\rangle}{\partial\lambda}, \tag{3.12}$$

with E an elliptic integral of the second kind, $k^2 = 2\varepsilon\lambda/[1-(1-\varepsilon)\lambda]$, and $\varepsilon = r/R$ with r the minor radius. The equation to be solved is first rewritten as

$$2(Z+1)\frac{\partial}{\partial\lambda}\left[\lambda\langle\xi\rangle\frac{\partial}{\partial\lambda}\left(\frac{\bar{h}_p}{f_0}\right)\right] + \frac{T_e x^3}{m_e B_0 v^2 f_0}\left\langle\frac{B}{\xi}\right\rangle\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}\left(\frac{\bar{h}_p}{f_0}\right)\right] = -\frac{\langle B^2\rangle v x^3}{I B_0 \nu_\ell}. \tag{3.13}$$

Only the lowest -order v dependence is required. Inserting

$$\frac{\bar{h}_p}{f_0} = \frac{\langle B^2\rangle v x^3}{I B_0 \nu_\ell}\varLambda(\lambda), \tag{3.14}$$

leads to

$$2(Z+1)\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\varLambda}{\partial\lambda}\right) + \frac{T_e\varLambda}{m_e B_0 v^3 f_0}\left\langle\frac{B}{\xi}\right\rangle\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}(v x^3)\right] = -1. \tag{3.15}$$

Using

$$\frac{T_e}{m_e v^3 f_0}\frac{\partial}{\partial v}\left[\frac{v^2 f_0}{x^3}\frac{\partial}{\partial v}(v x^3)\right] = \frac{1}{2x^3 e^{-x^2}}\frac{\partial}{\partial x}\left(\frac{e^{-x^2}}{x}\frac{\partial x^4}{\partial x}\right) = \frac{2}{x^3 e^{-x^2}}\frac{\partial}{\partial x}(x^2 e^{-x^2}) = -4 + \frac{4}{x^2}, \tag{3.16}$$

then for $x \gg 1$ drag dominates over energy scattering and the equation to be solved reduces to

$$\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\varLambda}{\partial\lambda}\right) + \frac{4}{Z+1}\frac{\partial\langle\xi\rangle}{\partial\lambda}\varLambda = -\frac{1}{2(Z+1)}. \tag{3.17}$$

The Cordey (1976) eigenfunctions $\varLambda_j$ and associated eigenvalues κ_j of the Sturm–Liouville differential equation

$$\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\varLambda_j}{\partial\lambda}\right) = \kappa_j\frac{\partial\langle\xi\rangle}{\partial\lambda}\varLambda_j = -\frac{\kappa_j}{2}\left\langle\frac{B}{B_0\xi}\right\rangle\varLambda_j, \tag{3.18}$$

are used to obtain a solution. Expanding in the eigenfunctions $\varLambda_j$, which satisfy $\varLambda_j(\lambda = 0) = 1$, $\varLambda_j(\lambda = B_0/B_{\max}) = 0$ and, for $j \neq k$, the orthogonality condition

$$\int_0^{B_0/B_{max}} \mathrm{d}\lambda \varLambda_k \varLambda_j \frac{\partial\langle\xi\rangle}{\partial\lambda} = 0, \tag{3.19}$$

by inserting

$$\varLambda = \sum_{j=1}^{\infty} A_j \varLambda_j, \tag{3.20}$$

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into the differential equation leads to

$$\sum_{j=1}^{\infty} A_j \left(\kappa_j + \frac{4}{Z+1} \frac{\partial \langle \xi \rangle}{\partial \lambda} \right) \Lambda_j = -\frac{1}{2(Z+1)}. \tag{3.21}$$

Multiplying by Λ_k and integrating over λ, yields the coefficients A_j to be given by

$$2\alpha_j[(Z+1)\kappa_j + 4]A_j = \beta_j, \tag{3.22}$$

where

$$\alpha_j = -\int_0^{B_0/B_{max}} \mathrm{d}\lambda \Lambda_j^2 \frac{\partial \langle \xi \rangle}{\partial \lambda} \underset{\varepsilon\to 0}{\to} \frac{1}{4j-1}, \tag{3.23}$$

and

$$\beta_j = \int_0^{B_0/B_{max}} \mathrm{d}\lambda \Lambda_j \underset{\varepsilon\to 0}{\to} \frac{2}{3}. \tag{3.24}$$

Fortunately, aspect ratio expansions of the preceding eigenvalues and coefficients are available (Hsu, Catto & Sigmar 1990; Xiao, Catto & Molvig 2007; Parker & Catto 2012). In particular, the lowest eigenvalue is $\kappa_1 \simeq 1 + 1.46\sqrt{\varepsilon} + 1.48\varepsilon + 0.13\varepsilon^{3/2}$ so always of order unity, with the others increasing in size as for all Sturm–Liouville problems. Moreover, as ε increases so do all of the κ_j and the β_j/α_j. These prior investigations find that only the leading few eigenfunctions are required, as might be expected from how quickly the eigenvalues increase. Being more explicit by using results from Hsu *et al.* (1990),

$$\kappa_j - (2j^2 - j) = 1.46\sqrt{\varepsilon}\frac{4j-1}{3}\left[\frac{(2j-1)!!}{(2j-2)!!}\right]^2 = 1.46\sqrt{\varepsilon}\begin{cases} 1 & j=1 \\ 21/4 & j=2 \\ 825/64 & j=3 \end{cases}, \tag{3.25}$$

where $(2j-1)!! = 1\cdot 3\cdot 5\ldots(2j-1)$ and $(2j-2)!! = 2\cdot 4\cdot 6\ldots(2j-2)$ (and both equal 1 for $j=1$). Consequently, only the first couple of β_j and α_j are required. For β_j

$$\begin{aligned} \beta_j = \int_0^{B_0/B_{\max}} \mathrm{d}\lambda \Lambda_j &= \frac{2}{3}(1-\varepsilon)\left[a_{j1} - (-1)^{j+1}\frac{(2j-1)!!}{(2j-2)!!}\varepsilon\right] \\ &+ O(\varepsilon^{3/2}) \to \frac{2}{3}(1-\varepsilon)\begin{cases} a_{11} - \varepsilon & j=1 \\ a_{21} + 3\varepsilon/2 & j=2 \end{cases}, \end{aligned} \tag{3.26}$$

where based on appendix B and the fits in table 1 of Hsu *et al.* (1990)

$$a_{jj} = 1 - \sum_{j\neq m} a_{jm} \underset{j=1}{\to} a_{11} = 1 - 0.62\sqrt{\varepsilon} + 1.33\varepsilon, \tag{3.27}$$

and

$$\begin{aligned} a_{jm} \simeq &-\frac{(-1)^{j+m}(4m-1)1.46\sqrt{\varepsilon}}{3[2m^2 - m - (2j^2 - j)]}\left[\frac{(2j-1)!!(2m-1)!!}{(2j-2)!!(2m-2)!!}\right] \\ &+ O(\varepsilon) \underset{\substack{j=2 \\ m=1}}{\to} a_{21} = -0.44\sqrt{\varepsilon} - 0.13\varepsilon. \end{aligned} \tag{3.28}$$

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In addition,

$$\alpha_j = -\int_0^{B_0/B_{\max}} \mathrm{d}\lambda \Lambda_j^2 \frac{\partial\langle\xi\rangle}{\partial\lambda} = \sum_{k=1}^{\infty} \frac{a_{jk}^2}{4k-1} - \frac{4}{3}\varepsilon\left[\frac{(2j-1)!!}{(2j-2)!!}\right]^2$$
$$+ O(\varepsilon^{3/2}) = \begin{cases} a_{11}^2/3 + O(\varepsilon) & j=1 \\ 1/7 + O(\varepsilon) & j=2 \end{cases}, \tag{3.29}$$

giving

$$\beta_1/\alpha_1 \simeq 2/a_{11} + O(\varepsilon) = 2/(1-0.62\sqrt{\varepsilon}) + O(\varepsilon), \tag{3.30}$$

and

$$\beta_2/\alpha_2 \simeq 14a_{21}/3 + O(\varepsilon) = -2.04\sqrt{\varepsilon} + O(\varepsilon). \tag{3.31}$$

Using the preceding results gives the lowest -order adjoint solution to be

$$\frac{\bar{h}_p}{f_0} = \frac{\langle B^2\rangle v x^3}{2IB_0\nu_\ell} \sum_{j=1}^{\infty} \frac{\beta_j \Lambda_j(\lambda)}{[(Z+1)\kappa_j + 4]\alpha_j}$$
$$\simeq \frac{vx^3}{R\nu_\ell}\left\{\frac{(1+0.62\sqrt{\varepsilon})\Lambda_1(\lambda)}{[(Z+1)(1+1.46\sqrt{2\varepsilon})+4]} - \frac{1.02\sqrt{\varepsilon}\Lambda_2(\lambda)}{[(Z+1)(7+7.66\sqrt{2\varepsilon})+4]}\right\}, \tag{3.32}$$

which will be used to evaluate the lower hybrid driven current. Notice that even the $j=2$ term is small and can be ignored. The preceding solution also ignores $x^{-3} \ll 1$ corrections as small.

As $\bar{h}_t = 0$ only the passing electrons contribute to 2.20, it simplifies to

$$\left\langle B\int \mathrm{d}^3 v v_{||} f_1\right\rangle \simeq \frac{B_0 I}{2\pi q R}\int \mathrm{d}^3 v \frac{v_{||}\bar{h}_p}{Bf_0}\tau_p \overline{Q\{f_0\}}, \tag{3.33}$$

where large aspect ratio is assumed and the passing transit time for a full poloidal circuit is

$$\tau_p = \oint_p \mathrm{d}\tau = \oint_p \mathrm{d}\vartheta/v_{||}\boldsymbol{n}\cdot\nabla\vartheta \simeq 4qR\sqrt{(1-\varepsilon)k^2 + 2\varepsilon}K(k)/v\sqrt{2\varepsilon}, \tag{3.34}$$

with $\varepsilon = r/R$, $k^2 = 2\varepsilon\lambda/[1-(1-\varepsilon)\lambda]$, and K an elliptic integral of the first kind. The parallel current will next be formed by evaluating (3.33).

4. Current in a tokamak for a correlated QL diffusivity

The recently derived form for the correlated diffusivity (Catto & Tolman 2021) for the electrons in a tokamak is

$$\bar{D} = \frac{2\pi^3 q^2 R^2 e^2 |\boldsymbol{e}_m\cdot\boldsymbol{n}|^2}{m_e^2 v^2 \tau_p}\sum_\ell \delta\left(\oint_p \mathrm{d}\tau \Lambda - 2\pi\ell\right)\Theta(v,k), \tag{4.1}$$

where the argument of the delta function is transit averaged with

$$\oint_p \mathrm{d}\tau \Lambda \simeq \omega\tau_p - 2\pi\sigma(qn-m), \tag{4.2}$$

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allowing aspect ratio modifications to be properly evaluated. The phase integral Θ is defined as

$$\Theta \equiv \left| \oint_p \frac{\mathrm{d}\vartheta}{2\pi} \,\mathrm{e}^{-i\int_{\tau_0}^{\tau} \mathrm{d}\tau' \Lambda(\tau')} \right|^2 \leq 1, \tag{4.3}$$

with

$$\int_{\tau_0}^{\tau} \mathrm{d}\tau' \Lambda(\tau') = \omega(\tau - \tau_0) - \sigma(qn - m)\vartheta(\tau) = \omega q R \int_0^{\vartheta} \mathrm{d}\vartheta / v_{||} - \sigma(qn - m)\vartheta, \tag{4.4}$$

where $\vartheta(\tau_0) = 0$ and

$$\mathrm{d}\vartheta/\mathrm{d}\tau = v_{||}\boldsymbol{n} \cdot \nabla\vartheta. \tag{4.5}$$

For notational simplicity, only the poloidal mode number (m) is shown as a subscript on the Fourier coefficients of the applied rf parallel electric field $\boldsymbol{e}_m \cdot \boldsymbol{n}$. The frequency ($\omega$), toroidal mode number (n) and radial mode index (s) subscripts are suppressed.

For a Maxwellian the transit averaged QL operator is

$$\overline{Q\{f_0\}} = \sum_{\substack{\omega,m\\ n,s}} \frac{1}{\tau_p} \frac{\partial}{\partial E} \left(\tau_p v^2 \bar{D} \frac{\partial f_0}{\partial E} \right) = \sum_{\substack{\omega,m\\ n,S}} \frac{1}{\tau_p v} \left.\frac{\partial}{\partial v}\right|_{\mu} \left(\tau_p v \bar{D} \frac{\partial f_0}{\partial v} \right). \tag{4.6}$$

with ℓ the bounce harmonic index as sucessive passes are correlated. As only the passing contribute, the large aspect ratio form of the flow becomes

$$\begin{aligned} \left\langle B \int d^3 v v_{||} f_1 \right\rangle &\simeq \frac{B_0 I}{2\pi q R} \sum_{\substack{\omega,m\\ n,s}} \mathrm{d}^3 v \frac{v_{||} \bar{h}_p}{B f_0 v} \left.\frac{\partial}{\partial v}\right|_{\mu} \left(v \tau_p \bar{D} \left.\frac{\partial f_0}{\partial v}\right|_{\mu} \right) \\ &= \frac{m_e B_0 I}{2\pi T_e q R} \sum_{\substack{\omega,m\\ n,s}} \int \mathrm{d}^3 v \frac{v_{||} \tau_p}{B v} \bar{D} f_0 \left.\frac{\partial}{\partial v}\right|_{\mu} \left(\frac{v^2 \bar{h}_p}{f_0} \right). \end{aligned} \tag{4.7}$$

To proceed it is convenient to perform the v integral first by writing

$$\bar{D} = v^{-2} \tau_p^{-1} \sum_{\ell} d_\ell \Theta(v, k) \delta(\oint_p \mathrm{d}\tau \Lambda - 2\pi\ell), \tag{4.8}$$

with

$$d_\ell \equiv 2\pi^3 q^2 R^2 e^2 m_e^{-2} |\boldsymbol{e}_m \cdot \boldsymbol{n}|^2. \tag{4.9}$$

Then

$$\left\langle B \int \mathrm{d}^3 v v_{||} f_1 \right\rangle = \frac{m_e B_0 I}{2\pi T_e q R} \sum_{\substack{\omega,m\\ n,s,\ell}} d_\ell \int \mathrm{d}^3 v \frac{v_{||} f_0}{B v^3} \Theta(v, k) \delta(\oint_p \mathrm{d}\tau \Lambda - 2\pi\ell) \left.\frac{\partial}{\partial v}\right|_{\mu} \left(\frac{v^2 \bar{h}_p}{f_0} \right). \tag{4.10}$$

Taylor expanding the argument of the delta function leads to

$$\delta(\oint_p \mathrm{d}\tau \Lambda - 2\pi\ell) = \frac{\delta(v - v_k)}{\omega |\partial\tau_p / \partial v|_{v_k}} = \frac{v_k^2 \delta(v - v_k)}{\omega v \tau_p}, \tag{4.11}$$

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with the speed at resonance defined as

$$v_k \equiv \omega v\tau_p/2\pi(\ell + |qn - m|). \tag{4.12}$$

As a result,

$$\left\langle B\int \mathrm{d}^3v v_\| f_1\right\rangle = \frac{m_e B_0 I}{2\pi T_e qR}\sum_{\substack{\omega,m\\ n,s,\ell}} d_\ell \int \mathrm{d}^3v \frac{v_\| f_0}{Bv}\frac{\Theta(v,k)}{\omega v\tau_p}\delta(v - v_k)\left.\frac{\partial}{\partial v}\right|_\mu \left(\frac{v^2\bar{h}_p}{f_0}\right). \tag{4.13}$$

Using $\nabla_v v = \boldsymbol{v}/v$, $\nabla_v\lambda = (2B_0/Bv^2)\boldsymbol{v}_\perp - (2\lambda/v^2)\boldsymbol{v}$, and $\nabla_v\varphi = v_\perp^{-2}\boldsymbol{n}\times\boldsymbol{v}$, gives $\mathrm{d}^3v = \mathrm{d}v\,\mathrm{d}\lambda\mathrm{d}\varphi/\nabla_v v\times\nabla_v\lambda\boldsymbol{\cdot}\nabla_v\varphi = \mathrm{d}v\,\mathrm{d}\lambda\,\mathrm{d}\varphi Bv^3/2B_0 v_\|$. The LH parallel electron flow must be driven in the negative direction to sustain the poloidal magnetic field, making $\langle B\int \mathrm{d}^3v v_\| f_1\rangle < 0$. Hence, a minus sign must be inserted as the velocity space integration is only over $v_\| < 0$ electrons with $\bar{D}$ vanishing for $v_\| > 0$. Consequently, in the resonance condition, $\sigma = -1$ and $m > qn$. Therefore, the magnitude of the parallel wave number $|k_\|| \equiv |qn - m|/qR$, is used as $k_\| < 0$.

Multiplying the velocity integral by 1/2 and inserting the diffusivity

$$\begin{aligned}\left\langle B\int \mathrm{d}^3v v_\| f_1\right\rangle &= \frac{-m_e I}{4T_e qR}\sum_{\substack{\omega,m\\ n,s,\ell}} d_\ell \int_0^{B_0/B_{\max}} \mathrm{d}\lambda \int_0^\infty \mathrm{d}v\Theta(v,k)\delta(v - v_k)\frac{v^2 f_0}{\omega v\tau_p}\left.\frac{\partial}{\partial v}\right|_\mu\left(\frac{v^2\bar{h}_p}{f_0}\right)\\ &= -\frac{m_e I}{4T_e qR}\sum_{\substack{\omega,m\\ n,s,\ell}} d_\ell \int_0^{B_0/B_{\max}} \mathrm{d}\lambda\Theta(v_k,k)\frac{v^2 d_\ell f_0}{\omega v\tau_p}\left.\frac{\partial}{\partial v}\right|_\mu \left.\left(\frac{v^2\bar{h}_p}{f_0}\right)\right|_{v=v_k}.\end{aligned} \tag{4.14}$$

The preceding equation is a convenient form of the parallel flow and can be used to evaluate the driven parallel electron current for a more general like particle collision operator than was considered in the preceding section.

Defining

$$x_k^2 = m_e v_k^2/2T_e = m_e(\omega v\tau_p)^2/8\pi^2(\ell + |qn - m|)^2 T_e, \tag{4.15}$$

with $x_k^2 \gg 1$ in $f_0 \propto \mathrm{e}^{-x_k^2}$, the $v\tau_p \propto K(k) \underset{k\to 1}{\rightarrow} \infty$ indicates that the freely passing will dominate the driven current. Consequently, $x_k^2 \gg 1$ is used to integrate by parts by first noting

$$\left.\frac{\partial f_0}{\partial\lambda}\right|_v = -f_0\left.\frac{\partial x_k^2}{\partial\lambda}\right|_v = -f_0\frac{m_e\omega^2 v\tau_p}{4\pi^2 T_e(\ell + |qn - m|)^2}\left.\frac{\partial(v\tau_p)}{\partial\lambda}\right|_v. \tag{4.16}$$

Then ignoring $x_k^{-2} \ll 1$ corrections

$$\begin{aligned}\left\langle B\int \mathrm{d}^3v v_\| f_1\right\rangle &= -\frac{\pi^2 I}{qR}\sum_{\substack{\omega,m\\ n,s,\ell}} \frac{d_\ell(\ell + |qn - m|)^2 f_0}{\omega(\omega\tau_p)^2\partial(v\tau_p)/\partial\lambda}\Theta(x_k, k=0)\\ &\quad\times\left.\left[\left.\frac{\partial}{\partial v}\right|_\lambda\left(\frac{v^2\bar{h}_p}{f_0}\right) - 2\lambda v\left.\frac{\partial}{\partial\lambda}\right|_v\left(\frac{\bar{h}_p}{f_0}\right)\right]\right|_{\substack{v=v_k\\ \lambda=0=k}} + \cdots.\end{aligned} \tag{4.17}$$

Changing to v and λ velocity variables using

$$\left.\frac{\partial}{\partial v}\right|_\mu = \left.\frac{\partial}{\partial v}\right|_\lambda + \left.\frac{\partial\lambda}{\partial v}\right|_\mu\left.\frac{\partial}{\partial\lambda}\right|_{v_\|} = \left.\frac{\partial}{\partial v}\right|_\lambda - \frac{2\lambda}{v}\left.\frac{\partial}{\partial\lambda}\right|_v, \tag{4.18}$$

https://doi.org/10.1017/S0022377821000568 Published online by Cambridge University Press

gives

$$\left.\frac{\partial}{\partial v}\right|_{\mu}\left(\frac{v^2\bar{h}_p}{f_0}\right) = \left.\frac{\partial}{\partial v}\right|_{\lambda}\left(\frac{v^2\bar{h}_p}{f_0}\right) - 2\lambda v\left.\frac{\partial}{\partial \lambda}\right|_{v}\left(\frac{\bar{h}_p}{f_0}\right). \tag{4.19}$$

Therefore,

$$\left\langle B\int \mathrm{d}^3 v v_{||} f_1\right\rangle = -\frac{\pi^2 I}{qR}\sum_{\substack{\omega,m\\ n,s,\ell}}\frac{d_\ell(\ell+|qn-m|)^2 f_0}{\omega(\omega\tau_p)^2\partial(v\tau_p)/\partial\lambda}\Theta(x_k,k=0)$$
$$\times\left[\left.\frac{\partial}{\partial v}\right|_{\lambda}\left(\frac{v^2\bar{h}_p}{f_0}\right) - 2\lambda v\left.\frac{\partial}{\partial \lambda}\right|_{v}\left(\frac{\bar{h}_p}{f_0}\right)\right]\Bigg|_{\substack{v=v_k\\ \lambda=0=k}} + \cdots. \tag{4.20}$$

As the pitch angle derivative of $\bar{h}_p$ is well behaved at $\lambda=0$ the last term vanishes leading to a lower hybrid driven current $J_{||}$ of

$$J_{||}\equiv -eB_0^{-1}\left\langle B\int \mathrm{d}^3 v v_{||} f_1\right\rangle \simeq -\frac{\pi^2 e}{q}\sum_{\substack{\omega,m\\ n,s,\ell}}\frac{d_\ell(\ell+|qn-m|)^2 f_0}{\omega(\omega\tau_p)^2\partial(v\tau_p)/\partial\lambda}\Theta(x_k,k=0)\left.\frac{\partial}{\partial v}\right|_{\lambda}\left(\frac{v^2\bar{h}_p}{f_0}\right)\Bigg|_{\substack{v=v_k\\ \lambda=0=k}}. \tag{4.21}$$

To simplify further

$$(v\tau_p)_{k=0} = 2\pi qR, \tag{4.22}$$

is used along with

$$\left.\frac{1}{\tau_p}\frac{\partial\tau_p}{\partial\lambda}\right|_{\lambda=0=k} = \frac{[(1-\varepsilon)k^2+2\varepsilon]^2}{4\varepsilon k^2}\left[\frac{E}{(1-k^2)K} - K + \frac{(1-\varepsilon)k^2}{(1-\varepsilon)k^2+2\varepsilon}\right]\Bigg|_{k=0}$$
$$= \frac{\varepsilon}{k^2}\left[\frac{k^2}{2}+\frac{(1-\varepsilon)k^2}{2\varepsilon}\right]\simeq\frac{1}{2}. \tag{4.23}$$

Moreover, only $\ell=0$ contributes as

$$\Theta(k=0,x_k=x_{k=0})\equiv\left|\oint_p\frac{\mathrm{d}\vartheta}{2\pi}\,\mathrm{e}^{-\mathrm{i}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\varLambda(\tau')}\right|^2$$
$$\simeq\left|\oint_p\frac{\mathrm{d}\vartheta}{2\pi}\,\mathrm{e}^{-\mathrm{i}(\omega qR/v-|qn-m|)\vartheta}\right|^2\simeq\left|\oint_p\frac{d\vartheta}{2\pi}\,\mathrm{e}^{-\mathrm{i}\ell\vartheta}\right|^2=\delta_{0\ell}, \tag{4.24}$$

giving

$$v_{k=0}=\omega(v\tau_p)_{k=0}/2\pi|qn-m| = \omega qR/|qn-m|. \tag{4.25}$$

Consequently, the LH current becomes

$$J_{||}\simeq\frac{\pi^{1/2}n_e R}{2v_e^3}\sum_{\substack{\omega,m\\ n,s}}\frac{e^3|\boldsymbol{e}_m\cdot\boldsymbol{n}|^2}{m_e^2\omega}\,\mathrm{e}^{-\omega^2/k_{||}^2v_e^2}\left.\frac{\partial}{\partial v}\right|_{\lambda}\left(\frac{v^2\bar{h}_p}{f_0}\right)\Bigg|_{\substack{v=v_k\\ \lambda=0=k}}. \tag{4.26}$$

Keeping only the $j=1$ term in $\bar{h}_p$ yields the final result for the driven lower hybrid current in a tokamak to be

$$J_{||}\simeq\frac{4en_e(1+0.62\sqrt{\varepsilon})}{[(Z+1)(1+2.06\sqrt{\varepsilon})+4]\nu_{ee}}\sum_{\substack{\omega,m\\ n,s}}\frac{e^2|\boldsymbol{e}_m\cdot\boldsymbol{n}|^2\omega^4}{m_e^2|k_{||}|^5 v_e^6}\,\mathrm{e}^{-\omega^2/k_{||}^2v_e^2}. \tag{4.27}$$

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Importantly, the key result in the preceding expression is that it retains the proper leading order analytic dependence on the electron trapping parameter $\sqrt{\varepsilon}$ for the first time. Notice in particular that the aspect ratio dependence of the $Z+1$ pitch angle scattering factor differs from that of the electron-electron energy scattering factor 4. Moreover, the overall current drive efficiency is reduced somewhat by the inverse aspect ratio dependences as the increase in the numerator is unable to overcome the decrease from the denominator. Presumably similar behaviour persists for more general cross sections and will make off-axis current profile control slightly more difficult in spherical tokamaks. Perhaps not surprisingly, energy scattering is less affected by toroidal geometry that pitch angle scattering, which is the reason only the leading Cordey (1976) eigenfunction need be retained. As the freely passing dominate the electron response, (4.27) reduces to the standard $\sqrt{\varepsilon} \to 0$ result. The driven parallel current (4.27) will be used in the next section to evaluate the current drive efficiency.

5. RF power and current drive efficiency

The rf power required to drive the lower hybrid current, P_{cd}, is found from

$$P_{cd} = \frac{m_e}{2}\left\langle \int \mathrm{d}^3 v v^2 \overline{Q\{f_0\}} \right\rangle = \frac{m_e}{2}\sum_{\substack{\omega,m \\ n,S}} \left\langle \int \mathrm{d}^3 v v^2 \frac{1}{\tau_p v}\left.\frac{\partial}{\partial v}\right|_{\mu}\left(\tau_p v \bar{D}\frac{\partial f_0}{\partial v}\right)\right\rangle. \tag{5.1}$$

Only the passing electrons contribute as there is no resonance for the trapped since

$$\oint_t \mathrm{d}\tau \Lambda \simeq \omega\tau_t, \tag{5.2}$$

and

$$\Theta \propto |\oint_t \mathrm{d}\tau v_{||}\, \mathrm{e}^{-\mathrm{i}\omega(\tau-\tau_0)}|^2 \to 0. \tag{5.3}$$

Recalling $\mathrm{d}^3 v = \mathrm{d}v\, \mathrm{d}\lambda\, \mathrm{d}\varphi B v^3/2B_0 v_{\|}$, and using $\oint \mathrm{d}\vartheta/\boldsymbol{n}\cdot\nabla\vartheta \simeq 2\pi q R$

$$P_{cd} = \frac{B_0 m_e}{4\pi q R}\sum_{\substack{\omega,m \\ n,S}} \int \mathrm{d}^3 v \frac{v_{||} v}{B}\left.\frac{\partial}{\partial v}\right|_{\mu}\left(\tau_p v \bar{D}\frac{\partial f_0}{\partial v}\right) = \frac{B_0 m_e^2}{\pi T_e q R}\sum_{\substack{\omega,m \\ n,S}} \int \mathrm{d}^3 v \frac{v_{||}}{B}\tau_p v^2 \bar{D} f_0. \tag{5.4}$$

Multiplying by 1/2 since only $v_{\|} < 0$ contribute, the preceding becomes

$$P_{cd} = \frac{m_e^2}{2T_e q R}\sum_{\substack{\omega,m \\ n,S}} \int_0^{B_0/B_{\min}} \mathrm{d}\lambda \int_0^{\infty} \mathrm{d}v \tau_p v^5 \bar{D} f_0. \tag{5.5}$$

Inserting $\bar{D}$

$$\begin{aligned} P_{cd} &= \frac{m_e^2}{2T_e q R}\sum_{\substack{\omega,m \\ n,s,\ell}} \frac{d_\ell}{\omega}\int_0^{B_0/B_{\max}} \frac{\mathrm{d}\lambda}{v\tau_p}\int_0^{\infty} \mathrm{d}v v^5 f_0 \Theta(v,k)\delta(v-v_k) \\ &= \frac{m_e^2}{2T_e q R}\sum_{\substack{\omega,m \\ n,s,\ell}} \frac{d_\ell}{\omega}\int_0^{B_0/B_{\max}} \frac{\mathrm{d}\lambda}{v\tau_p} v^5 f_0 \Theta(v,k)|_{v=v_k} \\ &= -\frac{2\pi^2 m_e}{qR}\sum_{\substack{\omega,m \\ n,s}} \frac{d_{\ell=0}|qn-m|^2}{\omega^3}\int_0^{B_0/B_{\max}} \frac{\mathrm{d}\lambda}{(v\tau_p)^2}\frac{v^5\Theta(v,k)}{\partial(v\tau_p)/\partial\lambda}\left.\frac{\partial f_0}{\partial\lambda}\right|_{v=v_k}. \end{aligned} \tag{5.6}$$

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Integrating by parts and noting that only $\ell = 0$ contributes, the rf power density required to drive the lower hybrid current $J_{\|}$ is just

$$P_{cd} \simeq \frac{2\pi^2 m_e}{qR} \sum_{\substack{\omega,m\\n,s}} \frac{d_{\ell=0}|qn-m|^2 v^5}{\omega^3 (v\tau_p)^2 \partial(v\tau_p)/\partial\lambda} f_0|_{v=v_k} = \pi^{1/2} n_e m_e \sum_{\substack{\omega,m\\n,s}} \frac{e^2 |\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^2 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{m_e^2 |k_{||}|^3 v_e^3}. \tag{5.7}$$

The current drive efficiency is defined by forming the ratio $J_{\|}/P_{cd}$

$$\frac{J_{||}}{P_{cd}} = \frac{4e(1+0.62\sqrt{2\varepsilon}) \displaystyle\sum_{\substack{\omega,m\\n,s}} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^4 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^5 v_e^6}}{\sqrt{\pi} m_e [(Z+1)(1+2.06\sqrt{\varepsilon})+4]\nu_{ee} \displaystyle\sum_{\substack{\omega,m\\n,s}} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^2 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^3 v_e^3}}, \tag{5.8}$$

or in its normalized form

$$\frac{J_{||}/en_e v_e}{P_{cd}/n_e m_e v_e^2 \nu_{ee}} \equiv \eta = \frac{4(1+0.62\sqrt{2\varepsilon}) \displaystyle\sum_{\substack{\omega,m\\n,s}} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^4 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^5 v_e^5}}{\sqrt{\pi} [(Z+1)(1+2.06\sqrt{\varepsilon})+4] \displaystyle\sum_{\substack{\omega,m\\n,s}} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^2 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^3 v_e^3}}. \tag{5.9}$$

Various normalizations of $J_{\|}$ and P_{cd} (often because of a $\sqrt{2}$ difference in the definition of the electron thermal speed), and differing definitions of collision frequency appear in the LHCD literature, as well as incompletely defined notation.

For a single frequency and single toroidal mode number (5.9) leads to the dimensionless current drive efficiency with electron trapping retained of

$$\begin{aligned} \eta &= \frac{4(1+0.62\sqrt{2\varepsilon}) \displaystyle\sum_{m,s} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^5 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^5 v_e^5}}{\sqrt{\pi} [(Z+1)(1+2.06\sqrt{\varepsilon})+4] \displaystyle\sum_{m,s} \frac{|\boldsymbol{e}_m \cdot \boldsymbol{n}|^2 \omega^3 \,\mathrm{e}^{-\omega^2/k_{\|}^2 v_e^2}}{|k_{||}|^3 v_e^3}} \\ &\to \frac{4(1+0.62\sqrt{2\varepsilon})\omega^2}{\sqrt{\pi}[(Z+1)(1+2.06\sqrt{\varepsilon})+4]|k_{||}|^2 v_e^2}, \end{aligned} \tag{5.10}$$

where the sums over poloidal mode number and radial mode structure (either as a Fourier or eikonal representation) are retained except in the final form.

6. Discussion

The key results derived here are the analytic expressions for the LHCD (4.26) and (4.27) and the LHCD efficiency (5.9) and (5.10) in tokamak geometry that properly retain the trapped electron modifications to leading order. The procedure used here can be extended to include lower speed electrons by using the full Rosenbluth form of the electron–electron collision operator with a momentum conserving modification. Keeping additional Cordey (1976) eigenfunctions will only make minor, unimportant changes to the results here.

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However, generalizing the Cordey eigenfunctions to finite aspect ratio in a more realistic geometry will lead to improved results as the separatrix is approached. Such a treatment might be valuable as lower hybrid is viewed as an effective means to drive and control the off axis current profile (Bonoli 2014). Of course, near the separatrix and beyond the long mean free path limit will become inappropriate and the Cordey eigenfunctions are no longer useful.

The QL derivation of Catto & Tolman (2021) remains valid as long as the collisional boundary layer about the resonant electron trajectories defined in (4.11) and (4.12) remains narrow enough in velocity space to satisfy $1 \gg (\nu_{ee}/k_{\|} v_e)^{1/3} \propto n_e^{1/3}/T_e^{7/6}$. The QL treatment they derive and the results here are expected to fail once the applied rf amplitude substantially distorts the electron distribution function from Maxwellian (Catto 2020; Catto & Tolman 2021). This level of distortion is estimated by Catto & Tolman (2021) to occur once the applied rf becomes sufficiently strong that the nonlinear term in the linearized Fokker–Planck equation for the electron kinetic response no longer satisfies $e|\boldsymbol{e}_m \cdot \boldsymbol{n}|/m_e \ll \nu_{ee}^{2/3} k_{\|}^{1/3} v_e^{4/3} \propto n_e^{2/3}/T_e^{1/3}$ and integrating over unperturbed electron trajectories to obtain the QL operator becomes inappropriate. This estimate may provide a hint as to why LHCD becomes less efficient at higher densities (Bonoli 2014).

Acknowledgements

Work supported by the U.S. Department of Energy grant DE-FG02-91ER-54109 at MIT. The author is grateful for suggestions from Dr P. Bonoli.

Editor Per Helander thanks the referees for their advice in evaluating this article.

Declaration of interests

The author reports no conflict of interest.

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