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EXISTENCE AND BOX DIMENSION OF GENERAL RECURRENT FRACTAL INTERPOLATION FUNCTIONS

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Abstract

The notion of recurrent fractal interpolation functions (RFIFs) was introduced by Barnsley *et al.* [‘Recurrent iterated function systems’, *Constr. Approx.* **5** (1989), 362–378]. Roughly speaking, the graph of an RFIF is the invariant set of a recurrent iterated function system on $\mathbb{R}^2$. We generalise the definition of RFIFs so that iterated functions in the recurrent system need not be contractive with respect to the first variable. We obtain the box dimensions of all self-affine RFIFs in this general setting.

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1. Introduction

Let $(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N) \in \mathbb{R}^2$ be given data, where $x_0 < x_1 < \cdots < x_N$. There are different functions supported on $\mathbb{R}$ that map each x_i exactly to y_i. These functions are called *interpolation functions*. For example, Lagrangian polynomial interpolation gives a unique polynomial function satisfying the interpolation conditions. However, polynomials or other smooth functions might not be well suited for approaching ‘fractal’ curves such as coastlines or electrocardiograms. Fractal interpolation functions (FIFs for short), introduced by Barnsley [3], provide an effective tool in this situation.

In 1989, Barnsley *et al.* [4] introduced recurrent fractal interpolation functions as follows. (In the original version, the corresponding iterated function systems are composed of affine maps.) Let $N \geqslant 2$ and $\{\ell(i), r(i)\}_{i=1}^N \subset \{0, 1, \dots, N\}$ with $\ell(i) < r(i)$. For $1 \leqslant i \leqslant N$, we set $I_i = [x_{i-1}, x_i]$, $D_i = [x_{\ell(i)}, x_{r(i)}]$ and assume that:

(1) $L_i : D_i \to I_i$ is a homeomorphism;
(2) $F_i : D_i \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists β_i $(0 < \beta_i < 1)$ with

$$|F_i(x, y') - F_i(x, y'')| \leqslant \beta_i |y' - y''| \quad \text{for all } x \in D_i, \quad y', y'' \in \mathbb{R};$$

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(3) if we define $\omega_i : D_i \times \mathbb{R} \to I_i \times \mathbb{R}$ by $\omega_i(x, y) = (L_i(x), F_i(x, y))$, then ω_i maps $(x_{\ell(i)}, y_{\ell(i)})$ to (x_{i-1}, y_{i-1}) and $(x_{r(i)}, y_{r(i)})$ to (x_i, y_i) (or $(x_{\ell(i)}, y_{\ell(i)})$ to (x_i, y_i) and $(x_{r(i)}, y_{r(i)})$ to (x_{i-1}, y_{i-1})).

Set $\mathcal{I}(i) := \{j : I_j \subset D_i\}$. For any function f supported on $I = [x_0, x_N]$, we let $\Gamma(f)$ be the graph of f, that is, $\Gamma(f) = \{(x, f(x)) : x \in I\}$, and $f|_{I_j}$ the restriction of f to I_j.

THEOREM 1.1 [4]. *Suppose that* $x_{r(i)} - x_{\ell(i)} > x_i - x_{i-1}$ *for each i and that there exists* $\alpha_i \in (0, 1)$ *such that*

$$|L_i(x') - L_i(x'')| \le \alpha_i |x' - x''| \quad \textit{for all } x', x'' \in D_i.$$

Then there is a unique function $f \in C(I)$ *such that* $f(x_0) = y_0, \ldots, f(x_N) = y_N$ *and*

$$\Gamma(f|_{I_i}) = \bigcup_{j \in \mathcal{I}(i)} \omega_i(\Gamma(f|_{I_j})) \quad \textit{for } 1 \leqslant i \leqslant N.$$

Such an f is called the *recurrent fractal interpolation function* (RFIF for short) determined by $\{\omega_1, \ldots, \omega_N\}$. FIFs and RFIFs have been widely used in applications (see, for example, [13]).

There is great interest in the calculation of fractal dimensions, especially the box dimension and the Hausdorff dimension, of graphs of FIFs and RFIFs. Given any bounded set $E \subset \mathbb{R}^n$, the lower and upper box dimensions of E are given by

$$\underline{\dim}_{\mathrm{B}} E = \varliminf_{\varepsilon \to 0} \frac{\log \mathcal{N}_E(\varepsilon)}{-\log \varepsilon}, \quad \overline{\dim}_{\mathrm{B}} E = \varlimsup_{\varepsilon \to 0} \frac{\log \mathcal{N}_E(\varepsilon)}{-\log \varepsilon},$$

where $\mathcal{N}_E(\varepsilon)$ is the number of ε-mesh cubes intersecting E. If these two limits coincide, we call the common value the *box dimension* of E and denote it by $\dim_{\mathrm{B}} E$.

Once we interpolate some rough data by a fractal function, it is reasonable to say that fractal dimensions of the fractal function reflect the fractal properties of the rough data. Barnsley *et al.* [4] obtained the box dimension of the graphs of self-affine RFIFs. See also [5–8, 10–12, 14] for constructions of FIFs and RFIFs in general settings (for example, rectangular domains) and the calculations of box dimensions of their graphs. For results and techniques on the Hausdorff dimension of the graphs of fractal interpolation functions, see the survey papers [1, 2].

The following question arises naturally.

QUESTION 1.2. Is the assumption $x_{r(i)} - x_{\ell(i)} > x_i - x_{i-1}$, or equivalently $|D_i| > |I_i|$, essential for the existence of RFIFs? If the assumption is not essential, can we determine the box dimensions of the graphs in this general setting?

We try to answer this question. By using the classical method of [3], we show that the assumption can be removed, giving extra flexibility in the construction of RFIFs. In practice, this helps in finding more effective methods for modelling natural shapes.

THEOREM 1.3. *There always exists a unique function $f \in C(I)$ satisfying $f(x_i) = y_i$ for $0 \leqslant i \leqslant N$ and*

$$\Gamma(f|_{I_i}) = \bigcup_{j \in \mathcal{I}(i)} \omega_i(\Gamma(f|_{I_j})) \quad \text{for } 1 \leqslant i \leqslant N. \tag{1.1}$$

The second part of the question is more involved and we restrict ourselves to affine cases. To be precise, suppose that there are $a_i, c_i, d_i, e_i, f_i \in \mathbb{R}$ for $1 \leqslant i \leqslant N$ such that

$$L_i(x) = a_i x + e_i, \quad F_i(x, y) = c_i x + d_i y + f_i \quad \text{for } x \in D_i, y \in \mathbb{R},$$

where $d_i \in (-1, 1)$. Clearly, $|a_i| = |I_i|/|D_i|$ for all i. Thus,

$$\omega_i(x, y) = (a_i x + e_i, c_i x + d_i y + f_i) \quad \text{for } (x, y) \in D_i \times \mathbb{R}. \tag{1.2}$$

In this case, the RFIF determined by $\{\omega_i\}_{i=1}^N$ is called a *self-affine RFIF*.

In order to obtain the box dimensions of self-affine RFIFs, we will mainly use the methods in [4, 11, 14]. Before stating our result, we introduce some basic concepts. For any $s \in \mathbb{R}$, let $Q(s)$ be an $N \times N$ matrix defined by $Q(s)_{ij} = |d_i||a_i|^{s-1}$ if $I_j \subset D_i$ and $Q(s)_{ij} = 0$ otherwise. For simplicity, we also denote $Q(1)$ by Q if there is no confusion. To calculate $\dim_B \Gamma(f)$, we define a directed graph $G := G_Q$ as follows:

(1) the vertex set of G consists of $1, 2, \ldots, N$;
(2) there is an edge from j to i ($1 \leqslant i, j \leqslant N$) if and only if $Q_{ij} > 0$, that is, $I_j \subset D_i$ and $d_i \neq 0$.

We say that G is *strongly connected* if there is a path from i to j for $1 \leqslant i, j \leqslant N$, that is, there exists a finite sequence $\{v_0 = i, v_1, \ldots, v_n = j\}$ in $\{1, \ldots, N\}$ such that there is an edge from v_{k-1} to v_k for $1 \leqslant k \leqslant n$. The subgraph induced by $V' \subset \{1, \ldots, N\}$ is called a *strongly connected component* of G if it is itself strongly connected, but the subgraph induced by U is not strongly connected whenever $\{1, \ldots, N\} \supset U \supsetneqq V'$. It is well known that Q is irreducible if and only if the corresponding directed graph G is strongly connected (see, for example, [9]). For brevity, we also say that $\{1, \ldots, N\}$ is strongly connected if G is strongly connected, and $V \subset \{1, \ldots, N\}$ is a strongly connected component if V is the vertex set of a strongly connected component of G.

DEFINITION 1.4 [11]. For $1 \leqslant i \leqslant N$, we call i *degenerate* if:

(1) points in $\{(x_k, y_k) : \ell(i) \leqslant k \leqslant r(i)\}$ are collinear; and
(2) points in $\{(x_k, y_k) : \ell(j) \leqslant k \leqslant r(j)\}$ are collinear when there is a path from j to i.

In addition, a strongly connected component $V \subset \{1, \ldots, N\}$ is called *degenerate* if every $i \in V$ is degenerate and otherwise *nondegenerate*. Clearly, if some $i \in V$ is not degenerate, then any other $i' \in V$ is also not degenerate. In other words, if V is nondegenerate, then each $i \in V$ is not degenerate.

For any $N \times N$ matrix A and any $V \subset \{1, \ldots, N\}$, we denote by A_V the principal submatrix of A indexed by $V \times V$, that is, A_V is obtained by deleting all the ith rows and jth columns of A for $i, j \notin V$.

Now we can present our result on the box dimension of the graphs of self-affine RFIFs.

THEOREM 1.5. *Let f be the self-affine RFIF determined by $\{\omega_i\}_{i=1}^N$ in (1.2), and $V_1, \ldots, V_m$ be nondegenerate strongly connected components of $\{1, \ldots, N\}$. Let s_t be the unique real number satisfying $\rho(Q(s_t)_{V_t}) = 1$ for $1 \le t \le m$, where $\rho(\cdot)$ represents the spectral radius. Then* $\dim_B \Gamma(f) = \max\{s_1, \ldots, s_m, 1\}$.

The paper is organised as follows. In Section 2 we establish the existence and uniqueness of an RFIF without the assumption that $x_{r(i)} - x_{\ell(i)} > x_i - x_{i-1}$ and present some basic estimates. In Section 3 we prepare the essential ingredients for Section 4, where we prove Theorem 1.5.

2. Unique existence and some basic estimates

The proof of Theorem 1.3 is classical.

PROOF OF THEOREM 1.3. Set $C_*(I) = \{g \in C(I) : g(x_0) = y_0, \ldots, g(x_N) = y_N\}$. It is clear that $C_*(I)$ is a closed subset of $(C(I), \|\cdot\|_\infty)$ and thus a complete metric space. Given a function $g \in C_*(I)$, we define

$$Tg(x) = F_i(L_i^{-1}(x), g(L_i^{-1}(x))) \quad \text{for all } x \in I_i \text{ and } 1 \leqslant i \leqslant N.$$

It is easy to check that Tg is well defined and $Tg \in C_*(I)$. For any $g, h \in C_*(I)$, any i and any $x \in I_i$,

$$\begin{aligned} |Tg(x) - Th(x)| &= |F_i(L_i^{-1}(x), g(L_i^{-1}(x))) - F_i(L_i^{-1}(x), h(L_i^{-1}(x)))| \\ &\leqslant |d_i||g(L_i^{-1}(x)) - h(L_i^{-1}(x))| \leqslant \Big(\max_{1\leqslant i\leqslant N} |d_i|\Big)\|g - h\|_\infty. \end{aligned}$$

Hence, T is a contractive map and has a unique fixed point $f \in C_*(I)$ for which

$$f(x) = Tf(x) = F_i(L_i^{-1}(x), f(L_i^{-1}(x))) \quad \text{for all } x \in I_i \text{ and } 1 \leqslant i \leqslant N. \tag{2.1}$$

Thus,

$$\begin{aligned} \Gamma(f|_{I_i}) &= \{(x, f(x)) : x \in I_i\} = \{(x, F_i(L_i^{-1}(x), f(L_i^{-1}(x)))) : x \in I_i\} \\ &= \{(L_i(x), F_i(x, f(x))) : x \in D_i\} = \{\omega_i(x, f(x)) : x \in D_i\} = \bigcup_{j\in\mathcal{I}(i)} \omega_i(\Gamma(f|_{I_j})). \end{aligned}$$

On the other hand, if there is another function $g \in C_*(I)$ satisfying (1.1), then

$$\{(x, g(x)) : x \in I_i\} = \{(L_i(x), F_i(x, g(x))) : x \in D_i\} = \{(x, F_i(L_i^{-1}(x), g(L_i^{-1}(x)))) : x \in I_i\}$$

for $1 \leqslant i \leqslant N$, so that $g(x) = Tg(x)$ for all $x \in I$. Hence, $g = f$. □

The following proposition is well known and the proof is straightforward.

PROPOSITION 2.1. *Let f be the self-affine RFIF determined by $\{\omega_i\}_{i=1}^N$ in (1.2). If $d_i = 0$ for some i with $1 \leqslant i \leqslant N$, then $f|_{I_i}$ is linear, so that* $\dim_B \Gamma(f|_{I_i}) = 1$.

PROOF. If $d_i = 0$, then, from (2.1), $f|_{I_i}(x) = c_i L_i^{-1}(x) + f_i = a_i^{-1} c_i (x - e_i) + f_i$. □

DEFINITION 2.2 [4, 14]. For any bounded closed interval $E = [c, d]$ and $0 < \varepsilon < |E|$, we call $\{\tau_\ell\}_{\ell=0}^m$ an *ε-partition* of E if $\tau_0 = c$, $\tau_m = d$ and $\varepsilon/2 < \tau_{\ell+1} - \tau_\ell \leqslant \varepsilon$ holds for $0 \leqslant \ell \leqslant m-1$.

It is easy to see that if $\{\tau_\ell\}_{\ell=0}^m$ is an *ε-partition* of $E = [c, d]$, then $m\varepsilon/2 < d - c \leqslant m\varepsilon$. For any real function f defined on E and any $\tau \in \mathbb{R}$, we denote the oscillation of f on $[\tau, \tau+\varepsilon] \cap E$ by

$$O(f, \tau, \varepsilon) = \sup\{f(x') - f(x'') : x', x'' \in [\tau, \tau+\varepsilon] \cap E\}$$

and define

$$\mathcal{N}^*_{\Gamma(f)}(\varepsilon) = \inf\Big\{\varepsilon^{-1} \sum_{\ell=0}^{m-1} O(f, \tau_\ell, \varepsilon) : \{\tau_\ell\}_{\ell=0}^m \text{ is an } \varepsilon\text{-partition of } E\Big\},$$

where the infimum is taken over all ε-partitions of E. By the following lemma, we can use $\mathcal{N}^*_{\Gamma(f)}(\varepsilon)$ instead of $\mathcal{N}_{\Gamma(f)}(\varepsilon)$ to estimate the box dimension of $\Gamma(f)$.

LEMMA 2.3 [14]. *For any continuous function f on a closed interval $E = [c, d]$, there exist a constant $C > 0$ independent of f and another constant $\gamma > 0$ dependent on f such that*

$$C^{-1}\mathcal{N}^*_{\Gamma(f)}(\varepsilon) \leqslant \mathcal{N}_{\Gamma(f)}(\varepsilon) \leqslant \gamma\varepsilon^{-1} + C\mathcal{N}^*_{\Gamma(f)}(\varepsilon), \quad 0 < \varepsilon < d - c.$$

The method in the proof of [14, Lemma 2] yields the following estimate.

LEMMA 2.4. *Let f be the self-affine RFIF determined by $\{\omega_i\}_{i=1}^N$ in (1.2). Then there exist two positive constants $\beta, \varepsilon_0 > 0$ such that*

$$\Big|\mathcal{N}^*_{\Gamma(f|_{I_i})}(\varepsilon) - \Big|\frac{d_i}{a_i}\Big| \sum_{j\in\mathcal{I}(i)} \mathcal{N}^*_{\Gamma(f|_{I_j})}\Big(\frac{\varepsilon}{|a_i|}\Big)\Big| \leqslant \beta\varepsilon^{-1}$$

for $1 \leqslant i \leqslant N$ and $0 < \varepsilon < \varepsilon_0$.

PROOF. Define $\varepsilon_0 = \min\{|a_i||I_j| : 1 \leqslant i, j \leqslant N\}$. For $0 < \varepsilon < \varepsilon_0$, we have $\varepsilon/|a_i| < |I_j|$ for all i, j. We remark that $\min\{|a_i| : 1 \leqslant i \leqslant N\} \leqslant 1$ since $|a_i| = |I_i|/|D_i|$. Fix i with $1 \leqslant i \leqslant N$. If $d_i = 0$, then $f|_{I_i}$ is a linear function, so that there is a constant $\beta_i > 0$ such that $\mathcal{N}^*_{\Gamma(f|_{I_i})}(\varepsilon) \leqslant \beta_i\varepsilon^{-1}$ for all $\varepsilon \in (0, \varepsilon_0)$.

Suppose that $d_i \neq 0$. Given $\varepsilon \in (0, \varepsilon_0)$, we let $\{\eta_{\ell,j}\}_{\ell=0}^{m_j}$ be arbitrary $\varepsilon/|a_i|$-partitions of I_j for $j \in \mathcal{I}(i)$. Then $\{L_i(\eta_{\ell,j}) : 0 \leqslant \ell \leqslant m_j, j \in \mathcal{I}(i)\}$ is an ε-partition of I_i.

Define $q_i(x) = c_i x + f_i$, $x \in D_i$. By (2.1),

$$f(x') - f(x'') = d_i(f(L_i^{-1}(x')) - f(L_i^{-1}(x''))) + q_i(L_i^{-1}(x')) - q_i(L_i^{-1}(x'')) \tag{2.2}$$

for $x', x'' \in I_i$. Hence,

$$\sum_{j\in\mathcal{I}(i)} \sum_{\ell=0}^{m_j-1} O(f|_{I_i}, L_i(\eta_{\ell,j}), \varepsilon) \leqslant |d_i| \sum_{j\in\mathcal{I}(i)} \sum_{\ell=0}^{m_j-1} O\Big(f|_{I_j}, \eta_{\ell,j}, \frac{\varepsilon}{|a_i|}\Big) + \sum_{j\in\mathcal{I}(i)} \sum_{\ell=0}^{m_j-1} O\Big(q_i|_{I_j}, \eta_{\ell,j}, \frac{\varepsilon}{|a_i|}\Big).$$

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For each $j \in \mathcal{I}(i)$, since $\{\eta_{\ell,j}\}_{\ell=0}^{m_j}$ is an $\varepsilon/|a_i|$-partition of I_j, it follows that $m_j\varepsilon/|a_i| < 2(x_j - x_{j-1})$. Combining this with the linearity of q_i,

$$\sum_{\ell=0}^{m_j-1} O\Big(q_i|_{I_j}, \eta_{\ell,j}, \frac{\varepsilon}{|a_i|}\Big) \leqslant \frac{|c_i|m_j\varepsilon}{|a_i|} \leq 2|c_i|(x_j - x_{j-1}), \tag{2.3}$$

so that $\sum_{j\in\mathcal{I}(i)} \sum_{\ell=0}^{m_j-1} O(q_i|_{I_j}, \eta_{\ell,j}, \varepsilon/a_i) \leqslant 2|c_i|(x_{r(i)} - x_{\ell(i)})$. Thus, there exists a constant $\beta_{i,1} > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\mathcal{N}^*_{\Gamma(f|_{I_i})}(\varepsilon) \leqslant \left|\frac{d_i}{a_i}\right| \sum_{j\in\mathcal{I}(i)} \mathcal{N}^*_{\Gamma(f|_{I_j})}\Big(\frac{\varepsilon}{|a_i|}\Big) + \beta_{i,1}\varepsilon^{-1}.$$

On the other hand, given an arbitrary ε-partition $\{\tau_\ell\}_{\ell=0}^{m}$ of I_i, we denote

$$A_j := \{L_i^{-1}(\tau_\ell) : 0 \leqslant \ell \leqslant m\} \cap \Big(x_{j-1} + \frac{\varepsilon}{2|a_i|}, x_j - \frac{\varepsilon}{2|a_i|}\Big), \quad j \in \mathcal{I}(i).$$

From (2.2), for each $j \in \mathcal{I}(i)$,

$$\sum_{\ell:\, L_i^{-1}(\tau_\ell)\in A_j} O\Big(f|_{I_j}, L_i^{-1}(\tau_\ell), \frac{\varepsilon}{|a_i|}\Big) \leqslant \frac{1}{|d_i|} \sum_{\ell:\, L_i^{-1}(\tau_\ell)\in A_j} \Big(O(f|_{I_i}, \tau_\ell, \varepsilon) + O\Big(q_i|_{I_j}, L_i^{-1}(\tau_\ell), \frac{\varepsilon}{|a_i|}\Big)\Big).$$

Summing over $j \in \mathcal{I}(i)$,

$$\sum_{j\in\mathcal{I}(i)} \sum_{\ell:\, L_i^{-1}(\tau_\ell)\in A_j} O\Big(f|_{I_j}, L_i^{-1}(\tau_\ell), \frac{\varepsilon}{|a_i|}\Big) \leqslant \frac{1}{|d_i|}\Big(\sum_{\ell=0}^{m-1} O(f|_{I_i}, \tau_\ell, \varepsilon) + \sum_{\ell=0}^{m-1} O\Big(q_i|_{D_i}, L_i^{-1}(\tau_\ell), \frac{\varepsilon}{|a_i|}\Big)\Big).$$

One can add x_{j-1}, x_j and at most two other points into A_j such that the resulting set, denoted by A_j^*, is an $\varepsilon/|a_i|$-partition of I_j. This implies that

$$\sum_{j\in\mathcal{I}(i)} \sum_{\tau\in A_j^*} O\Big(f|_{I_j}, \tau, \frac{\varepsilon}{|a_i|}\Big) \leqslant \frac{1}{|d_i|}\Big(\sum_{\ell=0}^{m-1} O(f|_{I_i}, \tau_\ell, \varepsilon) + \sum_{\ell=0}^{m-1} O\Big(q_i|_{D_i}, L_i^{-1}(\tau_\ell), \frac{\varepsilon}{|a_i|}\Big)\Big) + 8N\|f\|_\infty.$$

As before, $\sum_{\ell=0}^{m-1} O(q_i|_{D_i}, L_i^{-1}(\tau_\ell), \varepsilon/|a_i|) \leqslant |a_i|^{-1}|c_i|m\varepsilon \leq 2|c_i|(x_{r(i)} - x_{\ell(i)})$. Thus, there exists a constant $\beta_{i,2} > 0$ such that

$$\mathcal{N}^*_{\Gamma(f|_{I_i})}(\varepsilon) \geqslant \left|\frac{d_i}{a_i}\right| \sum_{j\in\mathcal{I}(i)} \mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon/|a_i|) - \beta_{i,2}\varepsilon^{-1}$$

for all $\varepsilon \in (0, \varepsilon_0)$. Let $\beta_i = \max\{\beta_{i,1}, \beta_{i,2}\}$.

The lemma holds with $\beta = \max\{\beta_i : 1 \leqslant i \leqslant N\}$. □

3. Analysis on strongly connected components

The following proposition is a natural property of degenerate elements.

PROPOSITION 3.1. *Let f be the self-affine RFIF determined by $\{\omega_i\}_{i=1}^N$ in (1.2). If i_0 is degenerate, then $\Gamma(f|_{I_{i_0}})$ is a line segment.*

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PROOF. Let $\Omega := \{j :$ there is a path from j to $i_0\}$ and let T and $C_*(I)$ be as in the proof of Theorem 1.3. Denote by $C^*(I)$ the collection of $g \in C_*(I)$ such that $\Gamma(g|_{I_k})$ is a line segment connecting (x_{k-1}, y_{k-1}) and (x_k, y_k) for every $k \in \Omega \cup \{i_0\}$. It suffices to show that T maps $C^*(I)$ into itself.

For $k \in \Omega \cup \{i_0\}$ and $x \in I_k$, there exists $\theta \in [0, 1]$ such that $x = \theta x_{k-1} + (1 - \theta)x_k$. Without loss of generality, we assume that ω_k maps $(x_{\ell(k)}, y_{\ell(k)})$ to (x_{k-1}, y_{k-1}) and $(x_{r(k)}, y_{r(k)})$ to (x_k, y_k). For any $g \in C^*(I)$, since $L_k^{-1}(x) = \theta x_{\ell(k)} + (1 - \theta)x_{r(k)}$,

$$\begin{aligned} Tg(\theta x_{k-1} + (1 - \theta)x_k) &= F_k(L_k^{-1}(x), g(L_k^{-1}(x))) \\ &= c_k L_k^{-1}(x) + d_k g(L_k^{-1}(x)) + f_k \\ &= \theta F_k(x_{\ell(k)}, g(x_{\ell(k)})) + (1 - \theta)F_k(x_{r(k)}, g(x_{r(k)})) \\ &= \theta Tg(x_{k-1}) + (1 - \theta)Tg(x_k), \end{aligned}$$

which implies that $Tg \in C^*(I)$. □

DEFINITION 3.2. We say that $\{i_1, \ldots, i_m\} \subset \{1, \ldots, N\}$, where $i_1, \ldots, i_m$ are distinct, *forms a cycle* if (by rearrangement if necessary) $D_{i_k} = I_{i_{k+1}}$ for each $1 \leqslant k \leqslant m$, where $i_{m+1} = i_1$.

Note that if $V \subset \{1, \ldots, N\}$ forms a cycle, then V is clearly degenerate.

DEFINITION 3.3. For $n \in \mathbb{Z}^+ \cup \{0\}$, we call $\mathbf{i} = \{i_k\}_{k=0}^n$ an *n-chain* if $i_k \in \{1, 2, \ldots, N\}$ for all $0 \leqslant k \leqslant n$ and $i_{k+1} \in \mathcal{I}(i_k)$ for each $0 \leqslant k \leqslant n-1$; the *weight* of $\mathbf{i}$ is defined by $a(\mathbf{i}) = \prod_{k=0}^n |a_{i_k}|$.

PROPOSITION 3.4. *If no subsets of* $\{1, \ldots, N\}$ *can form a cycle, then*

$$\lim_{n\to\infty} \max\{a(\mathbf{i}) : \mathbf{i} \text{ is an } n\text{-chain}\} = 0.$$

PROOF. The key observation is that for any N-chain $\{i_k\}_{k=0}^N$, there must exist some k_0 with $0 \leqslant k_0 \leqslant N-1$ such that $\#\mathcal{I}(i_{k_0}) \geqslant 2$. In fact, by the pigeonhole principle, we can always find s, t with $0 \leqslant s < t \leqslant N$ such that $i_s = i_t$. It follows that there exists $k_0 \in \{s, s+1, \ldots, t\}$ such that $\mathcal{I}(i_{k_0}) \supsetneq \{i_{k_0+1}\}$ since otherwise $\{i_s, i_{s+1}, \ldots, i_{t-1}\}$ would form a cycle, which leads to a contradiction. Since $i_s = i_t$, we can choose some k_0 from $\{s, s+1, \ldots, t-1\}$ so that $k_0 \leqslant N-1$. Denote

$$\alpha = \max\Big\{\frac{|I_j|}{|D_i|} : j \in \mathcal{I}(i),\ \#\mathcal{I}(i) \geqslant 2\Big\}.$$

Then $0 < \alpha < 1$ is a constant only dependent on the iterated function system $\{\omega_i\}_{i=1}^N$. For $0 \leqslant k \leqslant N-1$, we have $i_{k+1} \in \mathcal{I}(i_k)$, so that $|I_{i_{k+1}}|/|D_{i_k}| \leqslant 1$. Further, $I_{i_{k_0+1}} \subsetneq D_{i_{k_0}}$, so that $|I_{i_{k_0+1}}|/|D_{i_{k_0}}| \leqslant \alpha$.

Given $n \in \mathbb{Z}^+$ and an n-chain $\mathbf{i} = \{i_k\}_{k=0}^n$, we have $n = m(N+1) + p$, where $0 \leqslant p \leqslant N$ and m is a nonnegative integer. Note that $\{i_k\}_{k=0}^N, \{i_k\}_{k=N+1}^{2N+1}, \ldots, \{i_k\}_{k=(m-1)(N+1)}^{m(N+1)-1}$ are all N-chains. From the above argument, $\prod_{k=q(N+1)}^{(q+1)(N+1)-2}(|I_{i_{k+1}}|/|D_{i_k}|) \leqslant \alpha$ for $0 \leqslant q \leqslant m-1$, so that

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$$\prod_{k=0}^{m(N+1)-1} |a_{i_k}| = \prod_{k=0}^{m(N+1)-1} \frac{|I_{i_k}|}{|D_{i_k}|} = \frac{|I_{i_0}|}{|D_{i_{m(N+1)-1}}|} \prod_{k=0}^{m(N+1)-2} \frac{|I_{i_{k+1}}|}{|D_{i_k}|} \leqslant \frac{|I|}{\min_{1\leqslant j\leqslant N} |D_j|}\alpha^m.$$

Similarly, $\prod_{k=m(N+1)}^{n} |a_{i_k}| \leqslant |I|/\min_{1\leqslant j\leqslant N} |D_j|$. Thus, $a(\mathbf{i}) \leqslant |I|^2\alpha^m/\min_{1\leqslant j\leqslant N} |D_j|^2$, which proves the proposition. □

We denote by CH_n the set of all n-chains. Given $V \subset \{1, \ldots, N\}$, we denote by $\mathrm{CH}_n(V)$ the set of all n-chains $\mathbf{i} = \{i_k\}_{k=0}^n$ with $i_k \in V$ for $0 \leqslant k \leqslant n$. For convenience, we also use $i_0 i_1 \cdots i_n$ to denote the chain $\mathbf{i} = \{i_k\}_{k=0}^n$.

REMARK 3.5. For any nondegenerate strongly connected component V, one can argue as above to see that $\lim_{n\to\infty} \max\{a(\mathbf{i}) : \mathbf{i} \in \mathrm{CH}_n(V)\} = 0$.

LEMMA 3.6 (Perron–Frobenius theorem). *Let $A = (a_{ij})_{n\times n}$ be an irreducible nonnegative matrix.*

(1) *The spectral radius of A, denoted by $\rho(A)$, is an eigenvalue of A and has a positive eigenvector.*
(2) *$\rho(A)$ strictly increases as any a_{ij} strictly increases.*

The following simple result will be used in the proof of Proposition 3.8.

LEMMA 3.7. *Let E be a closed interval and $f \in C(E)$. For any $\varepsilon > 0$ and ε-partition $\{\tau_\ell\}_{\ell=0}^m$ of E,*

$$\sum_{\ell=0}^{m-1} O(f, \tau_\ell, \varepsilon) \geqslant \max_{x\in E} f(x) - \min_{x\in E} f(x). \tag{3.1}$$

PROOF. Set $G_\ell = f([\tau_\ell, \tau_\ell + \varepsilon] \cap E)$ for $0 \leqslant \ell \leqslant m - 1$. By the continuity of f, we see that G_ℓ is a bounded closed interval for each ℓ and $\bigcup_{\ell=0}^{m-1} G_\ell = f(E)$. Then (3.1) follows from the facts that $O(f, \tau_\ell, \varepsilon) = \mathcal{L}^1(G_\ell)$ and $\max_{x\in E} f(x) - \min_{x\in E} f(x) = \mathcal{L}^1(f(E))$, where $\mathcal{L}^1$ represents the Lebesgue measure on $\mathbb{R}$. □

Now we can obtain the following result by using a similar method to the proof of [4, Lemma 4.3]. We remark that in the proof of that lemma, the irreducibility of C should be replaced by $CS(1)$, where C is the connection matrix.

PROPOSITION 3.8. *Let f be the self-affine RFIF determined by $\{\omega_i\}_{i=1}^N$ in (1.2). Suppose that $V \subset \{1, \ldots, N\}$ is a nondegenerate strongly connected component and $\rho(Q_V) > 1$. Then*

$$\lim_{\varepsilon\to 0} \varepsilon \mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) = \infty \quad \text{for all } j \in V.$$

PROOF. Denote $\lambda := \rho(Q_V) > 1$ and suppose that $V = \{i_1, \ldots, i_M\}$. We may assume for notational simplicity that $V = \{1, \ldots, M\}$. Since V is a nondegenerate strongly connected component, $\Gamma(f|_{I_j})$ is not a line segment for each $j \in V$. This allows us to find $p_j, q_j, r_j \in \mathbb{R}$ with $x_{j-1} < p_j < q_j < r_j < x_j$ such that $f(p_j), f(q_j), f(r_j)$ are not collinear. Thus, if we let

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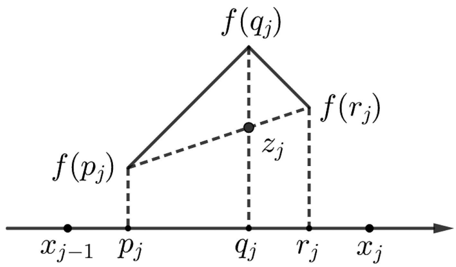

FIGURE 1. p_j, q_j and r_j.

$$z_j := \frac{f(r_j) - f(p_j)}{r_j - p_j}(q_j - p_j) + f(p_j),$$

then $\gamma_j := |f(q_j) - z_j| > 0$, as one can see in Figure 1. Notice that

$$\max\{|f(r_j) - f(q_j)|, |f(p_j) - f(q_j)|\} \geq |f(z_j) - f(q_j)|.$$

From Lemma 3.7, $\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant \varepsilon^{-1}\gamma_j$. Denote $\delta = \min_{1 \leqslant j \leqslant M} \min\{p_j - x_{j-1}, x_j - r_j\}$.

By the Perron–Frobenius theorem, we can find a positive eigenvector of Q_V associated with the eigenvalue λ, say $v = (v_1, \ldots, v_M)$, such that $0 < v_j \leqslant \gamma_j$ for $1 \leqslant j \leqslant M$. By the definitions of Q and Q_V, we have $|d_j| \sum_{k \in V \cap \mathcal{I}(j)} v_k = \lambda v_j$ for $j = 1, 2, \ldots, M$.

Since $\Gamma(f|_{I_j}) \supset \bigcup_{k \in V \cap \mathcal{I}(j)} \omega_j(\Gamma(f|_{I_k}))$, from Lemma 3.7 and the affinity of ω_j,

$$\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant \varepsilon^{-1} \sum_{k \in V \cap \mathcal{I}(j)} |d_j| \gamma_k \geqslant \varepsilon^{-1} \sum_{k \in V \cap \mathcal{I}(j)} |d_j| v_k = \varepsilon^{-1} \lambda v_j$$

for $0 < \varepsilon < \min\{\delta |a_i| : i \in V\}$. By induction, $\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant \varepsilon^{-1}\lambda^n v_j$ for all $n \in \mathbb{Z}^+$, provided $0 < \varepsilon < \min\{\delta a(\mathbf{i}) : \mathbf{i} \in \mathrm{CH}_n(V)\}$. Combining this with Remark 3.5 shows that the proposition holds. □

4. Proof of Theorem 1.5

If some $V \subset \{1, \ldots, N\}$ forms a cycle, then, from Proposition 3.1, $\bigcup_{i \in V} \Gamma(f|_{I_i})$ is a union of line segments. Hence, we may assume that no subset of $\{1, \ldots, N\}$ can form a cycle. In view of Proposition 3.4, we set

$$N_* := \min\{n \geqslant 0 : a(\mathbf{i}) < 1 \text{ for all } m\text{-chains } \mathbf{i} \text{ with } m \geq n\}.$$

In particular, if $N_* = 0$, then $|a_i| < 1$ for all $1 \leqslant i \leqslant N$. We also denote

$$\overline{a} = \max\{a(\mathbf{i}) : \mathbf{i} \text{ is an } N_*\text{-chain}\}, \quad \underline{a} = \min\{a(\mathbf{i}) : \mathbf{i} \text{ is an } N_*\text{-chain}\}.$$

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Clearly, $0 < \underline{a} \leqslant \overline{a} < 1$. For any $v = (v_1, \ldots, v_N)^T$, it is not difficult to see that $(Q(s))^n v = (v_1^{(n)}, v_2^{(n)}, \ldots, v_N^{(n)})^T$, where

$$v_k^{(n)} = \sum_{kj_1\cdots j_n \in \mathrm{CH}_n} |d_k d_{j_1} \cdots d_{j_{n-1}}||a_k a_{j_1} \cdots a_{j_{n-1}}|^{s-1} v_{j_n}, \quad k = 1, 2, \ldots, N.$$

For each matrix A and $n \in \mathbb{Z}^+$, we have $\rho(A) = 1$ if and only if $\rho(A^n) = 1$. Since each nonzero entry of the matrix $(Q(s)_{V_t})^{N_*+1}$ decreases as s increases, it follows from the Perron–Frobenius theorem that there exists a unique real number s_t such that $\rho(Q(s_t)_{V_t}) = 1$.

For $1 \leqslant i, j \leqslant N$, we denote $i \sim j$ if i and j belong to the same strongly connected component of the graph G. Otherwise, $i \nsim j$. Let

$$\mathcal{A}(i) = \{j : j \nsim i \text{ but there is a path from } j \text{ to } i\}.$$

As in [11], we define the *position* $P(i)$ of each i $(1 \leqslant i \leqslant N)$ recursively as follows: $P(i) = 1$ if $\mathcal{A}(i) = \emptyset$, otherwise $P(i) = 1 + \max\{P(j) : j \in \mathcal{A}(i)\}$. Clearly, $P(i) \leqslant N$ for each i. Also, $P(i) = P(j)$ if $i \sim j$ since in this case $\mathcal{A}(i) = \mathcal{A}(j)$.

PROOF OF THEOREM 1.5. From Lemma 2.4, there exist two positive constants β_* and ε_* such that

$$\left| \mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) - \sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}} \left| \frac{d_j d_{j_1} \cdots d_{j_{N_*}}}{a_j a_{j_1} \cdots a_{j_{N_*}}} \right| \mathcal{N}^*_{\Gamma(f|_{I_{j_{N_*+1}}})}\left(\frac{\varepsilon}{|a_j a_{j_1} \cdots a_{j_{N_*}}|} \right) \right| \leq \beta_* \varepsilon^{-1} \tag{4.1}$$

for $1 \leq j \leq N$ and $0 < \varepsilon < \varepsilon_*$. Let $s_* := \max\{s_1, \ldots, s_m, 1\}$. The proof is divided into two parts.

Step 1. We show that $\underline{\dim}_{\mathrm{B}} \Gamma(f) \geqslant s_*$. Suppose that $\max\{s_1, \ldots, s_m\} = s_{i_0}$ for some i_0 with $1 \leqslant i_0 \leqslant m$. It is clear that $\underline{\dim}_{\mathrm{B}} \Gamma(f) \geqslant 1$, so we may assume that $s_{i_0} > 1$. Also, for notational simplicity, we assume that $V_{i_0} = \{1, \ldots, M\}$. By the Perron–Frobenius theorem, $\rho((Q(s_{i_0})_{V_{i_0}})^{N_*+1}) = \rho(Q(s_{i_0})_{V_{i_0}}) = 1$, so $\lambda := \rho((Q_{V_{i_0}})^{N_*+1}) > 1$. Furthermore, there are positive eigenvectors $v = (v_1, \ldots, v_M)^T$ of $(Q_{V_{i_0}})^{N_*+1}$ associated with λ, and $w = (w_1, \ldots, w_M)^T$ of $(Q(s_{i_0})_{V_{i_0}})^{N_*+1}$ associated with 1. That is, for all $j \in V_{i_0}$,

$$\sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_{i_0})} |d_j d_{j_1} \cdots d_{j_{N_*}}| v_{j_{N_*+1}} = \lambda v_j, \tag{4.2}$$

$$\sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_{i_0})} |d_j d_{j_1} \cdots d_{j_{N_*}}||a_j a_{j_1} \cdots a_{j_{N_*}}|^{s_{i_0}-1} w_{j_{N_*+1}} = w_j. \tag{4.3}$$

Choose $B > 0$ such that $Bv_j \geqslant \beta_*$ for every $j \in V_{i_0}$. From (4.1),

$$\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant \sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_{i_0})} \left| \frac{d_j d_{j_1} \cdots d_{j_{N_*}}}{a_j a_{j_1} \cdots a_{j_{N_*}}} \right| \mathcal{N}^*_{\Gamma(f|_{I_{j_{N_*+1}}})}\left(\frac{\varepsilon}{|a_j a_{j_1} \cdots a_{j_{N_*}}|} \right) - Bv_j \varepsilon^{-1}$$

for $j \in V_{i_0}$ and $0 < \varepsilon < \varepsilon_*$. By Proposition 3.8, we can choose ε_* so small that $\varepsilon \mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) > Bv_j/(\lambda - 1)$ for all $j \in V_{i_0}$ and $\varepsilon \in (0, \varepsilon_*)$. Select a constant $B' > 0$ small enough such that for all $\varepsilon \in [\underline{a}\overline{a}\varepsilon_*, \varepsilon_*)$,

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$$\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant B'\varepsilon^{-s_{i_0}} w_j + \frac{B}{\lambda - 1} v_j \varepsilon^{-1} \quad \text{for all } j \in V_{i_0}. \tag{4.4}$$

We prove by induction that (4.4) holds for $0 < \varepsilon < \varepsilon_*$. In fact, if $\varepsilon \in [\underline{a}\overline{a}^2\varepsilon_*, \underline{a}\overline{a}\varepsilon_*)$, since $\varepsilon/|a_j a_{j_1} \cdots a_{j_{N_*}}| \in [\underline{a}\overline{a}\varepsilon_*, \varepsilon_*)$ when $jj_1 \cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}$, it follows that

$$\begin{aligned}\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \geqslant & \sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_{i_0})} \left|\frac{d_j d_{j_1} \cdots d_{j_{N_*}}}{a_j a_{j_1} \cdots a_{j_{N_*}}}\right| \left(B'\left(\frac{\varepsilon}{|a_j a_{j_1} \cdots a_{j_{N_*}}|}\right)^{-s_{i_0}} w_{j_{N_*+1}} \right.\\ &\left. + \frac{B}{\lambda - 1} v_{j_{N_*+1}} \left(\frac{\varepsilon}{|a_j a_{j_1} \cdots a_{j_{N_*}}|}\right)^{-1}\right) - Bv_j\varepsilon^{-1} \\ = & B'\varepsilon^{-s_{i_0}} w_j + \frac{B\lambda}{\lambda - 1} v_j\varepsilon^{-1} - Bv_j\varepsilon^{-1} = B'\varepsilon^{-s_{i_0}} w_j + \frac{B}{\lambda - 1} v_j \varepsilon^{-1}\end{aligned}$$

for $j \in V_{i_0}$. Similarly, (4.4) holds for $\varepsilon \in [\underline{a}\overline{a}^n\varepsilon_*, \varepsilon_*)$ and $n = 1, 2, \ldots$ and hence for all $\varepsilon \in (0, \varepsilon_*)$. This implies that $\underline{\dim}_B \Gamma(f) \geqslant s_{i_0} = s_*$.

Step 2. We show that $\overline{\dim}_B \Gamma(f) \leqslant s_*$. This will be proved by induction on the position of i $(1 \leqslant i \leqslant N)$.

Suppose that $P(i) = 1$. We may assume that i belongs to a strongly connected component V. Otherwise, there exists no path from any j with $1 \leqslant j \leqslant N$ to i and hence $d_i = 0$ and, by Proposition 2.1, $\dim_B \Gamma(f|_{I_i}) = 1 \leqslant s_*$. For notational simplicity, assume that $V = \{1, \ldots, n\}$. By Proposition 3.1, we can assume that V is nondegenerate, that is, there is some t with $1 \leqslant t \leqslant m$ such that $V = V_t$.

Arbitrarily pick $\delta > 0$. Let $\lambda' := \rho((Q(s_* + \delta)_{V_t})^{N_*+1})$. It follows from $s_* + \delta > s_t$ that $\lambda' < 1$. Choose $p = (p_1, \ldots, p_n)^T$ to be a positive eigenvector of $(Q(s_t)_{V_t})^{N_*+1}$ associated with eigenvalue 1 and $q = (q_1, \ldots, q_n)^T$ of $(Q(s_* + \delta)_{V_t})^{N_*+1}$ associated with λ'. Choose $\beta' > 0$ such that $\beta' q_j > \beta_*$ for all $j \in V_t$.

Since $P(i) = 1$, we can see for all $j \in V_t$ that $jj_1 \cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}$ if and only if $jj_1 \cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_t)$. Without loss of generality, we may assume that $\varepsilon_* < 1$. Combining the fact that $s_* \geqslant 1$ and (4.1),

$$\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \leqslant \sum_{jj_1\cdots j_{N_*+1} \in \mathrm{CH}_{N_*+1}(V_t)} \left|\frac{d_j d_{j_1} \cdots d_{j_{N_*}}}{a_j a_{j_1} \cdots a_{j_{N_*}}}\right| \mathcal{N}^*_{\Gamma(f|_{I_{j_{N_*+1}}})}\left(\frac{\varepsilon}{|a_j a_{j_1} \cdots a_{j_{N_*}}|}\right) + \beta'\varepsilon^{-s_*-\delta} q_j \tag{4.5}$$

for all $j \in V_t$ and all $\varepsilon \in (0, \varepsilon_*)$. Choose $B_1 > 0$ large enough such that for all $\varepsilon \in [\underline{a}\overline{a}\varepsilon_*, \varepsilon_*)$,

$$\mathcal{N}^*_{\Gamma(f|_j)}(\varepsilon) \leqslant B_1 \varepsilon^{-s_t} p_j + \frac{\beta'\varepsilon^{-s_*-\delta}}{1 - \lambda'} q_j \quad \text{for all } j \in V_t. \tag{4.6}$$

By applying an analogous induction argument as before, we can show that this holds for all $\varepsilon \in (0, \varepsilon_0)$, which implies that $\overline{\dim}_B \Gamma(f|_{I_j}) \leqslant s_* + \delta$ for each $j \in \mathcal{I}(i)$. Since $\delta > 0$ is chosen arbitrarily, $\overline{\dim}_B \Gamma(f|_{I_j}) \leqslant s_*$ for each $j \in \mathcal{I}(i)$ and, in particular, $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_*$.

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Assume that $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_*$ for all i with positions strictly less than P. For any i such that $P(i) = P$, we may still assume that i is not degenerate. If there exist no strongly connected components to which i belongs, then $P(j) < P$ for any $j \in \mathcal{I}(i)$. By Lemma 2.3, for arbitrary $\delta > 0$, we have $\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \leqslant \varepsilon^{-s_*-\delta}$ for all small ε and all $j \in \mathcal{I}(i)$. Combining this with Lemma 2.4, we can easily see that $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_* + \delta$ and hence $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_*$.

Now there is only one case left. Suppose that i belongs to a strongly connected component V. By Proposition 3.1, we may assume that V is nondegenerate, that is, $V = V_t$ for some $1 \leqslant t \leqslant m$. We will abuse notation again and just assume that $V_t = \{1, \ldots, n\}$. Arbitrarily pick $\delta > 0$. Let $\lambda' := \rho((Q(s_* + \delta)_{V_t})^{N_*+1}) < 1$. Choose $p = (p_1, \ldots, p_n)^T$ to be a positive eigenvector of $(Q(s_t)_{V_t})^{N_*+1}$ associated with eigenvalue 1 and $q = (q_1, \ldots, q_n)^T$ of $(Q(s_* + \delta)_{V_t})^{N_*+1}$ associated with λ'.

By Lemma 2.4 and the inductive assumption, we can find two positive constants, still denoted by β and ε_0, such that for each $j \in V_t$ and $0 < \varepsilon < \varepsilon_0$,

$$\mathcal{N}^*_{\Gamma(f|_{I_j})}(\varepsilon) \leqslant \left|\frac{d_j}{a_j}\right| \sum_{k \in \mathcal{I}(j) \cap V_t} \mathcal{N}^*_{\Gamma(f|_{I_k})}\left(\frac{\varepsilon}{|a_j|}\right) + \beta \varepsilon^{-s_*-\delta}.$$

As before, in this case, there also exist two positive constants β' and ε_{**} such that (4.5) holds for all $j \in V_t$ and all $\varepsilon \in (0, \varepsilon_{**})$. Thus, using a similar argument again, we can find a constant $B_1 > 0$ such that (4.6) holds for all $j \in V_t$ and all $\varepsilon \in (0, \varepsilon_{**})$. As a result, $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_* + \delta$. Hence, $\overline{\dim}_B \Gamma(f|_{I_i}) \leqslant s_*$. □

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