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Loewner evolution of hedgehogs and 2-conformal measures of circle maps

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Abstract. Let f be a germ of a holomorphic diffeomorphism with an irrationally indifferent fixed point at the origin in $\mathbb{C}$ (i.e. $f(0)=0$, $f'(0)=e^{2\pi i\alpha}$, $\alpha\in\mathbb{R}-\mathbb{Q}$). Pérez-Marco [Fixed points and circle maps. *Acta Math.* **179**(2) (1997), 243–294] showed the existence of a unique continuous monotone one-parameter family of non-trivial invariant full continua containing the fixed point called Siegel compacta, and gave a correspondence between germs and families (g_t) of circle maps obtained by conformally mapping the complement of these compacts to the complement of the unit disk. The family of circle maps (g_t) is the orbit of a locally defined semigroup (Φ_t) on the space of analytic circle maps, which we show has a well-defined infinitesimal generator X. The explicit form of X is obtained by using the Loewner equation associated to the family of hulls (K_t). We show that the Loewner measures (μ_t) driving the equation are 2-conformal measures on the circle for the circle maps (g_t).

Key words: Loewner evolution, conformal measures, indifferent fixed points, circle maps
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1. *Introduction*

A holomorphic diffeomorphism $f(z)=e^{2\pi i\alpha}z+O(z^2)$, $\alpha\in\mathbb{R}-\mathbb{Q}/\mathbb{Z}$ defined in a neighbourhood of the origin in $\mathbb{C}$ is said to be *linearizable* if it is analytically conjugate to the rigid rotation $R_\alpha(z)=e^{2\pi i\alpha}z$. The number α is called the rotation number of f and the maximal domain of linearization is called the Siegel disk of f. The linearizability of f is dependent on the arithmetic of α and there is an optimal arithmetic condition for linearizability in this setting given by the well-known Brjuno condition $\mathcal{B}\subset\mathbb{R}-\mathbb{Q}$, such that if $\alpha\in\mathcal{B}$, then any map f with rotation number α is linearizable, while if $\alpha\notin\mathcal{B}$, then there exists a map f with rotation number α which is non-linearizable (see [**Sie42**, **Brj71**, **Yoc95**]).

Irrespective of whether or not f is linearizable, Pérez-Marco proved the existence of a unique, strictly increasing Hausdorff continuous family $(K_t)_{t>0}$ of non-trivial, totally invariant (meaning that $f(K_t) = f^{-1}(K_t) = K_t$), full continua containing the fixed point called *Siegel compacts* [**PM97**, **PM96**], where $K_t \to \{0\}$ as $t \to 0$, and K_t can be described as the connected component containing the origin of the set of non-escaping points in the closed disk $\overline{\mathbb{D}}_t$ of radius t around the origin. When f is non-linearizable, these are called *hedgehogs*. The topology and dynamics of hedgehogs have been studied by Pérez-Marco [**PM94**, **PM96**], who also developed techniques using 'tube-log Riemann surfaces' [**BPM15a**, **BPM15b**, **BPM13**] for the construction of interesting examples [**PM93**, **PM95**, **PM00**] of indifferent germs and hedgehogs, which were also used by the author [**Bis05**, **Bis08**, **Bis16**] and Chéritat [**Ch11**] to construct further examples.

Notation. Throughout, $\mathbb{D}_\infty$ will denote the complement in $\hat{\mathbb{C}}$ of the closed unit disk, $\mathbb{D}_\infty = \hat{\mathbb{C}} - \overline{\mathbb{D}}$, $r : \mathbb{C}^* \to \mathbb{C}^*$ will denote complex reflection in the unit circle, $r(\xi) = 1/\overline{\xi}$ and d^+/dt will denote the right-hand derivative.

The construction of Pérez-Marco from [**PM97**] associates to a pair (f, K) an analytic circle diffeomorphism g, where f is a germ with an irrationally indifferent fixed point at the origin and K is a Siegel compact of f, by considering a conformal map ψ from the complement of K to the complement of the closed unit disk, $\psi : \hat{\mathbb{C}} - K \to \mathbb{D}_\infty$, such that $\psi(\infty) = \infty$. Conjugating f by ψ gives a holomorphic diffeomorphism $g = \psi \circ f \circ \psi^{-1}$ in an annulus in $\mathbb{D}_\infty$ having S^1 as a boundary component, and g is shown to extend across S^1 to an analytic circle diffeomorphism defined in a neighbourhood of S^1 such that the rotation numbers of g and f are equal, $\rho(g) = \rho(f) = \alpha$.

Invariant compacts for g containing S^1 then correspond to invariant compacts for f containing K, and the theorem on existence and uniqueness of Siegel compacts then gives the existence and uniqueness of a continuous strictly increasing one-parameter family of *Herman compacts* $(A_t)_{0 \leq t < \epsilon}$ for the circle map g with $A_0 = S^1$, where a Herman compact for g is a connected totally invariant compact A containing S^1 such that $\hat{\mathbb{C}} - A$ has two components and $r(A) = A$. The construction which produces g from the pair (f, K) can similarly be applied to each pair (g, A_t) to give a one-parameter family of analytic circle diffeomorphisms $(g_t)_{t \geq 0}$ with $g_0 = g$, by conjugating g by a conformal map $\psi_t : \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A_t) \to \mathbb{D}_\infty$, normalized so that $\psi_t(\infty) = \infty$, $\psi_t'(\infty) > 0$. As before the map $g_t = \psi_t \circ g \circ \psi_t^{-1}$ defined in a one-sided neighbourhood of S^1 extends across to give an analytic circle diffeomorphism g_t with rotation number equal to that of g.

The family A_t of Herman compacts can be shown to be Hausdorff continuous by the same argument as in [**PM96**] used to show Hausdorff continuity of the family of Siegel compacts K_t. Since the family A_t is Hausdorff continuous and strictly increasing, it is possible to reparametrize the family so that the compacts $\overline{\mathbb{D}} \cup A_t$ have logarithmic capacity t, meaning that the expansion of ψ_t near $z = \infty$ is of the form

$$\psi_t(z) = e^{-t}z + a_0(t) + \frac{a_1(t)}{z} + \cdots .$$

Let $\mathrm{Diff}^\omega(S^1)$ and $\mathrm{Diff}^\omega_\alpha(S^1)$ denote the space of analytic, orientation-preserving circle diffeomorphisms and the subspace of those diffeomorphisms with fixed rotation number $\alpha \in (\mathbb{R} - \mathbb{Q})/\mathbb{Z}$, respectively. We obtain for $t \geq 0$ a locally defined family of maps $\Phi_t : \mathcal{D}_t \subset \mathrm{Diff}^\omega_\alpha(S^1) \to \mathrm{Diff}^\omega_\alpha(S^1)$ mapping a circle map g to the circle map g_t obtained from the above construction. Here the domain $\mathcal{D}_t$ of Φ_t consists of those circle maps in $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$ which extend to an annular neighbourhood of S^1 and have a Herman compact A in that neighbourhood such that $\overline{\mathbb{D}} \cup A$ has logarithmic capacity t (such a Herman compact is independent of the choice of annular neighbourhood; see Lemma 5.1). It is not hard to show that in fact the maps Φ_t form a continuous semigroup, namely $\Phi_0 = \mathrm{id}$,

$$\Phi_s \circ \Phi_t = \Phi_{s+t}$$

on the domain of Φ_{s+t} (which is contained in the domain of $\Phi_s \circ \Phi_t$; see Proposition 5.2) and the orbits $t \mapsto \Phi_t(g)$ give continuous curves in $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$ for the topology of uniform convergence on S^1 (see §5 for the proofs of these assertions).

We show in fact that the semigroup $(\Phi_t)_{t\geq 0}$ has an infinitesimal generator X, meaning that the curves $t \mapsto \Phi_t(g)$ are right-hand differentiable in t and

$$\frac{d^+}{dt}\Phi_t(g) = X(\Phi_t(g)).$$

Since the space $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$ does not carry any obvious differentiable structure, the sense in which these assertions hold is made precise in the statements of the theorems below.

The form of the infinitesimal generator X is obtained by studying the *Loewner equation* associated to the family of hulls $\overline{\mathbb{D}} \cup A_t$ (we recall in §3 the basic facts about the Loewner equation which we will be needing). The maps $\phi_t := \psi_t^{-1} : \mathbb{D}_\infty \to \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A_t)$ form a *Loewner chain* and it is known [**Pom75**, Lemma 6.1] that $t \mapsto \phi_t(z)$ is absolutely continuous for each fixed $z \in \mathbb{D}_\infty$. Moreover, for almost every (a.e.) t, the right-hand derivatives

$$\chi_t(z) = \frac{d^+}{ds}_{|s=t}(\phi_t^{-1} \circ \phi_s)(z)$$

exist and the functions $H_t(z) = \chi_t(z)/z$ are given by the Herglotz transforms on $\mathbb{D}_\infty$ of a family of probability measures on the unit circle (μ_t), called the *driving* or *Loewner measures* of the Loewner equation. Here by the Herglotz transform on $\mathbb{D}_\infty$ of a probability measure μ on S^1 we mean the holomorphic function $H = \mathcal{H}\mu$ in $\mathbb{D}_\infty$ with positive real part and satisfying $H(\infty) = 1$, defined by

$$(\mathcal{H}\mu)(z) = \int_{S^1} \frac{1/\xi + 1/z}{1/\xi - 1/z}\, d\mu(\xi)$$

(the classical Herglotz theorem asserts that any holomorphic function H on $\mathbb{D}_\infty$ satisfying $\mathrm{Re}\, H > 0$, $H(\infty) = 1$ is of this form for a unique probability measure μ on S^1). Douady and Yoccoz have shown in [**DY99**] the existence and uniqueness of an s-conformal measure $\mu = \mu_{s,g}$ for any C^2 circle diffeomorphism g with irrational rotation number and any $s \in \mathbb{R}$. By an s-conformal measure for a circle diffeomorphism g we mean a probability measure μ on S^1 such that

$$\mu(g(E)) = \int_E |g'(x)|^s\, d\mu(x)$$

for all measurable sets $E \subset S^1$.

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We show that for any g in $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$, the associated Loewner chain (ϕ_t) is differentiable for *all* t uniformly for z in any compact subset of $\mathbb{D}_\infty$, and the associated Loewner measures are the 2-conformal measures of the circle diffeomorphisms $g_t = \Phi_t(g)$.

THEOREM 1.1. (Loewner measures are 2-conformal measures) *For any g in $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$ and any $t \geq 0$, the right-hand derivative of the Loewner chain (ϕ_t) exists (uniformly for z in any compact subset of $\mathbb{D}_\infty$) and is given by*

$$\chi_t(z) = \frac{d^+}{ds}_{|s=t} (\phi_t^{-1} \circ \phi_s)(z) = z \cdot (\mathcal{H}\mu_{2,g_t})(z),$$

where μ_{2,g_t} is the unique 2-conformal measure of g_t.

We remark that the map $\phi_t^{-1} \circ \phi_s$ is only defined on $\mathbb{D}_\infty$ for $s > t$, so we only consider right-hand derivatives in the above theorem (even though it is true that for each fixed $z \in \mathbb{D}_\infty$ and $t > 0$, there is a two-sided neighbourhood of t such that $\phi_t^{-1} \circ \phi_s$ is defined near z for s in this neighbourhood of t). The existence and form of the infinitesimal generator X of the semigroup (Φ_t) are stated as follows.

THEOREM 1.2. (Infinitesimal generator of the semigroup) *For any $g \in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$, there exists a function $X(g)$ holomorphic in a neighbourhood V of S^1 such that for $t > 0$ small the circle maps $\Phi_t(g)$ are defined in V and*

$$\lim_{t \to 0^+} \frac{\Phi_t(g) - g}{t} = X(g) \tag{1}$$

uniformly on compacts in V. The holomorphic function $X(g)$ is given (in $V \cap \mathbb{D}_\infty$, a one-sided neighbourhood of S^1) by

$$X(g) = g' \cdot \chi - \chi \circ g, \tag{2}$$

where $\chi(z) = z \cdot (\mathcal{H}\mu_{2,g})(z)$.

We can think of the curves $(t \mapsto \Phi_t(g))$ as integral curves of a vector field X on the space $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$. It is worth noting that while by equation (2) above the holomorphic function $X(g)$ is defined a priori only in a one-sided neighbourhood $V \cap \mathbb{D}_\infty$ of S^1, by equation (1) above (which holds on V) the function $X(g)$ does in fact extend to a holomorphic function in a full neighbourhood V of S^1, so the differential equation $dg/dt = X(g)$ does make sense on S^1. Moreover, for each $g \in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$, there exists an integral curve for this ordinary differential equation, namely the curve $t \mapsto \Phi_t(g)$ constructed using the Herman compacts A_t. The next theorem asserts the uniqueness in forward time for these integral curves.

THEOREM 1.3. (Uniqueness in forward time of integral curves) *If $(g_t)_{0 \leq t < \epsilon} \subset \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ is a family of circle diffeomorphisms holomorphic in a neighbourhood V of S^1, continuous in t (for the topology of uniform convergence on compacts in V), such that the right-hand derivatives exist and satisfy*

$$\frac{d^+}{ds}_{|s=t} g_s(z) = X(g_t)(z)$$

uniformly on compacts in V, then $g_t = \Phi_t(g_0)$ for $0 \leq t < \epsilon$.

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Definition 1.4. (Germ of integral curves) An integral curve (in the sense of the previous theorem) $(g_t)_{-\infty<t<t_0} \subset \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ defined for all times t less than some t_0 is said to be a *backward integral curve* of X. We say that two backward integral curves $(g^1_t)_{-\infty<t<t_1}, (g^2_t)_{-\infty<t<t_2}$ of X define the same germ of the integral curve of X if $g^1_t = g^2_t$ for all $t < t_0$ for some $t_0 < t_1, t_2$.

It follows from the above that if $(g^1_t)_{-\infty<t<t_0}, (g^2_t)_{-\infty<t<t_0}$ are two backward integral curves defining the same germ of an integral curve, then in fact $g^1_t = g^2_t$ for *all* $t < t_0$. Any germ f with an irrationally indifferent fixed point gives a family of Siegel compacts $(K_t)_{-\infty<t<t_0}$ (parametrizing the compacts K_t by their logarithmic capacities) and hence (by applying the germs to the circle map construction to the pairs (f, K_t)) gives a family of circle maps $(g_t = g^f_t)_{-\infty<t<t_0}$, which it is easy to see is a backward integral curve. We remark that unlike the uniqueness in forward time for integral curves, we cannot yet show uniqueness in backward time for integral curves, which is related to an open conjecture of Pérez-Marco's in [**PM97**, §V.3(a)]. We show that conversely any backward integral curve arises in this way from a germ f. Denoting the space of germs with rotation number α by $\mathrm{Diff}_{\alpha}(\mathbb{C}, 0)$, we have the following.

THEOREM 1.5. (Germs of diffeomorphisms and germs of integral curves) *For any backward integral curve* $(g_t)_{-\infty<t<t_0} \subset \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ *of* X*, we have* $g_t \to R_\alpha$ *uniformly on* S^1 *as* $t \to -\infty$.

The map $f \mapsto [(g^f_t)]$ *gives a one-to-one correspondence between* $\mathrm{Diff}_{\alpha}(\mathbb{C}, 0)$ *and germs of integral curves of* X.

Finally in §8 we describe some further results that can be obtained in the case of analytically linearizable circle maps and germs.

2. *Boundary values of the Herglotz transform*

The Herglotz transform of a probability measure μ on S^1 is the holomorphic function $\mathcal{H}\mu$ in $\mathbb{D}$ defined by

$$(\mathcal{H}\mu)(w) = \int_{S^1} \frac{\xi + w}{\xi - w}\, d\mu(\xi). \tag{3}$$

The real and imaginary parts of the Herglotz transform $\mathcal{H}\mu = \mathcal{P}\mu + i\mathcal{Q}\mu$ are given by the Poisson and conjugate Poisson transforms,

$$(\mathcal{P}\mu)(w) = \int_{S^1} \mathrm{Re}\, \frac{\xi + w}{\xi - w}\, d\mu(w) = \int_{S^1} \frac{1 - |w|^2}{|\xi - w|^2}\, d\mu(\xi),$$
$$(\mathcal{Q}\mu)(w) = \int_{S^1} \mathrm{Im}\, \frac{\xi + w}{\xi - w}\, d\mu(w) = \int_{S^1} \frac{2\,\mathrm{Im}\,\overline{\xi} w}{|\xi - w|^2}\, d\mu(\xi).$$

The radial limits of these harmonic functions exist for a.e. $\xi \in S^1$ with respect to Lebesgue measure λ; we will call the radial limits P and Q, respectively, and will call the function $P + iQ$ the boundary value of the function $\mathcal{H}\mu$. The decomposition of the measure μ into absolutely continuous and singular parts with respect to Lebesgue measure $\mu = f\, d\lambda + \mu_s$ can be recovered from these radial limits as follows.

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THEOREM 2.1. (Fatou) *For Lebesgue-a.e. $\xi \in S^1$, the radial limit P of $(\mathcal{P}\mu)(w)$ exists and equals $f(\xi)$.*

THEOREM 2.2. (Poltoratski [**Pol96**]) *As $t \to +\infty$, the measures $(\pi/2)t1_{\{|Q|>t\}}d\lambda$ converge weakly to μ_s (where Q denotes the radial limit of $\mathcal{Q}\mu$ on S^1).*

Since we will be dealing with functions defined in $\mathbb{D}_\infty$, we will refer to the holomorphic function in $\mathbb{D}_\infty$ defined by

$$H(z) = \int_{S^1} \frac{1/\xi + 1/z}{1/\xi - 1/z}\, d\mu(\xi)\ ,\ z \in \mathbb{D}_\infty$$

as the Herglotz transform of μ on $\mathbb{D}_\infty$. Then $H(z) = \overline{\tilde{H}(r(z))}$, where $r(z) = 1/\bar{z}$ and $\tilde{H} = \mathcal{H}\mu$ is the Herglotz transform defined by equation (3), which is holomorphic in $\mathbb{D}$. Appropriate versions of the above theorems hold for the boundary values of the function H. Let the boundary values of H, $\tilde{H}$ be $P + iQ$, $\tilde{P} + i\tilde{Q}$, respectively. Then $P + iQ = \tilde{P} - i\tilde{Q}$, so for $\mu = f d\lambda + \mu_s$ we have

$$f = \tilde{P} = P$$

and

$$\begin{aligned}\mu_s &= \lim_{t\to+\infty} \left(\frac{\pi}{2} t 1_{\{|\tilde{Q}|>t\}} d\lambda\right)\\ &= \lim_{t\to+\infty} \left(\frac{\pi}{2} t 1_{\{|Q|>t\}} d\lambda\right)\end{aligned}$$

weakly.

3. *The Loewner equation*

A *hull* is a connected, full, non-trivial compact K in $\mathbb{C}$ containing the origin. Its complement Ω in $\hat{\mathbb{C}}$ is then a simply connected domain containing ∞, so there is a conformal map from the complement of the closed unit disk, $\phi : \mathbb{D}_\infty \to \Omega = \hat{\mathbb{C}} - K$, such that $\phi(\infty) = \infty$, which is unique when normalized to satisfy $\phi'(\infty) > 0$. The map ϕ has an expansion around $z = \infty$ of the form

$$\phi(z) = e^{c(K)}z + a_0 + \frac{a_1}{z} + \cdots,$$

where the real number $c(K)$ is called the *logarithmic capacity* of the hull K (note that the closed disk of radius R then has logarithmic capacity $\log R$).

Given a strictly increasing family of hulls $(K_t)_{-\infty<t\leq t_0}$, if the domains $\Omega_t = \hat{\mathbb{C}} - K_t$ are continuous for the Carathéodory topology (which holds if the family of hulls is Hausdorff continuous for example) and $K_t \to \{0\}$ as $t \to -\infty$, then one can continuously reparametrize the family by logarithmic capacity so that the associated conformal maps $\phi_t : \mathbb{D}_\infty \to \Omega_t$ satisfy

$$\phi_t(z) = e^t z + a_0(t) + \frac{a_1(t)}{z} + \cdots.$$

The family of conformal maps (ϕ_t) is called a *Loewner chain* (strictly speaking, the classical notion of Loewner chain as presented in [**Pom75**, Ch. 6] considers conformal

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maps from the unit disk $\mathbb{D}$ with images simply connected domains containing $0 \in \mathbb{C}$, and the parameter t lies in $[0, \infty)$; what we consider here though is a version of this notion where the conformal maps are defined on the disk $\mathbb{D}_\infty$ with images simply connected domains containing $\infty \in \hat{\mathbb{C}}$, for which it is natural that the parameter t lies in $(-\infty, t_0])$.

For each compact $K \subset \mathbb{D}_\infty$, for each z in K the map $t \mapsto \phi_t(z)$ is C_K-Lipschitz in t for some constant C_K only depending on K [**Pom75**, §6.1]. Thus, for each z the derivative of $\phi_t(z)$ with respect to t exists for all t outside some null set E_z, but in fact one can choose a null set E independent of z such that the derivative with respect to t exists for all z in $\mathbb{D}_\infty$ [**Pom75**, Theorem 6.3]. Moreover, this derivative is of the form

$$\frac{d^+}{ds}_{|s=t} (\phi_t^{-1} \circ \phi_s)(z) = z H_t(z)$$

(only the right-hand derivative is considered since the maps $\phi_t^{-1} \circ \phi_s$ are only defined on $\mathbb{D}_\infty$ for $t \leq s$), where H_t is a holomorphic function on $\mathbb{D}_\infty$ satisfying $\operatorname{Re} H_t > 0$, $H_t(\infty) = 1$ and hence can be written as the Herglotz transform on $\mathbb{D}_\infty$ of a unique probability measure μ_t on S^1,

$$H_t(z) = \int_{S^1} \frac{1/\xi + 1/z}{1/\xi - 1/z} \, d\mu_t(\xi).$$

The measures (μ_t) are called the driving or Loewner measures for the Loewner chain (ϕ_t).

4. *Conformal measures of analytic circle diffeomorphisms*

A conformal measure of dimension δ for a holomorphic map f is a finite measure μ on an f-invariant compact K such that

$$\mu(f(E)) = \int_E |f'(z)|^\delta \, d\mu(z)$$

for all measurable sets $E \subset K$ or, when f is a diffeomorphism,

$$f^*\mu = |f'|^\delta \, d\mu,$$

where $f^*\mu$ is defined by $(f^*\mu)(E) = \mu(f(E))$. Conformal measures have been studied extensively for rational maps and Kleinian groups (as Patterson–Sullivan measures), where existence is usually proved by means of transfer operators for hyperbolic dynamical systems. While the machinery of transfer operators is not available for circle diffeomorphisms, Douady and Yoccoz [**DY99**] proved nonetheless the existence and uniqueness of s-conformal measures $\mu_{s,g}$ for any C^2 circle diffeomorphism g with irrational rotation number for every $s \in \mathbb{R}$. Moreover, the measures $\mu_{s,g}$ depend continuously on s, g for the weak topology on measures and the C^1 topology on circle diffeomorphisms.

5. *The semigroup* $(\Phi_t)_{t \geq 0}$ *on* $\mathrm{Diff}^\omega_\alpha(S^1)$

Given an analytic circle diffeomorphism g with irrational rotation number α, for $\epsilon > 0$ small, there is a unique, strictly increasing, Hausdorff continuous family of Herman compacts $(A_t)_{0 \leq t < \epsilon}$ totally invariant under g (and the reflection r) such that the hulls $\overline{\mathbb{D}} \cup A_t$ have logarithmic capacity t. Let $(\Omega_t)_{0 \leq t < \epsilon}$ be the family of decreasing simply connected domains $\Omega_t = \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A_t)$ and let $\phi_t : \mathbb{D}_\infty \to \Omega_t$ be conformal maps normalized so that $\phi_t(\infty) = \infty$, $\phi_t'(\infty) = e^t$. We let $\psi_t = \phi_t^{-1} : \Omega_t \to \mathbb{D}_\infty$.

The map $g_t := \psi_t \circ g \circ \psi_t^{-1}$ is holomorphic in an annulus contained in $\mathbb{D}_\infty$ with one boundary component equal to S^1 and (as shown in [**PM97**]) extends analytically across S^1 to give an analytic circle diffeomorphism with rotation number equal to α. This defines a map

$$\Phi_t : \mathcal{D}_t \subset \mathrm{Diff}^\omega_\alpha(S^1) \to \mathrm{Diff}^\omega_\alpha(S^1),$$
$$g \mapsto g_t,$$

where $\mathcal{D}_t$ is the set of g in $\mathrm{Diff}^\omega_\alpha(S^1)$ having a Herman compact A such that $\overline{\mathbb{D}} \cup A$ has logarithmic capacity t. To remove any ambiguity considering the possibility of different Herman compacts for holomorphic extensions of g to different open sets containing S^1, we prove the following lemma.

LEMMA 5.1. *Let $g \in \mathrm{Diff}^\omega_\alpha(S^1)$, where $\alpha \in \mathbb{R} - \mathbb{Q}$, and let $t_0 \geq 0$. For $i = 1, 2$, let U_i be a connected open set containing S^1 which is symmetric under the reflection r and such that g and g^{-1} extend holomorphically to U_i, and g has a Herman compact $A_i \subset U_i$ such that $\overline{\mathbb{D}} \cup A_i$ has logarithmic capacity t_0. Then $A_1 = A_2$.*

Proof. We will use the following fact, which follows from the results of [**PM96**]: if $W \supset S^1$ is a (topological) annulus with C^1 boundary, symmetric under the reflection r, such that g and g^{-1} are univalent in a neighbourhood of $\overline{W}$, then there exists a unique Herman compact $A = A(g, \overline{W})$ for g such that $A \subset \overline{W}$ and $A \cap \partial W \neq \emptyset$. Moreover, the compact $A(g, \overline{W})$ is given by the connected component containing S^1 of the set $\{z \in \overline{W} \mid g^n(z) \in \overline{W}$ for all $n \in \mathbb{Z}\}$.

For $i = 1, 2$, we can choose an annulus $W_i \supset S^1$ with C^1 boundary and symmetric under the reflection r such that $A_i \subset W_i$ and $\overline{W_i} \subset U_i$. Choosing a C^1 diffeomorphism $\phi_i : \{1/2 \leq |z| \leq 2\}$ such that $\phi \circ r = r \circ \phi$, the annulus W_i admits a filtration by closed sub-annuli $W_i^s := \phi(\{(1/2)^s \leq |z| \leq 2^s\})$, $0 \leq s \leq 1$. This gives a Hausdorff continuous, strictly increasing one-parameter family of Herman compacts $A_i^s := A(g, W_i^s)$ such that $A_i = A_i^{s_i}$, where $s_i = \inf\{s \in [0, 1] \mid A_i \subset W_i^s\}$. We can reparametrize the family (A_i^s) by logarithmic capacity, i.e. for $0 \leq s \leq s_i$, we let $t = t(s)$ be the logarithmic capacity of $\overline{\mathbb{D}} \cup A_i^s$, and set $A_{i,t} = A_i^s$ and thus obtain a Hausdorff continuous, strictly increasing family of Herman compacts $(A_{i,t})_{0 \leq t \leq t_0}$ parametrized by $t \in [0, t_0]$, with $A_{i,0} = S^1$ and $A_{i,t_0} = A_i$. Consider the set $C = \{t \in [0, t_0] \mid A_{1,t} = A_{2,t}\}$. Then $0 \in C$, and the set C is closed since if $t_n \in C$ converges to some t, then $A_{1,t_n} = A_{2,t_n}$ for all n implies that $A_{1,t} = A_{2,t}$ because $A_{i,t_n} \to A_{i,t}$ as $n \to \infty$ for $i = 1, 2$ in the Hausdorff topology.

Since C is closed, $\sup C \in C$. We claim that $\sup C = t_0$. Suppose to the contrary that $t_1 := \sup C < t_0$. Then $A_{1,t_1} = A_{2,t_1} = A$, say. Let $V = U_1 \cap U_2$; then V is an open neighbourhood of A and we can find an annulus W with C^1 boundary such that $A \subset W$, $\overline{W} \subset V$ and W is symmetric with respect to the reflection r. As before we can choose a filtration of W by closed sub-annuli W^s, $0 \leq s \leq 1$, with $W^0 = S^1$ and $W^1 = W$. By Hausdorff continuity, since $A \subset W$, we can choose $t_2 \in (t_1, t_0)$ such that for $t \in [t_1, t_2]$, we have $A_{i,t} \subset W$, $i = 1, 2$. For $t \in [t_1, t_2]$ and $i = 1, 2$, let $s_i(t) = \inf\{s \in [0, 1] \mid A_{i,t} \subset W^s\}$; then $t \mapsto s_i(t)$ is a continuous, strictly increasing function and $A_{i,t} = A(g, W^{s_i(t)})$

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for $t \in [t_1, t_2]$. For $t \in (t_1, t_2)$ close enough to t_1, we have $s_1(t) < s_2(t_2)$ (since $s_2(t_2) > s_2(t_1) = s_1(t_1)$), so there exists $t' \in (t_1, t_2)$ such that $s_1(t) = s_2(t')$ and hence $A_{1,t} = A(g, W^{s_1(t)}) = A(g, W^{s_2(t')}) = A_{2,t'}$. In particular, the logarithmic capacities of $\overline{\mathbb{D}} \cup A_{1,t}$ and $\overline{\mathbb{D}} \cup A_{2,t'}$ are equal and hence $t = t'$. It follows that for $t > t_1$ close enough to t_1, we have $A_{1,t} = A_{2,t}$, so $t \in C$, contradicting the fact that $t_1 = \sup C$. Hence, $\sup C = t_0 \in C$, so $A_1 = A_{1,t_0} = A_{2,t_0} = A_2$. □

For g in $\mathrm{Diff}^{\omega}_{\alpha}(S^1)$ and $t \geq 0$, let $A_t(g)$ be a Herman compact for g such that $\overline{\mathbb{D}} \cup A_t(g)$ has logarithmic capacity t (if such a Herman compact exists), let $\Omega_t(g) := \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A_t(g))$ and let $\psi_t(g) : \Omega_t(g) \to \mathbb{D}_\infty$ be the corresponding normalized conformal map.

PROPOSITION 5.2. *For $s, t \geq 0$, $\Phi_t(\mathcal{D}_{s+t}) \subset \mathcal{D}_s$ and $\Phi_s \circ \Phi_t = \Phi_{s+t}$ on $\mathcal{D}_{t+s}$. Moreover, if $g \in \mathcal{D}_{t_0}$ for some $t_0 > 0$, then the curve $t \in [0, t_0] \mapsto \Phi_t(g) \in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ is continuous (for the topology of uniform convergence on S^1).*

Proof. For $g \in \mathcal{D}_{s+t}$, where $s, t \geq 0$, $\psi_t(g)$ maps $\Omega_{s+t}(g)$ conformally to a domain Ω_1 such that the compact A_1 given by the complement of Ω_1 and its reflection in the unit circle is invariant under $\Phi_t(g)$ and hence $\Omega_1 = \Omega_{s'}(\Phi_t(g))$ for some $s' \geq 0$. The map $\psi_{s'}(\Phi_t(g))$ maps Ω_1 conformally to $\mathbb{D}_\infty$, so the composition $\psi := \psi_{s'}(\Phi_t(g)) \circ \psi_t(g)$ maps $\Omega_{s+t}(g)$ conformally to $\mathbb{D}_\infty$ and satisfies the normalizations $\psi(\infty) = \infty$, $\psi'(\infty) = e^{s'} e^t = e^{s'+t} > 0$. It follows from uniqueness of the normalized conformal mapping that $\psi = \psi_{s+t}(g)$, so $e^{s'+t} = e^{s+t}$ and hence $s' = s$. Thus, $\psi_s(\Phi_t(g)) \circ \psi_t(g) = \psi_{s+t}(g)$, from which it follows that $(\Phi_s \circ \Phi_t)(g) = \Phi_{s+t}(g)$.

Now suppose that $g \in \mathcal{D}_{t_0}$ for some $t_0 > 0$. Let $t_n \in [0, t_0]$ be a sequence converging to some $t \in [0, t_0]$. Then the compacts $A_{t_n}(g)$ converge to $A_t(g)$ in the Hausdorff topology, so the domains $\Omega_{t_n}(g)$ converge to the domain $\Omega_t(g)$ in the sense of Carathéodory kernel convergence. By the Carathéodory kernel convergence theorem, the normalized conformal mappings $\psi_{t_n}(g)$ and their inverses $\psi_{t_n}^{-1}(g)$ converge uniformly on compacts of $\Omega_t(g)$ and $\mathbb{D}_\infty$ to the maps $\psi_t(g)$ and $\psi_t^{-1}(g)$, respectively. Let V be an annular neighbourhood of S^1 containing $A_{t_0}(g)$ such that g extends holomorphically to V. We can choose a Jordan curve $\gamma \subset V \cap \mathbb{D}_\infty$ which separates $A_{t_0}(g)$ from ∞. Then the Jordan curves $\beta_n := \psi_{t_n}(g)(\gamma) \subset \mathbb{D}_\infty$ converge to the Jordan curve $\beta := \psi_t(g)(\gamma) \subset \mathbb{D}_\infty$, so we can choose $R > 1$ close to one such that the annulus $\{1/R \leq |z| \leq R\}$ is contained in the annuli A_n bounded by β_n and $r(\beta_n)$ for all n. Each map $\Phi_{t_n}(g)$, $\Phi_t(g)$ then extends holomorphically to a neighbourhood of the annulus $\{1/R \leq |z| \leq R\}$ and $\Phi_{t_n}(g) = \psi_{t_n}(g) \circ g \circ \psi_{t_n}^{-1}(g)$ on the circle $\{|z| = R\}$; hence, $\Phi_{t_n}(g) \to \Phi_t(g) = \psi_t(g) \circ g \circ \psi_t^{-1}(g)$ uniformly on $\{|z| = R\}$ as $n \to \infty$. Since these maps commute with the reflection r, this implies uniform convergence of $\Phi_{t_n}(g)$ to $\Phi_t(g)$ on $\{|z| = 1/R\}$ and hence by the maximum principle we have uniform convergence on S^1. □

6. *Infinitesimal generator of the semigroup*

Let g be an analytic circle diffeomorphism with irrational rotation number α. Let $(A_t)_{0 \leq t < \epsilon}$ be the family of Herman compacts of g and $(\mathcal{H}_t = \overline{\mathbb{D}} \cup A_t)_{0 \leq t < \epsilon}$ the associated family of hulls. Let $\Omega_t = \hat{\mathbb{C}} - \mathcal{H}_t$ and let $\phi_t : \mathbb{D}_\infty \to \Omega_t$, $\psi_t = \phi_t^{-1}$ be normalized conformal

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mappings. Let $g_t \in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ be the family of analytic circle diffeomorphisms obtained from g using the Herman compacts A_t as in the previous section (so $g_t = \psi_t \circ g \circ \psi_t^{-1}$ in an annulus in $\mathbb{D}_\infty$ with one boundary component equal to S^1). The functions $\{(\phi_t - \mathrm{id})/t : 0 \le t < \epsilon\}$ form a normal family [**Pom75**, §6.1].

LEMMA 6.1. *Let $\{t_n\}$ be a sequence decreasing monotonically to 0 such that $(\phi_{t_n} - \mathrm{id})/t_n$ converges uniformly on compacts of $\mathbb{D}_\infty$ to a holomorphic function χ. Then $(\psi_{t_n} - \mathrm{id})/t_n$ converges uniformly on compacts in $\mathbb{D}_\infty$ to $(-\chi)$.*

Proof. Fix a compact $K \subset \mathbb{D}_\infty$; then, for n large, $K \subset \Omega_{t_n}$ and the maps ψ_{t_n} are defined on K. We have $\phi_{t_n} = \mathrm{id} + t_n\chi + o(t_n)$ uniformly on K. Then $\mathrm{id} = \phi_{t_n} \circ \psi_{t_n}$ gives

$$\mathrm{id} = (\mathrm{id} + t_n\chi + o(t_n)) \circ \psi_{t_n} = \psi_{t_n} + t_n\chi \circ \psi_{t_n} + o(t_n),$$

so

$$\frac{\psi_{t_n} - \mathrm{id}}{t_n} = -\chi \circ \psi_{t_n} + o(1)$$

uniformly on K and the lemma follows since $\psi_{t_n} \to \mathrm{id}$ uniformly on K as $n \to \infty$. □

LEMMA 6.2. *Let $\{t_n\}$ be a sequence decreasing monotonically to 0 such that $(\phi_{t_n} - \mathrm{id})/t_n$ converges uniformly on compacts of $\mathbb{D}_\infty$ to a holomorphic function χ. Then $(g_{t_n} - g)/t_n$ converges uniformly on compacts in a neighbourhood V of S^1 to a holomorphic function $\dot{g}$ given (in $V \cap \mathbb{D}_\infty$) by*

$$\dot{g} = g' \cdot \chi - \chi \circ g.$$

Moreover, the functions $\{(g_t - g)/t : 0 \le t < \epsilon\}$ form a normal family on V.

Proof. By the previous lemma, the equality $\psi_{t_n} \circ g = g_{t_n} \circ \psi_{t_n}$ gives

$$(\mathrm{id} - t_n\chi + o(t_n)) \circ g = g_{t_n} \circ (\mathrm{id} - t_n\chi + o(t_n))$$

uniformly on a circle $\{|z| = R\} \subset \mathbb{D}_\infty$, so

$$g - t_n\chi \circ g + o(t_n) = g_{t_n} - g'_{t_n} t_n\chi + o(t_n);$$

thus,

$$\frac{g_{t_n} - g}{t_n} = g'_{t_n} \cdot \chi - \chi \circ g + o(1) \to g' \cdot \chi - \chi \circ g$$

uniformly on $\{|z| = R\}$. Since the maps g_{t_n}, g are circle maps, $(g_{t_n} - g)/t_n$ converges uniformly on the circle $\{|z| = 1/R\} \subset \mathbb{D}$ and hence by the maximum principle we have uniform convergence on the closed annulus $V = \{1/R \le |z| \le R\}$.

To see that $\{(g_t - g)/t : 0 \le t < \epsilon\}$ is a normal family, it suffices to show that for any sequence t_n converging to 0, the sequence $(g_{t_n} - g)/t_n$ has a normally convergent subsequence (note for t bounded away from 0, the functions $(g_t - g)/t$ are uniformly bounded on V since g_t, g are uniformly bounded on V). Given a sequence t_n converging to 0, since the family $\{(\phi_t - \mathrm{id})/t\}$ is normal, there is a subsequence of the sequence $\{(\phi_{t_n} - \mathrm{id})/t_n\}$ which converges normally, but then by the first part of the lemma, the same subsequence of the sequence $\{(g_{t_n} - g)/t_n\}$ converges normally. □

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THEOREM 6.3. *Let* $\{t_n\}$ *be a sequence decreasing monotonically to* 0 *such that* $(\phi_{t_n} - \mathrm{id})/t_n$ *converges uniformly on compacts of* $\mathbb{D}_\infty$ *to a holomorphic function* χ. *Then* $H = \chi/\mathrm{id}$ *satisfies* $\mathrm{Re}\, H > 0$, $H(\infty) = 1$ *and is the Herglotz transform* $\mathcal{H}\mu$ *on* $\mathbb{D}_\infty$ *of* $\mu = \mu_{2,g}$, *where* $\mu_{2,g}$ *is the unique* 2*-conformal measure of* g.

Proof. Let $\chi_n(z) = (\phi_{t_n}(z) - z)/t_n$ and let $H_n(z) = \chi_n(z)/z$ for $|z| > 1$. Since for $t > 0$, Ω_t is a proper subdomain of $\mathbb{D}_\infty$, by the Schwarz lemma $|\phi_t(z)/z| > 1$ for all $t > 0$. The Moebius map $w = (z-1)/(z+1)$ maps $\mathbb{D}_\infty$ conformally to the half-plane $\{\mathrm{Re}\, w > 0\}$, so

$$\mathrm{Re}\left(\frac{t_n z H_n(z)}{\phi_{t_n}(z) + z}\right) = \mathrm{Re}\left(\frac{\phi_{t_n}(z) - z}{\phi_{t_n}(z) + z}\right) > 0$$

and, letting $n \to \infty$, it follows that (using $H_n(z) \to H(z)$, $z/(\phi_{t_n}(z) + z) \to 1/2$) $\mathrm{Re}\, H(z) \geq 0$ for all z. Moreover, $H_n(\infty) = (e^{t_n} - 1)/t_n$ and H_n converges uniformly to H on compacts in $\mathbb{D}_\infty$, so $H(\infty) = 1$. If $\mathrm{Re}\, H(z_0) = 0$ for some z_0, then by the open mapping theorem we must have $H(z) \equiv H(z_0) \in i\mathbb{R}$, contradicting $H(\infty) = 1$ and hence $\mathrm{Re}\, H(z) > 0$ for all z. By the Herglotz theorem, H is the Herglotz transform on $\mathbb{D}_\infty$ of a probability measure μ on S^1. It suffices to show that μ is 2-conformal for g.

Let $\mu = f\, d\lambda + \mu_s$ be the decomposition of μ into absolutely continuous and singular parts with respect to Lebesgue measure.

By Fatou's theorem, the radial limits of $\mathrm{Re}\, H$ exist for a.e. $\xi \in S^1$ and equal f. From Lemma 6.2, we have $\dot{g} + \chi \circ g = g' \cdot \chi$ and so

$$\frac{\dot{g}}{g}(z) + (H \circ g)(z) = \frac{z g'(z)}{g(z)} H(z).$$

By Lemma 6.2, the function $\dot{g}$ is holomorphic in a neighbourhood of S^1 and, since $g_{t_n}(\xi)$ converges to $g(\xi)$ along S^1 for ξ in S^1, we have $\mathrm{Re}\, (\dot{g}/g)(\xi) = 0$. Moreover, $\xi g'(\xi)/g(\xi) = |g'(\xi)|$ since g is a circle map, so taking real parts in the equation above as z tends to $\xi \in S^1$ radially gives $(f \circ g)(\xi) = |g'(\xi)| f(\xi)$; thus,

$$g^*(f d\lambda) = (f \circ g)|g'|\, d\lambda = |g'|^2 (f\, d\lambda)$$

and thus the absolutely continuous part $f\, d\lambda$ of μ is 2-conformal.

It remains to show that the singular part μ_s of μ is 2-conformal. Taking imaginary parts in $(\dot{g}/g)(z) + (H \circ g)(z) = (z g'(z)/g(z)) H(z)$ as z tends to a point $\xi \in S^1$ radially gives (recalling that the boundary value of H is given by $H = P + iQ$)

$$(\dot{g}/g)(\xi) + i(Q \circ g)(\xi) = |g'(\xi)| i Q(\xi).$$

The function $\dot{g}/g$ is holomorphic in a neighbourhood of S^1 and hence bounded on S^1, so $||Q \circ g| - |g'||Q|| < M$ for some $M > 0$.

For $t \gg 1$, let ν_t denote the measure $(\pi/2) t 1_{|Q| > t} d\lambda$, so ν_t converges weakly to μ_s as $t \to +\infty$.

Given a small $\epsilon > 0$, by uniform continuity of g' on S^1 there exists $\delta = \delta(\epsilon) > 0$ such that if $U \subset S^1$ is any interval of length less than δ, then for any $\xi, \xi_0 \in U$, we have $(1 - \epsilon) < |g'(\xi)/g'(\xi_0)| < (1 + \epsilon)$. For $t \gg 1$ such that $t/(t - M) < 1 + \epsilon$,

$t/(t+M) > 1-\epsilon$, we then have

$$\begin{aligned}\nu_t(g(U)) &= \frac{\pi}{2}t\lambda(g(U)\cap\{|Q|>t\}) \\ &= \frac{\pi}{2}t\lambda(g(\{\xi\in U, |Q(g(\xi))|>t\})) \\ &\leq \frac{\pi}{2}t(1+\epsilon)|g'(\xi_0)|\lambda(\{\xi\in U, |Q(g(\xi))|>t\}) \\ &\leq \frac{\pi}{2}t(1+\epsilon)|g'(\xi_0)|\lambda(\{\xi\in U, |g'(\xi)||Q(\xi)|>t-M\}) \\ &\leq t(1+\epsilon)|g'(\xi_0)|\frac{1}{T}\nu_T(U) \qquad \left(\text{where } T=\frac{t-M}{(1+\epsilon)|g'(\xi_0)|}\right) \\ &\leq (1+\epsilon)^3|g'(\xi_0)|^2\nu_T(U).\end{aligned}$$

Similarly, we have

$$\nu_t(g(U)) \geq (1-\epsilon)^3|g'(\xi_0)|^2\nu_{T'}(U),$$

where $T'=(t+M)/(1-\epsilon)|g'(\xi_0)|$.

Douady and Yoccoz [DY99] showed that if $\log|g'|$ is of bounded variation (which is the case if g is C^2), then any conformal measure for g has no atoms. In particular, μ_s has no atoms and hence $\nu_t(U)\to\mu_s(U)$ for any interval U. From the above we then have that given ϵ, there is a $\delta=\delta(\epsilon)>0$ such that if U is any interval of length less than δ, and $\xi_0\in U$, then

$$(1-\epsilon)^3|g'(\xi_0)|^2\mu_s(U) \leq \mu_s(g(U)) \leq (1+\epsilon)^3|g'(\xi_0)|^2\mu_s(U).$$

Given any interval V in S^1 and $\epsilon>0$, let $U_1,\ldots,U_n$ be a partition of V into intervals of length less than $\delta(\epsilon)$ centred around points $\xi_1,\ldots,\xi_n$. We may assume that the variation of $|g'|^2$ on each U_i is less than ϵ. Then

$$\begin{aligned}\mu_s(g(V)) &= \sum_i \mu_s(g(U_i)) \\ &\leq \sum_i (1+\epsilon)^3|g'(\xi_i)|^2\mu_s(U_i) \\ &\leq (1+\epsilon)^3\sum_i\int_{U_i}(|g'(\xi)|^2+\epsilon)\,d\mu_s(\xi) \\ &= (1+\epsilon)^3\left(\int_V |g'(\xi)|^2\,d\mu_s(\xi)+\epsilon\right).\end{aligned}$$

Letting ϵ tend to 0 gives $\mu_s(g(V)) \leq \int_V |g'|^2\,d\mu_s$ and similarly $\mu_s(g(V)) \geq \int_V |g'|^2\,d\mu_s$. It follows that μ_s is 2-conformal for g and hence μ is 2-conformal for g and $\mu=\mu_{2,g}$. □

7. *Proofs of main results*

We can now prove the main results from the Introduction.

Proof of Theorem 1.1. Given $g\in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$, by uniqueness of the 2-conformal measure $\mu_{2,g}$ of g, it follows from Theorem 6.3 that any normal limit χ of the functions

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$(\phi_t - t)/t, t > 0$ (where $\phi_t : \mathbb{D}_\infty \to \Omega_t$ is the normalized Riemann mapping as before) is given by $\chi(z) = z \cdot H(z)$, where H is the Herglotz transform on $\mathbb{D}_\infty$ of $\mu_{2,g}$ (recall that the family $\{(\phi_t - \mathrm{id})/t\}$ is normal by [**Pom75**, §6.1]). Since the normal limit is unique,

$$\frac{\phi_t(z) - z}{t} \to z \cdot H(z)$$

uniformly on compacts in $\mathbb{D}_\infty$ as $t \to 0$.

For any $t > 0$ and $s > t$, the map $\phi_t^{-1} \circ \phi_s$ is the normalized Riemann mapping from $\mathbb{D}_\infty$ to the complement of the unique hull of logarithmic capacity $s - t$ associated to the circle map g_t, so it follows from the same argument as above (applied to g_t) that

$$\frac{\phi_t^{-1} \circ \phi_s(z) - z}{s - t} \to z \cdot H_t(z)$$

uniformly on compacts in $\mathbb{D}_\infty$ as $s \to t$, where H_t is the Herglotz transform on $\mathbb{D}_\infty$ of μ_{2,g_t}. □

Proof of Theorem 1.2. Given $g \in \mathrm{Diff}^\omega_\alpha(S^1)$ and $g_t = \Phi_t(g)$, since $(\phi_t - \mathrm{id})/t$ converges uniformly on compacts in $\mathbb{D}_\infty$ to a unique function χ as $t \to 0$, it follows from Lemma 6.2 that for any sequence $\{t_n\}$ converging to 0, the functions $(g_{t_n} - g)/t_n$ converge uniformly on compacts in a neighbourhood V of S^1 to a holomorphic function $\dot{g}$ on V satisfying (in $V \cap \mathbb{D}_\infty$)

$$\dot{g} = g' \cdot \chi - \chi \circ g.$$

So, all normal limits of the normal family $\{(g_t - g)/t\}$ coincide (recall that the family is normal by Lemma 6.2) and thus

$$\frac{g_t - g}{t} \to g' \cdot \chi - \chi \circ g$$

as $t \to 0$, where $\chi(z) = z \cdot H(z)$, H being the Herglotz transform on $\mathbb{D}_\infty$ of $\mu_{2,g}$. □

Proof of Theorem 1.3. Let $g \in \mathrm{Diff}^\omega_\alpha(S^1)$ and let $\{g_t\}_{0 \le t < \epsilon} \subset \mathrm{Diff}^\omega_\alpha(S^1)$ be such that $g_0 = g$ and such that the right-hand derivatives exist uniformly on a neighbourhood V of S^1 and satisfy (on $V \cap \mathbb{D}_\infty$)

$$\dot{g}_t := \frac{d^+}{ds}_{s=t} g_s(z) = X(g_t)(z) = g_t'(z) \cdot \chi_t(z) - \chi_t \circ g_t(z),$$

where $\chi_t(z) = zH_t(z)$, $H_t(z)$ being the Herglotz transform on $\mathbb{D}_\infty$ of the measure μ_{2,g_t}. The function $p(z, t) := H_t(z)$ satisfies $\mathrm{Re}\, p(z, t) > 0$, $p(\infty, t) = 1$. We are assuming that the maps g_t depend continuously on t with respect to the topology of uniform convergence in a neighbourhood of S^1 and, hence, being analytic maps, with respect to C^1 convergence on S^1, so by [**DY99**] the measures μ_{2,g_t} depend continuously on t for the weak topology and hence $p(z, t) = H_t(z)$ depends continuously on t for fixed z. It follows [**Pom75**, Theorem 6.3] that there exists a Loewner chain $(\phi_t)_{0 \le t < \epsilon}$, where the maps ϕ_t are normalized conformal mappings from $\mathbb{D}_\infty$ onto a decreasing family of simply connected domains Ω_t, such that $\phi_t(z) = e^t z + O(1)$ near $z = \infty$, $\phi_0 = \mathrm{id}$ and the

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following right-hand derivatives exist uniformly on compacts of $\mathbb{D}_\infty$:

$$\frac{d^+}{ds}_{|s=t} \phi_t^{-1} \circ \phi_s = \chi_t.$$

Let $(A_t)_{0\leq t<\epsilon}$ be the increasing family of annular compacts containing S^1 given by the complement in $\hat{\mathbb{C}}$ of Ω_t and its reflection in S^1, so that the hulls $\mathcal{H}_t := \hat{\mathbb{C}} - \Omega_t$ are given by $\mathcal{H}_t = \overline{\mathbb{D}} \cup A_t$.

Let $h_t = \phi_t \circ g_t \circ \phi_t^{-1}$; then h_t depends continuously on t with respect to the topology of uniform convergence on compacts in a neighbourhood of S^1 and hence so does h_t'. We will show that in fact h_t does not depend on t. Let $U = V \cap \mathbb{D}_\infty$ be a one-sided neighbourhood of S^1. For $s > t$, let $h = s - t$; then as $h \to 0$, we have, uniformly on any compact in U,

$$\begin{aligned}
h_s' &= h_t' + o(1), \\
\phi_s &= \phi_t \circ (\mathrm{id} + h\chi_t + o(h)) = \phi_t + h\phi_t'\chi_t + o(h), \\
g_s &= g_t + h\dot{g}_t + o(h),
\end{aligned}$$

so the equation $h_s \circ \phi_s = \phi_s \circ g_s$ gives

$$\begin{aligned}
& h_s \circ (\phi_t + h\phi_t'\chi_t + o(h)) = (\phi_t + h\phi_t'\chi_t + o(h)) \circ (g_t + h\dot{g}_t + o(h)) \\
\Rightarrow\ & h_s \circ \phi_t + h(h_s' \circ \phi_t)(\phi_t'\chi_t) + o(h) = \phi_t \circ g_t + h(\phi_t' \circ g_t)(\chi_t \circ g_t + \dot{g}_t) + o(h) \\
\Rightarrow\ & h_s \circ \phi_t + h(h_t' \circ \phi_t)(\phi_t'\chi_t) + o(h) = h_t \circ \phi_t + h(\phi_t' \circ g_t)(g_t'\chi_t) + o(h) \\
\Rightarrow\ & (h_s - h_t) \circ \phi_t + h(h_t \circ \phi_t)'\chi_t + o(h) = h(\phi_t \circ g_t)'\chi_t + o(h) \\
\Rightarrow\ & (h_s - h_t) \circ \phi_t + h(h_t \circ \phi_t)'\chi_t + o(h) = h(h_t \circ \phi_t)'\chi_t + o(h),
\end{aligned}$$

from which it follows that for any z in U, the right-hand derivative $(d^+/dt)h_t(z) = 0$ for all t. Since $t \mapsto h_t(z)$ is continuous, we have $h_t(z) = h_0(z) = g(z)$ for all t (since a continuous function on an interval whose right-hand derivative exists and vanishes everywhere is constant).

The map h_t, being the conjugate of the circle map g_t by the map ϕ_t on U, maps the annulus $\phi_t(U)$ to the annulus $\phi_t(g_t(U))$. Both these annuli have $\partial\mathcal{H}_t$ as one boundary component, and $h_t = g$ extends analytically across S^1 to be univalent in a neighbourhood of S^1 containing $\mathcal{H}_t$ and hence g leaves A_t invariant. It follows that A_t is the unique annular compact of g such that the hull $\mathcal{H}_t = \overline{\mathbb{D}} \cup A_t$ has logarithmic capacity t and hence $g_t = \Phi_t(g)$. □

Proof of Theorem 1.5. Let f be a germ with rotation number α. Let $(K_t)_{-\infty<t<t_0}$ be the one-parameter family of Siegel compacts of f parametrized by their logarithmic capacities and let $\phi_t : \mathbb{D}_\infty \to \hat{\mathbb{C}} - K_t$ be normalized conformal mappings such that $\phi_t(\infty) = \infty$, $\phi_t(z) = e^t z + O(1)$ near $z = \infty$. Fix disks $\mathbb{D}_r \subset \mathbb{D}_{r_0}$ with $0 < r < r_0$ such that f maps $\mathbb{D}_r$ univalently into $\mathbb{D}_{r_0}$. Then, for $t \ll -1$, the circle map g_t^f (given by conjugating f by ϕ_t^{-1}) is univalent on $\phi_t^{-1}(\mathbb{D}_r - K_t)$, which is an annulus in $\mathbb{D}_\infty$ with modulus tending to $+\infty$ as $t \to -\infty$. Therefore, the family (g_t^f) forms a normal family and any normal limit of g_t^f as $t \to -\infty$ must be a circle map univalent on $\mathbb{D}_\infty$ and hence equal to R_α, so $g_t^f \to R_\alpha$ as $t \to -\infty$.

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If two germs of integral curves $(g_t^{f_1}), (g_t^{f_2})$ are equal, then there are a $t_0 \in \mathbb{R}$ and neighbourhoods D_1, D_2 of the origin such that for $t < t_0$ there is a map h_t univalent on $\hat{\mathbb{C}} - K_t(f_1)$ with $h_t'(\infty) = 1$ conjugating f_1 on $D_1 - K_t(f_1)$ to f_2 on $D_2 - K_t(f_2)$ (where $K_t(f_i), i = 1, 2$ denotes the Siegel compact of f_i of logarithmic capacity t). The maps $(h_t)_{t<t_0}$ form a normal family, any normal limit of which is univalent on $\mathbb{C}^*$, takes values in $\mathbb{C}^*$ and has derivative one at ∞ and hence must be the identity. Thus, $h_t \to \text{id}$ as $t \to -\infty$ and $f_1 = f_2$.

Finally, given a backward integral curve $(g_t)_{-\infty<t\leq c}$, let $\{t_n\}$ be a sequence in $(-\infty, c]$ decreasing to $-\infty$. Pérez-Marco showed in [**PM97**] that for the circle map g_{t_0}, there exists a germ f_{t_0} with a Siegel compact K such that the fundamental construction of [**PM97**] applied to the pair (f_{t_0}, K) gives the circle map g_{t_0}. Conjugating by a scaling if necessary, we may assume that K has logarithmic capacity t_0, so $K = K_{t_0}(f_{t_0})$ and $g_{t_0} = g_{t_0}^{f_{t_0}}$. By Theorem 1.3, for $t \in [t_0, c]$, $g_t = \Phi_{t-t_0}(g_{t_0})$, i.e. g_t arises from g_{t_0} by applying the fundamental construction of §5 to g_{t_0} and a Herman compact of g_{t_0}; then pulling back this Herman compact for g_{t_0} to the plane of f_{t_0} gives a Siegel compact $K_t(f_{t_0})$ for f_{t_0} of logarithmic capacity t such that $g_t = g_t^{f_{t_0}}$.

Similarly, for any $n \geq 0$, we obtain a germ f_{t_n} such that $g_t = g_t^{f_{t_n}}$ for all $t \in [t_n, c]$. In particular, this holds for $t = c$, so f_{t_n} is given, outside a Siegel compact $K_c(f_{t_n})$, by conjugating g_c by a conformal map $\phi_n : \mathbb{D}_\infty \to \hat{\mathbb{C}} - K_c(f_{t_n})$ with $\phi_n(z) = e^c z + O(z)$ near $z = \infty$. The maps $\{\phi_n\}$ are conformal mappings on $\mathbb{D}_\infty$ with fixed derivative at ∞ and hence form a normal family. Fix an annulus $U \subset \mathbb{D}_\infty$ with boundary components S^1 and a Jordan curve γ such that g_c is univalent on a neighbourhood of $\overline{U}$. Let D_n be the Jordan domain bounded by $\phi_n(\gamma)$. We may assume that the map f_{t_n} is defined and univalent on the whole domain D_n as follows: the map $\phi_n \circ g_c \circ \phi_n^{-1}$ is defined and univalent on the domain $W_n := D_n - K_c(f_{t_n})$ and we can choose a connected neighbourhood U_n of $K_c(f_{t_n})$ such that f_{t_n} is defined and univalent on U_n and such that $U_n \subset W_n$; then the maps $\phi_n \circ g_c \circ \phi_n^{-1}$ and f_{t_n} on the domains W_n and U_n agree on the intersection $W_n \cap U_n$ and hence they may be glued together to give a holomorphic map on $W_n \cup U_n = D_n$, which we continue to denote by f_{t_n}. The maps $\{f_{t_n}\}$ are then normalized univalent functions on the Jordan domains D_n bounded by $\phi_n(\gamma)$ and we may pass to a subsequence such that the Jordan curves $\phi_n(\gamma)$ converge (since $\{\phi_n\}$ is a normal family); then the maps $\{f_{t_n}\}$ form a normal family on the kernel of the domains D_n. Any normal limit f then satisfies $g_t = g_t^f$ for all $t \leq c$ (by continuity of the fundamental construction of [**PM97**]). □

The fact [**PM97**] that any circle map g arises from a pair (f, K), where f is a germ with a Siegel compact K, also gives the following.

PROPOSITION 7.1. *For any* $g \in \text{Diff}_\alpha^\omega(S^1)$*, there exists a backward integral curve* $(g_t)_{-\infty<t\leq c}$ *with* $g_c = g$.

Proof. Given a pair (f, K) which gives rise to the circle map g, let $(K_t)_{-\infty<t\leq c}$ be the unique family of Siegel compacts of f parametrized by logarithmic capacity, with $K_c = K$. Applying the fundamental construction of [**PM97**] to each pair (f, K_t) gives a backward integral curve $(g_t)_{-\infty<t\leq c}$ with $g_c = g$. □

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8. *Linearizable maps and conformal radius of linearization domains*

Let $g \in \mathrm{Diff}^{\omega}_{\alpha}(S^1)$ be a circle map which is analytically linearizable. For such a map we have uniqueness in both forward and backward time for any integral curve with initial condition g.

THEOREM 8.1. *Let $(g^i_t)_{-\infty<t<c} \subset \mathrm{Diff}^{\omega}_{\alpha}(S^1)$, $i = 1, 2$, be two integral curves of X such that $g^1_{c_0} = g^2_{c_0} = g$ for some $c_0 < c$. Then $g^1_t = g^2_t$ for all $t < c$.*

Proof. By Theorem 1.3, we have $g^1_t = g^2_t$ for $c_0 \le t < c$. Consider a $t < c_0$; then we have $g = \Phi_s(g^1_t) = \Phi_s(g^2_t)$, where $s = c_0 - t > 0$. Let ϕ^1, ϕ^2 be the normalized conformal mappings defined on $\mathbb{D}_\infty$ conjugating g to g^1_t, g^2_t, respectively (implicitly, here we fix a holomorphic extension of g to a neighbourhood N of S^1 and modify extensions of g^1_t, g^2_t appropriately so that their domains contain the images of $N \cap \mathbb{D}_\infty$ under ϕ^1, ϕ^2, respectively). Let $U \subset \mathbb{D}_\infty$ be an invariant annulus for g with boundary components equal to S^1 and a Jordan curve $\gamma \subset \mathbb{D}_\infty$ such that g is univalent in a neighbourhood of $\overline{U}$. For $i = 1, 2$, $\phi^i(\gamma)$ is an invariant Jordan curve for g^i_t in $\mathbb{D}_\infty$ and hence so is $r(\phi^i(\gamma))$; letting V_i denote the annulus with boundary components $\phi^i(\gamma)$ and $r(\phi^i(\gamma))$, it follows that V_i is an invariant annulus for g^i_t containing S^1, so g^i_t is analytically linearizable. Since g^i_t is analytically linearizable, the g^i_t-invariant annulus V_i is filtered by a monotone increasing one-parameter family of g^i_t-invariant sub-annuli containing S^1, whose closures give the family of Herman compacts of g^i_t contained in V_i. So, the Herman compact for g^i_t which gives rise to g (on conjugating by ϕ^i) is given by one such g^i_t-invariant sub-annulus containing S^1, with boundary components equal to an invariant Jordan curve $\gamma^i \subset \mathbb{D}_\infty$ and its reflection $r(\gamma^i)$. Let A^i be the g^i_t-invariant annulus with boundary components S^1 and γ^i; then ϕ^i maps $\mathbb{D}_\infty$ conformally to $\hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A^i)$ and conjugates the action of g on S^1 to that of g^i_t on γ^i.

Let $L = \phi^2 \circ (\phi^1)^{-1} : \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A^1) \to \hat{\mathbb{C}} - (\overline{\mathbb{D}} \cup A^2)$; then $L(\infty) = \infty$, $L'(\infty) = 1$ and L conjugates the action of g^1_t on γ^1 to that of g^2_t on γ^2. Fix a point $z_1 \in \gamma^1$ and let $z_2 = L(z_1) \in \gamma^2$. For $i = 1, 2$, let η^i be a conformal map from a round annulus $\{r_i < |z| < 1\}$ to A^i mapping S^1 to γ^i such that $\eta^i(z_i) = 1$; then η^i conjugates the rotation R_α to g^i_t.

Suppose that $r_1 \ge r_2$. Then the holomorphic map $\nu := \eta^2 \circ (\eta^1)^{(-1)}$ maps A^1 into A^2, conjugates the action of g^1_t on γ^1 to that of g^2_t on γ^2, and $\nu(z_1) = L(z_1) = z_2$; hence, $\nu = L$ on γ^1 (since the maps ν, L differ on γ^1 by post-composition with a homeomorphism of γ^2 commuting with g^2_t, which must be the identity if it fixes a point of γ^2).

It follows that L extends to a continuous map from $\mathbb{D}_\infty$ into $\mathbb{D}_\infty$ by setting $L = \nu$ on $A^1 \cup \gamma^1$, and this map is holomorphic on $\mathbb{D}_\infty - \gamma^1$ and hence also on $\mathbb{D}_\infty$ since the curve γ^1 is analytic and hence removable. Since $L'(\infty) = 1$, the Schwarz lemma implies that $L = \mathrm{id}$ and hence $g^1_t = g^2_t$.

A similar argument works if $r_2 \ge r_1$. □

We recall that the *conformal radius* $r(D, z_0)$ of a simply connected domain $D \ne \mathbb{C}$ with a base point $z_0 \in D$ is defined by $r(D, z_0) = h'(0) > 0$, where $h : \mathbb{D} \to D$ is a conformal map from the unit disk to D satisfying the normalizations $h(0) = z_0$, $h'(0) > 0$. Note that a disk of radius R centred around z_0 has conformal radius R. If D is a simply connected

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domain in $\hat{\mathbb{C}}$ with base point $z_0 = \infty$, and such that the complement of D in $\mathbb{C}$ has at least two points, then the conformal radius is defined to be $r = e^{-t}$, where t is the logarithmic capacity of the hull $\hat{\mathbb{C}} - D$ (so the domain $\{|z| > R\}$ has conformal radius $1/R$).

Let f be a linearizable germ with irrational rotation number and let $(K_t)_{-\infty<t<c}$ be the family of Siegel compacts of f parametrized by logarithmic capacity. Let D be a linearization domain of f and let $h : \mathbb{D} \to D$ be the normalized conformal mapping satisfying $h(0) = 0$, $h'(0) > 0$. For some $t_0 \leq c$, the interiors of the Siegel compacts K_t for $t < t_0$ are linearization domains $D_t \subset D$ for f bounded by analytic Jordan curves γ_t. Let $r(t) = r(D_t, 0)$ be the conformal radius of D_t and let $R = r(D, 0)$ be the conformal radius of the linearization domain D. The normalized conformal mappings of the domains D_t are given by the maps $h_t : \mathbb{D} \to D_t$, $w \mapsto h((r(t)/R)w)$. Let $\Omega_t = \hat{\mathbb{C}} - K_t$ and let $\phi_t : \mathbb{D}_\infty \to \Omega_t$ be the normalized conformal map satisfying $\phi_t(\infty) = \infty$, $\phi_t'(\infty) > 0$.

Since $\gamma_t = \partial D_t = \partial\Omega_t$ is an analytic Jordan curve, the maps h_t, ϕ_t extend analytically across S^1 and define an associated 'welding homeomorphism', which is the analytic circle map $w_t := h_t^{-1} \circ \phi_{t|S^1} : S^1 \to S^1$. The analytic circle map $k_t := w_t^{-1}$ conjugates the rotation R_α to the circle map $g_t = g_t^f$ (arising from the pair (f, K_t)).

LEMMA 8.2. *For $t < t_0$:*

(i) *the conformal radius $r = r(t)$ of the interior D_t of γ_t depends smoothly on the conformal radius e^{-t} of the exterior Ω_t of γ_t;*

(ii) *the map k_t depends smoothly on t.*

Proof. (i) For $r \in (0, r(t_0))$, let $t = t(r) \in (-\infty, t_0)$ be the logarithmic capacity of the hull $K_t = h(\{|w| \leq r/R\})$. Fix a $\beta \in (0, 1)$ and let $C^{1,\beta}(S^1)$ denote the Schauder space of C^1 complex functions on S^1 whose derivative is β-Hölder continuous. The parametrizations $\xi \in S^1 \mapsto h(\xi r/R)$ of the Jordan curves γ_t depend smoothly on r in the space $C^{1,\beta}(S^1)$ and hence the boundary values $\phi_{t|S^1}$ of the normalized conformal mappings ϕ_t depend smoothly on r as well in $C^{1,\beta}(S^1)$ (see [**dCP03**, Theorem 3.4] and [**dCR00**, Theorem 5.4]). Since $\phi_t'(\infty) = e^t$ is given in terms of these boundary values by Cauchy's integral formula, it follows that $t = t(r)$ depends smoothly on r. We claim that moreover $t'(r) > 0$ for all r.

Fix an $r_1 \in (0, r(t_0))$ and let $r \in (r_1, r(t_0))$. Let $t_1 = t(r_1)$ and $t = t(r)$. Note that the map $\phi_{t_1}^{-1} \circ \phi_t$ is holomorphic in a neighbourhood of $\overline{\mathbb{D}_\infty}$ and $(\phi_{t_1}^{-1} \circ \phi_t)(z) = e^{t-t_1}z + O(1)$ near $z = \infty$, so by Cauchy's integral formula

$$e^{t-t_1} = \frac{1}{2\pi i}\int_{S^1} (\phi_{t_1}^{-1} \circ \phi_t)(\xi)\frac{1}{\xi}\frac{d\xi}{\xi}. \tag{4}$$

In a neighbourhood of S^1 we have $\phi_{t_1} = h_{t_1} \circ w_{t_1}$, $\phi_t = h_t \circ w_t$ and so for r close enough to r_1, in a neighbourhood of S^1 we can write

$$\begin{aligned} w_{t_1} \circ \phi_{t_1}^{-1} \circ \phi_t(\xi) &= h_{t_1}^{-1} \circ h_t \circ w_t(\xi) \\ &= \frac{r}{r_1} w_t(\xi) \end{aligned}$$

(where we have used $h_{t_1}(\xi) = h((r_1/R)\xi)$, $h_t(\xi) = h((r/R)\xi)$; the above equation holds taking analytic extensions of w_{t_1}, w_t to a neighbourhood of S^1). For $\xi \in S^1$, taking right-hand derivatives in the equation above with respect to r at $r = r_1$ gives, on writing $\dot{w}_{t_1}(\xi) = \frac{d}{dr}_{|r=r_1} w_t(\xi)$,

$$w'_{t_1}(\xi) \cdot \frac{d^+}{dr}_{|r=r_1}(\phi_{t_1}^{-1} \circ \phi_t)(\xi) = \frac{1}{r_1} w_{t_1}(\xi) + \dot{w}_{t_1}(\xi).$$

Now taking right-hand derivatives with respect to r at $r = r_1$ in equation (4) and using the above equation gives

$$\begin{aligned} t'(r_1) &= \frac{1}{2\pi i} \int_{S^1} \frac{d^+}{dr}_{|r=r_1}(\phi_{t_1}^{-1} \circ \phi_t)(\xi) \frac{1}{\xi} \frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i} \int_{S^1} \frac{1}{r_1} \frac{w_{t_1}(\xi)}{\xi w'_{t_1}(\xi)} + \frac{\dot{w}_{t_1}(\xi)}{\xi w'_{t_1}(\xi)} \frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i} \int_{S^1} \frac{1}{r_1} \frac{1}{|w'_{t_1}(\xi)|} + \frac{1}{|w'_{t_1}(\xi)|} \frac{\dot{w}_{t_1}(\xi)}{w_{t_1}(\xi)} \frac{d\xi}{\xi} \\ &> 0 \end{aligned}$$

(where we have used the facts that $\xi w'_{t_1}(\xi)/w_{t_1}(\xi) = |w'_{t_1}(\xi)|$ since w_{t_1} is an analytic circle map and that Re $\dot{w}_{t_1}(\xi)/w_{t_1}(\xi) = 0$ since the maps w_t are all circle maps). This proves that $t'(r) > 0$ for all $r \in (0, r(t_0))$ and hence the inverse mapping $t \in (-\infty, t_0) \mapsto r = r(t) \in (0, r(t_0))$ is smooth.

(ii) Since the boundary values $\phi_{t|S^1}$ depend smoothly on r and $t = t(r)$ is smooth by (i), these boundary values depend smoothly on t as well. By [**dCP03**, Theorem 3.9], the welding maps w_t depend smoothly on t and hence so do their inverses k_t. □

THEOREM 8.3. *Let H_t be the Herglotz transform on $\mathbb{D}_\infty$ of the measure μ_{2,g_t} and let $P_t + iQ_t$ be the boundary values on S^1 of H_t (defined as radial limits almost everywhere as in §2). Then, for $\xi \in S^1$, we have*

$$(P_t \circ k_t)(\xi) = \frac{r'(t)}{r(t)} \cdot |k'_t(\xi)|$$

and

$$\frac{\dot{k}_t(\xi)}{k_t(\xi)} + iQ_t(k_t(\xi)) = 0$$

(where $\dot{k}_t(\xi)$ denotes the derivative with respect to t).

Proof. For $s = t + \epsilon$ with $\epsilon > 0$ small, and $w \in \mathbb{D}_\infty$ close to S^1, we have

$$(k_t^{-1} \circ \phi_t^{-1} \circ \phi_s \circ k_s)(w) = \frac{r(s)}{r(t)}(w)$$

(where the above equation holds taking analytic extensions of the maps k_t^{-1}, k_s to a neighbourhood of S^1; it is not hard to show that in fact k_t can be defined on the annulus $\{r(t)/R < |z| < R/r(t)\}$). As $\epsilon \to 0$, we have $\phi_t^{-1} \circ \phi_s = \mathrm{id} + \epsilon\chi_t + o(\epsilon)$, where

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$\chi_t(z) = zH_t(z)$, and $k_s = k_t + \epsilon \dot{k}_t + o(\epsilon)$, $r(s)/r(t) = 1 + \epsilon r'(t)/r(t) + o(\epsilon)$; hence,

$$k_t^{-1} \circ (\mathrm{id} + \epsilon\chi_t) \circ (k_t + \epsilon\dot{k}_t)(w) + o(\epsilon) = \left(1 + \epsilon\frac{r'(t)}{r(t)}\right) w + o(\epsilon)$$
$$\Rightarrow k_t^{-1} \circ (k_t + \epsilon(\dot{k}_t + \chi_t \circ k_t))(w) + o(\epsilon) = \left(1 + \epsilon\frac{r'(t)}{r(t)}\right) w + o(\epsilon)$$
$$\Rightarrow (\mathrm{id} + \epsilon((k^{-1})'_t \circ k_t)(\dot{k}_t + \chi_t \circ k_t))(w) + o(\epsilon) = \left(1 + \epsilon\frac{r'(t)}{r(t)}\right) w + o(\epsilon);$$

thus,

$$\frac{\dot{k}_t(w) + (\chi_t \circ k_t)(w)}{k'_t(w)} = \frac{r'(t)}{r(t)} w,$$

from which we obtain

$$\frac{\dot{k}_t(w)}{k_t(w)} + (H_t \circ k_t)(w) = \frac{r'(t)}{r(t)} \cdot \frac{k'_t(w)}{k_t(w)} \cdot w.$$

Since g_t is analytically linearizable, the measure μ_{2,g_t} is absolutely continuous with respect to Lebesgue measure and has a smooth density. So, for any $\xi \in S^1$, letting w tend to ξ radially in the equation above and taking real and imaginary parts gives the equalities asserted in the theorem (using the facts that $\dot{k}_t(\xi)/k_t(\xi)$ is purely imaginary and $\xi k'_t(\xi)/k_t(\xi) = |k'_t(\xi)|$ since the maps k_t are circle maps). □

We obtain as a corollary the following formula relating the conformal radius of linearization domains to the conformal radius of their complements.

COROLLARY 8.4. *The conformal radius of the linearization domains D_t satisfies*

$$\frac{r'(t)}{r(t)} = \int_{S^1} \frac{d\mu_{2,g_t}}{d\lambda}\, d\mu_{0,g_t},$$

where μ_{0,g_t} is the invariant probability measure of the circle map g_t. The above equality can also be written more symmetrically as

$$\frac{r'(t)}{r(t)} = \int_{S^1} \frac{d\mu_{2,g_t}}{d\lambda} \frac{d\mu_{0,g_t}}{d\lambda}\, d\lambda,$$

Proof. We have $P_t = d\mu_{2,g_t}/d\lambda$ by Fatou's theorem, so the equality above follows by integrating the equality $(P_t \circ k_t)(\xi) = r'(t)/r(t) \cdot |k'_t(\xi)|$ with respect to normalized Lebesgue measure $(1/2\pi)\lambda$, since $\int_{S^1} |k'_t(\xi)|(d\lambda/2\pi) = 1$ and $(k_t)_*(\lambda/2\pi) = \mu_{0,g_t}$. □

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