J. Appl. Prob. **55,** 900–919 (2018)
*doi:*10.1017/jpr.2018.57

FROM TREES TO GRAPHS: COLLAPSING CONTINUOUS-TIME BRANCHING PROCESSES

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Abstract

Continuous-time branching processes (CTBPs) are powerful tools in random graph theory, but are not appropriate to describe real-world networks since they produce trees rather than (multi)graphs. In this paper we analyze collapsed branching processes (CBPs), obtained by a collapsing procedure on CTBPs, in order to define multigraphs where vertices have fixed out-degree $m \geq 2$. A key example consists of preferential attachment models (PAMs), as well as generalized PAMs where vertices are chosen according to their degree and age. We identify the degree distribution of CBP s, showing that it is closely related to the limiting distribution of the CTBP before collapsing. In particular, this is the first time that CTBPs are used to investigate the degree distribution of PAMs beyond the tree setting.

Keywords: Branching process; preferential attachment; ageing

2010 Mathematics Subject Classification: Primary 05C80
Secondary 60B05; 60B10

1. Introduction and main results

1.1. Our model and main result

The main result of this paper is the definition of multigraphs from continuous-time branching processes (CTBPs), through a procedure we call *collapsing*. We analyze the case where we collapse a fixed number $m \in \mathbb{N}$ of individuals. Our heuristic approach is to consider the tree defined by the branching process, and collapse or merge together m different nodes in the tree to create a vertex in the multigraph. Throughout this paper we will consider an *individual* to be a node in the tree of the branching process, while a *vertex* is a node in the multigraph obtained by collapsing. We call the number of vertices in a graph the *size of the graph*.

We now recall some notation for CTBPs. For a more detailed introduction, we refer the reader to Section 2.1. We consider a branching process $\boldsymbol{\xi}$ defined by a birth process $(\xi_t)_{t\geq 0}$. In these models, individuals produce children according to independent and identically distributed (i.i.d.) copies of the process $(\xi_t)_{t\geq 0}$. Usually, individuals in the branching populations are denoted by $x = \varnothing x_1 \cdots x_k$ (see Definition 4). In this paper we will not denote individuals with their position in the genealogical tree, but rather by their birth order. Denote the sequence of birth times of individuals in the branching population by $(\tau_n)_{n\in\mathbb{N}}$.

Fix $m \in \mathbb{N}$. Denote $(n, j) = m(n - 1) + j$ for $j = 1, \ldots, m$. We now state the precise definition of the collapsed branching process.

Received 13 November 2017; revision received 3 May 2018.
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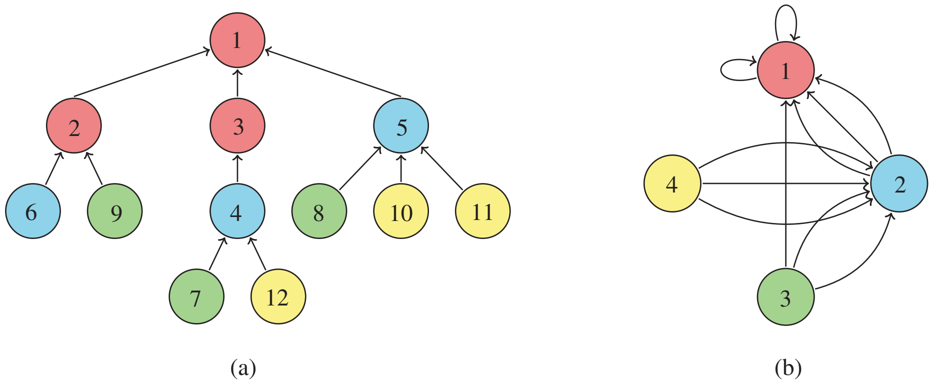

FIGURE 1: (a) A branching process tree, and (b) an example of a CBP where vertices have fixed out-degree $m = 3$.

Definition 1. (*Collapsed branching process.*) Consider a branching process $\boldsymbol{\xi}$. Then, a *collapsed branching process* (CBP) is a random process $(\mathrm{CBP}_t^{(m)})_{t\geq 0}$, for which, for every $t \geq 0$, $\mathrm{CBP}_t^{(m)}$ is a directed multigraph with adjacency matrix $(g_{x,y}(t))_{x,y\in\mathbb{N}}$, where

$$g_{x,y}(t) = \sum_{j=1}^{m} \mathbf{1}_{\{(x,j)\to(y,1),\ldots,(y,m)\}} \mathbf{1}_{\mathbb{R}^+}(t - \tau_{(x,j)}),$$

and $\{(x, j) \to (y, 1), \ldots, (y, m)\}$ is the event that there is a directed edge between individual (x, j) and one of the individuals $(y, 1), \ldots, (y, m)$ in the tree defined by the branching process at time t. We denote the size of $\mathrm{CBP}_t^{(m)}$ by $N^{(m)}(t)$.

As we can see from the definition, the collapsing procedure combines m individuals together with their edges to create a vertex, and there is an edge between two vertices if and only if there is an edge between a pair of individuals collapsed to create the two vertices. A $\mathrm{CBP}_t^{(m)}$ is a graph where every vertex (except vertex 1) has out-degree m. Self-loops and multiple edges are allowed; see Figure 1 for an example of a CBP.

We consider the birth time of the vertex n in the multigraph to be $\tau_{(n,1)} = \tau_{m(n-1)+1}$. Thus, vertex n in $\mathrm{CBP}^{(m)}$ is considered alive when $(n, 1)$ is alive in $\boldsymbol{\xi}$. Note that when n is born, it has only one out-edge, since the other individuals $(n, 2), \ldots, (n, m)$ are not yet alive. Clearly, the in-degree at time t of a vertex n in $\mathrm{CBP}^{(m)}$ is

$$D_n^{(\mathrm{in})}(t) = \sum_{j=1}^{m} \xi_{t-\tau_{(n,j)}}^{(n,j)}.$$

The main difference between CBPs and preferential attachment models (PAMs) is that CBPs are defined in continuous time, while time in PAMs is discrete. Heuristically, discrete time in PAMs is described as the time unit at which a new vertex is added to the graph (see, for example, [1], [9], and [26, Chapter 8]), while in CBPs time is continuous and new vertices are born at an exponential rate; see [23, Theorem 5.4], [24, Theorem A], and Theorem 3.

1.2. Results

Our main goal is to show that we can define a multigraph from a CTBP, and analyze its rate of growth as well as the limiting degree distribution.

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Our results are a first attempt to create a link between trees and multigraphs in continuous time. The collapsing procedure creates difficulties though. For instance, we consider different individuals to create a vertex, each one of them having its own birth time. This has to be taken into account in order to investigate the degree evolution of a vertex in CBP.

We state the result on the limiting degree distribution of CBPs, relying on properties of CTBPs as formulated in Theorem 4.

Theorem 1. (Limiting degree distribution of CBPs.) *Consider a branching process* $\boldsymbol{\xi}$*, and fix* $m \in \mathbb{N}$*. Denote the size of* $\mathrm{CBP}_t^{(m)}$ *by* $N^{(m)}(t)$ *and the number of vertices with degree* k *by* $N_k^{(m)}(t)$*. Under the hypotheses of Theorem 4, as* $t \to \infty$*,*

$$\frac{N_k^{(m)}(t)}{N^{(m)}(t)} \xrightarrow{\mathbb{P}} p_k^{(m)} = \mathbb{P}(\xi^1_{T_{\alpha^*}} + \cdots + \xi^m_{T_{\alpha^*}} = k),$$

where $(\xi_t^1)_{t\geq 0}, \ldots, (\xi_t^m)_{t\geq 0}$ *are* m *independent copies of the birth process* $(\xi_t)_{t\geq 0}$*,* α^* *is the Malthusian parameter of* $\boldsymbol{\xi}$*, and* T_{α^*} *is an exponentially distributed random variable with parameter* α^**. We denote convergence in probability by '*$\xrightarrow{\mathbb{P}}$*'.*

The hypotheses of Theorem 4 are technical, and they are deferred until later. Theorem 1 is part of Theorem 4, which is more general and requires notation from CTBPs theory, and we introduce this in Section 2.1.

1.3. Embedding PAMs

In discrete time, PAMs are defined as a sequence of random graphs $(\mathcal{G}_n)_{n\in\mathbb{N}}$, where at every step a new vertex is introduced in the graph. In general, the attachment rule is given in terms of a function of the degree f that we call the *preferential attachment (PA) function* or *weight*. Conditionally on the graph $\mathcal{G}_{(n,j)}$, where the jth edge of the nth vertex has been added,

$$\mathbb{P}\big(n \overset{j+1}{\to} i \mid \mathcal{G}_{(n,j)}\big) = \frac{f(D_i(n,j))}{\sum_{h=1}^{n} f(D_h(n,j))}, \tag{1}$$

where $D_i(n, j)$ denotes the degree of the vertex i in $\mathcal{G}_{(n,j)}$. When f is affine, it is possible to define the model with out-degree $m \geq 2$ from the tree case where the out-degree is 1; we refer the reader to [26, Chapter 8, Section 8.2] for the precise definition. In particular, the collapsing procedure introduced in Definition 1 mimics the construction of PAMs with affine attachment function.

CTBPs have used to investigate the degree distribution of PA trees; see [2], [4], [6], and [24]. In particular, embedding theorems were proved between discrete and continuous time; see [2, Theorem 3.3] and [4, Theorem 2.1]. These results are based on the fact that all intervals between two jumps in every copy of the birth process $(\xi_t)_{t\geq 0}$ are exponentially distributed. This means that, conditionally on the present state of the tree, the probability that a new vertex is attached to the ith vertex already present is just the ratio between the PA function of the degree of vertex i and the total weight of the tree. Also PAMs with out-degree $m \geq 2$ have been investigated, but not through embeddings of CTBPs.

It is possible to construct a CBP that embeds PAMs with an affine attachment function. We need to define a suitable birth process in order to do so.

Definition 2. (*Embedding birth process.*) Consider a sequence of positive numbers $(\lambda_k)_{k\in\mathbb{N}}$. Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of independent and exponentially distributed random variables with $E_k \sim E(\lambda_k)$ and $E_{-1} = 0$. We call $(\xi_t)_{t\geq 0}$ the *embedding birth process*, where

$$\xi_t = k \quad \text{if } t \in [E_{-1} + \cdots + E_{k-1}, E_{-1} + \cdots + E_k).$$

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This construction was used in [2], [4], and [24]. It allows us to embed PA trees in continuous time where the PA function is given by $f(k) = \lambda_k$. Embedding birth processes allows us to describe PAMs with out-degree $m \geq 2$ and affine f using CBPs. In fact, an immediate application of [2, Theorem 3.3] and [4, Theorem 2.1] is that it is enough to prove that the transition probability in a CBP from $\mathrm{CBP}^{(m)}_{\tau_{(n,j)}}$ to $\mathrm{CBP}^{(m)}_{\tau_{(n,j+1)}}$ is exactly given by (1), with the restriction that the first edge of every vertex cannot be a self-loop. In particular, this yields the next result, proved in several other papers through entirely different means; see below for a discussion.

Corollary 1. (Continuous-time PAM.) *Fix $m \geq 2$ and $\delta > -m$. Let $(\xi_t)_{t\geq 0}$ be an embedding birth process defined by the sequence $(k + 1 + \delta/m)_{k\in\mathbb{N}}$. Then the corresponding CBP embeds the PAM in continuous time with attachment rule $f(k) = k + \delta$ and satisfies Theorem 1 (and Theorem 4). As a consequence, the limiting degree distribution is*

$$p^{(m)}_k = \left(2 + \frac{\delta}{m}\right)\frac{\Gamma(2 + \delta/m + m + \delta)}{\Gamma(m + \delta)}\frac{\Gamma(k + m + \delta)}{\Gamma(k + m + \delta + 3 + \delta/m)}. \tag{2}$$

The limiting degree distributions of PAMs is already known in the literature. Interestingly, (2) is the limiting degree disribution for several versions of PAMs. Bollobás *et al.* [9] proved it for $\delta = 0$, i.e. the original Barabási–Albert model. For other works related to the degree sequence of several other versions of PAMs, we refer the reader to [6], [8], [13], [16], [22], and [25].

Corollary 1 is the application of Theorem 1 to the case of the CTBPs that embed PAMs in continuous time. Indeed, the CBP observed at times $(\tau_n)_{n\in\mathbb{N}}$ (the sequence of birth times of the CTBP) corresponds to the discrete-time PAM. However, since the ratio $N^{(m)}_k(t)/N^{(m)}(t)$ converges in probability, Theorem 1 does not imply the convergence along the sequence $(\tau_n)_{n\in\mathbb{N}}$. To prove that the convergence also holds in discrete time, an extra argument is needed, therefore we state it as a separate result.

Theorem 2. (Discrete-time PAMs.) *Fix $m \geq 2$ and $\delta > -m$. Let $(\xi_t)_{t\geq 0}$ be an embedding birth process defined by the sequence $(k + 1 + \delta/m)_{k\in\mathbb{N}}$. Consider the corresponding discrete-time PAM defined as $\mathrm{PA}_{n,j}(m, \delta) = \mathrm{CBP}^{(m)}_{\tau_{(n,j)}}$ for $n \in \mathbb{N}$ and $j \in [m]$. Then, for every $k \in \mathbb{N}$, the fraction of vertices with degree k in $\mathrm{PA}_{n,j}(m, \delta)$ converges in probability to $p^{(m)}_k$ as in (2).*

While CTBP arguments have been used extensively in the context of PA trees (for which $m = 1$), Theorem 2 provides the first example where it is applied beyond the tree setting. Thus, our results offer the opportunity to use the powerful CTBP tools in order to study PAMs.

To show the universality of our collapsing construction, we apply Theorem 1 to another classical random graph model. A random recursive tree (RRT) is a sequence of PA trees where the attachment function f is equal to 1. At every step, a vertex is added to the tree and attached uniformly to one existing vertex; see [27] for an introduction. We also consider a graph version of the RRT. In this case we obtain the following result, which could be interpreted as the $\delta = \infty$ version of Theorem 2, in which a graph is grown by uniform attachments.

Corollary 2. (Random recursive graph.) *Fix $m \geq 2$. Let $(\xi_t)_{t\geq 0}$ be an embedding birth process defined by the sequence $\lambda_k = 1$ for every $k \in \mathbb{N}$. Then the corresponding CBP defines*

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a sequence of random graphs with transition probabilities

$$\mathbb{P}\big(n \overset{j+1}{\to} i \mid \mathrm{CBP}^{(m)}_{\tau_{(n,j)}}\big) = \begin{cases} \dfrac{1}{(n-1)+j/m} & \text{if } i \neq n, \\ \dfrac{j/m}{(n-1)+j/m} & \text{if } i = n. \end{cases} \tag{3}$$

We call the sequence of random graphs defined by (3) a random recursive graph (RRG). As a consequence, the limiting degree distribution is

$$p_k^{(m)} = \frac{1}{m+1}\left(1+\frac{1}{m}\right)^{-k}. \tag{4}$$

Consequently, the same result also holds in discrete time.

In this case the CBP can be seen as the generalization of the RRT to the case where the out-degree is $m \geq 2$. In particular, when $m = 1$ the distribution in (4) reduces to $p_k^{(1)} = 2^{-(k+1)}$, which is the known limiting degree distribution for the RRT; see [21]. The result also holds in discrete time by Theorem 2 since the argument is general; see Section 6.

An extension of the PAM was proposed by Garavaglia *et al.* [14], where fitness and ageing in PA trees was introduced. The methodology used in the present work is applicable to the case with ageing only. The fitness case is not tractable, and we explain the reason for this in Section 1.4. A PA tree with ageing is given in terms of a CTBP where we introduce the effect of ageing, i.e. the probability of generating a child decreases with age. For a precise definition of such processes, we refer the reader to [14, Section 2.2].

Definition 3. (*Ageing birth process.*) Consider a sequence of positive numbers $(\lambda_k)_{k\in\mathbb{N}}$, and the corresponding embedding birth process as in Definition 2. Consider a function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$, called the *ageing function*, such that $\int_0^\infty g(t)\,\mathrm{d}t < \infty$. Defining $G(t) = \int_0^t g(s)\,\mathrm{d}s$, we call $(\xi_{G(t)})_{t\geq 0}$ an *ageing birth process*.

The assumption on the integrability of g is not necessary, but as shown in [14] this is the nontrivial case of the ageing effect. Garavaglia *et al.* [14] proved that a CTBP defined by an ageing birth process has a limiting degree distribution $(p_k^{(1)})_{k\in\mathbb{N}}$ with exponential tail, under the condition that $\lim_{t\to\infty}\mathbb{E}[\xi_{G(t)}] > 1$. The result of the present paper can also be applied to the ageing birth processes, leading to our next result.

Corollary 3. (Ageing PAMs.) *Fix $m \geq 2$, $\delta > -m$, and define the sequence $(k+1+\delta/m)_{k\in\mathbb{N}}$. Denote the corresponding embedding birth process by $(\xi_t)_{t\geq 0}$. Let g be an ageing function as in Definition 3 such that $g(t) \leq \bar{g}$ for some constant $\bar{g} > 0$ and for every $t \geq 0$. Assume that $\lim_{t\to\infty}\mathbb{E}[\xi_{G(t)}] > 1$. Then the CBP obtained by the CTBP defined by the ageing process satisfies Theorem 1 (and Theorem 4). As a consequence, the limiting degree distribution $(p_k^{(m)})_{k\in\mathbb{N}}$ satisfies*

$$p_k^{(m)} = \frac{\Gamma(k+m+\delta)}{\Gamma(k+1)}\mathrm{e}^{-Ck}(1+o(1)), \tag{5}$$

where $C = |\log(1-\exp(-\int_0^\infty g(t)\,\mathrm{d}t))|$.

In particular, it is possible to show that the transition probabilities of the discrete-time version $(\mathrm{CBP}^{(m)}_{\tau_{(n,j)}})_{n\in\mathbb{N},\, j\in[m]}$ of a CBP defined by an ageing process satisfies

$$\mathbb{P}\big(n \overset{j+1}{\to} i \mid \mathrm{CBP}^{(m)}_{\tau_{(n,j)}}, \tau_{(n,j+1)}\big) \approx \frac{(D_i(\tau_{(n,j)})+\delta)g(\tau_{(n,j+1)}-\tau_{(i,1)})}{\sum_{h=1}^n (D_h(\tau_{(n,j)})+\delta)g(\tau_{(n,j+1)}-\tau_{(h,1)})}, \tag{6}$$

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where $D_i(t)$ denotes the total degree of vertex i in $\mathrm{CBP}_t^{(m)}$ and the approximation is due to the fact that we consider $\tau_{(i,1)}$ as the birth time of all the m individuals collapsed to generate vertex i. The expression in (6) for the attachment rule in the presence of ageing resembles those in the literature on ageing in PAMs; see [12], [15], [17], [18], [28], and [29].

1.4. Discussion and open problems

1.4.1. *Neighborhoods in CBP.* A CBP with fixed out-degree $m \geq 2$ is a continuous-time random graph model in which the size of the graph grows exponentially in time (see (9)), and we are able to describe its limiting degree distribution. In particular, we can view a branching tree as a special case of a CBP with $m = 1$. This is an attempt to translate properties from a CTBP to multigraphs. As a consequence, we might ask what other topological properties a CBP might inherit from the underlying CTBP. As an example, PAMs are known to be *locally tree-like* graphs (see [5]), prompting the question whether this is true because PAMs can be defined as CBPs.

For example, a tree in a CBP with depth k and vertices of minimum degree m is generated by *chains* of individuals in the corresponding CTBP. Caravenna *et al.* [11] proved that the number of such trees in a PAM diverges as the size of the graph increases. In terms of the CTBP, it is necessary to look for structures similar to the one in Figure 2. It would be interesting to investigate the topological properties of the neighborhoods of vertices in a CBP, to see if and how they depend on the corresponding CTBP. It would also be interesting to compare the local structure of a CTBP with the result of [5] in terms of local weak convergence.

1.4.2. *Random out-degree.* An interesting extension of the present work is the case of *random out-degree* graphs. Instead, our CBPs have *fixed out-degree* $m \geq 2$. However, the collapsing procedure is well defined for any sequence of out-degrees $(m_n)_{n\in\mathbb{N}}$, both deterministic or random. Results are known for PAMs with random out-degree (see [13]), suggesting that a CBP with random out-degrees is the continuous-time version of a PAM with random out-degrees. In our CBP setting there are some difficulties, for instance, in Proposition 2. Since m is now random, all the indeces $(n, 1), \dots, (n, m)$ in (21) corresponding to vertex n in the CBP are all random variables. This requires care when carrying out the conditional expectation in (21).

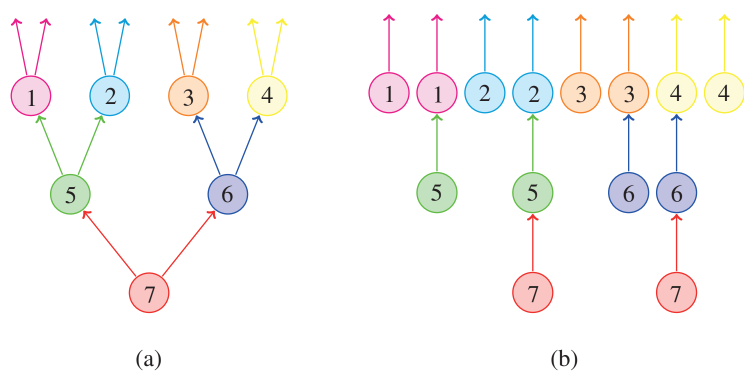

FIGURE 2: (a) An example of minimum degree tree in a CBP with $m = 2$, and (b) a realization of the corresponding structure in the CTBP that generates it. Note that different realizations in the CTBP can generate the same graph in the CBP.

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1.4.3. *More general PA functions and fitnesses.* When collapsing, the degree $D_n^{(\mathrm{in})}(t)$ of a vertex n in the CBP is distributed as the sum of m independent birth process $(\xi_t)_{t\geq 0}$. When we consider an affine PA function of the type $f(k) = ak + b$, with $a \geq 0$ and $b > 0$, the sum of the m weights corresponding to the m individuals becomes $a(D_1 + \cdots + D_m) + mb$, i.e. the collapsed individuals become *indistinguishable*. This still holds when we consider an affine PA function f and ageing g, due to the linearity of f and the fact that the error given by the difference in birth times is negligible.

This no longer holds when the PA function is not affine and/or in the presence of fitness. In fitness models, every individual x is assigned an independent realization η_x from a fitness distribution, and it produces children according to the sequence of PA weights $(\eta_x f(k))_{k\in\mathbb{N}}$; see [7], [10], and [14]. In this case, individuals with different fitness values are no longer indistinguishable. Assigning the same fitness value to m different individuals would define a process that is not a CTBP in the sense of Definition 4. To overcome this problem, in the case of discrete-valued fitness, we might collapse individuals according to their fitness values and not according to their birth order. This might be applied also to CTBPs with fitness and ageing as introduced in [14]. This is a topic for future work.

2. Overview of the proof of Theorem 4

2.1. General branching process theory

We recall the main results on branching processes that we will use in this paper. CTBPs are models where a population is composed of individuals that produce children according to i.i.d. copies of a birth process $(\xi_t)_{t\geq 0}$. The formal definition of a CTBP is as follows.

Definition 4. (*Branching process.*) We define the set of individuals in the population as

$$\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n.$$

Consider a point process ξ. Then the CTBP is described by

$$(\Omega, \mathcal{A}, \mathbb{P}) = \prod_{x\in\mathcal{N}} (\Omega_x, \mathcal{A}_x, \mathbb{P}_x),$$

where $(\Omega_x, \mathcal{A}_x, \mathbb{P}_x)$ are probability spaces and $(\xi^x)_{x\in\mathcal{N}}$ are i.i.d. copies of ξ. For $x \in \mathbb{N}^n$ and $k \in \mathbb{N}$, we denote the kth child of x by $xk \in \mathbb{N}^{n+1}$. More generally, for $x \in \mathbb{N}^n$ and $y \in \mathbb{N}^m$, we denote the y descendant of x by xy. We call the branching process the triplet $(\Omega, \mathcal{A}, \mathbb{P})$ and the sequence of point processes $(\xi^x)_{x\in\mathcal{N}}$. We denote the branching process by $\boldsymbol{\xi}$.

The behavior of CTBPs is determined by the properties of the birth process. Consider a jump process ξ on $\mathbb{R}^+$, i.e. an integer-valued random measure on $\mathbb{R}^+$. Denote the time of the kth jump of $(\xi_t)_{t\geq 0}$ by T_k. Then we say that ξ is *supercritical* when there exists $\alpha^* > 0$ such that

$$\mathcal{L}(\mathbb{E}\xi(\mathrm{d}\cdot))(\alpha^*) = \int_0^\infty \mathrm{e}^{-\alpha^* t}\mathbb{E}\xi(\mathrm{d}t) = 1. \tag{7}$$

Here $\mathbb{E}\xi(\mathrm{d}x)$ denotes the density of the *averaged* measure $\mathbb{E}[\xi([0, t])]$.

A second fundamental property for the analysis of branching processes is the Malthusian property. Consider a point process ξ. Take the parameter α^* that satisfies (7). Then the process ξ is *Malthusian* with Malthusian parameter α^* if

$$\mu := -\frac{\mathrm{d}}{\mathrm{d}\alpha}(\mathcal{L}(\mathbb{E}\xi(\mathrm{d}\cdot)))(\alpha)\bigg|_{\alpha^*} = \int_0^\infty t\mathrm{e}^{-\alpha^* t}\mathbb{E}\xi(\mathrm{d}t) < \infty.$$

An important class of functions of branching processes are random characteristics.

Definition 5. (*Random characteristic.*) A *random characteristic* is a real-valued process $\Phi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ such that $\Phi(\omega, s) = 0$ for any $s < 0$, and $\Phi(\omega, s) = \Phi(s)$ is a deterministic bounded function for every $s \geq 0$ that depends only on ω through the birth time of the individual as well as the birth process of its children.

Let $\mathcal{L}(f(\cdot))(\alpha)$ denote the Laplace transform of a function f evaluated in $\alpha > 0$. We are now ready to state the main result on CTBPs.

Theorem 3. *Consider a point process ξ and the corresponding branching process $\boldsymbol{\xi}$. Let ξ be supercritical and Malthusian with parameter α^*, and suppose that there exists $\bar{\alpha} < \alpha^*$ such that*

$$\int_0^\infty \mathrm{e}^{-\bar{\alpha}t}\mathbb{E}\xi(\mathrm{d}t) < \infty.$$

Then, the following properties hold:

(i) *there exists a random variable Θ such that, for any random characteristic Ψ as $t \to \infty$,*

$$\mathrm{e}^{-\alpha^* t}\boldsymbol{\xi}_t^{\Psi} \to \frac{1}{\mu}\mathcal{L}(\mathbb{E}[\Psi(\cdot)])(\alpha^*)\Theta, \quad \mathbb{P}\text{-}a.s. \tag{8}$$

(where we abbreviate $\mathbb{P}$-almost surely to $\mathbb{P}$-a.s.);

(ii) *on the event $\{\boldsymbol{\xi}_t^{\mathbf{1}_{\mathbb{R}^+}} \to \infty\}$, $\mathbb{P}(\Theta > 0) = 1$ and $\mathbb{E}[\Theta] = 1$.*

These results, first proved by Nerman [23], are classical, and proved in great generality in the extensive work by Jagers and Nerman; see [19] and [20]. From (8), it follows immediately that for any two random characteristics Φ and Ψ as $t \to \infty$,

$$\frac{\boldsymbol{\xi}_t^{\Phi}}{\boldsymbol{\xi}_t^{\Psi}} \to \frac{\mathcal{L}(\mathbb{E}[\Phi(\cdot)])(\alpha^*)}{\mathcal{L}(\mathbb{E}[\Psi(\cdot)])(\alpha^*)}, \quad \mathbb{P}\text{-a.s.}$$

As a consequence, the ratio between the branching process evaluated with the two characteristics $\mathbf{1}_{\{k\}}$ and $\mathbf{1}_{\mathbb{R}^+}$, which is the fraction of individuals with k children, converges to a deterministic limit. We denote this limit by $(p_k^{(1)})_{k\in\mathbb{N}}$, where

$$p_k^{(1)} = \alpha^*\mathcal{L}(\mathbb{P}(\xi(\cdot) = k))(\alpha^*) = \alpha^*\int_0^\infty \mathrm{e}^{-\alpha^* t}\mathbb{P}(\xi(t) = k)\,\mathrm{d}t = \mathbb{E}[\mathbb{P}(\xi(u) = k)_{u=T_{\alpha^*}}].$$

Here T_{α^*} is an exponential random variable with rate α^* independent of ξ. Then $(p_k^{(1)})_{k\in\mathbb{N}}$ is called the *limiting degree distribution* for the branching process $\boldsymbol{\xi}$. The notation $p_k^{(1)}$ underlines the fact that the CTBP can be seen as a CBP with fixed $m = 1$.

2.2. Structure of the proof of Theorem 4

Our main result requires the next condition.

Condition 1. (Lipschitz.) *Assume that a birth process $(\xi_t)_{t\geq 0}$ is supercritical and Malthusian. The Lipschitz condition is that, for every $k \in \mathbb{N}$, there exists a constant $0 < \ell(k) < \infty$ such that the function $P_k[\xi](t) = \mathbb{P}(\xi_t = k)$ is Lipschitz with constant $\ell(k)$.*

Condition 1 requires that the functions $(P_k[\xi](t))_{k\in\mathbb{N}}$ associated to the birth process $(\xi_t)_{t\geq 0}$ are smooth, in the sense that they do not have dramatic changes over time. We can now state the main result of the paper.

Theorem 4. *Let $(\xi_t)_{t\geq 0}$ be a supercritical and Malthusian birth process that satisfies Condition 1. Let $(\mathrm{CBP}_t^{(m)})_{t\geq 0}$ be the corresponding CBP. Let Θ and μ be as in Theorem 3. Denote the size of $\mathrm{CBP}_t^{(m)}$ by $N^{(m)}(t)$, and the number of vertices with degree k by $N_k^{(m)}(t)$. Then, as $t \to \infty$,*

$$m\mathrm{e}^{-\alpha^* t} N^{(m)}(t) \to \frac{1}{\mu\alpha^*}\Theta, \quad \mathbb{P}\text{-a.s.} \tag{9}$$

Further, for every $k \in \mathbb{N}$, there exists $p_k^{(m)}$ such that

$$m\mathrm{e}^{-\alpha^* t} N_k^{(m)}(t) \xrightarrow{\mathbb{P}} \frac{1}{\mu\alpha^*} p_k^{(m)}\Theta. \tag{10}$$

As a consequence,

$$\frac{N_k^{(m)}(t)}{N^{(m)}(t)} \xrightarrow{\mathbb{P}} p_k^{(m)}. \tag{11}$$

The sequence $(p_k^{(m)})_{k\in\mathbb{N}}$ is called the *limiting degree distribution* of $(\mathrm{CBP}_t^{(m)})_{t\geq 0}$, i.e.

$$p_k^{(m)} = \alpha^* \mathcal{L}(P[\xi](\cdot)_k^{*m})(\alpha^*) = \mathbb{E}[P[\xi](T_{\alpha^*})_k^{*m}],$$

where $P_k[\xi](t) = \mathbb{P}(\xi_t = k)$, T_{α^*} is an exponentially distributed random variable with parameter α^*, and

$$P[\xi](t)_k^{*m} = \sum_{k_1+\cdots+k_m=k} P_{k_1}[\xi](t)\cdots P_{k_m}[\xi](t)$$

is the kth element of the m-fold convolution of the sequence $(P_k[\xi](t))_{k\in\mathbb{N}}$.

We now comment on Theorem 4 (for comparison with CTBPs, see Theorem 3). Equation (9) ensures that the size of a CBP grows at an exponential rate α^* as does the underlying CTBP. Even the size of $\mathrm{CBP}_t^{(m)}$, up to the constant m, scales exactly as the size of the CTBP, and the limiting random variable Θ is the same. This means that the collapsing procedure does not destroy the exponential growth of the graph.

Equation (10) ensures that, for every $k \in \mathbb{N}$, the number of vertices with in-degree k scales exponentially and also in this case we have a limiting random variable. Equation (11) tells us that there exists a *deterministic* limiting degree distribution for a CBP.

The expression for $(p_k^{(m)})_{k\in\mathbb{N}}$ can be explained in terms of CTBPs. In fact, for a CTBP $\boldsymbol{\xi}$, the limiting degree distribution is $p_k^{(1)} = \mathbb{E}[P_k[\xi](T_{\alpha^*})]$ with α^* the Malthusian parameter of $\boldsymbol{\xi}$. We can view T_{α^*} as a *time unit* that a process $(\xi_t)_{t\geq 0}$ takes to generate, on average, one individual. Then $p_k^{(1)}$ can be seen as the probability that $(\xi_t)_{t\geq 0}$ generates k individuals instead of the, on average, one. Using the same heuristic, the limiting degree distribution of a CBP can be seen as the probability that m different individuals produce k children in total in the time unit T_{α^*}. Note that in the expression of $(p_k^{(m)})_{k\in\mathbb{N}}$, the Malthusian parameter α^* is that of the branching process $\boldsymbol{\xi}$.

Unfortunately, the size of a CBP and the number of vertices with degree $k \in \mathbb{N}$ are not the result of a CTBP with a random characteristic as in Definition 5. For example, the degree of a vertex in a CBP is the sum of the degrees of m different individuals. The solution for the size of a CBP and the number of vertices with degree k is different. From Definition 1, it is obvious that

$$N^{(m)}(t) = \left\lceil \frac{\boldsymbol{\xi}_t^{\mathbf{1}_{\mathbb{R}^+}}}{m} \right\rceil. \tag{12}$$

Using (8), the proof of (9) is immediate.

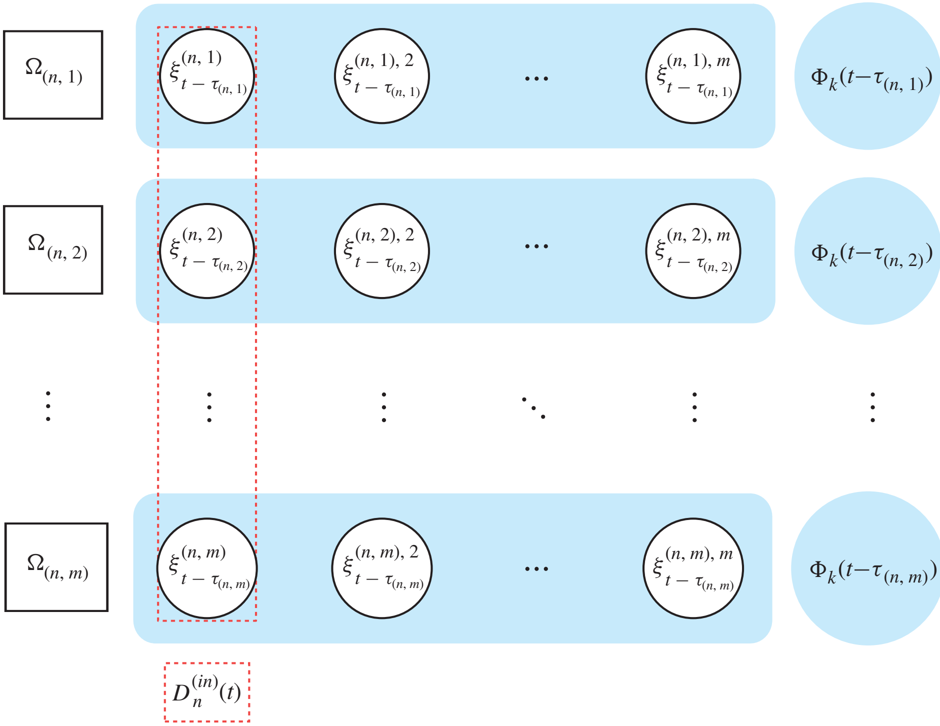

FIGURE 3: Heuristic of the random characteristic $\Phi_k^{(m)}$. For a CTBP, to m individuals $(n, 1), \dots, (n, m)$ we assign different probability spaces $\Omega_{(n,1)}, \dots, \Omega_{(n,m)}$ with the birth processes $\xi^{(n,1)}_{t-\tau_{(n,1)}}, \dots, \xi^{(n,m)}_{t-\tau_{(n,m)}}$, respectively. The degree of vertex n in the CBP by the sum of the processes as indicated by the circles within the dashed rectangle. We artificially add the remaining processes in the shaded area to define the random characteristic $\Phi_k^{(m)}$.

The proof of (10) is not so immediate and requires a conditional second-moment method on $N_k^{(m)}(t)$. Before stating the result, we need a preliminary discussion. We use artificial randomness added to the branching process in order to rewrite the degree of a vertex in a CBP in terms of a random characteristic. In the population space in the definition of CTBPs, we consider a single birth process $(\xi_t^x)_{t\geq 0}$ for every individual x in the population. We instead consider on every Ω_x a vector of birth processes $(\xi_t^{x,1}, \dots, \xi_t^{x,m})$, where $\xi_t^{x,1}, \dots, \xi_t^{x,m}$ are i.i.d. copies of the birth process, defined on the space corresponding to the individual x. With this notation, the standard branching processes defined by $(\xi_t)_{t\geq 0}$ is the branching process where we consider the birth process as the first component of every vector associated to every individual. Figure **??** explains how the additional processes are assigned to individuals.

Now, for $k \in \mathbb{N}$, we consider the random characteristic

$$\Phi_k^{(m)}(t) = \mathbf{1}_{\{k\}}(\xi^{x,1}_{t-\tau_x} + \cdots + \xi^{x,m}_{t-\tau_x}),$$

which corresponds to the event that the sum of the components of the vector associated to the individual x when its age $t - \tau_x$ is equal to k. This is a random characteristic that depends only on the randomness defined on the space Ω_x.

The crucial observation is that

$$\begin{aligned}\mathbb{P}(D_n^{(\mathrm{in})}(t) = k) &= \mathbb{P}\big(\xi_{t-\tau_{(n,1)}}^{(n,1)} + \cdots + \xi_{t-\tau_{(n,m)}}^{(n,m)} = k\big) \\ &\approx \frac{1}{m}\sum_{j=1}^{m} \mathbb{P}\big(\xi_{t-\tau_{(n,j)}}^{(n,j),1} + \cdots + \xi_{t-\tau_{(n,j)}}^{(n,j),m} = k\big) \\ &= \frac{1}{m}\sum_{j=1}^{m} \mathbb{E}[\Phi_k^{(m)}(t-\tau_{(n,j)})] + (\text{error}), \end{aligned} \tag{13}$$

when we assume that the difference between the birth times $\tau_{(n,1)}, \tau_{(n,2)}, \ldots, \tau_{(n,m)}$ is very small. The approximation in (13) can be explained by the fact that all the components of the vectors $(\xi_t^{n,1}, \ldots, \xi_t^{n,m})$ are i.i.d. and $\tau_{(n,1)} \approx \tau_{(n,m)}$. In fact, on the left-hand side of (13) we have the probability that the sum of m independent copies of $(\xi_t)_{t\geq 0}$, evaluated at different times, is equal to k. Assuming that the differences between the birth times $\tau_{(n,1)}, \tau_{(n,2)}, \ldots, \tau_{(n,m)}$ are small, we can just evaluate the m different processes at time $\tau_{(n,1)}$, with a negligible error.

The proof of this, based on Condition 1, can be found in Proposition 2. It provides the bound on the error term with the difference between the birth times of the individuals collapsed to generate the vertex, i.e. the error term is bounded by $m\ell|\tau_{(n,m)} - \tau_{(n,1)}|$, where $\ell = \max_{i\in[k]}\{\ell(i)\}$.

The use of artificial randomness might not seem intuitive. The point is that the equality in expectation between the random characteristic $\Phi_k^{(m)}(t-\tau_{(n,1)})$ and $D_n^{(\mathrm{in})}(t)$ is enough. This relies on the fact that, conditionally on the first stages of the branching process, the contribution to the number of vertices with degree k given by the latter individuals is almost deterministic. We now formalize this idea.

Definition 6. (*x-bulk filtration.*) Consider a branching process $\boldsymbol{\xi}$ and its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider an increasing function $x(t)\colon \mathbb{R}^+ \to \mathbb{R}^+$. We call $(\mathcal{F}_{x(t)})_{t\geq 0}$ the *x-bulk filtration* of $\boldsymbol{\xi}$. At every time $t \geq 0$, a random variable measurable with respect to $\mathcal{F}_{x(t)}$ is called *x-bulk-measurable*.

If we consider $x(t)$ to be $o(t)$ then the x-bulk filtration heuristically contains information only on the early stage of the CTBP. Nevertheless, the information contained in $\mathcal{F}_{x(t)}$ is enough to estimate the behavior of the CTBP.

Proposition 1. (Conditional moments of $N_k^{(m)}(t)$.) *Assume that x is a monotonic function such that, as $t \to \infty$, $x(t) \to \infty$ and $x(t) = o(t)$. Then, under the conditions of Theorem 4, as $t \to \infty$,*

$$m\mathrm{e}^{-\alpha^* t}\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] \to \frac{1}{\mu}\mathcal{L}(\Phi_k^{(m)}(\cdot))(\alpha^*)\Theta, \quad \mathbb{P}\text{-a.s.}, \tag{14}$$

$$\mathrm{e}^{-2\alpha^* t}\mathbb{E}[N_k^{(m)}(t)^2 \mid \mathcal{F}_{x(t)}] \leq (\mathrm{e}^{-\alpha^* t}\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}])^2 + o(1), \quad \mathbb{P}\text{-a.s.} \tag{15}$$

We point out that if $X \leq Y + o(1)$ then $o(1)$ is a term that converges almost surely to 0. The proof of Proposition 1 can be found in Section 4. With Proposition 1 in hand, we can prove (10). We bound $|m\mathrm{e}^{-\alpha^* t}N_k^{(m)}(t) - (1/\mu)\mathcal{L}(\Phi_k^{(m)}(\cdot))(\alpha^*)\Theta|$ by

$$\begin{aligned} &|m\mathrm{e}^{-\alpha^* t}N_k^{(m)}(t) - m\mathrm{e}^{-\alpha^* t}\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}]| \\ &\quad + \left| m\mathrm{e}^{-\alpha^* t}\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] - \frac{1}{\mu}\mathcal{L}(\Phi_k^{(m)}(\cdot))(\alpha^*)\Theta \right|. \end{aligned} \tag{16}$$

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As a consequence, (10) holds if both terms in (16) converge $\mathbb{P}$-a.s. to 0, denoted by $o_{\text{a.s.}}(1)$. For the second term, this holds by (14). For the first term, we use (14) and (15) to conclude that $\text{var}(m\text{e}^{-\alpha^* t} N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}) = o_{\text{a.s.}}(1)$, so that

$$|m\text{e}^{-\alpha^* t} N_k^{(m)}(t) - m\text{e}^{-\alpha^* t} \mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}]| \xrightarrow{\mathbb{P}} 0.$$

This concludes the proof of (10). Equation (11) follows immediately.

Remark 1. (*Times and bulk sigma-field.*) Proposition 1 was proved (and, thus, Theorem 4) by looking at the CTBP at time t, and considering the $x(t)$-bulk sigma-field. We can extend the argument as follows. Consider $s \geq 0$, and let $y\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a monotonic function of s such that $y(s)/s \to \infty$ as $s \to \infty$. In this case, looking at the graph at time $y(s)$ and considering the s-bulk sigma-field, Proposition 1 still holds. More generally, as we see from (27) below, conditionally on the s-bulk sigma-field, the evolution of a CTBP is almost deterministic. This implies that Proposition 1 holds even when we consider a *random* process $Y(s)$ such that $Y(s)/s \xrightarrow{\text{a.s.}} \infty$, under the assumption that $Y(s)$ is s-bulk-measurable for every $s \geq 0$. These observations will be useful when extending our results to the discrete-time setting in Section 6.

3. Preliminaries on birth times

3.1. Bound on the difference in time

In this section we prove the fact that the error term in (13) can be bounded by the difference in the birth times of the considered individuals. We now introduce the definition of convolution, as well as the bound we are interested in.

Definition 7. (*Convolution.*) We define the convolution between two sequences $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ as

$$(a * b)_k := \sum_{l=0}^{k} a_l b_{k-l}.$$

Lemma 1. (Difference in times.) *Consider the sequence of functions $(P[\xi]_k(t))_{k\in\mathbb{N}}$. If $(\xi_t)_{t\geq 0}$ satisfies Condition 1 then, for every $x \in \mathbb{R}^+$ and for every $h_i \leq x$ for $i \in [m]$,*

$$|(P[\xi](x-h_1) * \cdots * P[\xi](x-h_m))_k - (P[\xi](x-h_1)^{*m})_k| \leq \ell \sum_{j=2}^{m} |h_1 - h_j|, \tag{17}$$

where $\ell = \max_{i\in[k]} \ell(i)$.

Proof. Without loss of generality, assume that $0 \leq h_1 \leq \cdots \leq h_n$. We prove Lemma 1 by induction on m. We start the induction with $m = 2$, so

$$(P[\xi](x-h_1) * P[\xi](t-h_2))_k = \sum_{l=0}^{k} P[\xi]_l(x-h_1) P[\xi]_{k-l}(x-h_2). \tag{18}$$

We now use Condition 1 to bound $|P[\xi]_{k-l}(x-h_2) - P[\xi]_{k-l}(x-h_1)| \leq \ell(k-l)(h_2-h_1)$. Using this in (18), we then obtain, for $\ell = \max_{i\in[k]} \ell(i)$,

$$\begin{aligned} &|(P[\xi](x-h_1) * P[\xi](t-h_2))_k - (P[\xi](x-h_1)^{*2})_k| \\ &\quad \leq \ell \sum_{l=0}^{k} P[\xi]_{k-l}(x-h_1)|(h_2-h_1)|. \end{aligned} \tag{19}$$

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Since $\sum_{l=0}^{k} P_l[\xi](x-h_1) = P[\xi]_{\le k}(x-h_1) \le 1$,

$$|(P[\xi](x-h_1) * P[\xi](t-h_2))_k - (P[\xi](x-h_1)^{*2})_k| \le \ell |h_2 - h_1|,$$

so (17) holds for $m=2$. We now advance the induction hypothesis, so suppose that (17) holds for $m-1$. We can write

$$\begin{aligned}(P[\xi](x-h_1) * \cdots * P[\xi](x-h_m))_k \\ = \sum_{l=0}^{k} (P[\xi](x-h_1) * \cdots * P[\xi](x-h_{m-1}))_l P[\xi]_{k-l}(x-h_m).\end{aligned} \tag{20}$$

Note that we can apply (17) to the first terms in the sum in (20) thanks to the induction hypothesis, since it is now the convolution of $m-1$ functions. We just need to replace $P[\xi]_{k-l}(x-h_m)$ by $P[\xi]_{k-l}(x-h_1)$. This is straightforward by using a similar argument used to prove the bound in (19), which implies again the use of Condition 1. To conclude, we have

$$|(P[\xi](x-h_1) * \cdots * P[\xi](x-h_m))_k - (P[\xi](x-h_1)^{*m})_k| \le \ell \sum_{j=2}^{m-1} |h_1 - h_j| + \ell |h_m - h_1|,$$

where the $m-1$ terms come from the induction hypothesis, and the last one from the approximation of $P[\xi]_{k-l}(x-h_m)$. This completes the proof. □

Lemma 1 holds for every time x and $h_1, \dots, h_m$ that we consider. We can now prove the bound on the error term in (13).

Proposition 2. (Approximation at fixed time.) *Consider* $(\mathrm{CBP}_t^{(m)})_{t\ge 0}$ *obtained from a branching process* $\boldsymbol{\xi}$. *Assume that* $(\xi_t)_{t\ge 0}$ *satisfies Condition 1. Then, for every* $k \in \mathbb{N}$, *with* ℓ *as in Lemma 1,* $\mathbb{P}$*-a.s. for every* $n \in \mathbb{N}$,

$$|\mathbb{P}(D_n^{(\mathrm{in})}(t) = k \mid \tau_{(n,1)}, \dots, \tau_{(n,m)}) - (P[\xi](t-\tau_{(n,1)})^{*m})_k| \le \ell m |\tau_{(n,m)} - \tau_{(n,1)}|. \tag{21}$$

Proof. Conditionally on the birth times, the processes $(\xi_t^{(n,1)})_{t\ge 0}, \dots, (\xi_t^{(n,m)})_{t\ge 0}$ are independent. As a consequence,

$$\mathbb{P}(D_n^{(\mathrm{in})}(t) = k \mid \tau_{(n,1)}, \dots, \tau_{(n,m)}) = (P[\xi](t-\tau_{(n,1)}) * \cdots * P[\xi](t-\tau_{(n,m)}))_k.$$

Then (21) follows immediately from Lemma 1, where we consider $h_1 = \tau_{(n,1)}, \dots, h_m = \tau_{(n,m)}$, and the fact that $\tau_{(n,j)} - \tau_{(n,1)} \le \tau_{(n,m)} - \tau_{(n,1)}$ for every $j = 1, \dots, m$. □

3.2. Replacing birth times with $\mathcal{F}_t$-measurable approximations

Recall that $\mathcal{F}_t$ denotes the natural filtration of the CTBP up to time t. We can write (8) as

$$n e^{-\alpha^* \tau_n} \to \frac{1}{\mu \alpha^*} \Theta, \qquad \mathbb{P}\text{-a.s.}$$

As a consequence, as $n \to \infty$,

$$-\tau_n + \frac{1}{\alpha^*} \log n \to \frac{1}{\alpha^*} \log\left(\frac{1}{\mu \alpha^*} \Theta\right), \qquad \mathbb{P}\text{-a.s.} \tag{22}$$

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Note that on the event $\{\xi_t^{\mathbf{1}_{\mathbb{R}^+}} \to \infty\}$, Θ is positive with probability 1, so $\log((1/\mu\alpha^*)\Theta)$ is well defined. Define, for $n \geq \xi_t^{\mathbf{1}_{\mathbb{R}^+}}$,

$$\sigma_n(t) := \frac{1}{\alpha^*}\log n - \frac{1}{\alpha^*}\log\biggl(\frac{1}{\mu\alpha^*}\Theta_t\biggr), \tag{23}$$

where $\Theta_t = \mu\alpha^* \mathrm{e}^{-\alpha^* t}\xi_t^{\mathbf{1}_{\mathbb{R}^+}}$. Then $\sigma_n(t)$ is an approximation of τ_n, given the information up to time t, where the factor Θ_t includes the stochastic fluctuation of the size of the branching process. What is interesting is that the random variable $\sigma_n(t)$ is an approximation of τ_n *measurable with respect to* $\mathcal{F}_t$. We now prove that $(\sigma_n(t))_{t\geq 0}$ is an acceptable approximation of τ_n.

Lemma 2. (Error of $(\sigma_n(t))_{t\geq 0}$.) *It holds that,* $\mathbb{P}$*-a.s. as* $t \to \infty$*,*

$$\sup_{n\geq \xi_t^{\mathbf{1}_{\mathbb{R}^+}}} |\sigma_n(t) - \tau_n| \to 0. \tag{24}$$

Proof. For every $t \geq 0$ and $n \geq \xi_t^{\mathbf{1}_{\mathbb{R}^+}}$, we have

$$\begin{aligned} |\sigma_n(t) - \tau_n| \leq {} & \biggl|\frac{1}{\alpha^*}\log n - \tau_n - \frac{1}{\alpha^*}\log\biggl(\frac{1}{\mu\alpha^*}\Theta\biggr)\biggr| \\ & + \frac{1}{\alpha^*}\biggl|\log\biggl(\frac{1}{\mu\alpha^*}\Theta\biggr) - \log\biggl(\frac{1}{\mu\alpha^*}\Theta_t\biggr)\biggr|. \end{aligned} \tag{25}$$

Using (25) in (24), we can bound

$$\begin{aligned} \sup_{n\geq \xi_t^{\mathbf{1}_{\mathbb{R}^+}}} |\sigma_n(t) - \tau_n| \leq {} & \frac{1}{\alpha^*}\biggl|\log\biggl(\frac{1}{\mu\alpha^*}\Theta\biggr) - \log\biggl(\frac{1}{\mu\alpha^*}\Theta_t\biggr)\biggr| \\ & + \sup_{n\geq \xi_t^{\mathbf{1}_{\mathbb{R}^+}}} \biggl|\frac{1}{\alpha^*}\log n - \tau_n - \frac{1}{\alpha^*}\log\biggl(\frac{1}{\mu\alpha^*}\Theta\biggr)\biggr|. \end{aligned} \tag{26}$$

First, from (8) we know that $\Theta_t/\mu\alpha^* = \mathrm{e}^{-\alpha^* t}\xi_t^{\mathbf{1}_{\mathbb{R}^+}} \to \Theta/\mu\alpha^*$. As a consequence, the first term on the right-hand side of (26) converges $\mathbb{P}$-a.s. to 0. For the second term, we use (22) and the fact that the supremum decreases as $\xi_t^{\mathbf{1}_{\mathbb{R}^+}} \to \infty$. This completes the proof. □

From Lemma 2, conditionally on $\mathcal{F}_t$, we can replace the birth sequence $(\tau_n)_{n\geq\xi_t^{\mathbf{1}}}$ with the sequence $(\sigma_n(t))_{n\geq\xi_t^{\mathbf{1}}}$ when evaluating random characteristics.

4. Second-moment method: proof of Proposition 1

4.1. First conditional moment asymptotics

In this section we investigate the first conditional moment of $N_k^{(m)}(t)$ with respect to the bulk filtration. In particular, we consider a function x such that, as $t \to \infty$, $x(t) \to \infty$ and $x(t) = o(t)$. Heuristically, we want to show that

$$m\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] \approx N^{(m)}(x(t))\mathbb{E}\bigl[\xi_{t-x(t)}^{\Phi_k^{(m)}}\bigr]. \tag{27}$$

From (27), conditionally on the information up to time $x(t)$, at time t we have $N^{(m)}(x(t))$ processes, each one producing the expected number of vertices with degree k at time $t - x(t)$. This follows from the fact that all the individual processes in $\boldsymbol{\xi}$ are independent from each other once we condition on the birth times.

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We start by writing $N_k^{(m)}(t)$ as the sum of indicator functions, i.e.

$$\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] = \mathbb{E}\bigg[\sum_{n=1}^{N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\mathrm{in})}(t)=k\}} + \sum_{n=N^{(m)}(x(t))+1}^{\infty} \mathbf{1}_{\{D_n^{(\mathrm{in})}(t)=k\}} \;\bigg|\; \mathcal{F}_{x(t)}\bigg].$$

We can ignore the first sum in the conditional expectation, since

$$0 \le \mathrm{e}^{-\alpha^* t}\mathbb{E}\bigg[\sum_{n=1}^{N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\mathrm{in})}(t)=k\}} \;\bigg|\; \mathcal{F}_{x(t)}\bigg] \le \mathrm{e}^{-\alpha^* t} N^{(m)}(x(t)), \tag{28}$$

and using Theorem 3 and the fact that $x(t) = o(t)$,

$$\mathrm{e}^{-\alpha^*(t-x(t))}\mathrm{e}^{-\alpha^* x(t)} N^{(m)}(x(t)) \to 0, \quad \mathbb{P}\text{-a.s.} \tag{29}$$

We now consider the sequence $(\sigma_n(x(t)))_{n\in\mathbb{N}}^{t\ge 0}$ as defined in Section 3.2. This is a sequence of random variables that approximates $(\tau_n)_{n\in\mathbb{N}}$ and measurable with respect to the bulk filtration. This means that we can write, for any $n \ge N^{(m)}(x(t))$,

$$D_n^{(\mathrm{in})}(t) = \xi^{(n,1)}(t - \sigma_{(n,1)}(x(t))) + \cdots + \xi^{(n,m)}(t - \sigma_{(n,m)}(x(t))) + o_{\mathrm{a.s.}}(1).$$

Now, conditionally on the birth times $\sigma_{(n,1)}(x(t)), \dots, \sigma_{(n,m)}(x(t))$, the m processes related to the nth vertex $(\xi_t^{(n,1)})_{t\ge 0}, \dots, (\xi_t^{(n,m)})_{t\ge 0}$ are independent, so the probability that the sum is equal to k is

$$(P[\xi](t - \sigma_{(n,1)}(x(t))) * \cdots * P[\xi](t - \sigma_{(n,m)}(x(t))))_k,$$

which is an x-bulk-measurable random variable. As a consequence,

$$\begin{aligned}\mathbb{E}\bigg[&\sum_{n=N^{(m)}(x(t))+1}^{\infty} \mathbf{1}_{\{D_n^{(\mathrm{in})}(t)=k\}} \;\bigg|\; \mathcal{F}_{x(t)}\bigg] \\ &= \sum_{n=N^{(m)}(x(t))+1}^{\infty} (P[\xi](t - \sigma_{(n,1)}(x(t))) * \cdots * P[\xi](t - \sigma_{(n,m)}(x(t))))_k.\end{aligned} \tag{30}$$

For any $k \in \mathbb{N}$, the function $u \mapsto P_k[\xi](u)$ is 0 for the negative argument. As a consequence, the sum in (30) is taken only over indeces n such that $\sigma_{(n,j)}(x(t)) < t$. From the definition of $\sigma_{(n,j)}(x(t))$ as in (23) and the fact that $(n, j) = m(n-1) + j$, it follows that $\sigma_{(n,j)}(x(t)) < t$ if and only if

$$n < 1 - \frac{j}{m} + \frac{\mathrm{e}^{\alpha^*(t-x(t))}\boldsymbol{\xi}_{x(t)}^{\mathbf{1}_{\mathbb{R}^+}}}{m} = \mathrm{e}^{\alpha^*(t-x(t))} N^{(m)}(x(t))(1 + o_{\mathrm{a.s.}}(1)), \tag{31}$$

where, again, $o_{\mathrm{a.s.}}(1)$ denotes a term that converges $\mathbb{P}$-a.s. to 0. Using (31) and then applying Proposition 2 for ℓ as in Lemma 1, we obtain

$$\sum_{n=N^{(m)}(x(t))+1}^{N^{(m)}(x(t))\mathrm{e}^{\alpha^*(t-x(t))}} P[\xi](t - \sigma_{(n,1)}(x(t)))_k^{*m} + \ell m \sum_{n=N^{(m)}(x(t))+1}^{N^{(m)}(x(t))\mathrm{e}^{\alpha^*(t-x(t))}} \sigma_{(n,m)}(x(t)) - \sigma_{(n,1)}(x(t)), \tag{32}$$

where the difference between (30) and the first sum in (32) is bounded in absolute value by the second sum in (32).

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Consider the difference $t-\sigma_{(n,1)}(x(t))$. Using the definition of the sequence $(\sigma_n(x(t)))_{n\in\mathbb{N}}$, and recalling that $mN^{(m)}(x(t)) = \boldsymbol{\xi}_{x(t)}^{\mathbf{1}_{\mathbb{R}^+}}(1+o_{\text{a.s.}}(1))$ (see (12)), it follows that $t-\sigma_{(N^{(m)}(x(t)),1)}(x(t)) = (t-x(t))(1+o_{\text{a.s.}}(1))$. As a consequence,

$$\begin{aligned}
t-\sigma_{(n,1)}(x(t)) &= t-\sigma_{(N^{(m)}(x(t)),1)}(x(t)) - (\sigma_{(n,1)}(x(t)) - \sigma_{(N^{(m)}(x(t)),1)}(x(t))) \\
&= t - x(t) + \frac{1}{\alpha^*}\log\biggl(\frac{m(n-1)+1}{mN^{(m)}(x(t))}\biggr) + o_{\text{a.s.}}(1) \\
&= t - x(t) + \frac{1}{\alpha^*}\log\biggl(\frac{n}{N^{(m)}(x(t))}\biggr) + o_{\text{a.s.}}(1). \qquad (33)
\end{aligned}$$

The second sum on the right-hand side of (32) is bounded by a telescopic sum, since $\sigma_{(n,1)}(x(t)) \geq \sigma_{(n-1,m)}(x(t))$, which implies that we can bound it with the difference between the last and the first term. Using (33) in (32), for $s = t - x(t)$, leads to

$$\begin{aligned}
&\sum_{n=N^{(m)}(x(t))+1}^{N^{(m)}(x(t))\mathrm{e}^{\alpha^* s}} P[\xi]\biggl(s - \frac{1}{\alpha^*}\log\biggl(\frac{m(n-1)+1}{mN^{(m)}(x(t))}\biggr)\biggr)_k^{*m} + \frac{m\ell}{\alpha^*}\log\biggl(\frac{mN^{(m)}(x(t))\mathrm{e}^{\alpha^* s}}{mN^{(m)}(x(t))}\biggr) \\
&\quad = \sum_{p=1}^{\mathrm{e}^{\alpha^* s}} \sum_{q=1}^{N^{(m)}(x(t))} P[\xi]\biggl(s - \frac{1}{\alpha^*}\log\biggl(p + \frac{q}{N^{(m)}(x(t))}\biggr)\biggr)_k^{*m} + m\ell(t - x(t)) \\
&\quad = N^{(m)}(x(t)) \sum_{p=1}^{\mathrm{e}^{\alpha^* s}} P[\xi]\biggl(s - \frac{1}{\alpha^*}\log(p)\biggr)_k^{*m} + m\ell(t-x(t)) \\
&\quad = N^{(m)}(x(t)) \sum_{p=1}^{\mathrm{e}^{\alpha^* s}} \mathbb{E}\biggl[\Phi_k^{(m)}\biggl(s - \frac{1}{\alpha^*}\log(p)\biggr)\biggr] + m\ell(t-x(t)). \qquad (34)
\end{aligned}$$

The contribution of the $m\ell(t-x(t))$ term is negligible since $\mathrm{e}^{-\alpha^* t} m\ell(t-x(t)) = o(1)$. To analyze the remaining sum, we introduce two measures γ_1 and γ_2 on $\mathbb{R}^+$. For $v \geq 0$,

$$\begin{aligned}
\gamma_1([0,v]) &= \int_0^v \sum_{p\in\mathbb{N}} \delta_{\{1/\alpha^* \log p\}}(\mathrm{d}u) = \mathrm{e}^{\alpha^* v}, \\
\gamma_2([0,v]) &= \mathbb{E}\biggl[\int_0^v \sum_{n\in\mathbb{N}} \delta_{\{\tau_n\}}(\mathrm{d}u)\biggr] = \mathbb{E}[\boldsymbol{\xi}_v^{\mathbf{1}_{\mathbb{R}^+}}].
\end{aligned}$$

Note that γ_2 is the average measure of the random measure given by the branching process size. From Theorem 3, we know that $\gamma_2([0,v]) = \mathbb{E}[\boldsymbol{\xi}_v^{\mathbf{1}_{\mathbb{R}^+}}] = (1/\mu\alpha^*)\mathrm{e}^{\alpha^* v}(1+o(1))$. This means that, asymptotically in v, $\gamma_1([0,v]) = \mu\alpha^*\gamma_2([0,v])(1+o(1))$. Using these two measures, we can write

$$\begin{aligned}
\sum_{p=1}^{\mathrm{e}^{\alpha^* s}} \mathbb{E}\biggl[\Phi_k^{(m)}\biggl(s - \frac{1}{\alpha^*}\log(p)\biggr)\biggr] &= \int_0^s \mathbb{E}[\Phi_k^{(m)}(s-u)]\gamma_1(\mathrm{d}u) \\
&= \mu\alpha^* \int_0^s \mathbb{E}[\Phi_k^{(m)}(s-u)]\gamma_2(\mathrm{d}u) + o(1) \\
&= \mu\alpha^* \mathbb{E}[\boldsymbol{\xi}_s^{\Phi_k^{(m)}}] + o(1). \qquad (35)
\end{aligned}$$

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Using (35) in (34), we conclude that

$$\begin{aligned}\mathrm{e}^{-\alpha^* t}\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] &= \mathrm{e}^{-\alpha^* t}\mu\alpha^* N^{(m)}(x(t))\mathbb{E}\big[\boldsymbol{\xi}_{t-x(t)}^{\Phi_k^{(m)}}\big] + o_{\text{a.s.}}(1) \\ &= (\mu\alpha^*\mathrm{e}^{-\alpha^* x(t)}N^{(m)}(x(t)))\big(\mathrm{e}^{-\alpha^*(t-x(t))}\mathbb{E}\big[\boldsymbol{\xi}_{t-x(t)}^{\Phi_k^{(m)}}\big]\big) + o_{\text{a.s.}}(1).\end{aligned}$$

Applying (8), it follows that as $t \to \infty$, $\mu\alpha^*\mathrm{e}^{-\alpha^* x(t)}N(x(t))$ converges $\mathbb{P}$-a.s. to Θ, while $\mu\alpha^*\mathrm{e}^{-\alpha^*(t-x(t))}\mathbb{E}[\boldsymbol{\xi}_{t-x(t)}^{\Phi_k^{(m)}}]$ converges to $\mathcal{L}(\Phi_k^{(m)}(\cdot))(\alpha^*)/\mu$. This completes the proof of (14).

4.2. Conditional second-moment asymptotics

In this section we prove (15), i.e. the result on the conditional second moment of $N_k^{(m)}(t)$. We again write $N_k^{(m)}(t)$ as the sum of indicator functions, which means

$$\mathrm{e}^{-2\alpha^* t}\mathbb{E}[N_k^{(m)}(t)^2 \mid \mathcal{F}_{x(t)}] = \mathrm{e}^{-2\alpha^* t}\mathbb{E}\bigg[\sum_{n,n'\in\mathbb{N}} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}}\mathbf{1}_{\{D_{n'}^{(\text{in})}(t)=k\}} \,\bigg|\, \mathcal{F}_{x(t)}\bigg].$$

We now divide the sum into different sums, according to the indices n and n', as

$$\begin{aligned}&\sum_{n,n'\le N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}}\mathbf{1}_{\{D_{n'}^{(\text{in})}(t)=k\}} + \sum_{n,n'> N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}}\mathbf{1}_{\{D_{n'}^{(\text{in})}(t)=k\}} \\ &\quad + 2\sum_{n\le N^{(m)}(x(t)),\, n'>N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}}\mathbf{1}_{\{D_{n'}^{(\text{in})}(t)=k\}}. \end{aligned} \tag{36}$$

For the first sum in (36), we use (28) as a bound, and by (29) it is $o_{\text{a.s.}}(1)$. For the second sum in (36), we again use the sequence $(\sigma_n(x(t)))_{n\in\mathbb{N}}$ to approximate the birth times. Using similar arguments as in Section 4.1, and the fact that conditionally on the birth times all the birth processes are independent, we write, for $n \neq n'$ and $n, n' > N^{(m)}(x(t))$,

$$\begin{aligned}&\mathbb{P}(D_n^{(\text{in})}(t) = k, D_{n'}^{(\text{in})}(t) = k \mid \mathcal{F}_{x(t)}) \\ &\quad = [P[\xi](t - \sigma_{(n,1)}(x(t))) * \cdots * P[\xi](t - \sigma_{(n,m)}(x(t)))]_k \\ &\qquad \times [P[\xi](t - \sigma_{(n',1)}(x(t))) * \cdots * P[\xi](t - \sigma_{(n',m)}(x(t)))]_k. \end{aligned} \tag{37}$$

We can use (37) to bound the conditional expectation of the second sum in (36). In fact, adding the missing terms, we can write

$$\begin{aligned}&\mathbb{E}\bigg[\sum_{n,n'>N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}}\mathbf{1}_{\{D_{n'}^{(\text{in})}(t)=k\}} \,\bigg|\, \mathcal{F}_{x(t)}\bigg] \\ &\quad \le \bigg(\sum_{n>N^{(m)}(x(t))} (P[\xi](t-\sigma_{(n,1)}(x(t))) * \cdots * P[\xi](t-\sigma_{(n,m)}(x(t))))_k\bigg)^2 \\ &\qquad + \mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] \\ &\quad = \mathbb{E}\bigg[\sum_{n>N^{(m)}(x(t))} \mathbf{1}_{\{D_n^{(\text{in})}(t)=k\}} \,\bigg|\, \mathcal{F}_{x(t)}\bigg]^2 + \mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}] \\ &\quad \le \mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}]^2 + \mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}].\end{aligned}$$

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The third sum in (36) can be bounded easily by $2N^{(m)}(x(t))\mathbb{E}[N_k^{(m)}(t) \mid \mathcal{F}_{x(t)}]$. Combining the three bounds, we see that $e^{-2\alpha^* t}\mathbb{E}[N_k^{(m)}(t)^2 \mid \mathcal{F}_{x(t)}]$ is bounded by

$$e^{-2\alpha^* t}\mathbb{E}[N_k(t) \mid \mathcal{F}_{x(t)}]^2 + e^{-2\alpha^* t}(2N(x(t)) + 1)\mathbb{E}[N_k(t) \mid \mathcal{F}_{x(t)}] + o_{\text{a.s.}}(1). \tag{38}$$

The result follows since the second term in (38) is again $o_{\text{a.s.}}(1)$, similarly to the first term in (36).

5. Proofs of Corollaries 1–3

Proof of Corollaries 1 and 2. In Section 1.2 we showed that CBPs defined by birth processes as in Definition 2 embed the PAM in continuous time and what we called the RRG. We just need to show that Condition 1 is satisfied. In general, processes defined as in Definition 2 are differentiable and satisfy a recursive property (see [3, Section 3.2]), i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}P_k[\xi](t) = \begin{cases} -\lambda_0 P_0[\xi](t), & k = 0, \\ -\lambda_k P_k[\xi](t) + \lambda_{k-1}P_{k-1}[\xi](t), & k \geq 1. \end{cases} \tag{39}$$

Since, in general, we consider a nondecreasing sequence $(\lambda_k)_{k\in\mathbb{N}}$, it is possible to satisfy Condition 1 if we set $\ell(k) = \lambda_k$. Hence, the limiting degree distribution $(p_k^{(m)})_{k\in\mathbb{N}}$ is the distribution of the sum of m independent copies of $(\xi_t)_{t\geq 0}$ at exponential time T_{α^*} for an α^* Malthusian parameter of the CTBP.

In the case of PAM embedding, the sum of m birth processes is distributed as an embedding birth process defined by the PA rule $\bar{\lambda}_k = k + m + \delta$ (it is straightforward to prove this by induction over the distribution of birth times). This implies that we can use known results on this type of birth process (see [2] and [24]) to write

$$p_k^{(m)} = \mathbb{P}(\xi^1_{T_{\alpha^*}} + \cdots + \xi^m_{T_{\alpha^*}} = k) = \frac{\alpha^*}{\alpha^* + k + m + \delta}\prod_{i=0}^{k-1}\frac{i + m + \delta}{\alpha^* + i + m + \delta},$$

which can be written as in (2) using Γ functions, since in this case $\alpha^* = 1 + \delta/m$; see [24, Section 4.2] and [14, Proposition 3.15].

For the RRG, calculations are easier. It is easy to show that in this case $\alpha^* = 1$. Since the sum of m Poisson processes (PPs) with parameter 1 is a PP with parameter m, the limiting degree distribution is the distribution of a PP at an exponentially distributed time with parameter 1. Then

$$p_k^{(m)} = \mathbb{E}\bigg[e^{-mT_1}\frac{(mT_1)^k}{k!}\bigg] = \frac{1}{m+1}\bigg(1 + \frac{1}{m}\bigg)^{-k}.$$

As mentioned, for $m = 1$ (so without collapsing) the RRG reduces to the RRT, and the limiting distribution is just $p_k^{(1)} = 2^{-(k+1)}$; see [21]. □

Proof of Corollary 3. (The ageing case.) The result follows immediately from the proof of Corollary 1 and the definition of the ageing process. In fact, an ageing process is defined as $(\xi_{G(t)})_{t\geq 0}$, where $(\xi_t)_{t\geq 0}$ is an embedding process defined by the sequence $(k + 1 + \delta/m)_{k\in\mathbb{N}}$. As a simple consequence of the chain rule, from (39) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}P_k[\xi](G(t)) = \bigg(-\bigg(k + 1 + \frac{\delta}{m}\bigg)P_k[\xi](t) + \bigg(k + \frac{\delta}{m}\bigg)P_{k-1}[\xi](t)\bigg)g(t).$$

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Assuming that the ageing function g is bounded almost everywhere, Condition 1 is satisfied for $\ell = k \sup_{t\geq 0} |g(t)|$. The condition $\lim_{t\to\infty} \mathbb{E}[\xi_{G(t)}] > 1$ is necessary and sufficient for the existence of the Malthusian parameter α^*; see [14, Lemma 4.1].

Since the sum of m processes $\xi_t^1 + \cdots + \xi_t^m$ is distributed as a single embedding process defined by the sequence $(k+m+\delta)_{k\in\mathbb{N}}$, it follows that $\xi^1_{G(t)} + \cdots + \xi^m_{G(t)}$ is distributed as a single ageing process with the same ageing function g and sequence $(k+m+\delta)_{k\in\mathbb{N}}$. Equation (5) is then a consequence of [14, Proposition 5.2]. □

6. Discrete-time processes.

Proof of Theorem 2. The convergence result given in Theorem 1 ensures that in continuous time the proportion of vertices in a CBP with degree k converges in probability to $p_k^{(m)}$. When considering a CTBP in the presence of ageing, this result is enough since these types of CBP are defined only in continuous time.

When we instead consider embedding processes as in Definition 2, we can consider a discrete-time sequence of random graphs $(\mathrm{CBP}^{(m)}_{\tau_n})_{n\in\mathbb{N}}$, where $(\tau_n)_{n\in\mathbb{N}}$ is the sequence of birth times of the corresponding CTBP. This is the way the PAM is usually defined. In particular, the sequence $(\tau_n)_{n\in\mathbb{N}}$ corresponds to the sequence of times at which a new edge appears in the CBP. In this setting, the convergence in probability given in Theorem 1 does not imply the convergence in probability of $(m\mathrm{e}^{-\alpha^*\tau_n} N_k^{(m)}(\tau_n))_{n\in\mathbb{N}}$. Here, we will prove that $\mathrm{e}^{-\alpha^*\tau_n} N_k^{(m)}(\tau_n)$ converges in probability to $p_k^{(m)}\Theta/\mu\alpha^*$, and that this further implies that $N_k^{(m)}(\tau_{mn})/n$ converges in probability to $p_k^{(m)}$, as required.

Recall the t-bulk sigma-field. We denote, as in (23), for $n \geq \boldsymbol{\xi}_t^{\mathbf{1}}$,

$$\sigma_n = \sigma_n(t) = \frac{1}{\alpha^*}\log n - \frac{1}{\mu\alpha^*}\Theta_t.$$

Take $t = t_n = (\log n)^{1/2}$. Then define the sequence $(\tau_n')_{n\in\mathbb{N}}$, where $\tau_n' := \sigma_n(t_n)$. Note that τ_n' is t_n-bulk-measurable. Further, $\tau_n' \xrightarrow{\text{a.s.}} \infty$ and

$$\frac{t_n}{\tau_n'} = \frac{(\log n)^{1/2}}{(1/\alpha^*)\log n - (1/\mu\alpha^*)\log\Theta_{t_n}} = \frac{(\log n)^{1/2}}{\log n(1/\alpha^* - \log\Theta_{t_n}/\mu\alpha^*\log n)} \xrightarrow{\text{a.s.}} 0.$$

By Remark 1, Proposition 1 holds for $m\mathrm{e}^{-\alpha^*\tau_n'}N_k^{(m)}(\tau_n')$, so that

$$m\mathrm{e}^{-\alpha^*\tau_n'}N_k^{(m)}(\tau_n') \xrightarrow{\mathbb{P}} \frac{p_k^{(m)}\Theta}{\mu\alpha^*}.$$

The advantage of the sequence $(\tau_n')_{n\in\mathbb{N}}$, other than being t_n-bulk-measurable, is that it is a good approximation of the sequence $(\tau_n)_{n\in\mathbb{N}}$. Indeed,

$$|\tau_n - \tau_n'| \leq \left|\tau_n - \frac{1}{\alpha^*}\log n - \frac{1}{\mu\alpha^*}\log\Theta\right| + \left|\frac{1}{\mu\alpha^*}\log\Theta - \frac{1}{\mu\alpha^*}\log\Theta_{t_n}\right|,$$

so that $|\tau_n - \tau_n'| \xrightarrow{\text{a.s.}} 0$. As a consequence, we also have $m\mathrm{e}^{-\alpha^*\tau_n}N_k^{(m)}(\tau_n) \xrightarrow{\mathbb{P}} p_k^{(m)}\Theta/\mu\alpha^*$.

Further, by Theorem 4, we have $m\mathrm{e}^{-\alpha^* t}N^{(m)}(t) \xrightarrow{\text{a.s.}} \Theta/\mu\alpha^*$, so this holds also for $m\mathrm{e}^{-\alpha^*\tau_n}N^{(m)}(\tau_n)$. As a consequence,

$$\frac{m\mathrm{e}^{-\alpha^*\tau_n}N_k^{(m)}(\tau_n)}{m\mathrm{e}^{-\alpha^*\tau_n}N^{(m)}(\tau_n)} = \frac{N_k^{(m)}(\tau_n)}{N^{(m)}(\tau_n)} = \frac{m}{n}N_k^{(m)}(\tau_n) \xrightarrow{\mathbb{P}} p_k^{(m)}.$$

Consequently, $N_k^{(m)}(\tau_{mn})/n \xrightarrow{\mathbb{P}} p_k^{(m)}$. This completes the proof of Theorem 2. □

https://doi.org/10.1017/jpr.2018.57 Published online by Cambridge University Press

Acknowledgements

This work was supported in part by the Netherlands Organisation for Scientific Research (NWO) through the Gravitation NETWORKS grant 024.002.003. The work of RvdH was further supported by the NWO through VICI grant 639.033.806.

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