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Quasi-parallel propagation of solitary waves in magnetised non-relativistic electron–positron plasmas

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We study the propagation of nonlinear waves in non-relativistic electron–positron plasmas. The waves are assumed to propagate at small angles with respect to the equilibrium magnetic field. We derive the equation describing the wave propagation under the assumption that the waves are weakly dispersive and also can weakly depend on spatial variables orthogonal to the equilibrium magnetic field. We obtain solutions of the derived equation describing solitons. Then we study the stability of solitons with respect to transverse perturbations.

Key words: plasma waves, plasma nonlinear phenomena, plasma instabilities

1. Introduction

The problem of wave propagation in electron–positron plasmas attracted the attention of theorists for a few decades, first of all in relation to astrophysical applications. It is believed that in astrophysics electron–positron plasmas exist in pulsar magnetospheres (Sturrock 1971; Ruderman & Sutherland 1975; Chian & Kennel 1983; Arons & Barnard 1986; Aharonian, Bogovalov & Khangulyan 2012; Cerutti & Beloborodov 2017), active galactic nuclei (Ruffini, Vereshchagin & Xue 2010; El-Labany *et al.* 2013; Kawakatu, Kino & Takahara 2016) and the early universe (Gailis, Frankel & Dettmann 1995; Shukla 2003; Tatsuno *et al.* 2003). It is believed that large-amplitude low-frequency waves play and important role in such astrophysical processes as slowing down of pulsars, pulsar radiation and cosmic ray acceleration.

The linear theory of wave propagation in electron–positron plasmas was developed using both a hydrodynamic as well as a kinetic description (Sakai & Kawata 1980*a*; Arons & Barnard 1986; Stewart & Laing 1992). The nonlinear theory of waves in electron–positron plasmas has been also developed. The nonlinear Schrödinger equation was derived and used to study the modulational instability and envelope solitons (Chian & Kennel 1983; Cattaert, Kourakis & Shukla 2005; Rajib, Sultana & Mamun 2015). The Korteweg–de Vries (KdV) and modified Korteweg–de Vries (mKdV)

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equations were obtained and the dependence of the width and amplitude of solitons described by these equations on parameters of an unperturbed state was studied (Verheest & Lakhina 1996; Lakhina & Verheest 1997; Rajib *et al.* 2015).

We aim to study the propagation of nonlinear waves that is quasi-parallel with respect to the equilibrium magnetic field. In the case of electron–ion plasmas this problem was intensively studied during a few decades. It was shown that the one-dimensional quasi-parallel propagation of nonlinear waves is described by the derivative nonlinear Schrödinger (DNLS) equation (Rogister 1971; Mio *et al.* 1976*a*; Mjølhus 1976; Ruderman 2002). This equation was used to study the modulational instability of circularly polarised Alfvén waves (Mio *et al.* 1976*b*; Mjølhus 1976). The DNLS equation describes a few kinds of solitons as well as the generation of rogue waves (Ichikawa *et al.* 1980; Mjølhus & Hada 1997; Fedun, Ruderman & Erdélyi 2008). It was shown that the DNLS equation is completely integrable, the Lax pair for this equation was found and the inverse scattering method was used to obtain exact solutions (Kaup & Newell 1978; Kawata & Inoue 1978).

Later, an extension of the DNLS equations to two and three dimensions (three-dimensional DNLS) was derived (Mjølhus & Wyller 1986; Ruderman 1987; Mjølhus & Hada 1997). This extension is similar to that obtained by Kadomtsev & Petviashvili (1970) (the KP equation) for the KdV equation. The three-dimensional (3-D) DNLS was used to study the stability of solitons of the DNLS equation with respect to transvers perturbations (Ruderman 1987; Mjølhus & Hada 1997).

The propagation of large-amplitude Alfvén waves parallel to the external magnetic field has been also studied in an electron–positron plasma (Sakai & Kawata 1980*a*,*b*; Mikhailovskii, Onishchenko & Smolyakov 1985*a*; Mikhailovskii, Onishchenko & Tatarinov 1985*b*,*c*; Verheest 1996; Lakhina & Verheest 1997). It was shown that, in contrast to the electron–proton plasma, nonlinear waves propagating parallel to the magnetic field are described by the vector form of the mKdV equation.

In this paper we aim to extend this vector mKdV equation to two and three dimensions. First studies of waves in electron–positron plasmas were related to astrophysical applications. However, the progress of experimental physics then opened the possibility of creation of electron–positron plasmas in the laboratory (Surko, Leventhal & Passner 1989; Surko & Murphy 1990; Greaves, Tinkle & Surko 1994; Liang, Wilks & Tabak 1998; Gahn *et al.* 2000; Bell & Kirk 2008; Chen *et al.* 2009; Sarri *et al.* 2013). Another example is the semi-conductor plasma, where holes behave like positive charges with a mass equal to that of the electrons (Shukla *et al.* 1986). Although in astrophysical applications an electron–positron plasma is almost always relativistic, a non-relativistic electron–positron plasma is also of astrophysical interest. It can radiate very effectively by cyclotron emission. As a result, it cools and eventually becomes non-relativistic. As for laboratory plasmas, in many cases they can be described in the non-relativistic approximation. This observation inspired Iwamoto (1993) and Zank & Greaves (1995) to study waves in non-relativistic electron–positron plasmas.

The propagation of nonlinear waves and, in particular, solitons, for electron–ion plasmas were extensively studied in laboratory experiments (e.g. Ikezi 1973; Tran 1979; Lonngren 1983). To our knowledge up to now there have been no experimental studies of waves in electron–positron plasmas. Apparently, this is related to the substantial difficulty of creating electron–positron plasmas in the laboratory. Hence, theorists are ahead of experimentalists in studying waves in these plasmas. The state of affairs here is the same as was in the case of electron–ion plasmas, where nonlinear waves were studied theoretically much earlier than experimentally. There is no doubt that waves in electron–positron plasmas will be studied experimentally because they are of great importance for understanding physical phenomena both in astrophysical as well as in laboratory plasmas.

In this article we also use the non-relativistic approximation that strongly simplifies the derivation of the multi-dimensional generalisation of the mKdV equation. The article is organised as follows. In the next section we formulate the problem and present the governing equations. In § 3 we briefly discuss the linear theory. In § 4 we derive the equation describing small-amplitude weakly dispersive quasi-three-dimensional nonlinear waves. In § 5 we obtain the solutions describing planar one-dimensional solitons. In § 6 we study the soliton stability with respect to transvers perturbations. Section 7 contains the summary of the obtained results and conclusion.

2. Problem formulation and governing equations

We consider the propagation of nonlinear waves along the equilibrium magnetic field in a plasma that consists of electrons and positrons. We treat the electron and positron components as two charged fluids. We do not consider the annihilation or pair creation, meaning that the particle number is conserved. We use the non-relativistic approximation, meaning that the velocities of the two fluids are much smaller than the speed of light c, and the pressure of each fluid is much smaller than the density times c^2. We also assume that the phase speed of propagation of small perturbations is much smaller than c. The plasma motion is described by the mass conservation and momentum equations

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \boldsymbol{v}_s) = 0, \tag{2.1a}$$

$$\frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)\boldsymbol{v}_s + \frac{\nabla p_s}{m n_s} = \frac{q_s}{m}(\boldsymbol{E} + \boldsymbol{v}_s \times \boldsymbol{B}). \tag{2.1b}$$

In these equations n_s is the number density, $\boldsymbol{v}_s$ the velocity, p_s the pressure, m the electron mass and $s = +$ and $s = -$ refers to the positrons and electrons, respectively; $\boldsymbol{E}$ is the electrical field, $\boldsymbol{B}$ is the magnetic field, $q_+ = q$, $q_- = -q$ and q is the elementary charge. We assume that the motion is adiabatic and take

$$p_s = p_0 \left(\frac{n_s}{n_0} \right)^{\kappa}, \tag{2.2}$$

where n_0 and p_0 are the unperturbed number density and pressure (the same for the electrons and positrons), and $\kappa (= 5/3)$ is the adiabatic exponent. Equations (2.1a)–(2.2) must be supplemented with the Maxwell equations. Since we use the non-relativistic approximation, we can neglect the displacement current and write the Maxwell equations as

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \tag{2.3a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.3b}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.3c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}, \tag{2.3d}$$

where ε_0 is the permittivity of free space, μ_0 is the permeability of free space and the total electrical charge and current densities are determined by

$$\rho = \rho_+ + \rho_- = q(n_+ - n_-), \tag{2.4a}$$

$$\boldsymbol{j} = \boldsymbol{j}_+ + \boldsymbol{j}_- = q(n_+\boldsymbol{v}_+ - n_-\boldsymbol{v}_-). \tag{2.4b}$$

Recall that $\varepsilon_0\mu_0 = c^{-2}$.

We assume that in the equilibrium $n_+ = n_- = n_0$, $\boldsymbol{v}_+ = \boldsymbol{v}_- = 0$, $\boldsymbol{E} = 0$ and $\boldsymbol{B} = B_0\boldsymbol{e}_x$, where $\boldsymbol{e}_x$ is the unit vector along the x-axis of Cartesian coordinates x, y, z.

3. Linear theory

Here, we briefly describe the linear theory of wave propagation because below we use it as a guide for scaling when deriving the equation governing the propagation of nonlinear waves. Since below we study the nonlinear wave propagation along the magnetic field, we only consider linear wave propagation in the equilibrium magnetic field direction. We linearise (2.1) and (2.2) and then take perturbations of all quantities proportional to $\exp[\mathrm{i}(kx - \omega t)]$. As a result, we obtain two disconnected systems of algebraic equations. The first system is for the perturbations of the number density, pressure and x-components of the velocity and electric field. It describes the longitudinal wave mode. We do not study this mode in detail and only state that, in the long wavelength approximation, its phase speed is $a_0 = (\kappa p_0/mn_0)^{1/2}$. This speed can be considered as the sound speed.

The second system is for the y and z-components of the velocity, electric field and magnetic field perturbation. It describes transversal wave modes. Below we derive the equation describing the nonlinear transverse waves. Hence, here we present a more detailed study of this wave mode. The transverse waves are described by

$$\omega m\boldsymbol{v}_{\perp s} = \mathrm{i}q_s(\boldsymbol{E}_\perp - B_0\boldsymbol{e}_x \times \boldsymbol{v}_{\perp s}), \tag{3.1a}$$

$$k\boldsymbol{E}_\perp = -\omega\boldsymbol{e}_x \times \boldsymbol{B}_\perp, \tag{3.1b}$$

$$\mathrm{i}k\boldsymbol{e}_x \times \boldsymbol{B}_\perp = \mu_0 q n_0(\boldsymbol{v}_{\perp +} - \boldsymbol{v}_{\perp -}), \tag{3.1c}$$

where

$$\boldsymbol{v}_{\perp s} = (0, v_{ys}, v_{zs}), \tag{3.2a}$$

$$\boldsymbol{E}_\perp = (0, E_y, E_z), \quad \boldsymbol{B}_\perp = (0, B_y, B_z). \tag{3.2b}$$

Introducing the plasma bulk velocity and electrical current,

$$\boldsymbol{v}_\perp = \tfrac{1}{2}(\boldsymbol{v}_{\perp +} + \boldsymbol{v}_{\perp -}), \quad \boldsymbol{j} = qn_0(\boldsymbol{v}_{\perp +} - \boldsymbol{v}_{\perp -}), \tag{3.3a,b}$$

we obtain from (3.1a) and (3.1c)

$$\omega m n_0 \boldsymbol{v}_\perp = -\frac{\mathrm{i}}{2}B_0\boldsymbol{e}_x \times \boldsymbol{j}, \tag{3.4}$$

$$\boldsymbol{E}_\perp = B_0\boldsymbol{e}_x \times \boldsymbol{v}_\perp - \frac{\mathrm{i}\omega m\boldsymbol{j}}{2n_0q^2}, \tag{3.5}$$

$$\mathrm{i}k\boldsymbol{e}_x \times \boldsymbol{B}_\perp = \mu_0\boldsymbol{j}. \tag{3.6}$$

Equation (3.5) is Ohm's law. The second term on the right-hand side is similar to the Hall term in the Ohm's law for the electron–ion plasma. However, the Hall term would be proportional to $\boldsymbol{e}_x \times \boldsymbol{j}$ rather than $\boldsymbol{j}$ as in (3.5). This difference is related to the fact that the masses of positively and negatively charged particles are the same in the electron–positron plasma, while the mass of positively charged particles is much larger than the mass of negatively charged particles in the electron–ion plasma. The dispersion of waves propagating along the magnetic field in an electron–ion plasma is related to the account of ion inertia in the induction equation, while the electron inertia is neglected. In contrast, in an electron–positron plasma the inertia of both electrons and positrons is accounted for.

Equations (3.1*b*) and (3.4)–(3.6) constitute the system of linear homogeneous algebraic equations. It only has non-trivial solutions when its determinant is zero. This condition gives the dispersion equation

$$m\omega^2(mk^2 + 2\mu_0 q^2 n_0) = q^2 k^2 B_0^2. \tag{3.7}$$

For small values of k this dispersion equation reduces to the approximate form

$$\omega = kV(1 - k^2\ell^2), \tag{3.8}$$

where

$$V = \frac{B_0}{\sqrt{2\mu_0 m n_0}}, \quad \ell = \frac{1}{2q}\sqrt{\frac{m}{\mu_0 n_0}}. \tag{3.9a,b}$$

The wave dispersion is related to the presence of the second term in (3.5). If we neglect this term, then the dispersion relation reduces to $\omega = kV$.

The condition that k is small is written as $k\ell \ll 1$. In the non-relativistic approximation we must have a phase speed much smaller than the speed of light, $V \ll c$. This condition reduces to $B^2/\mu_0 \ll mn_0c^2$, that is, the magnetic energy density is much smaller than the rest density of the plasma. We note that the term describing the wave dispersion (the second term in the brackets in (3.8)) is proportional to k^2. In the case of an electron–ion plasma it is proportional to k.

4. Derivation of equation for small-amplitude nonlinear waves

We consider nonlinear waves propagating along the equilibrium magnetic field. We expect that the equation describing the nonlinear wave propagation will be similar to the 3-D DNLS equation describing quasi-parallel propagation of nonlinear waves in an ion–electron plasma with the only difference being that the term describing the wave dispersion will be different. This difference arises from the fact that, as we have already seen, the term describing the dispersion of waves in an electron–positron plasma is proportional to k^2, while it is proportional to k in the electron–ion plasma.

To derive the nonlinear equation describing the longitudinal propagation of nonlinear waves we use the reductive perturbation method (Kakutani *et al.* 1968; Taniuti & Wei 1968). In accordance with this method we introduce a dimensionless amplitude of the order of $\epsilon \ll 1$. In the linear theory the characteristic time is L/V, where L is the characteristic length of perturbation and V is the phase speed of very long waves. We assume that the ratio L/ℓ is ϵ^{-1}. The characteristic time of variation of the perturbation shape caused by the nonlinearity and dispersion is $\epsilon^{-2}L/V$. We also consider a weak dependence of the perturbations on y and z with the characteristic scale $\epsilon^{-2}L$. On a time scale much smaller than $\epsilon^{-2}L/V$ a perturbation propagates as a wave with permanent shape with all variables only depending on $x - Vt$. In accordance with the above analysis we introduce stretched variables

$$\xi = \epsilon(x - Vt), \quad \eta = \epsilon^2 y, \quad \zeta = \epsilon^2 z, \quad \tau = \epsilon^3 t. \tag{4.1a–d}$$

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With the aid of (2.4) we transform (2.1) and (2.3) in the new variables to

$$\epsilon^2\frac{\partial n_s}{\partial\tau} - V\frac{\partial n_s}{\partial\xi} + \frac{\partial(n_s v_{xs})}{\partial\xi} + \epsilon\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}(n_s\boldsymbol{v}_{\perp s}) = 0, \tag{4.2a}$$

$$\begin{aligned}\epsilon^2\frac{\partial v_{xs}}{\partial\tau} - V\frac{\partial v_{xs}}{\partial\xi} + v_{xs}\frac{\partial v_{xs}}{\partial\xi} + \epsilon\boldsymbol{v}_{\perp s}\boldsymbol{\cdot}\boldsymbol{\nabla}_\perp v_{xs} \\ + \frac{1}{mn_s}\frac{\partial p_s}{\partial\xi} = \epsilon^{-1}\frac{q_s}{m}[E_x + \boldsymbol{e}_x\boldsymbol{\cdot}(\boldsymbol{v}_{\perp s}\times\boldsymbol{B}_\perp)],\end{aligned} \tag{4.2b}$$

$$\begin{aligned}\epsilon^2\frac{\partial\boldsymbol{v}_{\perp s}}{\partial\tau} - V\frac{\partial\boldsymbol{v}_{\perp s}}{\partial\xi} + v_{xs}\frac{\partial\boldsymbol{v}_{\perp s}}{\partial\xi} + \epsilon(\boldsymbol{v}_{\perp s}\boldsymbol{\cdot}\boldsymbol{\nabla}_\perp)\boldsymbol{v}_{\perp s} + \epsilon\frac{\boldsymbol{\nabla}_\perp p_s}{mn_s} \\ = \epsilon^{-1}\frac{q_s}{m}[\boldsymbol{E}_\perp + \boldsymbol{e}_x\times(v_{xs}\boldsymbol{B}_\perp - B_x\boldsymbol{v}_{\perp s})],\end{aligned} \tag{4.2c}$$

$$\frac{\partial E_x}{\partial\xi} + \epsilon\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\boldsymbol{E}_\perp = \epsilon^{-1}\frac{q}{\varepsilon_0}(n_+ - n_-), \tag{4.2d}$$

$$\frac{\partial B_x}{\partial\xi} + \epsilon\boldsymbol{\nabla}_\perp\boldsymbol{\cdot}\boldsymbol{B}_\perp = 0, \tag{4.2e}$$

$$\epsilon^2\frac{\partial B_x}{\partial\tau} - V\frac{\partial B_x}{\partial\xi} = -\epsilon\,\boldsymbol{e}_x\boldsymbol{\cdot}\boldsymbol{\nabla}_\perp\times\boldsymbol{E}_\perp, \tag{4.2f}$$

$$\epsilon^2\frac{\partial\boldsymbol{B}_\perp}{\partial\tau} - V\frac{\partial\boldsymbol{B}_\perp}{\partial\xi} = -\boldsymbol{e}_x\times\left(\frac{\partial\boldsymbol{E}_\perp}{\partial\xi} - \epsilon\boldsymbol{\nabla}_\perp E_x\right), \tag{4.2g}$$

$$\epsilon\boldsymbol{e}_x\boldsymbol{\cdot}(\boldsymbol{\nabla}_\perp\times\boldsymbol{B}_\perp) = \epsilon^{-1}q\mu_0(n_+v_{x+} - n_-v_{x-}), \tag{4.2h}$$

$$\boldsymbol{e}_x\times\left(\frac{\partial\boldsymbol{B}_\perp}{\partial\xi} - \epsilon\boldsymbol{\nabla}_\perp B_x\right) = \epsilon^{-1}q\mu_0(n_+\boldsymbol{v}_{\perp+} - n_-\boldsymbol{v}_{\perp-}), \tag{4.2i}$$

where

$$\boldsymbol{\nabla}_\perp = \left(0,\ \frac{\partial}{\partial\eta},\ \frac{\partial}{\partial\zeta}\right). \tag{4.3}$$

Now we look for the solution in the form of expansions in power series with respect to ϵ,

$$\left.\begin{aligned}
p_s &= p_0 + \epsilon p_s^{(1)} + \epsilon^2 p_s^{(2)} + \epsilon^3 p_s^{(3)} + \cdots, \\
n_s &= n_0 + \epsilon n_s^{(1)} + \epsilon^2 n_s^{(2)} + \epsilon^3 n_s^{(3)} + \cdots, \\
v_{xs} &= \epsilon v_{xs}^{(1)} + \epsilon^2 v_{xs}^{(2)} + \epsilon^3 v_{xs}^{(3)} + \cdots, \\
\boldsymbol{v}_{\perp s} &= \epsilon\boldsymbol{v}_{\perp s}^{(1)} + \epsilon^2\boldsymbol{v}_{\perp s}^{(2)} + \epsilon^3\boldsymbol{v}_{\perp s}^{(3)} + \cdots, \\
B_x &= B_0 + \epsilon B_x^{(1)} + \epsilon^2 B_x^{(2)} + \epsilon^3 B_x^{(3)} + \cdots, \\
\boldsymbol{B}_\perp &= \epsilon\boldsymbol{B}_\perp^{(1)} + \epsilon^2\boldsymbol{B}_\perp^{(2)} + \epsilon^3\boldsymbol{B}_\perp^{(3)} + \cdots \\
E_x &= \epsilon E_x^{(1)} + \epsilon^2 E_x^{(2)} + \epsilon^3 E_x^{(3)} + \cdots, \\
\boldsymbol{E}_\perp &= \epsilon\boldsymbol{E}_\perp^{(1)} + \epsilon^2\boldsymbol{E}_\perp^{(2)} + \epsilon^3\boldsymbol{E}_\perp^{(3)} + \cdots
\end{aligned}\right\} \tag{4.4}$$

We impose the boundary conditions at $\xi\to\infty$,

$$\left.\begin{aligned}
&n_s\to n_0, &&p_s\to p_0, &&v_{xs}\to 0, &&B_x\to B_0, \\
&E_x\to 0, &&\boldsymbol{v}_{\perp s}\to 0, &&\boldsymbol{B}_\perp\to 0, &&\boldsymbol{E}_\perp\to 0.
\end{aligned}\right\} \tag{4.5}$$

It follows from (4.5) that all quantities with the upper indices 1, 2 and so on tend to zero as $\xi\to\infty$.

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4.1. *The zero-order approximation*

Substituting the expansions given by (4.4) in (4.2) and collecting terms of the order of unity in (4.2*b*)–(4.2*d*), (4.2*h*) and (4.2*i*) we easily obtain

$$n_+^{(1)} = n_-^{(1)} = n^{(1)}, \quad v_{x+}^{(1)} = v_{x-}^{(1)} = v_x^{(1)}, \tag{4.6a}$$

$$\boldsymbol{v}_{\perp+}^{(1)} = \boldsymbol{v}_{\perp-}^{(1)} = \boldsymbol{v}_\perp^{(1)}, \tag{4.6b}$$

$$E_x^{(1)} = 0, \quad \boldsymbol{E}_\perp^{(1)} = B_0\,\boldsymbol{e}_x \times \boldsymbol{v}_\perp^{(1)}. \tag{4.6c}$$

4.2. *The first-order approximation*

Collecting terms of the order of ϵ in (2.4*a*) and (4.2), and using (4.6) yields

$$V\frac{\partial n^{(1)}}{\partial \xi} = n_0 \frac{\partial v_x^{(1)}}{\partial \xi}, \quad p_s^{(1)} = \kappa p_0 \frac{n^{(1)}}{n_0}, \tag{4.7a}$$

$$\frac{1}{mn_0}\frac{\partial p_s^{(1)}}{\partial \xi} - V\frac{\partial v_x^{(1)}}{\partial \xi} = \frac{q_s}{m}[E_x^{(2)} + \boldsymbol{e}_x \boldsymbol{\cdot} (\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(1)})], \tag{4.7b}$$

$$V\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} = -\frac{q_s}{m}[\boldsymbol{E}_\perp^{(2)} + \boldsymbol{e}_x \times (v_x^{(1)}\boldsymbol{B}_\perp^{(1)} - B_x^{(1)}\boldsymbol{v}_\perp^{(1)} - B_0\boldsymbol{v}_{\perp s}^{(2)})], \tag{4.7c}$$

$$n_+^{(2)} = n_-^{(2)} = n^{(2)}, \quad v_{x+}^{(2)} = v_{x-}^{(2)} = v_x^{(2)}, \tag{4.7d}$$

$$V\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} = \boldsymbol{e}_x \times \frac{\partial \boldsymbol{E}_\perp^{(1)}}{\partial \xi}, \quad \frac{\partial B_x^{(1)}}{\partial \xi} = 0, \tag{4.7e}$$

$$\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} = qn_0\mu_0(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}). \tag{4.7f}$$

Equation (4.7*b*) represents two equations, one for $s = +$, and the other for $s = -$. Adding and subtracting these equations we obtain

$$\frac{1}{mn_0}\frac{\partial p_s^{(1)}}{\partial \xi} = V\frac{\partial v_x^{(1)}}{\partial \xi}, \tag{4.8a}$$

$$E_x^{(2)} = -\boldsymbol{e}_x \boldsymbol{\cdot} (\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(1)}). \tag{4.8b}$$

It follows from (4.7*a*), (4.8*a*), the second equation in (4.7*e*) and the boundary conditions (4.5) that

$$n^{(1)} = 0, \quad p_s^{(1)} = 0, \quad v_x^{(1)} = 0, \quad B_x^{(1)} = 0. \tag{4.9a–d}$$

Equation (4.7*c*) also represents two equations, one for $s = +$, and the other for $s = -$. Adding and subtracting these equations and using (4.9) we obtain

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} = \frac{qB_0}{2mV}\boldsymbol{e}_x \times (\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}), \tag{4.10a}$$

$$2\boldsymbol{E}_\perp^{(2)} = B_0\boldsymbol{e}_x \times (\boldsymbol{v}_{\perp+}^{(2)} + \boldsymbol{v}_{\perp-}^{(2)}). \tag{4.10b}$$

It follows from the first equation in (4.7*e*) and the last boundary condition in (4.5) that

$$\boldsymbol{E}_\perp^{(1)} = -V\boldsymbol{e}_x \times \boldsymbol{B}_\perp^{(1)}. \tag{4.11}$$

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Using the second equation in (4.6*c*) and (4.11) yields

$$B_0\boldsymbol{v}_\perp^{(1)} + V\boldsymbol{B}_\perp^{(1)} = 0. \tag{4.12}$$

Substituting (4.12) in (4.8*b*) and (4.10*a*) yields

$$E_x^{(2)} = 0, \qquad \boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)} = \frac{2mV^2}{qB_0^2}\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi}. \tag{4.13a,b}$$

Equation (4.7*f*) and the second equation in (4.13) constitute a linear homogeneous system of equations for $\partial\boldsymbol{B}_\perp^{(1)}/\partial\xi$ and $\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}$. It only has non-trivial solutions when its determinant is zero. This condition determines that V is given by (3.9).

4.3. *The second-order approximation*

Now we collect terms of the order of ϵ^2 in (2.2) and (4.2). As a result, we obtain

$$p_s^{(2)} = \kappa p_0 \frac{n^{(2)}}{n_0}, \qquad V\frac{\partial n^{(2)}}{\partial \xi} = n_0 \frac{\partial v_x^{(2)}}{\partial \xi} + n_0 \nabla_\perp \cdot \boldsymbol{v}_\perp^{(1)}, \tag{4.14a}$$

$$\frac{a_0^2}{n_0}\frac{\partial n^{(2)}}{\partial \xi} - V\frac{\partial v_x^{(2)}}{\partial \xi} = \frac{q_s}{m}[E_3^{(3)} + \boldsymbol{e}_x \cdot (\boldsymbol{v}_\perp^{(1)} \times \boldsymbol{B}_\perp^{(2)} + \boldsymbol{v}_{\perp s}^{(2)} \times \boldsymbol{B}_\perp^{(1)})], \tag{4.14b}$$

$$V\frac{\partial \boldsymbol{v}_{\perp s}^{(2)}}{\partial \xi} = -\frac{q_s}{m}[\boldsymbol{E}_\perp^{(3)} - \boldsymbol{e}_x \times (B_0\boldsymbol{v}_{\perp s}^{(3)} - v_x^{(2)}\boldsymbol{B}_\perp^{(1)} + B_x^{(2)}\boldsymbol{v}_\perp^{(1)})], \tag{4.14c}$$

$$\frac{\partial E_x^{(2)}}{\partial \xi} + \nabla_\perp \cdot \boldsymbol{E}_\perp^{(1)} = \frac{q}{\varepsilon_0}(n_+^{(3)} - n_-^{(3)}), \tag{4.14d}$$

$$\frac{\partial B_x^{(2)}}{\partial \xi} + \nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)} = 0, \qquad V\frac{\partial B_x^{(2)}}{\partial \xi} = \boldsymbol{e}_x \cdot \nabla_\perp \times \boldsymbol{E}_\perp^{(1)}, \tag{4.14e}$$

$$V\frac{\partial \boldsymbol{B}_\perp^{(2)}}{\partial \xi} = \boldsymbol{e}_x \times \frac{\partial \boldsymbol{E}_\perp^{(2)}}{\partial \xi}, \tag{4.14f}$$

$$\boldsymbol{e}_x \cdot \nabla_\perp \times \boldsymbol{B}_\perp^{(1)} = q\mu_0 n_0 (v_{x+}^{(3)} - v_{x-}^{(3)}) \tag{4.14g}$$

$$\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(2)}}{\partial \xi} = q\mu_0 n_0 (\boldsymbol{v}_{\perp+}^{(3)} - \boldsymbol{v}_{\perp-}^{(3)}). \tag{4.14h}$$

Using (4.12) we transform the second equation in (4.14*a*) to

$$\frac{\partial v_x^{(2)}}{\partial \xi} - \frac{V}{n_0}\frac{\partial n^{(2)}}{\partial \xi} = \frac{V}{B_0}\nabla_\perp \cdot \boldsymbol{B}_\perp^{(1)}. \tag{4.15}$$

Equation (4.14*b*) represents two equations, one for $s = +$, and the other for $s = -$. Adding these equations we obtain

$$\frac{a_0^2}{n_0}\frac{\partial n^{(2)}}{\partial \xi} - V\frac{\partial v_x^{(2)}}{\partial \xi} = \frac{q}{2m}\boldsymbol{e}_x \cdot (\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}) \times \boldsymbol{B}_\perp^{(1)}. \tag{4.16}$$

Using (4.13) we transform this equation to

$$\frac{a_0^2}{n_0}\frac{\partial n^{(2)}}{\partial \xi} - V\frac{\partial v_x^{(2)}}{\partial \xi} = -\frac{V^2}{2B_0^2}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi}. \tag{4.17}$$

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We find from (4.15) and (4.17)

$$\frac{\partial n^{(2)}}{\partial \xi} = \frac{n_0 V^2}{B_0(V^2 - a_0^2)} \left(\frac{1}{2B_0} \frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi} - \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{B}_\perp^{(1)} \right), \tag{4.18a}$$

$$\frac{\partial v_x^{(2)}}{\partial \xi} = \frac{V}{V^2 - a_0^2} \left(\frac{V^2}{2B_0^2} \frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi} - \frac{a_0^2}{B_0} \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{B}_\perp^{(1)} \right). \tag{4.18b}$$

Using (4.11) and the first equation in (4.13) we obtain from (4.14*d*)

$$q(n_+^{(3)} - n_-^{(3)}) = \varepsilon_0 V \boldsymbol{e}_x \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)}. \tag{4.19}$$

Finally, (4.14*c*) represents two equations, one for $s = +$, and the other for $s = -$. Subtracting the second equation from the first one yields

$$2\boldsymbol{E}_\perp^{(3)} - \boldsymbol{e}_x \times [B_0(\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)}) - 2v_x^{(2)} \boldsymbol{B}_\perp^{(1)} + 2B_x^{(2)} \boldsymbol{v}_\perp^{(1)}] = -\frac{mV}{q} \frac{\partial (\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)})}{\partial \xi}. \tag{4.20}$$

4.4. *The third-order approximation*

In the third-order approximation we collect the terms of the order of ϵ^3 in (4.2*c*), (4.2*g*) and (4.2*i*) to obtain

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \tau} + v_x^{(2)} \frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} - V \frac{\partial \boldsymbol{v}_{\perp s}^{(3)}}{\partial \xi} + (\boldsymbol{v}_\perp^{(1)} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp) \boldsymbol{v}_\perp^{(1)} + \frac{a_0^2}{n_0} \boldsymbol{\nabla}_\perp n^{(2)}$$
$$= \frac{q_s}{m} [\boldsymbol{E}_\perp^{(4)} + \boldsymbol{e}_x \times (v_x^{(2)} \boldsymbol{B}_\perp^{(2)} + v_{xs}^{(3)} \boldsymbol{B}_\perp^{(1)} - B_0 \boldsymbol{v}_{\perp s}^{(4)} - B_x^{(2)} \boldsymbol{v}_{\perp s}^{(2)} - B_x^{(3)} \boldsymbol{v}_\perp^{(1)})], \tag{4.21a}$$

$$\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} = \boldsymbol{e}_x \times \left(\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} - V \frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} \right) + \boldsymbol{\nabla}_\perp E_x^{(2)}, \tag{4.21b}$$

$$\boldsymbol{e}_x \times \left(\frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} - \boldsymbol{\nabla}_\perp B_x^{(2)} \right) = q\mu_0 [n_0(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}) + n^{(2)}(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)})]. \tag{4.21c}$$

Equation (4.21*a*) represents two equations, one for $s = +$, and the other for $s = -$. Adding these equations we obtain

$$\frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \tau} + v_x^{(2)} \frac{\partial \boldsymbol{v}_\perp^{(1)}}{\partial \xi} - \frac{V}{2} \frac{\partial (\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)})}{\partial \xi} + \frac{a_0^2}{n_0} \boldsymbol{\nabla}_\perp n^{(2)} + (\boldsymbol{v}_\perp^{(1)} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp) \boldsymbol{v}_\perp^{(1)}$$
$$= \frac{q}{2m} \boldsymbol{e}_x \times [\boldsymbol{B}_\perp^{(1)}(v_{x+}^{(3)} - v_{x-}^{(3)}) - B_x^{(2)}(\boldsymbol{v}_{\perp+}^{(2)} - \boldsymbol{v}_{\perp-}^{(2)}) - B_0(\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)})]. \tag{4.22}$$

Using (4.7*f*), (4.11), (4.13) and (4.14*g*) we transform (4.20) and (4.21*b*)–(4.22) to

$$2\boldsymbol{e}_x \times \frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} + B_0 \frac{\partial (\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)})}{\partial \xi} = \frac{mV}{qn_0\mu_0} \frac{\partial^3 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^3} + \frac{2}{B_0} \frac{\partial}{\partial \xi} [\boldsymbol{B}_\perp^{(1)}(VB_x^{(2)} + B_0 v_x^{(2)})], \tag{4.23a}$$

$$\frac{\partial \boldsymbol{E}_\perp^{(3)}}{\partial \xi} + V\boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} = \boldsymbol{e}_x \times \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau}, \tag{4.23b}$$

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$$\frac{\partial \boldsymbol{B}_\perp^{(3)}}{\partial \xi} + q\mu_0 n_0 \boldsymbol{e}_x \times (\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}) = \frac{n^{(2)}}{n_0}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} + \boldsymbol{\nabla}_\perp B_x^{(2)}, \tag{4.23c}$$

$$\begin{aligned}
&\frac{qB_0}{2m}\boldsymbol{e}_x \times (\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}) - \frac{V}{2}\frac{\partial(\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)})}{\partial \xi} \\
&\quad = \frac{V^2}{B_0^2}\left[B_x^{(2)}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} + (\boldsymbol{e}_x \times \boldsymbol{B}_\perp^{(1)})\boldsymbol{e}_x \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)}\right] + \frac{V}{B_0}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} + \frac{V v_x^{(2)}}{B_0}\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi} \\
&\quad - \frac{a_0^2}{n_0}\boldsymbol{\nabla}_\perp n^{(2)} - \frac{V^2}{B_0^2}(\boldsymbol{B}_\perp^{(1)} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp)\boldsymbol{B}_\perp^{(1)}.
\end{aligned} \tag{4.23d}$$

The system of (4.23) is the system of linear inhomogeneous algebraic equations for $\partial \boldsymbol{E}_\perp^{(3)}/\partial \xi$, $\partial \boldsymbol{B}_\perp^{(3)}/\partial \xi$, $\partial(\boldsymbol{v}_{\perp+}^{(3)} + \boldsymbol{v}_{\perp-}^{(3)})/\partial \xi$ and $\boldsymbol{v}_{\perp+}^{(4)} - \boldsymbol{v}_{\perp-}^{(4)}$. Using the expression for V it is straightforward to show that the determinant of this system is zero. Then the system of (4.23) has non-trivial solution only if the compatibility condition is satisfied. This condition is

$$\begin{aligned}
&\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} + V\ell^2\frac{\partial^3 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^3} + \frac{1}{2}\boldsymbol{B}_\perp^{(1)}\left(\frac{\partial v_x^{(2)}}{\partial \xi} + \frac{V}{B_0}\frac{\partial B_x^{(2)}}{\partial \xi}\right) + \frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \xi}\left(\frac{V}{B_0}B_x^{(2)} - \frac{Vn^{(2)}}{2n_0} + v_x^{(2)}\right) \\
&- \frac{V}{2}\boldsymbol{\nabla}_\perp B_x^{(2)} - \frac{a_0^2 B_0}{2n_0 V}\boldsymbol{\nabla}_\perp n^{(2)} + \frac{V}{2B_0}(\boldsymbol{e}_x \times \boldsymbol{B}_\perp^{(1)})\boldsymbol{e}_x \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)} - \frac{V}{2B_0}(\boldsymbol{B}_\perp^{(1)} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp)\boldsymbol{B}_\perp^{(1)} = 0.
\end{aligned} \tag{4.24}$$

The following identities can be verified by the direct calculation

$$(\boldsymbol{e}_x \times \boldsymbol{B}_\perp^{(1)})\boldsymbol{e}_x \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)} = -\boldsymbol{B}_\perp^{(1)} \times (\boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)}), \tag{4.25a}$$

$$(\boldsymbol{B}_\perp^{(1)} \boldsymbol{\cdot} \boldsymbol{\nabla}_\perp)\boldsymbol{B}_\perp^{(1)} = \tfrac{1}{2}\boldsymbol{\nabla}_\perp |\boldsymbol{B}_\perp^{(1)}|^2 - \boldsymbol{B}_\perp^{(1)} \times (\boldsymbol{\nabla}_\perp \times \boldsymbol{B}_\perp^{(1)}). \tag{4.25b}$$

Using (4.14*e*), (4.18*a*) and (4.18*b*) we obtain

$$\frac{\partial v_x^{(2)}}{\partial \xi} + \frac{V}{B_0}\frac{\partial B_x^{(2)}}{\partial \xi} = \frac{V^3}{B_0(V^2 - a_0^2)}\left(\frac{1}{2B_0}\frac{\partial |\boldsymbol{B}_\perp^{(1)}|^2}{\partial \xi} - \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{B}_\perp^{(1)}\right), \tag{4.26a}$$

$$v_x^{(2)} - \frac{Vn^{(2)}}{2n_0} + \frac{V}{B_0}B_x^{(2)} = \frac{V^3|\boldsymbol{B}_\perp^{(1)}|^2}{4B_0^2(V^2 - a_0^2)} - \frac{V^3 \Phi}{2B_0(V^2 - a_0^2)}, \tag{4.26b}$$

where Φ is defined by

$$\frac{\partial \Phi}{\partial \xi} = \boldsymbol{\nabla}_\perp \boldsymbol{\cdot} \boldsymbol{B}_\perp^{(1)}, \quad \Phi \to 0 \text{ as } \xi \to \infty. \tag{4.27}$$

Using (4.25)–(4.27) we transform (4.24) to

$$\frac{\partial \boldsymbol{B}_\perp^{(1)}}{\partial \tau} + \alpha\frac{\partial}{\partial \xi}[\boldsymbol{B}_\perp^{(1)}(|\boldsymbol{B}_\perp^{(1)}|^2 - 2B_0\Phi)] - \alpha B_0 \boldsymbol{\nabla}_\perp(|\boldsymbol{B}_\perp^{(1)}|^2 - 2B_0\Phi) + V\ell^2\frac{\partial^3 \boldsymbol{B}_\perp^{(1)}}{\partial \xi^3} = 0, \tag{4.28}$$

where

$$\alpha = \frac{V^3}{4B_0^2(V^2 - a_0^2)}. \tag{4.29}$$

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Introducing the notation

$$\boldsymbol{b}=\epsilon\boldsymbol{B}_{\perp}^{(1)}, \quad \widetilde{\boldsymbol{\nabla}}_{\perp}=\left(0,\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right), \quad \varphi=\epsilon^2\Phi, \tag{4.30a–c}$$

returning to the original independent variables and dropping the tilde we rewrite (4.27) and (4.28) as

$$\frac{\partial\varphi}{\partial x}=\boldsymbol{\nabla}_{\perp}\boldsymbol{\cdot}\boldsymbol{b}, \quad \varphi\to 0 \text{ as } x\to\infty, \tag{4.31}$$

$$\frac{\partial\boldsymbol{b}}{\partial t}+V\frac{\partial\boldsymbol{b}}{\partial x}+\alpha\frac{\partial}{\partial x}[\boldsymbol{b}(b^2-2B_0\varphi)]-\alpha B_0\boldsymbol{\nabla}_{\perp}(b^2-2B_0\varphi)+V\ell^2\frac{\partial^3\boldsymbol{b}}{\partial x^3}=0. \tag{4.32}$$

This equation only differs from the 3-D DNLS equation describing quasi-parallel propagation of magnetohydrodynamic (MHD) waves in an ion–electron plasma derived by Mjølhus & Wyller (1986) and Ruderman (1987) by the last term describing the dispersion. This difference is related to the difference in the dispersion relations for ion–electron and electron–positron plasmas as was pointed out in § 3.

When $\boldsymbol{b}$ is independent of y and z (4.32) reduces to the vector mKdV equation in a complete agreement with the result obtained by Verheest (1996) and Lakhina & Verheest (1997). In this equation the coefficient at the nonlinear term is α. Khanna & Rajaram (1982) derived the DNLS equation in a collisionless electron–ion plasma with anisotropic pressure. They used the Chew, Goldberger and Low equations (Chew, Goldberger & Low 1956) modified by including the account of the Hall current in the induction equations and terms related to the finite Larmor radius in the momentum equation (Yajima 1966). While the general form of the equation remains the same, the expressions for its coefficients are quite different. In particular, while $\alpha<0$ when $a_0>V$, in the case of plasma with anisotropic pressure the coefficient at the nonlinear term is negative only in a relatively narrow interval of parameters. It is possible that the account of plasma pressure anisotropy can cause a similar modification of the coefficient at the nonlinear term in the vector mKdV equation.

We emphasise that the system of (4.31) and (4.32) was derived under the assumption that the perturbations decay as $|x|\to\infty$. A natural question that arises is if this system of equations also describes perturbations periodic with respect to x. One-dimensional nonlinear sound waves are described by a very simple equation sometimes called the inviscid Burgers equation (e.g. Whitham 1974; Rudenko & Soluyan 1977). This equation also describes magnetosonic waves propagating at not very small angles with respect to the equilibrium magnetic field. It is valid both for perturbations decaying at infinity as well as for periodic perturbations. The same is true for its multi-dimensional generalisation, the Khokhlov–Zabolotskaya equation (Zabolotskaya & Khokhlov 1969). The generalisations of the inviscid Burgers and Khokhlov–Zabolotskaya equation taking into account either dissipation or dispersion, which are the Burgers, KdV and KP equation, also describe both perturbations decaying at infinity as well as spatially periodic perturbations. The general and very important property of all these equations is that the nonlinearity that they describe is quadratic.

In contrast, magnetohydrodynamic waves propagating either along or at small angles with respect to the equilibrium magnetic field are characterised by cubic nonlinearity. In the one-dimensional case they are described in the framework of ideal MHD by the

Cohen–Kulsrud equation (Cohen & Kulsrud 1974). Although this equation describing periodic waves is slightly different from that describing waves decaying at infinity, the former equation is easily reduced to the latter by a simple change of independent variables. The situation is the same with the extension of this equation to dissipative media, the so-called Cohen–Kulsrud–Burgers equation, and to dispersive media, which is the DNLS equation. Hence, we conclude that in the one-dimensional case both the periodic waves as well as the waves decaying at infinity are described by the same equation.

The situation is drastically different in the multi-dimensional case. Ruderman (1986) studied the quasi-longitudinal propagation of MHD waves in the multi-dimensional case. In this case the mean over the period of the transverse magnetic field magnitude squared cannot be eliminated from the equation describing the evolution of the magnetic field perturbation because, in general, this mean varies in the transverse direction. As a result, the equation describing periodic perturbations differs substantially from that describing perturbations decaying at infinity. Passot & Sulem (1993) investigated a similar problem, but using the Hall MHD. As a result they obtained the analogue of the 3-D DNLS equation valid for periodic perturbations. If we neglect the last term in the equation derived by Passot & Sulem (1993) (see their equation (2.33)), then their equation can be reduced to the equation similar to one derived by Ruderman (1986). However, this reduction is not straightforward. The problem is that Ruderman (1986) considered the spatial variation of waves. He assumed that they are driven at $x=0$ and propagate in the positive x-direction. Passot & Sulem (1993) considered the temporal evolution of the waves. However, the equation derived by them with the term describing dispersion dropped looks very similar to the equation derived by Ruderman (1986). On the basis of this similarity we can make a conjecture that we can obtain an analogue of (4.32) by changing the term describing dispersion in equation (2.33) in the paper by Passot & Sulem (1993). However, to prove this conjecture a formal derivation is needed.

5. Obliquely propagating solitary waves

We look for solitary waves propagating at a small angle with respect to the equilibrium magnetic field. In accordance with this we look for solutions to (4.32) that depends of $X = x + k_y y + k_z z - (C+V)t$, where C is a constant, and $|k_y| \ll 1$ and $|k_z| \ll 1$. It follows from (4.31) that

$$\varphi = \boldsymbol{k}_\perp \cdot \boldsymbol{b}, \quad \boldsymbol{k}_\perp = (0, k_y, k_z). \tag{5.1}$$

Using this result and the condition that $\boldsymbol{b} \to 0$ as $X \to \infty$ we obtain from (4.32)

$$V\ell^2 \boldsymbol{b}'' = C\boldsymbol{b} - \alpha(b^2 - 2B_0 \boldsymbol{k}_\perp \cdot \boldsymbol{b})(\boldsymbol{b} - B_0 \boldsymbol{k}_\perp), \tag{5.2}$$

where the prime indicates the derivative with respect to X. We can write down this equation in the Hamiltonian form,

$$g_y' = -\frac{\partial \mathcal{H}}{\partial b_y}, \quad g_z' = -\frac{\partial \mathcal{H}}{\partial b_z}, \quad h_y' = \frac{\partial \mathcal{H}}{\partial g_y}, \quad h_z' = \frac{\partial \mathcal{H}}{\partial g_z}, \tag{5.3a–d}$$

where $\boldsymbol{b} = (b_y, b_z)$, $g_y = b_y'$, $g_z = b_z'$ and the Hamiltonian $\mathcal{H}$ is given by

$$\mathcal{H} = \frac{1}{2}(g_y^2 + g_z^2) + \frac{1}{4V\ell^2}[\alpha(b^2 - 2B_0 \boldsymbol{k}_\perp \cdot \boldsymbol{b})^2 - 2Cb^2]. \tag{5.4}$$

Below we only consider solutions to the system of (5.3) describing planar solitary waves. In these solutions $\boldsymbol{b} \parallel \boldsymbol{k}_\perp$. In accordance with this we write

$$\boldsymbol{b} = \frac{\boldsymbol{k}_\perp}{k_\perp} h. \tag{5.5}$$

Since $\mathcal{H}$ is independent of X it follows that the energy equal to $\mathcal{H}$ is conserved. Since $\boldsymbol{b} \to 0$ and $\boldsymbol{b}' \to 0$ as $X \to \infty$, the energy conservation law is $\mathcal{H} = 0$. Then in the case of plane solitary waves we obtain

$$2V\ell^2 h'^2 = h^2[2C - \alpha(h - 2B_0 k_\perp)^2]. \tag{5.6}$$

In the one-dimensional planar case (4.32) reduces to the modified Korteweg–de Vries equation, which is completely integrable. This implies that planar solitary waves are solitons (recall that solitons are solitary waves that are solutions of completely integrable nonlinear equations). The integral curves of (5.4) corresponding to solitons must start and end at $h = 0$, which is a critical point in the phase plane. In addition, h must take either a maximum or minimum value, which implies that there should be the second critical point where the right-hand side of (5.4) is zero. The necessary condition of the existence of a solution to (5.6) describing a soliton is that its right-hand side must be non-negative when $|h|$ varies from zero to its maximum, which is defined by the condition that the right-hand side is zero. When $\alpha > 0$ this condition reduces to

$$C > 2\alpha B_0^2 k_\perp^2, \quad \alpha > 0, \tag{5.7}$$

while when $\alpha < 0$ it reduces to

$$2\alpha B_0^2 k_\perp^2 < C < 0, \quad \alpha < 0. \tag{5.8}$$

For $\alpha > 0$ there are two solitons. In one of them $h > 0$ and we call it the bright soliton, while in the other $h < 0$ and we call it the dark soliton. These solitons are described by

$$h = \frac{\pm 2(C - 2\alpha B_0^2 k_\perp^2)}{\sqrt{2\alpha C}\cosh(X/L + \Theta) \mp 2\alpha B_0 k_\perp}, \tag{5.9}$$

where the upper and lower signs correspond to the bright and dark solitons, respectively. When $\alpha < 0$ there is only one soliton, so we do not use the notion 'bright' or 'dark' in this case. It is described by (5.9) with the upper sign. The characteristic soliton thickness is given by

$$L = \ell\sqrt{\frac{V}{C - 2\alpha B_0^2 k_\perp^2}}. \tag{5.10}$$

Equation (5.10) is valid both for $\alpha > 0$ as well as for $\alpha < 0$. The phase shift Θ is defined by

$$\tanh\Theta = \begin{cases} \sqrt{1 - \dfrac{2\alpha B_0^2 k_\perp^2}{C}}, & \alpha > 0, \\ \sqrt{1 - \dfrac{C}{2\alpha B_0^2 k_\perp^2}}, & \alpha < 0. \end{cases} \tag{5.11}$$

The soliton amplitude is given by

$$A = \max|h| = \left|\sqrt{\frac{2C}{\alpha}} \pm 2B_0 k_\perp\right|, \tag{5.12}$$

where for $\alpha > 0$ the upper and lower signs correspond to the bright and dark solitons, respectively. For $\alpha < 0$ the bright soliton amplitude is given by (5.12) with the lower sign.

In this section we only obtained the solutions describing planar solitons. Although, at present, there is no rigorous study of the existence of non-planar solitary waves, we expect that there should be a whole three-parametric family of non-planar solitary waves. The two parameters are the same as in the planar solitons, that are C and $k_\perp$. The third parameter is the angle between the plane defined by $\boldsymbol{k}_\perp$ and $\boldsymbol{e}_x$ and the integral curve near the critical point corresponding to $|X| \to \infty$.

Verheest (1996) studied solitary waves of the vector mKdV equation with $\alpha > 0$. He showed that only a planar soliton exists. Below, we will call this soliton the standard mKdV soliton. However, Verheest considered solitary waves propagating exactly along the equilibrium magnetic field. His proof is not valid in the case of oblique propagation. It is straightforward to verify that both bright and dark solitons tend to the standard mKdV soliton as $k_\perp \to 0$.

Since the vector mKdV equation has some similarities to the DNLS equation it is expedient to compare solitons of the two equation. The DNLS equation possesses not only solitons that only depend on the linear combination of the spatial variable and time, but also solitons in the form of an envelope with the magnetic field vector rotating inside this envelope with constant angular velocity. Below, we only consider the first type of solitons. There are no solitons of this type propagating exactly along the equilibrium magnetic field. All of them propagate at some angle with respect to this field. And, in addition, all these solutions are non-planar. The family of solitons is three parametric (Ruderman 1987). The two parameters are $k_\perp$, determining the propagation direction, and the propagation velocity C. The third parameter, ϑ, determines the type of soliton. When $0 < \vartheta < \pi/2$ the component of the magnetic field orthogonal to the equilibrium magnetic field makes one full turn about the equilibrium magnetic field direction in the positive direction when $\alpha > 0$ and in the negative direction when $\alpha < 0$. In accordance with the nomenclature introduced by Ruderman this soliton is called the compression Alfvén soliton. When $\pi/2 < \vartheta < 2\pi/3$ the component of the magnetic field orthogonal to the equilibrium magnetic field makes one full turn about the equilibrium magnetic field direction in the negative direction when $\alpha > 0$ and in the positive direction when $\alpha < 0$. In accordance with the nomenclature introduced by Ruderman this soliton is called the rarefaction Alfvén soliton. Finally, when $2\pi/3 < \vartheta < \pi$ the component of the magnetic field orthogonal to the equilibrium magnetic field rotates by some angle and then returns back to the initial position. This soliton is called magnetosonic, fast when $\alpha > 0$ and slow when $\alpha < 0$.

We see that the properties of solitons of the DNLS equation are very much different from those of solitons of the vector mKdV equation. Ruderman (1987) showed that compression Alfvén solitons are stable with respect to transverse perturbations, while rarefaction Alfvén solitons and magnetosonic solitons are unstable.

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6. Soliton stability

In this section we study the stability of solitons described in the previous section with respect to transverse perturbations. This study is similar to those carried out for the stability of the KdV solitons by Kadomtsev & Petviashvili (1970) and for the stability of the DNLS solitons by Ruderman (1987). We write

$$\boldsymbol{b} = \boldsymbol{b}_s + \tilde{\boldsymbol{b}}, \quad \varphi = \boldsymbol{k}_\perp \cdot \boldsymbol{b}_s + \tilde{\varphi}, \tag{6.1}$$

where $\boldsymbol{b}_s$ corresponds to the soliton defined by (5.5) and (5.9). It describes either the bright or dark soliton. We substitute (6.1) in (4.32) and then linearise the obtained equation with respect to $\tilde{\boldsymbol{b}}$ and $\tilde{\varphi}$. This gives

$$\begin{aligned}\frac{\partial \tilde{\boldsymbol{b}}}{\partial t} + V\frac{\partial \tilde{\boldsymbol{b}}}{\partial x} + \alpha \frac{\partial}{\partial x}\left[\tilde{\boldsymbol{b}}(h^2 - 2B_0 k_\perp h) + \frac{2h}{k_\perp}\boldsymbol{k}_\perp\left(\frac{h}{k_\perp}\boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{b}} - B_0\tilde{\varphi}\right)\right] \\ - 2\alpha B_0 \nabla_\perp\left(\frac{h}{k_\perp}\boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{b}} - B_0\tilde{\varphi}\right) + V\ell^2\frac{\partial^3 \tilde{\boldsymbol{b}}}{\partial x^3} = 0.\end{aligned} \tag{6.2}$$

Equation (4.31) is transformed to

$$\frac{\partial \tilde{\varphi}}{\partial x} = \nabla_\perp \cdot \tilde{\boldsymbol{b}}, \quad \tilde{\varphi} \to 0 \text{ as } x \to \infty. \tag{6.3}$$

Equation (4.32) was derived under the assumption that the ratio of characteristic spatial scale in the y and z-directions to that in the x-direction is ϵ^{-1}. Now we assume that this ratio is even larger and is equal to $(\epsilon\delta)^{-1}$, where $\delta \ll 1$. We also study the stability with respect to normal modes and take $\tilde{\boldsymbol{b}} \propto \exp(\lambda t + \mathrm{i}\delta K_y y + \mathrm{i}\delta K_z z)$. Finally, we use the variable X instead of x. As a result, we transform (6.2) and (6.3) to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}X}\mathcal{L}(\tilde{\boldsymbol{b}}) &= -\lambda\tilde{\boldsymbol{b}} + 2\alpha B_0\left\{\boldsymbol{k}_\perp\left[\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{h}{k_\perp}(\tilde{\varphi} - \boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{b}})\right) - \mathrm{i}\delta B_0(\boldsymbol{K}\cdot\tilde{\boldsymbol{b}})\right]\right. \\ &\quad \left. + \mathrm{i}\delta\boldsymbol{K}\left(\frac{h}{k_\perp}(\boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{b}}) - B_0\tilde{\varphi}\right)\right\},\end{aligned} \tag{6.4}$$

$$\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}X} = \mathrm{i}\delta\boldsymbol{K}\cdot\tilde{\boldsymbol{b}} + \boldsymbol{k}_\perp \cdot \frac{\mathrm{d}\tilde{\boldsymbol{b}}}{\mathrm{d}X}, \tag{6.5}$$

where $\boldsymbol{K} = (0, K_y, K_z)$, and

$$\mathcal{L}(\tilde{\boldsymbol{b}}) = \tilde{\boldsymbol{b}}[\alpha h(h - 2k_\perp B_0) - C] + \frac{2\alpha}{k_\perp^2}\boldsymbol{k}_\perp(\boldsymbol{k}_\perp \cdot \tilde{\boldsymbol{b}})(h - k_\perp B_0)^2 + V\ell^2\frac{\mathrm{d}^2\tilde{\boldsymbol{b}}}{\mathrm{d}X^2}. \tag{6.6}$$

We look for the solution to (6.4) and (6.5) in the form of expansions

$$\tilde{\boldsymbol{b}} = \boldsymbol{b}_0 + \delta\boldsymbol{b}_1 + \delta^2\boldsymbol{b}_2 + \cdots, \quad \tilde{\varphi} = \varphi_0 + \delta\varphi_1 + \delta^2\varphi_2 + \cdots, \quad \lambda = \delta\lambda_1 + \delta^2\lambda_2 + \cdots \tag{6.7}$$

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6.1. *The zero-order approximation*

Substituting (6.7) in (6.4) and (6.5) and using the condition that $\tilde{\boldsymbol{b}} \to 0$ as $X \to -\infty$ we obtain in the zero-order approximation

$$\mathcal{L}(\boldsymbol{b}_0) = 0, \quad \varphi_0 = \boldsymbol{k}_\perp \cdot \boldsymbol{b}_0. \tag{6.8a,b}$$

Differentiating (5.2) and using the second equation in (6.8) we obtain that

$$\boldsymbol{b}_0 = \ell \frac{\boldsymbol{k}_\perp}{k_\perp} \frac{\mathrm{d}h}{\mathrm{d}X}, \quad \varphi_0 = \ell\, k_\perp \frac{\mathrm{d}h}{\mathrm{d}X}. \tag{6.9a,b}$$

The multiplier ℓ is introduced in the expression for $\boldsymbol{b}_0$ to have the same dimension of the left and right sides. We obtain the general solution to the first equation in (6.8) multiplying this expression by an arbitrary constant. Since we solve a linear problem we can take this constant equal to unity without loss of generality.

6.2. *The first-order approximation*

Now we collect the terms of the order of δ in (6.4) and (6.5) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}X}\mathcal{L}(\boldsymbol{b}_1) \;=\; & -\lambda_1 \boldsymbol{b}_0 + 2\alpha B_0 \left\{ \boldsymbol{k}_\perp \left[\frac{\mathrm{d}}{\mathrm{d}X} \left(\frac{h}{k_\perp} (\varphi_1 - \boldsymbol{k}_\perp \cdot \boldsymbol{b}_1) \right) \right.\right. \\ & \left.\left. - \mathrm{i} B_0 (\boldsymbol{K} \cdot \boldsymbol{b}_0) \right] + \mathrm{i}\boldsymbol{K} \left(\frac{h}{k_\perp} (\boldsymbol{k}_\perp \cdot \boldsymbol{b}_0) - B_0 \varphi_0 \right) \right\},\end{aligned} \tag{6.10a}$$

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}X} = \mathrm{i}\boldsymbol{K} \cdot \boldsymbol{b}_0 + \boldsymbol{k}_\perp \cdot \frac{\mathrm{d}\boldsymbol{b}_1}{\mathrm{d}X}. \tag{6.10b}$$

Using (6.9) we transform (6.10*b*) to

$$\varphi_1 - \boldsymbol{k}_\perp \cdot \boldsymbol{b}_1 = \frac{\mathrm{i}\ell h}{k_\perp} (\boldsymbol{k}_\perp \cdot \boldsymbol{K}). \tag{6.11}$$

With the aid of (6.9) and (6.11) we transform (6.10*a*) to

$$\mathcal{L}(\boldsymbol{b}_1) = -\ell \lambda_1 h \frac{\boldsymbol{k}_\perp}{k_\perp} + \mathrm{i}\ell\alpha B_0 h \left[\frac{2\boldsymbol{k}_\perp}{k_\perp^2} (h - k_\perp B_0)(\boldsymbol{k}_\perp \cdot \boldsymbol{K}) + \boldsymbol{K}(h - 2k_\perp B_0) \right]. \tag{6.12}$$

Differentiating (5.2) with respect to C and $\boldsymbol{k}_\perp$ we obtain

$$\mathcal{L}\left(\frac{\boldsymbol{k}_\perp}{k_\perp} \frac{\partial h}{\partial C} \right) = \frac{\boldsymbol{k}_\perp}{k_\perp} h, \tag{6.13a}$$

$$\mathcal{L}\left(\left(\boldsymbol{K} \cdot \frac{\partial}{\partial \boldsymbol{k}_\perp} \right) \frac{\boldsymbol{k}_\perp}{k_\perp} h \right) = \alpha B_0 h \left[\frac{2\boldsymbol{k}_\perp}{k_\perp^2} (h - B_0 k_\perp)(\boldsymbol{k}_\perp \cdot \boldsymbol{K}) + \boldsymbol{K}(h - 2B_0 k_\perp) \right]. \tag{6.13b}$$

It follows from (6.13*a*) and (6.13*b*) that the solution to (6.12) is given by

$$\boldsymbol{b}_1 = -\ell \lambda_1 \frac{\boldsymbol{k}_\perp}{k_\perp} \frac{\partial h}{\partial C} + \mathrm{i}\ell \left(\boldsymbol{K} \cdot \frac{\partial}{\partial \boldsymbol{k}_\perp} \right) \frac{\boldsymbol{k}_\perp}{k_\perp} h. \tag{6.14}$$

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6.3. *The second-order approximation*

Collecting the terms of the order of δ^2 in (6.4) and (6.5), and using (6.9) and (6.11), yields

$$\frac{\mathrm{d}}{\mathrm{d}X}\mathcal{L}(\boldsymbol{b}_2) = -\lambda_1\boldsymbol{b}_1 - \lambda_2\ell\frac{\boldsymbol{k}_\perp}{k_\perp}\frac{\mathrm{d}h}{\mathrm{d}X} + 2\alpha B_0\left\{\boldsymbol{k}_\perp\left[\frac{\mathrm{d}}{\mathrm{d}X}\left(\frac{h}{k_\perp}(\varphi_2 - \boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{b}_2)\right) - \mathrm{i}B_0(\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{b}_1)\right]\right.$$
$$\left. + \frac{\boldsymbol{K}}{k_\perp}[\mathrm{i}(h - B_0k_\perp)(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{b}_1) + \ell B_0 h(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})]\right\}, \qquad (6.15a)$$

$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}X} = \mathrm{i}\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{b}_1 + \boldsymbol{k}_\perp\boldsymbol{\cdot}\frac{\mathrm{d}\boldsymbol{b}_2}{\mathrm{d}X}. \qquad (6.15b)$$

The homogeneous counterpart of (6.15*a*) has a non-trivial solution

$$\boldsymbol{b}_2 = \ell\frac{\boldsymbol{k}_\perp}{k_\perp}\frac{\mathrm{d}h}{\mathrm{d}X}.$$

This implies that (6.15*a*) has solutions only if its right-hand side satisfies the compatibility condition. To obtain this condition we take the scalar product of (6.15*a*) with $(\boldsymbol{k}_\perp/k_\perp)h$ and integrate with respect to X. Using the integration by parts we obtain that the left-hand side is zero, which implies that the right-hand side must be equal to zero. Then, using (6.15*b*) and the integration by parts to transform the term containing φ_2 we obtain the compatibility condition

$$\frac{\lambda_1}{k_\perp}\int_{-\infty}^{\infty} h(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{b}_1)\,\mathrm{d}X = \alpha B_0\int_{-\infty}^{\infty} h\,[\mathrm{i}(h - 2k_\perp B_0)(\boldsymbol{K}\boldsymbol{\cdot}\boldsymbol{b}_1)$$
$$+ \frac{2\mathrm{i}}{k_\perp^2}(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})(h - k_\perp B_0)(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{b}_1) + \frac{2\ell}{k_\perp^2}B_0(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})^2 h\Big]\,\mathrm{d}X. \qquad (6.16)$$

Now we introduce the notation

$$I_1 = \int_{-\infty}^{\infty} h^2\,\mathrm{d}X, \quad I_2 = \int_{-\infty}^{\infty} h^3\,\mathrm{d}X. \qquad (6.17a,b)$$

Then, using (6.14) and the identity

$$\boldsymbol{K}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{k}_\perp}\left[\frac{\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K}}{k_\perp}(I_2 - 2k_\perp B_0 I_1)\right] = (\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})\left(\frac{1}{k_\perp}\boldsymbol{K}\boldsymbol{\cdot}\frac{\partial I_2}{\partial\boldsymbol{k}_\perp} - 2B_0\boldsymbol{K}\boldsymbol{\cdot}\frac{\partial I_1}{\partial\boldsymbol{k}_\perp}\right)$$
$$- 2B_0K^2I_1 + \left(K^2 - \frac{(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})^2}{k_\perp^2}\right)\frac{I_2}{k_\perp}, \qquad (6.18)$$

we transform (6.16) to

$$\lambda_1^2\frac{\partial I_1}{\partial C} - \mathrm{i}\lambda_1\left[\boldsymbol{K}\boldsymbol{\cdot}\frac{\partial I_1}{\partial\boldsymbol{k}_\perp} + \frac{2\alpha}{k_\perp}B_0(\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})\left(\frac{\partial I_2}{\partial C} - 2k_\perp B_0\frac{\partial I_1}{\partial C}\right)\right]$$
$$- 2\alpha B_0\boldsymbol{K}\boldsymbol{\cdot}\frac{\partial}{\partial\boldsymbol{k}_\perp}\left[\frac{\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K}}{k_\perp}(I_2 - 2k_\perp B_0 I_1)\right] = 0. \qquad (6.19)$$

When the discriminant of quadratic equation (6.19) is positive it has two complex roots, and one of these roots has the positive real part. This implies that in this

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case the soliton is unstable. On the other hand, when the discriminant is negative, equation (6.17) has two purely imaginary roots and the soliton is neutrally stable. Hence, the instability condition is written as

$$8\alpha B_0 \frac{\partial I_1}{\partial C} \boldsymbol{K} \cdot \frac{\partial}{\partial \boldsymbol{k}_\perp} \left[\frac{\boldsymbol{k}_\perp \cdot \boldsymbol{K}}{k_\perp} (I_2 - 2k_\perp B_0 I_1) \right] > \left[\boldsymbol{K} \cdot \frac{\partial I_1}{\partial \boldsymbol{k}_\perp} + \frac{2\alpha}{k_\perp} B_0 (\boldsymbol{k}_\perp \cdot \boldsymbol{K}) \left(\frac{\partial I_2}{\partial C} - 2k_\perp B_0 \frac{\partial I_1}{\partial C} \right) \right]^2 . \tag{6.20}$$

Now we consider two cases, one with $\alpha > 0$, and the other with $\alpha < 0$. First we assume that $\alpha > 0$. It is shown in appendix A that I_1 and I_2 are given by

$$I_1 = 4\ell B_0 k_\perp \sqrt{\frac{2V}{\alpha}} F_1(\sigma), \quad I_2 = 4\ell B_0^2 k_\perp^2 \sqrt{\frac{2V}{\alpha}} F_2(\sigma), \tag{6.21a,b}$$

where $\sigma = C(2\alpha B_0^2 k_\perp^2)^{-1}$ and

$$F_1(\sigma) = \sqrt{\sigma - 1} \pm \frac{\pi}{2} + \arctan \frac{1}{\sqrt{\sigma - 1}}, \tag{6.22a}$$

$$F_2(\sigma) = (\sigma + 2) \left(\pm \frac{\pi}{2} + \arctan \frac{1}{\sqrt{\sigma - 1}} \right) + 3\sqrt{\sigma - 1}. \tag{6.22b}$$

Using (6.21) and (6.22) we transform (6.20) to

$$D_\pm \equiv [k_\perp^2 K^2 - (\boldsymbol{k}_\perp \cdot \boldsymbol{K})^2] Q_\pm(\sigma) - (\boldsymbol{k}_\perp \cdot \boldsymbol{K})^2 S_\pm(\sigma) > 0, \tag{6.23}$$

where

$$Q_\pm(\sigma) = \sigma \left(\pm \frac{\pi}{2} + \arctan \frac{1}{\sqrt{\sigma - 1}} \right) + \sqrt{\sigma - 1}, \tag{6.24a}$$

$$S_\pm(\sigma) = \frac{2\sigma}{\sqrt{\sigma - 1}} \left(\pm \frac{\pi}{2} + \arctan \frac{1}{\sqrt{\sigma - 1}} \right)^2 + 2\sqrt{\sigma - 1}. \tag{6.24b}$$

Obviously $Q_+(\sigma) > 0$ meaning that $D_+ > 0$ when $k_\perp^{-1} K^{-1} |\boldsymbol{k}_\perp \cdot \boldsymbol{K}|$ is sufficiently small. This implies that the bright soliton is unstable.

Now we note that $Q_-(0) = 0$ and

$$\frac{\mathrm{d}Q_-}{\mathrm{d}\sigma} = \arctan \frac{1}{\sqrt{\sigma - 1}} - \frac{\pi}{2} < 0, \tag{6.25}$$

which implies that $Q_-(\sigma) < 0$. Since $S_-(\sigma) > 0$ it follows that $D_- < 0$ implying that the dark soliton is stable.

Next, we proceed to the case where $\alpha < 0$. It is shown in appendix A that, now, I_1 and I_2 are given by

$$I_1 = 4\ell B_0 k_\perp \sqrt{\frac{2V}{|\alpha|}} G_1(\sigma), \quad I_2 = 4\ell B_0^2 k_\perp^2 \sqrt{\frac{2V}{|\alpha|}} G_2(\sigma), \tag{6.26a,b}$$

where

$$G_1(\sigma) = \frac{1}{2}\ln\frac{1+\sqrt{1-\sigma}}{1-\sqrt{1-\sigma}} - \sqrt{1-\sigma}, \tag{6.27a}$$

$$G_2(\sigma) = \left(1+\frac{\sigma}{2}\right)\ln\frac{1+\sqrt{1-\sigma}}{1-\sqrt{1-\sigma}} - 3\sqrt{1-\sigma}. \tag{6.27b}$$

Using (6.26) and (6.27) we transform (6.21) to

$$D \equiv [k_\perp^2 K^2 - (\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})^2]Q(\sigma) - (\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K})^2 S(\sigma) > 0, \tag{6.28}$$

where

$$Q(\sigma) = \sigma\ln\frac{1-\sqrt{1-\sigma}}{1+\sqrt{1+\sigma}} + 2\sqrt{1-\sigma}, \tag{6.29a}$$

$$S(\sigma) = \left(\ln\frac{1-\sqrt{1-\sigma}}{1+\sqrt{1+\sigma}}\right)^2 + 4\sqrt{1-\sigma}. \tag{6.29b}$$

Since $Q(0)=0$ and

$$\frac{\mathrm{d}Q}{\mathrm{d}\sigma} = \ln\frac{1-\sqrt{1-\sigma}}{1+\sqrt{1+\sigma}} < 0, \tag{6.30}$$

it follows that $Q(\sigma) > 0$. This implies that $D > 0$ when $k_\perp^{-1}K^{-1}|\boldsymbol{k}_\perp\boldsymbol{\cdot}\boldsymbol{K}|$ is sufficiently small. Consequently, the soliton existing when $\alpha < 0$ is unstable.

As we have already pointed out in § 5, both the bright and dark solitons become the standard mKdV soliton propagating exactly along the equilibrium magnetic field when $\kappa_\perp \to 0$. This soliton only exists when $\alpha > 0$. It is obvious that the previous stability analysis is not valid for $\kappa_\perp = 0$. Hence, the stability of solitons propagating along the equilibrium magnetic field must be studied separately. However, while the expression describing the standard mKdV soliton is simpler than those describing the obliquely propagating soliton, the study of stability of this soliton with respect to transverse perturbations turns out to be much more involved. The complexity of this study is related to the fact that, while obliquely propagating solitons are two parametric, the standard soliton is only one parametric. As a result, while we can obtain the relation similar to (6.13*a*) for the standard mKdV soliton, we cannot obtain an analogue of (6.13*b*). Hence, we cannot get a relatively simple expression for $\boldsymbol{b}_1$ similar to one given by (6.14). To calculate $\boldsymbol{b}_1$ we need to solve a second-order ordinary differential equation with variable coefficients. At present it is even not clear that the analytical expression for $\boldsymbol{b}_1$ can be obtained. Quite possible that this problem can be only solved numerically.

7. Summary and conclusions

In this article we studied the propagation of nonlinear waves along the equilibrium magnetic field in a non-relativistic electron–positron plasma. We assumed that the waves can weakly depend on the spatial coordinates orthogonal to the equilibrium magnetic field. Using the reductive perturbation method we derived the three-dimensional generalisation of the vector modified Korteweg–de Vries equation. We call this equation the 3-D vector mKdV equation.

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We obtained solutions to the 3-D vector mKdV equation in the form of one-dimensional planar solitons propagating at a small angle with respect to the equilibrium magnetic field. The propagation direction is defined by the vector $\boldsymbol{e}_x + \boldsymbol{k}_\perp$, where $\boldsymbol{e}_x$ is the unit vector in the direction of the equilibrium magnetic field, $\boldsymbol{k}_\perp \perp \boldsymbol{e}_x$ and $k_\perp \ll 1$. In planar solitons the magnetic field perturbation is everywhere in the direction of $\boldsymbol{k}_\perp$. We found that in the case where the Alfvén speed V is larger than the sound speed a_0 there are two kinds of soliton, bright and dark. In the bright solitons the magnetic field perturbation is parallel to $\boldsymbol{k}_\perp$, and in the dark solitons it is antiparallel to $\boldsymbol{k}_\perp$. In the case where $V < a_0$ there is only one kind of solitons with the magnetic field parallel to $\boldsymbol{k}_\perp$.

We used the 3-D vector mKdV equation to study the soliton stability with respect to transverse perturbations similar to that carried out by Kadomtsev & Petviashvili (1970) for solitons described by the KdV equation. We found that only the dark solitons are stable, while both the bright solitons in the case where $V > a_0$ as well as solitons in the case where $V < a_0$ are unstable.

Appendix A. Calculation of I_1 and I_2

In this appendix we calculate I_1 and I_2. We start from the case where $\alpha > 0$. Using (5.9) we obtain

$$I_1 = \int_{-\infty}^{\infty} \frac{4(C - 2\alpha B_0^2 k_\perp^2)^2 \, \mathrm{d}X}{[\sqrt{2\alpha C}\cosh(X/L + \Theta) \pm 2\alpha B_0 k_\perp]^2}. \tag{A 1}$$

Using the variable substitution

$$u = \exp(X/L + \Theta) \pm B_0 k_\perp \sqrt{2\alpha/C} \tag{A 2}$$

we transform (A 1) to

$$\begin{aligned} I_1 &= \frac{8L(C - 2\alpha B_0^2 k_\perp^2)^2}{\alpha C} \left(\int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{u \, \mathrm{d}u}{(u^2 + 1 - 2\alpha B_0^2 k_\perp^2/C)^2} \right. \\ &\quad \left. \mp B_0 k_\perp \sqrt{\frac{2\alpha}{C}} \int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{\mathrm{d}u}{(u^2 + 1 - 2\alpha B_0^2 k_\perp^2/C)^2} \right). \end{aligned} \tag{A 3}$$

We easily obtain

$$\int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{u \, \mathrm{d}u}{(u^2 + 1 - 2\alpha B_0^2 k_\perp^2/C)^2} = \frac{1}{2}. \tag{A 4}$$

Next, we calculate the second integral in (A 3). Using the variable substitution

$$w = u \sqrt{\frac{C}{C - 2\alpha B_0^2 k_\perp^2}} \tag{A 5}$$

we transform it to

$$\int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{\mathrm{d}u}{(u^2+1-2\alpha B_0^2 k_\perp^2/C)^2} = \left(\frac{C}{C-2\alpha B_0^2 k_\perp^2}\right)^{3/2} \int_{\pm B_0 k_\perp \sqrt{C/(C-2\alpha B_0^2 k_\perp^2)}}^{\infty} \frac{\mathrm{d}w}{(1+w^2)^2}. \tag{A 6}$$

Then, the integration by parts yields

$$\int \frac{\mathrm{d}w}{(1+w^2)^2} = \int \frac{\mathrm{d}w}{1+w^2} - \int \frac{w^2\,\mathrm{d}w}{(1+w^2)^2} = \frac{w}{2(1+w^2)} - \frac{1}{2}\int \frac{\mathrm{d}w}{1+w^2} = \frac{w}{2(1+w^2)} + \frac{1}{2}\arctan w, \tag{A 7}$$

where we dropped the arbitrary constant. Using (A 4), (A 6), and (A 7) we obtain from (A 3)

$$I_1 = \frac{4L(C-2\alpha B_0^2 k_\perp^2)^2}{\alpha} \left[1 \mp \sqrt{\frac{2\alpha B_0^2 k_\perp^2}{C-2\alpha B_0^2 k_\perp^2}} \left(\frac{\pi}{2} \mp \arctan\sqrt{\frac{2\alpha B_0^2 k_\perp^2}{C-2\alpha B_0^2 k_\perp^2}}\right)\right]. \tag{A 8}$$

Substituting $C = 2\alpha\sigma B_0^2 k_\perp^2$ in this expression we eventually obtain the first expression in (6.21).

Now we proceed to the calculation of I_2. Using (5.9) we obtain

$$I_2 = \pm \int_{-\infty}^{\infty} \frac{8(C-2\alpha B_0^2 k_\perp^2)^3\,\mathrm{d}X}{[\sqrt{2\alpha C}\cosh(X/L+\Theta) \mp 2\alpha b_0 k_\perp]^3}. \tag{A 9}$$

Using the variable substitution defined by (A 2) we transform (A 9) to

$$I_2 = \pm \frac{64L(C-2\alpha B_0^2 k_\perp^2)^3}{(2\alpha C)^{3/2}} \int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{u^2 \mp 2uB_0 k_\perp \sqrt{2\alpha/C} + 2\alpha B_0^2 k_\perp^2/C}{(u^2+1-2\alpha B_0^2 k_\perp^2/C)^3}\,\mathrm{d}u. \tag{A 10}$$

We easily obtain

$$\int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{u\,\mathrm{d}u}{(u^2+1-2\alpha B_0^2 k_\perp^2/C)^3} = \frac{1}{4}. \tag{A 11}$$

Using the variable substitution defined by (A 5) yields

$$\int_{\pm B_0 k_\perp (2\alpha/C)^{1/2}}^{\infty} \frac{u^2+2\alpha B_0^2 k_\perp^2/C}{(u^2+1-2\alpha B_0^2 k_\perp^2/C)^3}\,\mathrm{d}u = \left(\frac{C}{C-2\alpha B_0^2 k_\perp^2}\right)^{3/2} \times \int_{\pm B_0 k_\perp \sqrt{C/(C-2\alpha B_0^2 k_\perp^2)}}^{\infty} \frac{w^2+2\alpha B_0^2 k_\perp^2 (C-2\alpha B_0^2 k_\perp^2)^{-1}}{(1+w^2)^3}\,\mathrm{d}w. \tag{A 12}$$

Using integration by parts and (A 7) yields

$$\int \frac{w^2 + 2\alpha B_0^2 k_\perp^2 (C - 2\alpha B_0^2 k_\perp^2)^{-1}}{(1+w^2)^3}\,\mathrm{d}w = \frac{2\alpha B_0^2 k_\perp^2}{C - 2\alpha B_0^2 k_\perp^2} \int \frac{\mathrm{d}w}{(1+w^2)^2}$$
$$- \frac{C - 4\alpha B_0^2 k_\perp^2}{C - 2\alpha B_0^2 k_\perp^2} \int \frac{w^2\,\mathrm{d}w}{(1+w^2)^3} = -\frac{C - 4\alpha B_0^2 k_\perp^2}{4(C - 2\alpha B_0^2 k_\perp^2)} \frac{w}{(1+w^2)^2}$$
$$+ \frac{C + 4\alpha B_0^2 k_\perp^2}{4(C - 2\alpha B_0^2 k_\perp^2)} \int \frac{\mathrm{d}w}{(1+w^2)^2} = \frac{C + 4\alpha B_0^2 k_\perp^2}{8(C - 2\alpha B_0^2 k_\perp^2)} \arctan w$$
$$+ \frac{w[(C + 4\alpha B_0^2 k_\perp^2) w^2 - C + 12\alpha B_0^2 k_\perp^2]}{8(C - 2\alpha B_0^2 k_\perp^2)(1+w^2)^2}. \tag{A 13}$$

With the aid of this result and (A 11)–(A 13) we obtain from (A 10)

$$I_2 = \frac{2\ell}{\alpha}\sqrt{\frac{2V}{\alpha}} \left[(C + 4\alpha B_0^2 k_\perp^2) \left(\pm\frac{\pi}{2} - \arctan\sqrt{\frac{2\alpha B_0^2 k_\perp^2}{C - 2\alpha B_0^2 k_\perp^2}} \right) \right.$$
$$\left. - 3\sqrt{2\alpha B_0^2 k_\perp^2 (C - 2\alpha B_0^2 k_\perp^2)} \right]. \tag{A 14}$$

Substituting $C = 2\alpha\sigma B_0^2 k_\perp^2$ in this expression we arrive at the second expression in (6.21).

Now we consider the case where $\alpha < 0$. I_1 is given by (A 1) with the upper sign. Using the variable substitution

$$u = e^{X/L} + \frac{1}{\sqrt{\sigma}} \tag{A 15}$$

we transform the expression for I_1 to

$$I_1 = -\frac{8\varsigma^3\sqrt{-VC}}{\alpha} \int_{1/\sqrt{\sigma}}^{\infty} \frac{u - 1/\sqrt{\sigma}}{(u^2 - \varsigma^2)^2}\,\mathrm{d}u, \tag{A 16}$$

where $\varsigma = \sqrt{1/\sigma - 1}$. Using the expansion

$$\frac{u - \sqrt{\sigma}}{(u^2 - \varsigma^2)^2} = \frac{1}{4\varsigma^3\sqrt{\sigma}} \left(\frac{1}{u - \varsigma} - \frac{1}{u + \varsigma} \right) + \left(\frac{1}{4\varsigma} - \frac{1}{4\varsigma^3\sqrt{\sigma}} \right) \frac{1}{(u - \varsigma)^2}$$
$$- \left(\frac{1}{4\varsigma} + \frac{1}{4\varsigma^3\sqrt{\sigma}} \right) \frac{1}{(u + \varsigma)^2} \tag{A 17}$$

we obtain

$$\int_{1/\sqrt{\sigma}}^{\infty} \frac{u - \sqrt{\sigma}}{(u^2 - \varsigma^2)^2}\,\mathrm{d}u = \frac{1}{4\varsigma^3\sqrt{\sigma}} \ln\frac{1 + \sqrt{1-\sigma}}{1 - \sqrt{1-\sigma}} - \frac{1}{2\varsigma^2}. \tag{A 18}$$

Using (A 11), (A 16) and (A 18), and the expression for C and ς in terms of σ we obtain the first expression in (6.26).

Now we proceed to calculating I_2; I_2 is given by (A 9) with the upper sign. Using the variable substitution defined by (A 15) we transform it to

$$I_2 = \frac{16\ell C\varsigma^5}{\alpha}\sqrt{\frac{2V}{|\alpha|}}\int_{1/\sqrt{\sigma}}^{\infty}\frac{(u-1/\sqrt{\sigma})^2}{(u^2-\varsigma^2)^3}\,\mathrm{d}u. \tag{A 19}$$

Using the expansion

$$\begin{aligned}\frac{(u-1/\sqrt{\sigma})^2}{(u^2-\varsigma^2)^3} &= \left(1-\frac{3}{\varsigma^3\sqrt{\sigma}}\right)\left(\frac{1}{u+\varsigma}-\frac{1}{u-\varsigma}\right)+\frac{1}{16\varsigma^2}\left(1-\frac{2}{\varsigma\sqrt{\sigma}}-\frac{3}{\varsigma^2}\right)\\ &\quad\times\frac{1}{(u+\varsigma)^2}+\frac{1}{16\varsigma^2}\left(1+\frac{2}{\varsigma\sqrt{\sigma}}-\frac{3}{\varsigma^2}\right)\frac{1}{(u-\varsigma)^2}\\ &\quad-\frac{1}{8\varsigma}\left(1+\frac{1}{\varsigma\sqrt{\sigma}}\right)^2\frac{1}{(u+\varsigma)^3}+\frac{1}{8\varsigma}\left(1-\frac{1}{\varsigma\sqrt{\sigma}}\right)^2\frac{1}{(u-\varsigma)^3}\end{aligned} \tag{A 20}$$

we obtain

$$\begin{aligned}\int_{1/\sqrt{\sigma}}^{\infty}\frac{(u-1/\sqrt{\sigma})^2}{(u^2-\varsigma^2)^3}\,\mathrm{d}u &= \frac{1}{16\varsigma^3}\left(1-\frac{3}{\sigma\varsigma^2}\right)\ln\frac{1-\varsigma\sqrt{\sigma}}{1+\varsigma\sqrt{\sigma}}\\ &\quad+\frac{\sqrt{\sigma}}{16\varsigma^2}\left(1-\frac{2}{\varsigma\sqrt{\sigma}}-\frac{3}{\varsigma^2}\right)\frac{1}{1+\varsigma\sqrt{\sigma}}\\ &\quad+\frac{\sqrt{\sigma}}{16\varsigma^2}\left(1+\frac{2}{\varsigma\sqrt{\sigma}}-\frac{3}{\varsigma^2}\right)\frac{1}{1-\varsigma\sqrt{\sigma}}\\ &\quad-\frac{\sigma}{16\varsigma}\left(1+\frac{1}{\varsigma\sqrt{\sigma}}\right)^2\frac{1}{(1+\varsigma\sqrt{\sigma})^2}\\ &\quad+\frac{\sigma}{16\varsigma}\left(1-\frac{1}{\varsigma\sqrt{\sigma}}\right)^2\frac{1}{(1-\varsigma\sqrt{\sigma})^2}.\end{aligned} \tag{A 21}$$

Using (A 19) and (A 21), and the expression for C and ς in terms of σ we obtain the second expression in (6.26).

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