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k-quasi-convexity reduces to quasi-convexity

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The relation between quasi-convexity and k-quasi-convexity, $k \geqslant 2$, is investigated. It is shown that every smooth strictly k-quasi-convex integrand with p-growth at infinity, $p > 1$, is the restriction to kth-order symmetric tensors of a quasi-convex function with the same growth. When the smoothness condition is dropped, it is possible to prove an approximation result. As a consequence, lower semicontinuity results for kth-order variational problems are deduced as corollaries of well-known first-order theorems. This generalizes a previous work by Dal Maso *et al.*, in which the case where $k = 2$ was treated.

1. Introduction

We consider higher-order variational problems in which the energy functional is expressed by

$$u \mapsto \int_{\Omega} f(x, u, \nabla u, \dots, \nabla^k u)\, \mathrm{d}x, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, $N, k \geqslant 2$ are integer, and f is a scalar function satisfying suitable growth conditions. Although our treatment can be extended to the vectorial case, to keep the formulation as simple as possible we will treat the case of scalar functions $u\colon \Omega \to \mathbb{R}$. Functionals of this type appear in the study of elastic materials of grade k [21], in the theory of second-order structured deformations [19], in the Blake–Zisserman model for image segmentation in computer vision [4], in gradient theories of phase transitions within elasticity regimes [6, 13, 18] and in the description of equilibria of micromagnetic materials [5, 8, 18, 20]. In order to study lower semicontinuity of functionals of this type, Meyers [16] introduced the notion of k-quasi-convexity (see also [3, 11]), extending the definition of quasi-convexity given by Morrey [17].

Let

$$E_k \subset \overbrace{\mathbb{R}^N \times \cdots \times \mathbb{R}^N}^{k\ \text{times}} = \mathbb{R}^{N^k}$$

be the set of kth-order tensors of $\mathbb{R}^N$ that are symmetric with respect to all permutations of indices. In particular, E_2 coincides with the set of the symmetric $N \times N$ matrices. A function $f \in L^1_{\mathrm{loc}}(E_k)$ is said to be *k-quasi-convex* if

$$\int_Q [f(A + \nabla^k \phi) - f(A)]\, \mathrm{d}x \geqslant 0$$

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for every $A \in E_k$ and every $\phi \in C^k_c(Q)$, where $Q = (0,1)^N$ is the open unit cube in $\mathbb{R}^N$, and $C^k_c(Q)$ is the set of functions of class C^k with compact support in Q. We recall that a function $F \in L^1_{\mathrm{loc}}(\mathbb{R}^{N^k})$ is said to be 1-quasi-convex (or simply quasi-convex) if

$$\int_Q [F(A + \nabla\varphi) - F(A)]\,\mathrm{d}x \geqslant 0$$

for every $A \in \mathbb{R}^{N^k}$ and every $\varphi \in C^1_c(Q; \mathbb{R}^{N^{k-1}})$. In [16], Meyers proved that k-quasi-convexity is a necessary and sufficient condition for sequential lower semicontinuity of (1.1) with respect to weak convergence in the Sobolev space $W^{k,p}(\Omega)$, under appropriate p-growth and continuity conditions on the integrand f. This result was later extended to the case where f is a Carathéodory integrand by Fusco [11] and by Guidorzi and Poggiolini [12], for $p = 1$ and $p > 1$, respectively.

We investigate the relation between k-quasi-convexity and quasi-convexity. This problem has been studied for the case when $k = 2$ by Dal Maso *et al*. [7], who proved that every strictly 2-quasi-convex function (see theorem 1.1(a)) of class C^1, whose gradient is locally Lipschitz continuous, is the restriction to symmetric matrices of a 1-quasi-convex function. Here we extend this result to the case when $k > 2$.

THEOREM 1.1. *Let $k \in \mathbb{N}$, $k \geqslant 2$. Let $f \in C^1(E_k)$, and let $1 < p < \infty$, $\mu \geqslant 0$, $L > 0$ and $\nu > 0$. Assume that*

(a) (*strict k-quasi-convexity*)

$$\int_Q [f(A + \nabla^k\phi) - f(A)]\,\mathrm{d}x \geqslant \nu \int_Q (\mu^2 + |A|^2 + |\nabla^k\phi|^2)^{(p-2)/2} |\nabla^k\phi|^2\,\mathrm{d}x$$

for every $A \in E_k$ and every $\phi \in C^k_c(Q)$,

(b) (*Lipschitz condition for gradients*)

$$|\nabla f(A + B) - \nabla f(A)| \leqslant L(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B| \tag{1.2}$$

for every $A, B \in E_k$.

Then there exists a 1-quasi-convex function $F\colon \mathbb{R}^{N^k} \to \mathbb{R}$ such that

$$F(A) = f(A) \qquad \forall A \in E_k, \tag{1.3}$$

$$|F(A)| \leqslant c_f(1 + |A|^p) \quad \forall A \in \mathbb{R}^{N^k}, \tag{1.4}$$

for a suitable constant c_f depending on f.

Note that the above conditions together imply $L \geqslant \nu$ (see proposition 2.8). When $p \geqslant 2$, we also give an explicit expression for the function F (see formula (3.9)).

The proof of theorem 1.1 is based on two preliminary lemmas (lemma 3.1 for the case when $1 < p < 2$ and lemma 3.2 for the case when $p \geqslant 2$) which, roughly speaking, show that every strictly j-quasi-convex function is the restriction to the appropriate set of tensors of a *strictly* $(j-1)$-quasi-convex function, for each $j = 2, \dots, k$. The main feature of lemmas 3.1 and 3.2 is the fact that they can be applied iteratively, since they preserve the *strict* quasi-convexity (up to a ‘perturbation’, in the case where $1 < p < 2$). Then, theorem 1.1 easily follows (see the end of § 3).

It is not clear whether the theorem still holds true when condition (1.2) is weakened. However, if we substitute (1.2) with the milder (see proposition 2.9) condition (1.5), we obtain an approximation result for the function f. More precisely, we show that a strictly k-quasi-convex function with p-growth at infinity can be obtained as a pointwise limit of a sequence of 1-quasi-convex functions with the same growth (see [7, theorem 2] for the case where $k = 2$).

Theorem 1.2. *Let $k \in \mathbb{N}$, $k \geqslant 2$. Let $1 < p < \infty$, $\mu \geqslant 0$, $\nu > 0$, $M > 0$, and let $f\colon E_k \to \mathbb{R}$ be a measurable function such that*

(a) (*strict k-quasi-convexity*)

$$\int_Q [f(A + \nabla^k \phi) - f(A)]\,\mathrm{d}x \geqslant \nu \int_Q (\mu^2 + |A|^2 + |\nabla^k \phi|^2)^{(p-2)/2} |\nabla^k \phi|^2\,\mathrm{d}x$$

for every $A \in E_k$ and every $\phi \in C^k_{\mathrm{c}}(Q)$,

(b) (*p-growth condition*)

$$|f(A)| \leqslant M(1 + |A|^p) \tag{1.5}$$

for every $A \in E_k$.

Then there exists an increasing sequence $\{F_i\}_{i\in\mathbb{N}}$ of 1-quasi-convex functions

$$F_i\colon \mathbb{R}^{N^k} \to \mathbb{R}$$

such that

$$\lim_{i\to+\infty} F_i(A) = f(A) \qquad \forall A \in E_k, \tag{1.6}$$

$$|F_i(A)| \leqslant M_i(1 + |A|^p) \quad \forall A \in \mathbb{R}^{N^k},\ \forall i \in \mathbb{N}, \tag{1.7}$$

where $\{M_i\}_{i\in\mathbb{N}}$ is a sequence of positive constants depending on i and on the constants p, μ, ν and M, but not on the specific function f.

To show this, we use the property that every k-quasi-convex function with p-growth is locally Lipschitz. Here we give a proof of this fact (see proposition 2.7) that was already known for the cases $k = 1$ [15] and $k = 2$ [12]. Due to theorem 1.2, the study of lower semicontinuity of (1.1) reduces to a first-order problem. Thus, when f is a k-quasi-convex normal integrand (see theorem 1.3(a)) we can prove the following result (see [7, theorem 3] for the case where $k = 2$). Here we use the notation

$$E_1 := \mathbb{R}^N, \qquad E_{[k-1]} := \mathbb{R} \times E_1 \times \cdots \times E_{k-1}$$

and

$$\mathrm{SBH}^{(k)}(\Omega) := \{u \in W^{k-1,1}(\Omega)\colon \nabla^{k-1} u \in \mathrm{SBV}(\Omega; E_{k-1})\}.$$

Theorem 1.3. *Let $k \in \mathbb{N}$, $k \geqslant 2$. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let*

$$f\colon \Omega \times E_{[k-1]} \times E_k \to [0, +\infty)$$

be a measurable function such that

(a) *$f(x, \cdot, \cdot)$ is lower semicontinuous on $E_{[k-1]} \times E_k$ for $\mathcal{L}^N$ almost every (a.e.) $x \in \Omega$;*

(b) *$f(x, \boldsymbol{v}, \cdot)$ is k-quasi-convex on E_k for $\mathcal{L}^N$-a.e. $x \in \Omega$ and every $\boldsymbol{v} \in E_{[k-1]}$;*

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(c) *there exist a locally bounded function* $a\colon \Omega\times E_{[k-1]}\to[0,+\infty)$ *and a constant* $p>1$ *such that*

$$0\leqslant f(x,\boldsymbol{v},A)\leqslant a(x,\boldsymbol{v})(1+|A|^p)$$

for $\mathcal{L}^N$*-a.e.* $x\in\Omega$ *and every* $(\boldsymbol{v},A)\in E_{[k-1]}\times E_k$.

Then

$$\int_\Omega f(x,u,\nabla u,\dots,\nabla^k u)\,\mathrm{d}x\leqslant\liminf_{j\to+\infty}\int_\Omega f(x,u_j,\nabla u_j,\dots,\nabla^k u_j)\,\mathrm{d}x$$

for every $u\in\mathrm{SBH}^{(k)}(\Omega)$ *and any sequence* $\{u_j\}\subset\mathrm{SBH}^{(k)}(\Omega)$ *converging to* u *in* $W^{k-1,1}(\Omega)$ *and such that*

$$\sup_j\left(\|\nabla^k u_j\|_{L^p(\Omega)}+\int_{S(\nabla^{k-1}u_j)}\theta(|[\nabla^{k-1}u_j]|)\,\mathrm{d}\mathcal{H}^{N-1}\right)<+\infty,$$

where $\theta\colon[0,+\infty)\to[0,+\infty)$ *is a concave, non-decreasing function such that*

$$\lim_{t\to0^+}\frac{\theta(t)}{t}=+\infty,$$

$\nabla^k u$ *is the density of the absolutely continuous part of* $D(\nabla^{k-1}u)$ *with respect to the* N*-dimensional Lebesgue measure and* $[\nabla^{k-1}u_j]$ *denotes the jump of* $\nabla^{k-1}u_j$ *on the jump set* $S(\nabla^{k-1}u_j)$.

This extends to the kth order a lower semicontinuity property of 1-quasi-convex functions in $\mathrm{SBV}(\Omega;\mathbb{R}^d)$ due to Ambrosio [2], later generalized by Kristensen [14], and a lower semicontinuity theorem for 2-quasi-convex integrands in $\mathrm{SBH}(\Omega;\mathbb{R}^d)$ proven by Dal Maso *et al*. [7]. As a corollary, we recover [12, theorem 7.1].

Corollary 1.4. *Let* Ω, f, k *and* p *be as in theorem 1.3. Then*

$$\int_\Omega f(x,u,\nabla u,\dots,\nabla^k u)\,\mathrm{d}x\leqslant\liminf_{j\to\infty}\int_\Omega f(x,u_j,\nabla u_j,\dots,\nabla^k u_j)\,\mathrm{d}x$$

for every $u\in W^{k,p}(\Omega)$ *and any sequence* $\{u_j\}\subset W^{k,p}(\Omega)$ *weakly converging to* u *in* $W^{k,p}(\Omega)$.

We note that in [12], Guidorzi and Poggiolini require the function f to be locally Lipschitz continuous with respect to the last variable. As previously mentioned, we do not need this hypothesis, since we prove here that this is a direct consequence of k-quasi-convexity and p-growth.

Finally, we mention that proof of the analogue of theorem 1.3 for the case $p=1$, even when $k=2$, is still an open problem unless very special functions f are considered [9]. This will probably require new and original ideas. Indeed, we think that, for $p=1$, the fundamental Korn-type inequalities (see lemmas 2.13 and 2.14) used in the proofs of theorem 1.1 and theorem 1.2 fail, although we do not have any explicit counter-example.

This paper is organized as follows. In § 2 we give the setting of the problem. Section 3 contains the proof of theorem 1.1, while theorems 1.2 and 1.3 are proved in § 4. Finally, some auxiliary results that are used extensively in the paper can be found in Appendix A.

2. Setting

Throughout the paper N and k are fixed integer numbers, with $N, k \geqslant 2$. For this reason, we will often not indicate the explicit dependence on N and k. Also, $\Omega \subset \mathbb{R}^N$ is an open bounded set, and $Q = (0,1)^N$ denotes the open unit cube of $\mathbb{R}^N$.

Definition 2.1. Let $A \in \mathbb{R}^{N^k}$. We say that A is a *kth-order tensor in* $\mathbb{R}^N$.

The components of a tensor $A \in \mathbb{R}^{N^k}$ will be denoted by the symbols

$$A_{i_1 \cdots i_k}, \quad i_1, \dots, i_k = 1, \dots, N.$$

Moreover, the scalar product of two tensors $A, B \in \mathbb{R}^{N^k}$ is given by

$$A \cdot B := \sum_{i_1,\dots,i_k=1}^{N} A_{i_1 \cdots i_k} B_{i_1 \cdots i_k}.$$

Accordingly, the norm of a kth-order tensor $A \in \mathbb{R}^{N^k}$ is

$$|A| := \biggl[\sum_{i_1,\dots,i_k=1}^{N} |A_{i_1 \cdots i_k}|^2\biggr]^{1/2}.$$

Now let $s \in \{1, \dots, k-1\}$ be fixed. For any $\zeta \in C^s(Q; \mathbb{R}^{N^{k-s}})$, we can regard the sth-order gradient $\nabla^s \zeta$ of ζ as a kth-order tensor in $\mathbb{R}^N$ by setting

$$(\nabla^s \zeta)_{i_1 \cdots i_k} := \frac{\partial^s \zeta_{i_1 \cdots i_{k-s}}}{\partial x_{i_{k-s+1}} \cdots \partial x_{i_k}}, \quad i_1, \dots, i_k = 1, \dots, N.$$

Note also that $\nabla^s \zeta$ is symmetric with respect to every permutation of the last s indices. To take account of this property, we introduce some additional notation.

Definition 2.2. Let $A \in \mathbb{R}^{N^k}$ be a kth-order tensor in $\mathbb{R}^N$, and let

$$j, r \in \{1, \dots, k\}.$$

The *(j,r)-transpose of* A is the element $A^{\mathrm{T}^j_r}$ of $\mathbb{R}^{N^k}$ such that (assuming, for instance, $j \leqslant r$)

$$(A^{\mathrm{T}^j_r})_{i_1 i_2 \cdots i_k} = A_{i_1 i_2 \cdots i_{j-1} i_r i_{j+1} \cdots i_{r-1} i_j i_{r+1} \cdots i_k}, \quad i_1, \dots, i_k = 1, \dots, N.$$

We then set

$$E_k^{N^{k-s}} := \{A \in \mathbb{R}^{N^k} : A = A^{\mathrm{T}^r_j} \text{ for every } r, j = k-s+1, \dots, k\}.$$

In particular, we will make the identification $E_k^{N^{k-1}} = \mathbb{R}^{N^k}$. In this way, for every $\zeta \in C^s(Q; \mathbb{R}^{N^{k-s}})$, we have

$$\nabla^s \zeta \in E_k^{N^{k-s}}.$$

To include the case where $s = k$, we define

$$E_k^1 := \{A \in \mathbb{R}^{N^k} : A = A^{\mathrm{T}^r_j} \text{ for every } r, j = 1, \dots, k\}.$$

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Very often we will simply write E_k instead of E_k^1. Hence, we have that

$$\nabla^k \phi \in E_k$$

for every $\phi \in C^k(Q)$, using the notation

$$(\nabla^k \phi)_{i_1 \cdots i_k} := \frac{\partial^k \phi}{\partial x_{i_1} \cdots \partial x_{i_k}}, \quad i_1, \ldots, i_k = 1, \ldots, N.$$

Now we define the symmetric part of an element of $E_k^{N^{k-s}}$.

Definition 2.3. The *symmetrization operator* $\mathcal{S}_{s+1} \colon E_k^{N^{k-s}} \to E_k^{N^{k-s-1}}$ is defined by

$$\mathcal{S}_{s+1} A := \frac{1}{s+1} \sum_{r=k-s}^{k} A^{\mathrm{T}_r^{k-s}} = \frac{A + A^{\mathrm{T}_{k-s+1}^{k-s}} + \cdots + A^{\mathrm{T}_k^{k-s}}}{s+1} \quad \text{for every } A \in E_k^{N^{k-s}}.$$

We will say that $\mathcal{S}_{s+1}A$ is the *symmetric part* of A.

The subscript $s+1$ denotes the fact that the tensor $\mathcal{S}_{s+1}A$ is symmetric in the last $s+1$ entries.

Definition 2.4. Accordingly, we define the *antisymmetric part* of a tensor $A \in E_k^{N^{k-s}}$ as the tensor $\mathcal{A}_{s+1}A \in E_k^{N^{k-s}}$ given by

$$\mathcal{A}_{s+1}A := A - \mathcal{S}_{s+1}A = \frac{sA - (A^{\mathrm{T}_{k-s+1}^{k-s}} + \cdots + A^{\mathrm{T}_k^{k-s}})}{s+1}.$$

We will use the notation

$$\mathcal{A}_{s+1} E_k^{N^{k-s}} := \{\mathcal{A}_{s+1}A : A \in E_k^{N^{k-s}}\} \subset E_k^{N^{k-s}}.$$

The next proposition generalizes the well-known fact that symmetric and antisymmetric matrices define orthogonal spaces. For the convenience of the reader, the proof is given in Appendix A.

Proposition 2.5. *It holds that*

$$A \cdot B = 0 \quad \textit{for every } A \in E_k^{N^{k-s-1}} \textit{ and for every } B \in \mathcal{A}_{s+1} E_k^{N^{k-s}}.$$

We now give the definition of (higher-order) quasi-convexity.

Definition 2.6. Let $j \in \{1, \ldots, k\}$. A function $f \in L^1_{\mathrm{loc}}(E_k^{N^{k-j}})$ is said to be j-*quasi-convex* if

$$\int_Q [f(A + \nabla^j \phi) - f(A)] \, \mathrm{d}x \geqslant 0$$

for every $A \in E_k^{N^{k-j}}$ and for every $\phi \in C_{\mathrm{c}}^j(Q; \mathbb{R}^{N^{k-j}})$.

It is very well known that every convex function is locally Lipschitz. This property still holds true for j-quasi-convex functions with p-growth. Here we give a proof of this fact that is, in general, explicitly stated only for the case where $j = 2$ [12].

PROPOSITION 2.7. *Let $j \in \{2, \dots, k\}$, and let $f \in L^1_{\mathrm{loc}}(E_k^{N^{k-j}})$ be j-quasi-convex. Assume, in addition, that*

$$|f(A)| \leqslant M(1 + |A|^p) \quad \text{for every } A \in E_k^{N^{k-j}} \tag{2.1}$$

for some $M > 0$ and $1 < p < \infty$. Then, there exists a constant

$$L = L(N, M, k, j, p) > 0$$

such that

$$|f(A + B) - f(A)| \leqslant L(1 + |A|^{p-1} + |B|^{p-1})|B| \quad \text{for every } A, B \in E_k^{N^{k-j}}.$$

Proof. Let us set

$$X := \{b \otimes \overbrace{w \otimes \cdots \otimes w}^{j \text{ times}} \colon b \in \mathbb{R}^{N^{k-j}},\ w \in \mathbb{S}^{N-1}\} \subset E_k^{N^{k-j}},$$
$$m = m(N, k, j) := \dim E_k^{N^{k-j}}.$$

Here, for every $b \in \mathbb{R}^{N^{k-j}}$ and $w \in \mathbb{S}^{N-1}$, $b \otimes w \otimes \cdots \otimes w$ denotes the element of $\mathbb{R}^{N^k}$ such that

$$(b \otimes w \otimes \cdots \otimes w)_{i_1, \dots, i_k} = b_{i_1 \cdots i_{k-j}} w_{i_{k-j+1}} \cdots w_{i_k}, \quad i_1, \dots, i_k = 1, \dots, N.$$

It can be proven that the orthogonal complement of X in $E_k^{N^{k-j}}$ is zero, so that

$$\operatorname{span} X = E_k^{N^{k-j}}.$$

Now let $\{\omega_1, \dots, \omega_m\} \subset X$ be a (not necessarily orthonormal) basis for $E_k^{N^{k-j}}$, with $|\omega_i| = 1$ for $i = 1, \dots, m$, and let $c_1, \dots, c_m \in \mathbb{R}$ be such that

$$B = \sum_{i=1}^{m} c_i \omega_i.$$

We have

$$\begin{aligned}
|f(A + B) - f(A)| &= \left| f\left(A + \sum_{i=1}^{m} c_i \omega_i\right) - f(A) \right| \\
&\leqslant \left| f\left(A + \sum_{i=1}^{m} c_i \omega_i\right) - f\left(A + \sum_{i=1}^{m-1} c_i \omega_i\right) \right| \\
&\quad + \left| f\left(A + \sum_{i=1}^{m-1} c_i \omega_i\right) - f\left(A + \sum_{i=1}^{m-2} c_i \omega_i\right) \right| \\
&\quad + \cdots + |f(A + c_1 \omega_1) - f(A)|.
\end{aligned}$$

It will be sufficient to prove that there exists $C = C(N, M, k, j, p)$ such that, for every $l = 1, \dots, m$,

$$\left| f\left(c_l \omega_l + A + \sum_{i=0}^{l-1} c_i \omega_i\right) - f\left(A + \sum_{i=0}^{l-1} c_i \omega_i\right) \right| \leqslant C(1 + |A|^{p-1} + |B|^{p-1})|B|, \tag{2.2}$$

where we set $c_0 := 0$ and $\omega_0 := 0$. Then, the conclusion will follow by defining $L := mC$.

By [10, proposition 3.4 and example 3.10(d)], for every $R \in E_k^{N^{k-j}}$ and every $\omega \in X$, the function

$$t \to f(t\omega + R)$$

is convex in $\mathbb{R}$. Hence, defining

$$G(t) := f\biggl(tc_l\omega_l + A + \sum_{i=0}^{l-1} c_i\omega_i\biggr)$$

and using (2.1), for every $t \geqslant 1$ we have

$$\begin{aligned}
&\biggl| f\biggl(c_l\omega_l + A + \sum_{i=0}^{l-1} c_i\omega_i\biggr) - f\biggl(A + \sum_{i=0}^{l-1} c_i\omega_i\biggr)\biggr| \\
&\quad = G(1) - G(0) \leqslant \frac{G(t) - G(0)}{t} \\
&\quad = \frac{1}{t}\biggl(f(tc_l\omega_l + A + \sum_{i=0}^{l-1} c_i\omega_i) - f(A + \sum_{i=0}^{l-1} c_i\omega_i)\biggr) \\
&\quad \leqslant \frac{M}{t}\biggl(2 + |tc_l\omega_l + A + \sum_{i=0}^{l-1} c_i\omega_i|^p + \biggl|A + \sum_{i=0}^{l-1} c_i\omega_i\biggr|^p\biggr) \\
&\quad \leqslant \frac{M}{t}\biggl(2 + 2^{p-1}t^p|c_l|^p + (2^{p-1}+1)\biggl|A + \sum_{i=0}^{l-1} c_i\omega_i\biggr|^p\biggr) \\
&\quad \leqslant \frac{M}{t}(2 + 2^{p-1}t^p\|B\|^p + 2^{p-1}(2^{p-1}+1)|A|^p + 2^{p-1}(2^{p-1}+1)m^{p/2}\|B\|^p),
\end{aligned}$$

where we set

$$\|B\| := \biggl(\sum_{i=0}^{m} c_i^2\biggr)^{1/2}.$$

Let us now choose

$$t = \frac{(|A|^{p-1} + \|B\|^{p-1})^{1/(p-1)}}{\|B\|} \geqslant 1.$$

Noting that

$$t^{p-1}\|B\|^p = (|A|^{p-1} + \|B\|^{p-1})\|B\|, \qquad \frac{|A|^p}{t} \leqslant |A|^{p-1}\|B\|, \qquad \frac{\|B\|^p}{t} \leqslant \|B\|^p,$$

and using the fact that $\|\cdot\|$ and $|\cdot|$ are equivalent norms, we obtain (2.2). □

The next proposition shows that conditions (a) and (b) of theorem 1.1 necessarily imply $L \geqslant \nu$.

PROPOSITION 2.8. *Let $f \in C^1(E_k)$ satisfy conditions (a) and (b) of theorem 1.1 for some constants $\mu \geqslant 0$, $L, \nu > 0$ and $1 < p < \infty$. Then $L \geqslant \nu$.*

Proof. Let $A \in E_k$, $\phi \in C_c^k(Q)$, and let $x \in Q$. By the mean-value theorem,

$$f(A + \nabla^k\phi(x)) - f(A) = [\nabla f(A + t\nabla^k\phi(x)) - \nabla f(A)] \cdot \nabla^k\phi(x) + \nabla f(A) \cdot \nabla^k\phi(x)$$

for some $t \in [0,1]$. Integrating the last equality, since $\phi \in C_c^k(Q)$, we get

$$\int_Q [f(A + \nabla^k\phi(x)) - f(A)]\,dx = \int_Q [\nabla f(A + t\nabla^k\phi(x)) - \nabla f(A)] \cdot \nabla^k\phi(x)\,dx.$$

Hence, using property (b),

$$\begin{aligned}\int_Q [f(A + \nabla^k\phi(x)) - f(A)]\,dx &\leqslant L\int_Q (\mu^2 + |A|^2 + t^2|\nabla^k\phi(x)|^2)^{(p-2)/2} t|\nabla^k\phi(x)|^2\,dx \\ &\leqslant L\int_Q (\mu^2 + |A|^2 + |\nabla^k\phi(x)|^2)^{(p-2)/2}|\nabla^k\phi(x)|^2\,dx,\end{aligned}$$

since the function $t \mapsto (\mu^2 + |A|^2 + t^2|\nabla^k\phi|^2)^{(p-2)/2} t|\nabla^k\phi|^2$ is increasing. Comparing the last relation and condition (a), we conclude that $L \geqslant \nu$. □

We now prove that condition (1.2) is stronger than (1.5).

Proposition 2.9. *Let $j \in \{2,\dots,k\}$, let $L > 0$, $\mu \geqslant 0$, $1 < p < \infty$ and let $f \in C^1(E_k^{N^{k-j}})$ be such that*

$$|\nabla f(A + B) - \nabla f(A)| \leqslant L(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B| \quad \textit{for every } A, B \in E_k^{N^{k-j}}. \tag{2.3}$$

Then, there exists a positive constant c_f, depending on f, such that

$$|f(A)| \leqslant c_f(1 + |A|^p) \quad \forall A \in E_k^{N^{k-j}}.$$

Proof. Let $C \in E_k^{N^{k-j}} \setminus \{0\}$ be fixed. Then, by the mean-value theorem, for every $A \in E_k^{N^{k-j}}$ we have

$$f(A) = f(C) + [\nabla f(C + t(A - C)) - \nabla f(C)] \cdot (A - C) + \nabla f(C) \cdot (A - C),$$

for some $t \in [0,1]$. By (2.3),

$$\begin{aligned}|f(A)| &\leqslant |f(C)| + L(\mu^2 + |C|^2 + t^2|A - C|^2)^{(p-2)/2} t|A - C|^2 + |\nabla f(C)||A - C| \\ &\leqslant |f(C)| + L(\mu^2 + |C|^2 + |A - C|^2)^{(p-2)/2}|A - C|^2 + |\nabla f(C)||A - C|,\end{aligned} \tag{2.4}$$

since the function $t \mapsto (\mu^2 + |C|^2 + t^2|A - C|^2)^{(p-2)/2} t|A - C|^2$ is increasing. Concerning the last term, using Young's inequality we have

$$|\nabla f(C)||A - C| \leqslant \frac{|\nabla f(C)|^{p'}}{p'} + \frac{|A - C|^p}{p} \leqslant \frac{|\nabla f(C)|^{p'}}{p'} + \frac{2^{p-1}}{p}(|A|^p + |C|^p), \tag{2.5}$$

where $p' = p/(p-1)$. Since the function $r \mapsto (\mu^2 + |C|^2 + r)^{(p-2)/2} r$ is increasing in $\mathbb{R}$, using inequality

$$|A - C|^2 \leqslant 2|A|^2 + 2|C|^2,$$

we have

$$
\begin{aligned}
(\mu^2+|C|^2+|A-C|^2)^{(p-2)/2}|A-C|^2 & \\
&\leqslant 2(\mu^2+3|C|^2+2|A|^2)^{(p-2)/2}(|A|^2+|C|^2) \\
&\leqslant 2\max\{1,|C|^2\}(\mu^2+3|C|^2+2|A|^2)^{(p-2)/2}(1+|A|^2) \\
&\leqslant 2K^{(p-2)/2}\max\{1,|C|^2\}(1+|A|^2)^{p/2},
\end{aligned}
$$

where

$$
K=\begin{cases}\min\{\mu^2+3|C|^2,2\} & \text{if } 1<p<2,\\ \max\{\mu^2+3|C|^2,2\} & \text{if } p\geqslant 2.\end{cases}
$$

Thus, since

$$
(1+|A|^2)^{p/2}\leqslant C_p(1+|A|^p),
$$

for some positive constant C_p depending only on p, we have

$$
L(\mu^2+|C|^2+|A-C|^2)^{(p-2)/2}|A-C|^2\leqslant 2LC_pK^{(p-2)/2}\max\{1,|C|^2\}(1+|A|^p). \tag{2.6}
$$

Combining (2.4)–(2.6), the conclusion follows. □

We now state some important results concerning periodic functions.

DEFINITION 2.10. A function $w\colon\mathbb{R}^N\to\mathbb{R}^{N^{k-s}}$ is said to be *Q-periodic* if $w(x+e_i)=w(x)$ for a.e. $x\in\mathbb{R}^N$ and every $i=1,\dots,N$, where $\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$.

Let $d,r\in\mathbb{N}$. We will denote by $C^\infty_{\mathrm{per}}(\mathbb{R}^N;\mathbb{R}^d)$ the space of Q-periodic functions of $C^\infty(\mathbb{R}^N;\mathbb{R}^d)$. Moreover, we will use the notation $C^r_{\mathrm{c}}(Q;\mathbb{R}^d)$ for the space of functions of class C^r from Q to $\mathbb{R}^d$ with compact support in Q. The next lemma will be used extensively in the paper.

LEMMA 2.11 (Helmholtz decomposition). *For every $\varphi\in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{N^{k-s}})$, there exist two functions*

$$
\phi\in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{N^{k-s-1}})\quad\text{and}\quad \psi\in C^\infty_{\mathrm{per}}(Q;R^{N^{k-s}})
$$

such that

$$
\varphi_{i_1\cdots i_{k-s}}=(\nabla\phi)_{i_1\cdots i_{k-s}}+\psi_{i_1\cdots i_{k-s}}\quad\text{for } i_1,\dots,i_{k-s}=1,\dots,N,
$$

with

$$
\sum_{i_b=1}^{N}\frac{\partial\psi_{i_1\cdots i_{b-1}i_bi_{b+1}\cdots i_{k-s}}}{\partial x_{i_b}}=0\quad\text{for every } b\in\{1,\dots,k-s\}. \tag{2.7}
$$

Proof. By applying the usual Helmholtz decomposition lemma [7, lemma 1] to each component $\varphi_{i_1\cdots i_{k-s}}$ of the function φ, the lemma follows. □

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Before stating the next lemma, we need the following definition.

DEFINITION 2.12. The *s-divergence* is given by the operator

$$s\text{-div} : C^s(Q;\mathbb{R}^{N^k}) \to C(Q;\mathbb{R}^{N^{k-s}})$$

defined by

$$(s\text{-div}\,\xi)_{i_1\cdots i_{k-s}} := \sum_{i_{k-s+1},\dots,i_k=1}^{N} \frac{\partial^s \xi_{i_1 i_2\cdots i_k}}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_k}}, \quad i_1,\dots,i_{k-s}=1,\dots,N,$$

for every $\xi \in C^s(Q;\mathbb{R}^{N^k})$. The definition is analogous when ξ is a Sobolev function.

We are now ready to state a fundamental Korn-type estimate.

LEMMA 2.13. *For every $p>1$ there exists a constant $\gamma = \gamma(N,p,s) \geqslant 1$ such that*

$$\int_Q |\nabla^s \psi|^p \,\mathrm{d}x \leqslant \gamma \int_Q |\mathcal{A}_{s+1}\nabla^s\psi|^p \,\mathrm{d}x$$

for every Q-periodic function $\psi\colon \mathbb{R}^N \to \mathbb{R}^{N^{k-s}}$ of class C^∞ satisfying condition (2.7).

Proof. Note that, for every $r = k-s+1,\dots,k$, we have

$$\begin{aligned}
&\sum_{i_r=1}^{N} \frac{\partial}{\partial x_{i_r}} [(\nabla^s\psi)^{\mathrm{T}_r^{k-s}}]_{i_1 i_2\cdots i_k} \\
&\quad = \sum_{i_r=1}^{N} \frac{\partial}{\partial x_{i_r}} \left[\frac{\partial^s \psi_{i_1\cdots i_{k-s-1} i_r}}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_{r-1}}\partial x_{i_{k-s}}\partial x_{i_{r+1}}\cdots\partial x_{i_k}} \right] \\
&\quad = \frac{\partial^s}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_{r-1}}\partial x_{i_{k-s}}\partial x_{i_{r+1}}\cdots\partial x_{i_k}} \left[\sum_{i_r=1}^{N} \frac{\partial \psi_{i_1\cdots i_{k-s-1} i_r}}{\partial x_{i_r}} \right] \\
&\quad = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&(s+1)[s\text{-div}\,(\mathcal{A}_{s+1}\nabla^s\psi)]_{i_1\cdots i_{k-s}} \\
&\qquad = \sum_{i_{k-s+1},\dots,i_k=1}^{N} \frac{\partial^s}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_k}} \\
&\qquad\qquad \times [s\nabla^s\psi - ((\nabla^s\psi)^{\mathrm{T}_{k-s+1}^{k-s}} + \cdots + (\nabla^s\psi)^{\mathrm{T}_k^{k-s}})]_{i_1 i_2\cdots i_k} \\
&\qquad = s \sum_{i_{k-s+1},\dots,i_k=1}^{N} \frac{\partial^s}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_k}} (\nabla^s\psi)_{i_1 i_2\cdots i_k} \\
&\qquad = s \sum_{i_{k-s+1},\dots,i_k=1}^{N} \frac{\partial^s}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_k}} \left[\frac{\partial^s \psi_{i_1\cdots i_{k-s}}}{\partial x_{i_{k-s+1}}\cdots\partial x_{i_k}} \right] \\
&\qquad = s\Delta^s \psi_{i_1\cdots i_{k-s}},
\end{aligned}$$

where by Δ^s we denote the sth power of the Laplace operator. Hence,

$$\Delta^s \psi_{i_1\cdots i_{k-s}} = \frac{s+1}{s}[s\text{-div}\,(\mathcal{A}_{s+1}\nabla^s\psi)]_{i_1\cdots i_{k-s}}, \quad i_1,\dots,i_{k-s}=1,\dots,N.$$

The conclusion follows by applying theorem 10.5 and the following remark from [1]. □

We will also need the following generalization of lemma 2.13.

Lemma 2.14. *For every $p>1$, there exists a constant $\tau=\tau(N,p,s)\geqslant 1$ such that*

$$\int_Q (\mu^2+|\nabla^s\psi|^2)^{(p-2)/2}|\nabla^s\psi|^2\,\mathrm{d}x \leqslant \tau\int_Q (\mu^2+|\mathcal{A}_{s+1}\nabla^s\psi|^2)^{(p-2)/2}|\mathcal{A}_{s+1}\nabla^s\psi|^2\,\mathrm{d}x$$

for every constant $\mu\geqslant 0$ and every Q-periodic function $\psi\colon\mathbb{R}^N\to\mathbb{R}^{N^{k-s}}$ of class C^∞ satisfying condition (2.7).

Proof. The proof simply follows by adapting the proof of [7, lemma 11] and using lemma 2.13. □

We conclude this section by giving some definitions of higher-order bounded variation spaces. We set

$$\begin{aligned}\mathrm{BH}^{(k)}(\Omega) &:= \{u\in W^{k-1,1}(\Omega)\colon D^k u \text{ is a finite Radon measure}\}\\ &= \{u\in W^{k-1,1}(\Omega)\colon \nabla^{k-1}u\in \mathrm{BV}(\Omega;E_{k-1})\},\end{aligned}$$

where $D^k u$ stands for the kth-order distributional gradient of u, and

$$\begin{aligned}\mathrm{SBH}^{(k)}(\Omega) &:= \{u\in \mathrm{BH}^{(k)}(\Omega)\colon \nabla^{k-1}u\in \mathrm{SBV}(\Omega;E_{k-1})\}\\ &= \{u\in W^{k-1,1}(\Omega)\colon \nabla^{k-1}u\in \mathrm{SBV}(\Omega;E_{k-1})\}\subset \mathrm{BH}^{(k)}(\Omega).\end{aligned}$$

3. Proof of theorem 1.1

To prove theorem 1.1 we will first show that, for every $j=2,\dots,k$, every strictly j-quasi-convex function of class C^1 can be extended to a strictly $(j-1)$-quasi-convex function, provided we require the gradient to be Lipschitz continuous. In the case when $1<p<2$ that we present below, we actually have to consider a 'perturbed' strict j-quasi-convexity.

Lemma 3.1. *Let $j\in\{2,\dots,k\}$, $1<p<2$, $\mu\geqslant 0$, and let $M^{(j)}$, $\nu^{(j)}$ and ε be positive constants. Let $f^{(j)}\in C^1(E_k^{N^{k-j}})$ satisfy the following conditions:*

(a) (*strict j-quasi-convexity up to a perturbation*)

$$\begin{aligned}&\int_Q [f^{(j)}(A+\nabla^j\phi)-f^{(j)}(A)]\,\mathrm{d}x\\ &\qquad\geqslant -\varepsilon h^{(j)}(A)+\nu^{(j)}\int_Q(\mu^2+|A|^2+|\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x\end{aligned}$$

for every $A\in E_k^{N^{k-j}}$ and every $\phi\in C_{\mathrm{c}}^j(Q;\mathbb{R}^{N^{k-j}})$, where

$$h^{(j)}\colon E_k^{N^{k-j}}\to[0,+\infty);$$

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(b) (*Lipschitz condition for gradients*)

$$|\nabla f^{(j)}(A+B) - \nabla f^{(j)}(A)| \leqslant M^{(j)}(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|$$

for every $A, B \in E_k^{N^{k-j}}$.

Then there exist a function $F^{(j)} \in C^1(E_k^{N^{k-j+1}})$ *and a positive constant* $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$ *such that*

(a′) (*strict* $(j-1)$*-quasi-convexity up to a perturbation*)

$$\int_Q [F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A)]\,\mathrm{d}x \geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2\,\mathrm{d}x - \varepsilon(\mu^2 + |\mathcal{A}_j A|^2)^{(p-2)/2}|\mathcal{A}_j A|^2 - \varepsilon h^{(j)}(\mathcal{S}_j A)$$

for every $A \in E_k^{N^{k-j+1}}$ *and every* $\varphi \in C_{\mathrm{c}}^{j-1}(Q; \mathbb{R}^{N^{k-j+1}})$,

(b′) (*Lipschitz condition for gradients*)

$$|\nabla F^{(j)}(A+B) - \nabla F^{(j)}(A)| \leqslant L^{(j)}(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|$$

for every $A, B \in E_k^{N^{k-j+1}}$,

(c) ($F^{(j)}$ *extends* $f^{(j)}$)

$$F^{(j)}(A) = f^{(j)}(A) \quad \forall A \in E_k^{N^{k-j}}.$$

Proof. Let $\beta > 0$ be a constant to be chosen at the end of the proof and define $F^{(j)} \colon E_k^{N^{k-j+1}} \to \mathbb{R}$ as

$$F^{(j)}(A) := f^{(j)}(\mathcal{S}_j A) + \beta[(\mu^2 + |\mathcal{A}_j A|^2)^{p/2} - \mu^p] = f^{(j)}(\mathcal{S}_j A) + \beta[g(\mathcal{A}_j A) - \mu^p],$$

where g is given by relation (A 2) with $X = E_k^{N^{k-j+1}}$.

Relation (c) is clearly satisfied. Let us show that condition (a′) holds true for a good choice of β. Let

$$\varphi \in C^\infty_{\mathrm{per}}(Q; \mathbb{R}^{N^{k-j+1}}).$$

By lemma 2.11 we can write

$$\varphi = \nabla\phi + \psi,$$

where $\psi \in C^\infty_{\mathrm{per}}(Q; R^{N^{k-j+1}})$ satisfies condition (2.7) with $s = j-1$, and

$$\phi \in C^\infty_{\mathrm{per}}(Q; \mathbb{R}^{N^{k-j}}).$$

By differentiating the previous relation $j-1$ times, we get

$$\nabla^{j-1}\varphi = \nabla^j\phi + \nabla^{j-1}\psi,$$

with

$$\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C^\infty_{\mathrm{per}}(Q; E_k^{N^{k-j+1}}) \quad \text{and} \quad \nabla^j\phi \in C^\infty_{\mathrm{per}}(Q; E_k^{N^{k-j}}).$$

We have

$$\begin{aligned}\int_Q [F^{(j)}(A+\nabla^{j-1}\varphi) - F^{(j)}(A)]\,\mathrm{d}x \\ &= \int_Q [f^{(j)}(\mathcal{S}_j A + \nabla^j\phi + \mathcal{S}_j\nabla^{j-1}\psi) - f^{(j)}(\mathcal{S}_j A + \nabla^j\phi)]\,\mathrm{d}x \\ &\quad + \int_Q [f^{(j)}(\mathcal{S}_j A + \nabla^j\phi) - f^{(j)}(\mathcal{S}_j A)]\,\mathrm{d}x \\ &\quad + \beta\int_Q [g(\mathcal{A}_j A + \mathcal{A}_j\nabla^{j-1}\psi) - g(\mathcal{A}_j A)]\,\mathrm{d}x \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Note that $\nabla f^{(j)}(\mathcal{S}_j A) \in E_k^{N^{k-j}}$. Then, by proposition 2.5 and using the fact that ψ is Q-periodic,

$$\int_Q \nabla f^{(j)}(\mathcal{S}_j A)\cdot\mathcal{S}_j\nabla^{j-1}\psi\,\mathrm{d}x = \int_Q \nabla f^{(j)}(\mathcal{S}_j A)\cdot\nabla^{j-1}\psi\,\mathrm{d}x = 0.$$

Hence,

$$I_1 = \int_Q [f^{(j)}(\mathcal{S}_j A+\nabla^j\phi+\mathcal{S}_j\nabla^{j-1}\psi) - f^{(j)}(\mathcal{S}_j A+\nabla^j\phi) - \nabla f^{(j)}(\mathcal{S}_j A)\cdot\mathcal{S}_j\nabla^{j-1}\psi]\,\mathrm{d}x.$$

Applying lemma A.6 with $\varepsilon = \frac{1}{2}\nu^{(j)}$, there exists a positive constant

$$c_1 = c_1(\nu^{(j)}, p, M^{(j)}) > 0$$

such that

$$\begin{aligned} I_1 &\geqslant -\tfrac{1}{2}\nu^{(j)}\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x \\ &\quad - c_1\int_Q (\mu^2 + |\mathcal{S}_j\nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{S}_j\nabla^{j-1}\psi|^2\,\mathrm{d}x \\ &\geqslant -\tfrac{1}{2}\nu^{(j)}\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x \\ &\quad - \tau c_1\int_Q (\mu^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,\mathrm{d}x,\end{aligned}$$

where $\tau = \tau(N, p, j-1)$ is given by lemma 2.14. The perturbed strict j-quasi-convexity of $f^{(j)}$ gives

$$I_2 \geqslant \nu^{(j)}\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x - \varepsilon h^{(j)}(\mathcal{S}_j A),$$

so that

$$\begin{aligned} I_1 + I_2 &\geqslant \tfrac{1}{2}\nu^{(j)}\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &\quad - \tau c_1\int_Q (\mu^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,\mathrm{d}x.\end{aligned}$$

We now apply lemma A.3 to the first integral of the last expression with $\tilde{\mu}^2 = \mu^2 + |\mathcal{S}_j A|^2$, $x = \nabla^j \phi$ and $y = \nabla^{j-1}\psi$. Recalling that $\nabla^j\phi + \nabla^{j-1}\psi = \nabla^{j-1}\varphi$, we get

$$\begin{aligned} I_1 + I_2 \geqslant \tfrac{1}{4}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &- \tfrac{1}{2}\nu^{(j)} \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1}\psi|^2)^{(p-2)/2} |\nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &- \tau c_1 \int_Q (\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x. \end{aligned} \tag{3.1}$$

Using lemma 2.14 and the fact that $1 < p < 2$,

$$\begin{aligned} -\tfrac{1}{2}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1}\psi|^2)^{(p-2)/2} |\nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &\geqslant -\tfrac{1}{2}\nu^{(j)} \int_Q (\mu^2 + |\nabla^{j-1}\psi|^2)^{(p-2)/2} |\nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &\geqslant -\tfrac{1}{2}\tau\nu^{(j)} \int_Q (\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x. \end{aligned} \tag{3.2}$$

Hence, collecting (3.1) and (3.2),

$$\begin{aligned} I_1 + I_2 &\geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &\quad - \tau(c_1 + \tfrac{1}{2}\nu^{(j)}) \int_Q (\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &\geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x - \varepsilon h^{(j)}(\mathcal{S}_j A) \\ &\quad - \tau(c_1 + \tfrac{1}{2}\nu^{(j)}) \int_Q (\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x, \end{aligned}$$

where we once again use the fact that $1 < p < 2$. Since $\nabla g(\mathcal{A}_j A) \in \mathcal{A}_j E_k^{N^{k-j+1}}$ and ψ is Q-periodic,

$$\int_Q \nabla g(\mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\psi \,\mathrm{d}x = \int_Q \nabla g(\mathcal{A}_j A) \cdot \nabla^{j-1}\psi \,\mathrm{d}x = 0,$$

so that

$$I_3 = \beta \int_Q [g(\mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\psi) - g(\mathcal{A}_j A) - \nabla g(\mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\psi] \,\mathrm{d}x.$$

Let $0 < \delta < 1$ be chosen at the end of the proof. By lemma A.1,

$$\begin{aligned} I_3 &\geqslant \beta\theta_p \int_Q (\mu^2 + |\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &\geqslant \beta\theta_p \delta^{(2-p)/2} \int_Q (\mu^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ &\quad - \beta\theta_p \delta (\mu^2 + |\mathcal{A}_j A|^2)^{(p-2)/2} |\mathcal{A}_j A|^2, \end{aligned}$$

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where, in the second inequality, we used lemma A.3 with $X = E_k^{N^{k-j+1}}$, $\tilde{\mu} = \mu$, $x = \mathcal{A}_j A$ and $y = \mathcal{A}_j \nabla^{j-1}\psi$. Choosing $\beta = \beta^{(j)} > 0$ and $\delta = \delta^{(j)} \in (0,1)$ such that

$$\beta^{(j)}\theta_p(\delta^{(j)})^{(2-p)/2} \geqslant \tau(c_1 + \tfrac{1}{2}\nu^{(j)}), \qquad \beta^{(j)}\theta_p\delta^{(j)} \leqslant \varepsilon,$$

we obtain

$$I_1 + I_2 + I_3 \geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\ - \varepsilon h^{(j)}(\mathcal{S}_j A) - \varepsilon(\mu^2 + |\mathcal{A}_j A|^2)^{(p-2)/2}|\mathcal{A}_j A|^2,$$

so that (a′) holds. To check condition (b′), we observe that the function g satisfies the hypotheses of lemma A.5. Then, for every $A, B \in E_k^{N^{k-j+1}}$,

$$|\nabla g(A+B) - \nabla g(A)| \leqslant C_p(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|,$$

where C_p is a positive constant depending only on p. Using the above relation, condition (b) and the fact that $\beta^{(j)}$ depends on $\nu^{(j)}, \tau$ and c_1, we conclude that (b′) holds for some positive constant $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$. □

We pass now to the case $p \geqslant 2$.

LEMMA 3.2. *Let $j \in \{2, \dots, k\}$, $p \geqslant 2$, $\mu \geqslant 0$, $M^{(j)} > 0$, $\nu^{(j)} > 0$, and let θ_p and Θ_p be given by lemma A.1. Let $f^{(j)} \in C^1(E_k^{N^{k-j}})$ satisfy the following conditions:*

(a) (*strict j-quasi-convexity*)

$$\int_Q [f^{(j)}(A + \nabla^j\phi) - f^{(j)}(A)] \,\mathrm{d}x \geqslant \nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2 \,\mathrm{d}x$$

for every $A \in E_k^{N^{k-j}}$ and every $\phi \in C_c^j(Q; \mathbb{R}^{N^{k-j}})$;

(b) (*Lipschitz condition for gradients*)

$$|\nabla f^{(j)}(A+B) - \nabla f^{(j)}(A)| \leqslant M^{(j)}(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|$$

for every $A, B \in E_k^{N^{k-j}}$.

Then there exist a function $F^{(j)} \in C^1(E_k^{N^{k-j+1}})$ and a positive constant $L^{(j)} = L^{(j)}(p, \mu, M^{(j)}, \nu^{(j)}, j)$ such that

(a′) (*strict $(j-1)$-quasi-convexity*)

$$\int_Q [F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A)] \,\mathrm{d}x \\ \geqslant \nu^{(j)} \frac{\theta_p}{4\Theta_p} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2 \,\mathrm{d}x$$

for every $A \in E_k^{N^{k-j+1}}$ and every $\varphi \in C_c^{j-1}(Q; \mathbb{R}^{N^{k-j+1}})$,

(b′) (*Lipschitz condition for gradients*)

$$|\nabla F^{(j)}(A+B) - \nabla F^{(j)}(A)| \leqslant L^{(j)}(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|$$

for every $A, B \in E_k^{N^{k-j+1}}$,

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(c) ($F^{(j)}$ *extends* $f^{(j)}$)

$$F^{(j)}(A) = f^{(j)}(A) \quad \forall A \in E_k^{N^{k-j}}.$$

Proof. Let $\lambda \in (0, \nu^{(j)}/\Theta_p]$ and $\beta > 0$ be two constants to be determined at the end of the proof. We define $F^{(j)} \colon E_k^{N^{k-j+1}} \to \mathbb{R}$ as

$$F^{(j)}(A) := f^{(j)}(\mathcal{S}_j A) - \lambda(\mu^2 + |\mathcal{S}_j A|^2)^{p/2} + \lambda(\mu^2 + |\mathcal{S}_j A|^2 + \beta^2 |\mathcal{A}_j A|^2)^{p/2}.$$

Let g and g_β be defined by (A 2) and (A 3), respectively, with

$$X = E_k^{N^{k-j}} \quad \text{and} \quad Y = \mathcal{A}_j E_k^{N^{k-j+1}}.$$

Setting

$$f_\lambda^{(j)}(B) := f^{(j)}(B) - \lambda g(B)$$

for every $B \in E_k^{N^{k-j}}$, we have

$$F^{(j)}(A) = f_\lambda^{(j)}(\mathcal{S}_j A) + \lambda g_\beta(\mathcal{S}_j A, \mathcal{A}_j A).$$

Condition (c) is clear from the definition of $F^{(j)}$. In order to check (a′), let $\varphi \in C^\infty_{\text{per}}(Q; \mathbb{R}^{N^{k-j+1}})$. By repeating the argument of the previous proof, we can write

$$\nabla^{j-1}\varphi = \nabla^j \phi + \nabla^{j-1}\psi,$$

with

$$\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C^\infty_{\text{per}}(Q; E_k^{N^{k-j+1}}) \quad \text{and} \quad \nabla^j \phi \in C^\infty_{\text{per}}(Q; E_k^{N^{k-j}}),$$

where $\psi \in C^\infty_{\text{per}}(Q; R^{N^{k-j+1}})$ satisfies condition (2.7) with $s = j - 1$. Hence,

$$\begin{aligned}
\int_Q [F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A)] \,\mathrm{d}x \\
&= \int_Q [f_\lambda^{(j)}(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi) - f_\lambda^{(j)}(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi - \mathcal{S}_j \nabla^{j-1}\psi)] \,\mathrm{d}x \\
&\quad + \int_Q [f_\lambda^{(j)}(\mathcal{S}_j A + \nabla^j \phi) - f_\lambda^{(j)}(\mathcal{S}_j A)] \,\mathrm{d}x \\
&\quad + \lambda \int_Q [g_\beta(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi, \mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\varphi) - g_\beta(\mathcal{S}_j A, \mathcal{A}_j A)] \,\mathrm{d}x \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

Concerning the second integral, since, by periodicity,

$$\int_Q \nabla g(\mathcal{S}_j A) \cdot \nabla^j \phi \,\mathrm{d}x = 0,$$

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using condition (a) and lemma A.1 we have

$$\begin{aligned} I_2 &= \int_Q [f(\mathcal{S}_j A + \nabla^j \phi) - f(\mathcal{S}_j A)]\,\mathrm{d}x - \lambda \int_Q [g(\mathcal{S}_j A + \nabla^j \phi) - g(\mathcal{S}_j A)]\,\mathrm{d}x \\ &= \int_Q [f(\mathcal{S}_j A + \nabla^j \phi) - f(\mathcal{S}_j A)]\,\mathrm{d}x \\ &\qquad - \lambda \int_Q [g(\mathcal{S}_j A + \nabla^j \phi) - g(\mathcal{S}_j A) + \nabla g(\mathcal{S}_j A)\cdot \nabla^j \phi]\,\mathrm{d}x \\ &\geqslant (\nu^{(j)} - \lambda \Theta_p) \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2\,\mathrm{d}x \\ &\geqslant 0. \end{aligned} \tag{3.3}$$

Let us pass to the first integral. Noting that $\nabla f_\lambda^{(j)}(\mathcal{S}_j A) \in E_k^{N^{k-j}}$, due to proposition 2.5 and using the fact that ψ is Q-periodic,

$$\begin{aligned} \int_Q \nabla f_\lambda^{(j)}(\mathcal{S}_j A)\cdot \mathcal{S}_j \nabla^{j-1}\psi\,\mathrm{d}x &= \int_Q \nabla f_\lambda^{(j)}(\mathcal{S}_j A)\cdot \nabla^{j-1}\psi\,\mathrm{d}x \\ &= 0. \end{aligned}$$

Hence,

$$\begin{aligned} I_1 = -\int_Q [f_\lambda^{(j)}(&\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi - \mathcal{S}_j \nabla^{j-1}\psi) \\ &- f_\lambda^{(j)}(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi) - \nabla f_\lambda^{(j)}(\mathcal{S}_j A)\cdot \mathcal{S}_j \nabla^{j-1}\psi]\,\mathrm{d}x. \end{aligned}$$

As observed in the previous proof, the function g satisfies condition (A 6), and so by lemma A.5, condition (b) still holds for the function f_λ for a suitable constant $\tilde{M} = \tilde{M}(p, M^{(j)}, \lambda)$ in place of $M^{(j)}$. Thus, applying lemma A.6 with $\varepsilon = \frac{1}{2}\lambda\theta_p$, there exists a positive constant $\sigma = \sigma(p, M^{(j)}, \lambda)$ such that

$$\begin{aligned} I_1 \geqslant -\tfrac{1}{2}\lambda\theta_p &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2)^{(p-2)/2} |\mathcal{S}_j \nabla^{j-1}\varphi|^2\,\mathrm{d}x \\ &- \sigma(\mu^2 + |\mathcal{S}_j A|^2)^{(p-2)/2} \int_Q |\mathcal{S}_j \nabla^{j-1}\psi|^2\,\mathrm{d}x - \sigma \int_Q |\mathcal{S}_j \nabla^{j-1}\psi|^p\,\mathrm{d}x. \end{aligned}$$

Due to lemma 2.13 and using (3.3), we get

$$\begin{aligned} I_1 + I_2 \geqslant -\tfrac{1}{2}\lambda\theta_p &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2)^{(p-2)/2} |\mathcal{S}_j \nabla^{j-1}\varphi|^2\,\mathrm{d}x \\ &- \sigma\gamma(N,2,j-1)(\mu^2 + |\mathcal{S}_j A|^2)^{(p-2)/2} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^2\,\mathrm{d}x \\ &- \sigma\gamma(N,p,j-1) \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

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Since φ is Q-periodic,

$$0 = \int_Q \nabla g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) \cdot \nabla^{j-1}\varphi \,\mathrm{d}x$$
$$= \int_Q [\nabla_x g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\varphi + \nabla_y g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\varphi] \,\mathrm{d}x,$$

so that

$$I_3 = \lambda \int_Q [g_\beta(\mathcal{S}_j A + \mathcal{S}_j \nabla^{j-1}\varphi, \mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\varphi) - g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) - \nabla_x g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) \cdot \mathcal{S}_j \nabla^{j-1}\varphi - \nabla_y g_\beta(\mathcal{S}_j A, \mathcal{A}_j A) \cdot \mathcal{A}_j \nabla^{j-1}\varphi] \,\mathrm{d}x. \tag{3.5}$$

We shall now split I_3 into two terms. We will use the first term to balance the sum I_1+I_2, and the remaining one to get the strict $(j-1)$-quasi-convexity. Relation (A 5) of lemma A.2 gives

$$\tfrac{1}{2}I_3 \geqslant \tfrac{1}{2}\lambda\theta_p \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2)^{(p-2)/2} |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \,\mathrm{d}x + [\tfrac{1}{4}\lambda\theta_p\beta^2](\mu^2 + |\mathcal{S}_j A|^2)^{(p-2)/2} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x + \tfrac{1}{4}\lambda\theta_p\beta^p \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x. \tag{3.6}$$

If we choose $\beta = \beta^{(j)} > 0$ so large that

$$\tfrac{1}{4}\lambda\theta_p(\beta^{(j)})^2 \geqslant \sigma\gamma(N, 2, j-1) \quad \text{and} \quad \tfrac{1}{4}\lambda\theta_p(\beta^{(j)})^p \geqslant \sigma\gamma(N, p, j-1),$$

using relations (3.4) and (3.6), we have

$$I_1 + I_2 + I_3 \geqslant \tfrac{1}{2}I_3.$$

Let us estimate the last term. Without any loss of generality we can assume $\beta^{(j)} \geqslant 1$. Then, recalling (3.5) and using inequality (A 4),

$$\begin{aligned}
\int_Q [F^{(j)}(A + \nabla^{j-1}\varphi) - F^{(j)}(A)] \,\mathrm{d}x
&= I_1 + I_2 + I_3 \\
&\geqslant \tfrac{1}{2}\lambda\theta_p \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \\
&\qquad + (\beta^{(j)})^2 |\mathcal{A}_j A|^2 + (\beta^{(j)})^2 |\mathcal{A}_j \nabla^{j-1}\varphi|^2)^{(p-2)/2} \\
&\qquad \times (|\mathcal{S}_j \nabla^{j-1}\varphi|^2 + (\beta^{(j)})^2 |\mathcal{A}_j \nabla^{j-1}\varphi|^2) \,\mathrm{d}x \\
&\geqslant \tfrac{1}{2}\lambda\theta_p \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\
&= \nu^{(j)} \frac{\theta_p}{4\Theta_p} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x,
\end{aligned}$$

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where we choose

$$\lambda = \frac{\nu^{(j)}}{2\Theta_p}.$$

One can show that $F^{(j)}$ satisfies condition (b′) as in the proof of lemma 3.1. □

We can now pass to the proof of theorem 1.1.

Proof of theorem 1.1.

STEP 1 ($1 < p < 2$). To simplify the notation, for every $B \in \mathbb{R}^{N^k}$ we set

$$\mathcal{P}(B) := (\mu^2 + |B|^2)^{(p-2)/2}|B|^2, \qquad \mathcal{G}(B) := [(\mu^2 + |B|^2)^{p/2} - \mu^p].$$

Let $\varepsilon > 0$ be fixed. We start the proof by applying lemma 3.1 with $j = k$, $\nu^{(k)} = \nu$, $h^{(k)} \equiv 0$ and

$$f^{(k)}(A) = f(A) \quad \text{for every } A \in E_k.$$

We again apply lemma 3.1 $k-2$ times with $j = k-1, k-2, \dots, 2$, respectively, with

$$\nu^{(j)} = \frac{\nu}{4^{k-j}},$$

and

$$f^{(j)}(A) = F^{(j+1)}(A) \quad \text{for every } A \in E_k^{N^{k-j}},$$

while the functions $h^{(j)}\colon E_k^{N^{k-j}} \to [0, +\infty)$ will be chosen as

$$h^{(k-1)}(A) = \mathcal{P}(\mathcal{A}_k A),$$

$$h^{(j)}(A) = \mathcal{P}(\mathcal{A}_{j+1} A) + \sum_{r=j+2}^{k} \mathcal{P}(\mathcal{A}_r \mathcal{S}_{r-1} \cdots \mathcal{S}_{j+1} A), \quad j = k-2, \dots, 2.$$

In this way, after the last step, corresponding to $j = 2$, we obtain a function $F^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$F^{(2)}(A) := f(\mathcal{S}_k \mathcal{S}_{k-1} \cdots \mathcal{S}_2) + \beta^{(2)} \mathcal{G}(\mathcal{A}_2 A) + \sum_{r=3}^{k} \beta^{(r)} \mathcal{G}(\mathcal{A}_r \mathcal{S}_{r-1} \cdots \mathcal{S}_2 A). \tag{3.7}$$

Here, for every $j = 2, \dots, k$, the constant $\beta^{(j)}$ is given by the proof of lemma 3.1 with the corresponding index j. $F^{(2)}$ has the following properties:

(a′) (strict 1-quasi-convexity up to a perturbation)

$$\begin{aligned}\int_Q [F^{(2)}(A + \nabla\varphi) - F^{(2)}(A)]\,\mathrm{d}x &\geqslant -\varepsilon \mathcal{P}(\mathcal{A}_2 A) - \varepsilon h^{(2)}(\mathcal{S}_2 A) \\ &\quad + \frac{\nu}{4^{k-1}} \int_Q (\mu^2 + |A|^2 + |\nabla\varphi|^2)^{(p-2)/2} |\nabla\varphi|^2\,\mathrm{d}x\end{aligned}$$

for every $A \in \mathbb{R}^{N^k}$ and every $\varphi \in C^1_{\mathrm{c}}(Q; \mathbb{R}^{N^{k-1}})$;

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(b′) (Lipschitz condition for the gradient)

$$|\nabla F^{(2)}(A+B) - \nabla F^{(2)}(A)| \leqslant L(\mu^2 + |A|^2 + |B|^2)^{(p-2)/2}|B|$$

for every $A, B \in \mathbb{R}^{N^k}$, with $L = L(p, \mu, M, \nu)$;

(c) ($F^{(2)}$ extends f)

$$F^{(2)}(A) = f(A) \quad \forall A \in E_k.$$

Now let us define

$$F(A) := \inf\left\{ \int_Q F^{(2)}(A + \nabla\varphi(x))\,\mathrm{d}x \colon \varphi \in C^\infty_{\mathrm{per}}(Q; \mathbb{R}^{N^{k-1}}) \right\}$$

for every $A \in \mathbb{R}^{N^k}$. Property (a′) implies that, for every $A \in \mathbb{R}^{N^k}$,

$$F^{(2)}(A) - \varepsilon\mathcal{P}(\mathcal{A}_2 A) - \varepsilon h^{(2)}(\mathcal{S}_2 A) \leqslant F(A) \leqslant F^{(2)}(A). \tag{3.8}$$

Since, for every $A \in E_k$,

$$\mathcal{P}(\mathcal{A}_2 A) = h^{(2)}(\mathcal{S}_2 A) = 0,$$

from property (c) and relation (3.8) equality (1.3) follows. Let us check (1.4). By proposition 2.9, from condition (b′) we infer that there exists a positive constant c, depending on the function $F^{(2)}$ and in turn on f, such that

$$|F^{(2)}(A)| \leqslant c(1 + |A|^p) \quad \forall A \in \mathbb{R}^{N^k}.$$

Recalling the definitions of the functions $\mathcal{P}$ and $h^{(2)}$, the last relation and (3.8) give (1.4).

Step 2 ($p \geqslant 2$). Repeating the strategy used for the case where $1 < p < 2$, we first apply lemma 3.2 with $j = k$, $\nu^{(k)} = \nu$ and

$$f^{(k)}(A) = f(A) \quad \text{for every } A \in E_k.$$

Then, we again apply lemma 3.2 $k-2$ times with $j = k-1, k-2, \dots, 2$, respectively, with

$$\nu^{(j)} = \nu\left(\frac{\theta_p}{4\Theta_p}\right)^{k-j+1}$$

and

$$f^{(j)}(A) = F^{(j+1)}(A) \quad \text{for every } A \in E_k^{N^{k-j}}.$$

Finally, when $j = 2$, we obtain a function $F^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$F^{(2)}(A) = f(\mathcal{S}_k \cdots \mathcal{S}_2 A) + \mathcal{L}^{(2)}(\mathcal{S}_2 A, \mathcal{A}_2 A) + \sum_{r=3}^{k} \mathcal{L}^{(r)}(\mathcal{S}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_2 A, \mathcal{A}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_2 A), \tag{3.9}$$

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where we set

$$\mathcal{L}^{(r)}(A,B) := -\frac{\nu^{(r)}}{2\Theta_p}(\mu^2+|A|^2)^{p/2}+\frac{\nu^{(r)}}{2\Theta_p}(\mu^2+|A|^2+(\beta^{(r)})^2|B|^2)^{p/2}, \quad r=2,\dots,k,$$

and for every $j=2,\dots,k$, the constant $\beta^{(j)}$ is given by the proof of lemma 3.2 with the corresponding index j. The function $F^{(2)}$ just defined is such that

(a′) (strict 1-quasi-convexity)

$$\int_Q [F^{(2)}(A+\nabla\varphi)-F^{(2)}(A)]\,\mathrm{d}x \geqslant \frac{\nu}{4^k}\left(\frac{\theta_p}{\Theta_p}\right)^k \int_Q (\mu^2+|A|^2+|\nabla\varphi|^2)^{(p-2)/2}|\nabla\varphi|^2\,\mathrm{d}x$$

for every $A\in\mathbb{R}^{N^k}$ and every $\varphi\in C^1_{\mathrm{c}}(Q;\mathbb{R}^{N^{k-1}})$,

(b′) (Lipschitz condition for the gradient)

$$|\nabla F^{(2)}(A+B)-\nabla F^{(2)}(A)| \leqslant L(\mu^2+|A|^2+|B|^2)^{(p-2)/2}|B|$$

for every $A,B\in\mathbb{R}^{N^k}$, with $L=L(p,\mu,M,\nu)$,

(c) ($F^{(2)}$ extends f)

$$F^{(2)}(A)=f(A)\quad \forall A\in E_k.$$

We claim that the proof is concluded by setting $F:=F^{(2)}$. Indeed, condition (c) gives (1.3), while (1.4) follows by applying proposition 2.9 to $F^{(2)}$. □

4. Proof of theorem 1.2

To prove the theorem, first we need two preliminary lemmas.

LEMMA 4.1. *Let $j\in\{2,\dots,k\}$, $1<p<2$, $\mu\geqslant 0$, $\nu^{(j)}>0$, and let $\{M_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of positive constants. Let $\{f_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of functions*

$$f_i^{(j)}\colon E_k^{N^{k-j}}\to\mathbb{R}$$

satisfying the following conditions:

(a) (*strict j-quasi-convexity up to a perturbation*)

$$\int_Q [f_i^{(j)}(A+\nabla^j\phi)-f_i^{(j)}(A)]\,\mathrm{d}x \geqslant -h_i^{(j)}(A)+\nu^{(j)}\int_Q(\mu^2+|A|^2+|\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x$$

for every $A\in E_k^{N^{k-j}}$, for every $\phi\in C^j_{\mathrm{c}}(Q;\mathbb{R}^{N^{k-j}})$, and for every $i\in\mathbb{N}$, where $\{h_i^{(j)}\}_{i\in\mathbb{N}}$ is a sequence of functions

$$h_i^{(j)}\colon E_k^{N^{k-j}}\to[0,+\infty);$$

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(b) (*p-growth condition*)

$$|f_i^{(j)}(A)| \leqslant M_i^{(j)}(1+|A|^p) \quad \forall A \in E_k^{N^{k-j}},\ \forall i \in \mathbb{N}.$$

Then there exist an increasing sequence $\{F_i^{(j)}\}_{i\in\mathbb{N}}$ *of functions*

$$F_i^{(j)} : E_k^{N^{k-j+1}} \to \mathbb{R}$$

and two sequences $\{L_i^{(j)}\}_{i\in\mathbb{N}}$ *and* $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ *of positive numbers, depending on* $\nu^{(j)}$*,* $M_i^{(j)}$*,* j*,* p*,* μ*, such that*

(a′) (*strict* $(j-1)$*-quasi-convexity up to a perturbation*)

$$\begin{aligned}\int_Q [F_i^{(j)}(A+\nabla^{j-1}\varphi) - F_i^{(j)}(A)]\,\mathrm{d}x &\\ \geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2\,\mathrm{d}x &\\ -\frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)}|\mathcal{A}_j A|^p - h_i^{(j)}(\mathcal{S}_j A)&\end{aligned}$$

for every $A \in E_k^{N^{k-j+1}}$*, for every* $\varphi \in C_c^{j-1}(Q;\mathbb{R}^{N^{k-j+1}})$ *and for every* $i \in \mathbb{N}$*,*

(b′) (*p-growth condition*)

$$|F_i^{(j)}(A)| \leqslant L_i^{(j)}(1+|A|^p) \quad \forall A \in E_k^{N^{k-j+1}},\ \forall i \in \mathbb{N},$$

(c) ($F_i^{(j)}$ *extends* $f_i^{(j)}$)

$$F_i^{(j)}(A) = f_i^{(j)}(A) \quad \forall A \in E_k^{N^{k-j}},\ \forall i \in \mathbb{N}.$$

Proof. First we observe that, due to proposition 2.7, there exists a positive constant $L = L(M_i^{(j)}, j, p)$ (we do not stress the dependence on N and k here), such that

$$|f_i^{(j)}(A+B) - f_i^{(j)}(A)| \leqslant L(1+|A|^{p-1}+|B|^{p-1})|B| \quad \text{for every } A, B \in E_k^{N^{k-j}}. \tag{4.1}$$

Let $\beta > 0$ be a constant to be chosen at the end of the proof. For every $A \in E_k^{N^{k-j+1}}$, we define

$$F_i^{(j)}(A) := f_i^{(j)}(\mathcal{S}_j A) + \beta|\mathcal{A}_j A|^p. \tag{4.2}$$

Condition (c) is clearly satisfied. In order to show (a′), let us consider a function $\varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{N^{k-j+1}})$. We can write

$$\nabla^{j-1}\varphi = \nabla^j\phi + \nabla^{j-1}\psi,$$

with

$$\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C^\infty_{\mathrm{per}}(Q;E_k^{N^{k-j+1}}) \quad \text{and} \quad \nabla^j\phi \in C^\infty_{\mathrm{per}}(Q;E_k^{N^{k-j}}),$$

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where $\psi \in C^{\infty}_{\text{per}}(Q; R^{N^{k-j+1}})$ satisfies condition (2.7) with $s = j-1$. Hence,

$$\begin{aligned}\int_Q [F_i^{(j)}(A + \nabla^{j-1}\varphi) - F_i^{(j)}(A)]\,\mathrm{d}x \\ &= \int_Q [f_i^{(j)}(\mathcal{S}_j A + \nabla^j \phi + \mathcal{S}_j \nabla^{j-1}\psi) - f_i^{(j)}(\mathcal{S}_j A + \nabla^j \phi)]\,\mathrm{d}x \\ &\quad + \int_Q [f_i^{(j)}(\mathcal{S}_j A + \nabla^j \phi) - f_i^{(j)}(\mathcal{S}_j A)]\,\mathrm{d}x \\ &\quad + \beta \int_Q [|\mathcal{A}_j A + \mathcal{A}_j \nabla^{j-1}\psi|^p - |\mathcal{A}_j A|^p]\,\mathrm{d}x \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

By (4.1) and Young's inequality, for every $\delta > 0$ there exists a constant

$$C = C(M_i^{(j)}, j, p, \delta)$$

such that

$$\begin{aligned}I_1 &\geqslant -L \int_Q (1 + |\mathcal{S}_j A + \nabla^j \phi|^{p-1} + |\mathcal{S}_j \nabla^{j-1}\psi|^{p-1})|\mathcal{S}_j \nabla^{j-1}\psi|\,\mathrm{d}x \\ &\geqslant -\delta - \delta|\mathcal{S}_j A|^p - \delta \int_Q |\nabla^j \phi|^p\,\mathrm{d}x - C \int_Q |\mathcal{S}_j \nabla^{j-1}\psi|^p\,\mathrm{d}x.\end{aligned}$$

Using lemma 2.13,

$$I_1 \geqslant -\delta - \delta|\mathcal{S}_j A|^p - \delta \int_Q |\nabla^j \phi|^p\,\mathrm{d}x - C\gamma \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p\,\mathrm{d}x. \tag{4.3}$$

Due to lemma A.4, for every $0 < \varepsilon < 1$,

$$\begin{aligned}I_1 \geqslant{}& -\delta(1 + \varepsilon\mu^p) - \delta(1+\varepsilon)|\mathcal{S}_j A|^p - C\gamma \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p\,\mathrm{d}x \\ &- 8\delta\varepsilon^{(p-2)/p} \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2}|\nabla^j \phi|^2\,\mathrm{d}x.\end{aligned}$$

Then, applying lemma A.3 with $\tilde{\mu} = 0$, $x = \mathcal{A}_j A$ and $y = \mathcal{A}_j \nabla^{j-1}\psi$,

$$\begin{aligned}I_1 \geqslant{}& -\delta(1 + \varepsilon\mu^p) - \delta(1+\varepsilon)|\mathcal{S}_j A|^p - C\gamma\varepsilon^{p/2}|\mathcal{A}_j A|^p \\ &- C\gamma\varepsilon^{(p-2)/2} \int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{A}_j \nabla^{j-1}\psi|^2\,\mathrm{d}x \\ &- 8\delta\varepsilon^{(p-2)/p} \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2}|\nabla^j \phi|^2\,\mathrm{d}x.\end{aligned}$$

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Thus, there exists a sequence of positive numbers $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ such that, for every $i \in \mathbb{N}$,

$$\begin{aligned} I_1 \geqslant -\tfrac{1}{2}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x \\ &- \lambda_i^{(j)} \int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, \mathrm{d}x \\ &- \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)} |\mathcal{A}_j A|^p. \end{aligned}$$

Here, for every fixed $i \in \mathbb{N}$, $\lambda_i^{(j)} = \lambda_i^{(j)}(\nu^{(j)}, M_i^{(j)}, j, p, \mu)$. By condition (a),

$$I_2 \geqslant \nu^{(j)} \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x - h_i^{(j)}(\mathcal{S}_j A),$$

so that

$$\begin{aligned} I_1 + I_2 \geqslant \tfrac{1}{2}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x \\ &- \lambda_i^{(j)} \int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1}\psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1}\psi|^2 \, \mathrm{d}x \\ &- \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)} |\mathcal{A}_j A|^p - h_i^{(j)}(\mathcal{S}_j A). \end{aligned} \tag{4.4}$$

We focus now on the first term of the last expression. Applying lemma A.3 with $\tilde{\mu} = \mu^2 + |A|^2$, $x = \nabla^j \phi$ and $y = \nabla^{j-1}\psi$, and recalling that $\nabla^j \phi + \nabla^{j-1}\psi = \nabla^{j-1}\varphi$, we obtain

$$\begin{aligned} \tfrac{1}{2}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x \\ &\geqslant \tfrac{1}{2}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x \\ &\geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1} \varphi|^2)^{(p-2)/2} |\nabla^{j-1} \varphi|^2 \, \mathrm{d}x \\ &\quad - \tfrac{1}{2}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1} \psi|^2)^{(p-2)/2} |\nabla^{j-1} \psi|^2 \, \mathrm{d}x, \end{aligned}$$

where in the first line we used the fact that $1 < p < 2$. By lemma 2.14, the last inequality becomes

$$\begin{aligned} \tfrac{1}{2}\nu^{(j)} &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \, \mathrm{d}x \\ &\geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1} \varphi|^2)^{(p-2)/2} |\nabla^{j-1} \varphi|^2 \, \mathrm{d}x \\ &\quad - \tfrac{1}{2}\nu^{(j)}\tau \int_Q (\mu^2 + |A|^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, \mathrm{d}x \\ &\geqslant \tfrac{1}{4}\nu^{(j)} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1} \varphi|^2)^{(p-2)/2} |\nabla^{j-1} \varphi|^2 \, \mathrm{d}x \\ &\quad - \tfrac{1}{2}\nu^{(j)}\tau \int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j \nabla^{j-1} \psi|^2)^{(p-2)/2} |\mathcal{A}_j \nabla^{j-1} \psi|^2 \, \mathrm{d}x, \end{aligned} \tag{4.5}$$

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again exploiting the fact that $1 < p < 2$. Collecting (4.4) and (4.5), we have

$$\begin{aligned} I_1 + I_2 \geqslant \tfrac{1}{4}\nu^{(j)} &\int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2\,\mathrm{d}x \\ &- (\lambda_i^{(j)} + \tfrac{1}{2}\nu^{(j)}\tau)\int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,\mathrm{d}x \\ &- \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)}|\mathcal{A}_j A|^p - h_i^{(j)}(\mathcal{S}_j A). \end{aligned} \tag{4.6}$$

Concerning I_3, using the periodicity of ψ and lemma A.1 with $\mu = 0$,

$$\begin{aligned} I_3 &= \beta\int_Q [|\mathcal{A}_j A + \mathcal{A}_j\nabla^{j-1}\psi|^p - |\mathcal{A}_j A|^p - p|\mathcal{A}_j A|^{p-2}\mathcal{A}_j A\cdot\mathcal{A}_j\nabla^{j-1}\psi]\,\mathrm{d}x \\ &\geqslant \beta\theta_p\int_Q (|\mathcal{A}_j A|^2 + |\mathcal{A}_j\nabla^{j-1}\psi|^2)^{(p-2)/2}|\mathcal{A}_j\nabla^{j-1}\psi|^2\,\mathrm{d}x. \end{aligned} \tag{4.7}$$

Choosing $\beta = \beta_i^{(j)} > 0$ such that

$$\beta_i^{(j)}\theta_p \geqslant \lambda_i^{(j)} + \tfrac{1}{2}\nu^{(j)}\tau,$$

from (4.6) and (4.7) we obtain

$$\begin{aligned} I_1 + I_2 + I_3 \geqslant \tfrac{1}{4}\nu^{(j)} &\int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2\,\mathrm{d}x \\ &- \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p - \lambda_i^{(j)}|\mathcal{A}_j A|^p - h_i^{(j)}(\mathcal{S}_j A), \end{aligned}$$

so that (a′) holds. From (4.2), condition (b′) follows. □

The second lemma addresses the case where $p \geqslant 2$.

Lemma 4.2. *Let $j \in \{2,\dots,k\}$, $p \geqslant 2$, $\mu \geqslant 0$, $\nu^{(j)} > 0$, and let $\{M_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of positive constants. Moreover, let θ_p and Θ_p be given by lemma A.1, and let $\{f_i^{(j)}\}_{i\in\mathbb{N}}$ be a sequence of functions $f_i^{(j)} : E_k^{N^{k-j}} \to \mathbb{R}$ satisfying the following conditions:*

(a) (*strict j-quasi-convexity up to a perturbation*)

$$\begin{aligned} &\int_Q [f_i^{(j)}(A + \nabla^j\phi) - f_i^{(j)}(A)]\,\mathrm{d}x \\ &\qquad\geqslant -h_i^{(j)}(A) + \nu^{(j)}\int_Q (\mu^2 + |A|^2 + |\nabla^j\phi|^2)^{(p-2)/2}|\nabla^j\phi|^2\,\mathrm{d}x \end{aligned}$$

for every $A \in E_k^{N^{k-j}}$, for every $\phi \in C_{\mathrm{c}}^j(Q;\mathbb{R}^{N^{k-j}})$ and for every $i \in \mathbb{N}$, where $\{h_i^{(j)}\}_{i\in\mathbb{N}}$ is a sequence of functions $h_i^{(j)} : E_k^{N^{k-j}} \to [0,+\infty)$;

(b) (*p-growth condition*)

$$|f_i^{(j)}(A)| \leqslant M_i^{(j)}(1 + |A|^p) \quad \forall A \in E_k^{N^{k-j}},\ \forall i \in \mathbb{N}.$$

Then there exist an increasing sequence $\{F_i^{(j)}\}_{i\in\mathbb{N}}$ *of functions* $F_i^{(j)}: E_k^{N^{k-j+1}} \to \mathbb{R}$ *and a sequence* $\{L_i^{(j)}\}_{i\in\mathbb{N}}$ *of positive numbers depending on* $\nu^{(j)}, M_i^{(j)}, j, p, \mu$ *such that*

(a′) (*strict* $(j-1)$*-quasi-convexity up to a perturbation*)

$$\int_Q [F_i^{(j)}(A+\nabla^{j-1}\varphi) - F_i^{(j)}(A)]\,\mathrm{d}x \geqslant -h_i^{(j)}(\mathcal{S}_j A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p + \nu^{(j)}\frac{\theta_p}{4\Theta_p}\int_Q (\mu^2+|A|^2+|\nabla^{j-1}\varphi|^2)^{(p-2)/2}|\nabla^{j-1}\varphi|^2\,\mathrm{d}x$$

for every $A \in E_k^{N^{k-j+1}}$, *every* $\varphi \in C_{\mathrm{c}}^{j-1}(Q;\mathbb{R}^{N^{k-j+1}})$ *and every* $i \in \mathbb{N}$;

(b′) (*p-growth condition*)

$$|F_i^{(j)}(A)| \leqslant L_i^{(j)}(1+|A|^p) \quad \forall A \in E_k^{N^{k-j+1}},\ \forall i \in \mathbb{N};$$

(c) ($F_i^{(j)}$ *extends* $f_i^{(j)}$)

$$F_i^{(j)}(A) = f_i^{(j)}(A) \quad \forall A \in E_k^{N^{k-j}},\ \forall i \in \mathbb{N}.$$

Proof. Let $\alpha \in (0, \nu^{(j)}/\Theta_p]$ and $\beta > 0$ be determined at the end of the proof. We define

$$F_i^{(j)}(A) := f_i^{(j)}(\mathcal{S}_j A) - \alpha(\mu^2+|\mathcal{S}_j A|^2)^{p/2} + \alpha(\mu^2+|\mathcal{S}_j A|^2+\beta^2|\mathcal{A}_j A|^2)^{p/2}. \quad (4.8)$$

Condition (c) is clearly satisfied. Now let $\varphi \in C_{\mathrm{per}}^{\infty}(Q;\mathbb{R}^{N^{k-j+1}})$. As usual, we can write

$$\nabla^{j-1}\varphi = \nabla^j\phi + \nabla^{j-1}\psi,$$

with

$$\nabla^{j-1}\varphi, \nabla^{j-1}\psi \in C_{\mathrm{per}}^{\infty}(Q;E_k^{N^{k-j+1}}) \quad \text{and} \quad \nabla^j\phi \in C_{\mathrm{per}}^{\infty}(Q;E_k^{N^{k-j}}),$$

where $\psi \in C_{\mathrm{per}}^{\infty}(Q;R^{N^{k-j+1}})$ satisfies condition (2.7) with $s=j-1$. Setting

$$(f_i^{(j)})_\alpha(B) := f_i^{(j)}(B) - \alpha(\mu^2+|B|^2)^{p/2}$$

for every $B \in E_k^{N^{k-j}}$, we have

$$\begin{aligned}\int_Q [F_i^{(j)}(A+\nabla^{j-1}\varphi) - F_i^{(j)}(A)]\,\mathrm{d}x &= \int_Q [(f_i^{(j)})_\alpha(\mathcal{S}_j A+\nabla^j\phi+\mathcal{S}_j\nabla^{j-1}\psi) - (f_i^{(j)})_\alpha(\mathcal{S}_j A+\nabla^j\phi)]\,\mathrm{d}x \\ &\quad + \int_Q [(f_i^{(j)})_\alpha(\mathcal{S}_j A+\nabla^j\phi) - (f_i^{(j)})_\alpha(\mathcal{S}_j A)]\,\mathrm{d}x \\ &\quad + \alpha\int_Q [g_\beta(\mathcal{S}_j A+\mathcal{S}_j\nabla^{j-1}\varphi, \mathcal{A}_j A+\mathcal{A}_j\nabla^{j-1}\varphi) - g_\beta(\mathcal{S}_j A,\mathcal{A}_j A)]\,\mathrm{d}x \\ &=: I_1+I_2+I_3,\end{aligned}$$

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with g_β defined by (A 3), with

$$X = E_k^{N^{k-j}} \quad \text{and} \quad Y = \mathcal{A}_j E_k^{N^{k-j+1}}.$$

By repeating the chain of inequalities (3.3), one can show that $(f_i^{(j)})_\alpha$ is j-quasiconvex. In addition, applying lemma A.5 and proposition 2.9 to the function

$$B \mapsto \alpha(\mu^2 + |B|^2)^{p/2},$$

we have that $(f_i^{(j)})_\alpha$ satisfies condition (b), for some positive constant

$$\tilde{M}_i^{(j)} = \tilde{M}_i^{(j)}(\alpha, \mu, M_i^{(j)})$$

in place of $M_i^{(j)}$. Thus, by applying proposition 2.7, we can still conclude that relation (4.1) holds true for the function $(f_i^{(j)})_\alpha$, for a suitable constant

$$L = L(N, M_i^{(j)}, k, j, p, \alpha).$$

By repeating the same argument as in the previous proof, we get that, for every $\delta > 0$, there exists a positive constant

$$c = c(M_i^{(j)}, j, p, \alpha, \mu, \delta)$$

such that

$$\begin{aligned} I_1 &\geqslant -\delta - \delta|\mathcal{S}_j A|^p - \delta \int_Q |\nabla^j \phi|^p \,\mathrm{d}x - c\gamma \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x \\ &\geqslant -\delta - \delta|\mathcal{S}_j A|^p - \delta \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \,\mathrm{d}x \\ &\quad - c\gamma \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x. \end{aligned}$$

Hence, we can find a sequence of positive numbers $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ such that, for every $i \in \mathbb{N}$,

$$\begin{aligned} I_1 \geqslant -(\nu^{(j)} - \alpha\Theta_p) &\int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \,\mathrm{d}x \\ &- \lambda_i^{(j)} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p. \end{aligned}$$

Here $\lambda_i^{(j)} = \lambda_i^{(j)}(M_i^{(j)}, j, p, \alpha, \mu)$ for every fixed $i \in \mathbb{N}$. Adapting inequality (3.3) to the present situation, we get

$$\begin{aligned} I_2 &= \int_Q [(f_i^{(j)})_\alpha(\mathcal{S}_j A + \nabla^j \phi) - (f_i^{(j)})_\alpha(\mathcal{S}_j A)] \,\mathrm{d}x \\ &\geqslant -h_i^{(j)}(\mathcal{S}_j A) + (\nu^{(j)} - \alpha\Theta_p) \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\nabla^j \phi|^2)^{(p-2)/2} |\nabla^j \phi|^2 \,\mathrm{d}x. \end{aligned}$$

Moreover, assuming without any loss of generality that $\beta \geqslant 1$,

$$\begin{aligned} I_3 \geqslant{} & \tfrac{1}{2}\alpha\theta_p \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\ & + \tfrac{1}{2}\alpha\theta_p \int_Q (\mu^2 + |\mathcal{S}_j A|^2 + |\mathcal{S}_j \nabla^{j-1}\varphi|^2)^{(p-2)/2} |\mathcal{S}_j \nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\ & + \tfrac{1}{4}\alpha\theta_p \beta^2 (\mu^2 + |\mathcal{S}_j A|^2)^{(p-2)/2} \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^2 \,\mathrm{d}x \\ & + \tfrac{1}{4}\alpha\theta_p \beta^p \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x \\ \geqslant{} & \tfrac{1}{2}\alpha\theta_p \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\ & + \tfrac{1}{4}\alpha\theta_p \beta^p \int_Q |\mathcal{A}_j \nabla^{j-1}\psi|^p \,\mathrm{d}x. \end{aligned}$$

Now, choosing

$$\alpha = \alpha^{(j)} = \frac{\nu^{(j)}}{2\varTheta_p},$$

and $\beta = \beta_i^{(j)} > 0$ such that

$$\tfrac{1}{4}\alpha^{(j)}\theta_p (\beta_i^{(j)})^p \geqslant \lambda_i^{(j)},$$

we obtain

$$\begin{aligned} & \int_Q [F_i^{(j)}(A + \nabla^{j-1}\varphi) - F_i^{(j)}(A)] \,\mathrm{d}x \\ & \quad = I_1 + I_2 + I_3 \geqslant \nu^{(j)} \frac{\theta_p}{4\varTheta_p} \int_Q (\mu^2 + |A|^2 + |\nabla^{j-1}\varphi|^2)^{(p-2)/2} |\nabla^{j-1}\varphi|^2 \,\mathrm{d}x \\ & \qquad - h_i^{(j)}(\mathcal{S}_j A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_j A|^p, \end{aligned}$$

so that (a′) holds. Finally, condition (b′) follows by (4.8). □

We are now ready to prove theorem 1.2.

Proof of theorem 1.2.

Step 1 ($1 < p < 2$). We start by applying lemma 4.1 with $j = k$, $\nu^{(k)} = \nu$ and

$$f_i^{(k)}(A) = f(A), \quad h_i^{(k)}(A) = 0 \quad \text{for every } A \in E_k,\ i \in \mathbb{N}.$$

Then we again apply theorem 4.1 $k-2$ times with $j = k-1, k-2, \dots, 2$, respectively, where

$$\nu^{(j)} = \frac{\nu}{4^{k-j}}$$

and, for every $A \in E_k^{N^{k-j}}$ and $i \in \mathbb{N}$,

$$f_i^{(j)}(A) = F_i^{(j+1)}(A).$$

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Accordingly, the functions $h_i^{(j)}$ will be chosen as

$$h_i^{(k-1)}(A) = \frac{1}{i} + \frac{1}{i}|\mathcal{S}_k A|^p + \lambda_i^{(k)}|\mathcal{A}_k A|^p$$

and, for $j = k-2, \dots, 2$,

$$h_i^{(j)}(A) = \frac{k-j}{i} + \frac{1}{i}\sum_{r=j+1}^{k} |\mathcal{S}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_{j+1}A|^p + \lambda_i^{(j+1)}|\mathcal{A}_{j+1}A|^p + \sum_{r=j+2}^{k} \lambda_i^{(r)}|\mathcal{A}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_{j+1}A|^p,$$

where the sequences $\{\lambda_i^{(j)}\}_{i\in\mathbb{N}}$ are given by lemma 4.1. In this way, after the last step, corresponding to $j = 2$, we obtain a sequence $\{F_i^{(2)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$F_i^{(2)}(A) = f(\mathcal{S}_k\cdots\mathcal{S}_2A) + \beta_i^{(2)}|\mathcal{A}_2A|^p + \sum_{r=3}^{k}\beta_i^{(r)}|\mathcal{A}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_2A|^p.$$

Here, for $r = 2, \dots, k$, the sequence $\{\beta_i^{(r)}\}_{i\in\mathbb{N}}$ is as given in the proof of lemma 4.1. The functions $F_i^{(2)}$ just defined have the following properties:

(a′) (strict 1-quasi-convexity up to a perturbation)

$$\begin{aligned}\int_Q [F_i^{(2)}(A+\nabla\varphi) - F_i^{(2)}(A)]\,\mathrm{d}x &\geqslant -h_i^{(2)}(\mathcal{S}_2A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2A|^p \\ &\quad + \frac{\nu}{4^{k-1}}\int_Q (\mu^2 + |A|^2 + |\nabla\varphi|^2)^{(p-2)/2}|\nabla\varphi|^2\,\mathrm{d}x - \lambda_i^{(2)}|\mathcal{A}_2A|^p \\ &\geqslant -h_i^{(2)}(\mathcal{S}_2A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2A|^p - \lambda_i^{(2)}|\mathcal{A}_2A|^p\end{aligned}$$

for every $A \in \mathbb{R}^{N^k}$, every $\varphi \in C_c^1(Q;\mathbb{R}^{N^{k-1}})$ and every $i \in \mathbb{N}$;

(b′) (growth condition)

$$|F_i^{(2)}(A)| \leqslant L_i^{(2)}(1+|A|^p) \quad \forall A \in \mathbb{R}^{N^k},\ \forall i \in \mathbb{N},$$

with $L_i^{(2)} = L_i^{(2)}(\nu, M, p, \mu)$ for every fixed $i \in \mathbb{N}$;

(c) ($F_i^{(2)}$ extends f)

$$F_i^{(2)}(A) = f(A) \quad \forall A \in E_k,\ \forall i \in \mathbb{N}.$$

Now, for every $A \in \mathbb{R}^{N^k}$ and $i \in \mathbb{N}$, we set

$$F_i(A) := \inf\biggl\{\int_Q F_i^{(2)}(A + \nabla\varphi(x))\,\mathrm{d}x \colon \varphi \in C^\infty_{\mathrm{per}}(Q;\mathbb{R}^{N^{k-1}})\biggr\}.$$

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From property (a′) it follows that, for every $A \in \mathbb{R}^{N^k}$ and for every $i \in \mathbb{N}$,

$$F_i^{(2)}(A) - h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p - \lambda_i^{(2)}|\mathcal{A}_2 A|^p \leqslant F_i(A) \leqslant F_i^{(2)}(A). \tag{4.9}$$

Noting that, for every $A \in E_k$,

$$\lim_{i\to+\infty} h_i^{(2)}(\mathcal{S}_2 A) = \lim_{i\to+\infty}\left[\frac{k-2}{i} + \frac{1}{i}\sum_{r=3}^{k}|\mathcal{S}_r \cdots \mathcal{S}_2 A|^p\right] = 0,$$

from property (c) and (4.9) we have (1.6). Finally, (1.7) follows from (b′) and (4.9).

STEP 2 ($p \geqslant 2$). We first apply lemma 4.2 with $j = k$, $\nu^{(k)} = \nu$ and

$$f_i^{(k)}(A) = f(A), \quad h_i^{(k)}(A) = 0 \quad \text{for every } A \in E_k,\ i \in \mathbb{N}.$$

At this point, we again apply lemma 4.2 $k-2$ times with $j = k-1, k-2, \dots, 2$, respectively, where

$$\nu^{(j)} = \nu\left(\frac{\theta_p}{4\Theta_p}\right)^{k-j+1}$$

and, for every $A \in E_k^{N^{k-j}}$ and $i \in \mathbb{N}$,

$$f_i^{(j)}(A) = F_i^{(j+1)}(A), \qquad h_i^{(j)}(A) = \frac{k-j}{i} + \frac{1}{i}\sum_{r=j+1}^{k}|\mathcal{S}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_{j+1}A|^p.$$

Finally, when $j = 2$, we obtain a sequence $\{F_i^{(2)}\}_{i\in\mathbb{N}}$ of functions $F_i^{(2)} : \mathbb{R}^{N^k} \to \mathbb{R}$ given by

$$\begin{aligned} F_i^{(2)}(A) = f(\mathcal{S}_k \dots \mathcal{S}_2 A) + \sum_{r=3}^{k}\mathcal{L}_i^{(r)}(\mathcal{S}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_2 A, \mathcal{A}_r\mathcal{S}_{r-1}\cdots\mathcal{S}_2 A) \\ + \mathcal{L}_i^{(2)}(\mathcal{S}_2 A, \mathcal{A}_2 A), \end{aligned} \tag{4.10}$$

where we set

$$\mathcal{L}_i^{(r)}(A,B) := -\frac{\nu^{(r)}}{2\Theta_p}(\mu^2+|A|^2)^{p/2} + \frac{\nu^{(r)}}{2\Theta_p}(\mu^2+|A|^2+(\beta_i^{(r)})^2|B|^2)^{p/2}, \quad r = 2,\dots,k.$$

The functions $F_i^{(2)}$ just defined have the following properties:

(a′) (strict 1-quasi-convexity up to a perturbation)

$$\begin{aligned} &\int_Q [F_i^{(2)}(A+\nabla\varphi) - F_i^{(2)}(A)]\,\mathrm{d}x \\ &\qquad \geqslant -h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \\ &\qquad\quad + \frac{\nu}{4^k}\left(\frac{\theta_p}{\Theta_p}\right)^k \int_Q (\mu^2 + |A|^2 + |\nabla\varphi|^2)^{(p-2)/2}|\nabla\varphi|^2\,\mathrm{d}x \\ &\qquad \geqslant -h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \end{aligned}$$

for every $A \in \mathbb{R}^{N^k}$, every $\varphi \in C_c^1(Q; \mathbb{R}^{N^{k-1}})$ and every $i \in \mathbb{N}$;

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(b′) (growth condition)

$$|F_i^{(2)}(A)| \leqslant L_i^{(2)}(1+|A|^p) \quad \forall A \in \mathbb{R}^{N^k},\ i \in \mathbb{N};$$

(c) ($F_i^{(2)}$ extends f)

$$F_i^{(2)}(A) = f(A) \quad \forall A \in E_k,\ i \in \mathbb{N}.$$

For every $i \in \mathbb{N}$, we now define F_i as the quasi-convexification of the function $F_i^{(2)}$:

$$F_i(A) := \inf\left\{ \int_Q F_i^{(2)}(A + \nabla\varphi(x))\,\mathrm{d}x \colon \varphi \in C^\infty_{\mathrm{per}}(Q; \mathbb{R}^{N^{k-1}}) \right\}$$

for every $A \in \mathbb{R}^{N^k}$. From property (a′) and by the definition of F_i, we have

$$F_i^{(2)}(A) - h_i^{(2)}(\mathcal{S}_2 A) - \frac{1}{i} - \frac{1}{i}|\mathcal{S}_2 A|^p \leqslant F_i(A) \leqslant F_i^{(2)}(A) \quad \forall A \in \mathbb{R}^{N^k}. \tag{4.11}$$

Noting that

$$\lim_{i\to+\infty} h_i^{(2)}(\mathcal{S}_2 A) = 0 \quad \text{for all } A \in \mathbb{R}^{N^k},$$

from property (c) and (4.11) we have (1.6). Finally, (1.7) follows from (b′) and (4.11). □

4.1. Proof of theorem 1.3

To conclude the section, we give the proof of theorem 1.3.

Proof of theorem 1.3. It is sufficient to adapt the proof of [7, theorem 3] and to use theorem 1.3. □

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Appendix A.

This section contains some auxiliary results that are used in the rest of the paper. First, we give the proof of proposition 2.5.

Proof of proposition 2.5. Since $A \in E_k^{N^{k-s-1}}$ and $B \in \mathcal{A}_{s+1}E_k^{N^{k-s}}$, we can write $A = \mathcal{S}_{s+1}A$ and $B = \mathcal{A}_{s+1}C$, for some $C \in E_k^{N^{k-s}}$. As a first step, let us prove that, for every $r, l \in \{k-s+1, \dots, k\}$ with $r \neq l$, we have

$$A^{\mathrm{T}_r^{k-s}} \cdot C^{\mathrm{T}_l^{k-s}} = A \cdot C^{\mathrm{T}_r^{k-s}} = A^{\mathrm{T}_r^{k-s}} \cdot C. \tag{A 1}$$

To fix the ideas, let us assume $r < l$. By definition of transpose operators,

$$A^{\mathrm{T}_r^{k-s}}_{i_1 i_2 \cdots i_k} C^{\mathrm{T}_l^{k-s}}_{i_1 i_2 \cdots i_k} = A_{i_1 i_2 \cdots i_{k-s-1} i_r i_{k-s+1} \cdots i_{r-1} i_{k-s} i_{r+1} \cdots i_k} \times C_{i_1 i_2 \cdots i_{k-s-1} i_l i_{k-s+1} \cdots i_{l-1} i_{k-s} i_{l+1} \cdots i_k}$$

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for every $i_1, i_2, \ldots, i_k = 1, \ldots, N$. In the above expression, since $A \in E_k^{N^{k-s-1}}$ and $r, l > k-s$, we can exchange the indices in rth and lth positions in the first factor, obtaining

$$A^{\mathrm{T}_r^{k-s}}_{i_1 i_2 \cdots i_k} C^{\mathrm{T}_l^{k-s}}_{i_1 i_2 \cdots i_k} = A_{i_1 i_2 \cdots i_{k-s-1} i_r i_{k-s+1} \cdots i_{r-1} i_l i_{r+1} \cdots i_{l-1} i_{k-s} i_{l+1} \cdots i_k} \times C_{i_1 i_2 \cdots i_{k-s-1} i_l i_{k-s+1} \cdots i_{l-1} i_{k-s} i_{l+1} \cdots i_k}.$$

Summing the last relation with respect to $i_1, \ldots, i_k$ and renumbering the indices,

$$\begin{aligned} A^{\mathrm{T}_r^{k-s}} \cdot C^{\mathrm{T}_l^{k-s}} &= \sum_{i_1,\ldots,i_k}^{1,N} A^{\mathrm{T}_r^{k-s}}_{i_1 i_2 \cdots i_k} C^{\mathrm{T}_l^{k-s}}_{i_1 i_2 \cdots i_k} \\ &= \sum_{i_1,\ldots,i_k}^{1,N} A_{i_1 i_2 \cdots i_{k-s-1} i_r i_{k-s+1} \cdots i_{r-1} i_l i_{r+1} \cdots i_{l-1} i_{k-s} i_{l+1} \cdots i_k} \\ &\qquad \times C_{i_1 i_2 \cdots i_{k-s-1} i_l i_{k-s+1} \cdots i_{l-1} i_{k-s} i_{l+1} \cdots i_k} \\ &= \sum_{i_1,\ldots,i_k}^{1,N} A_{i_1 \cdots i_k} C_{i_1 i_2 \cdots i_{k-s-1} i_r i_{k-s+1} \cdots i_{r-1} i_{k-s} i_{r+1} \cdots i_k} \\ &= \sum_{i_1,\ldots,i_k}^{1,N} A_{i_1 i_2 \cdots i_k} C^{\mathrm{T}_r^{k-s}}_{i_1 i_2 \cdots i_k} \\ &= A \cdot C^{\mathrm{T}_r^{k-s}}. \end{aligned}$$

One can prove the second equality in (A 1) in the same way.

Let us now prove the proposition. We have

$$\begin{aligned} (s+1)^2 A \cdot B &= (s+1)^2 (\mathcal{S}_{s+1} A \cdot \mathcal{A}_{s+1} C) \\ &= (A + A^{\mathrm{T}_{k-s+1}^{k-s}} + \cdots + A^{\mathrm{T}_k^{k-s}}) \cdot [sC - (C^{\mathrm{T}_{k-s+1}^{k-s}} + \cdots + C^{\mathrm{T}_k^{k-s}})] \\ &= sA \cdot C - \sum_{r=k-s+1}^{k} A^{\mathrm{T}_r^{k-s}} \cdot C^{\mathrm{T}_r^{k-s}} - \sum_{r=k-s+1}^{k} A \cdot C^{\mathrm{T}_r^{k-s}} \\ &\quad + s \sum_{r=k-s+1}^{k} A^{\mathrm{T}_r^{k-s}} \cdot C - \sum_{l=k-s+1}^{k} \sum_{\substack{r \neq l \\ r=k-s+1}}^{k} A^{\mathrm{T}_r^{k-s}} \cdot C^{\mathrm{T}_l^{k-s}}. \end{aligned}$$

Since the sum of the first two terms is zero, using relation (A 1) we get

$$\begin{aligned} (s+1)^2 A \cdot B &= (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{\mathrm{T}_r^{k-s}} - \sum_{l=k-s+1}^{k} \sum_{\substack{r \neq l \\ r=k-s+1}}^{k} A \cdot C^{\mathrm{T}_r^{k-s}} \\ &= (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{\mathrm{T}_r^{k-s}} - (s-1) \sum_{r=k-s+1}^{k} A \cdot C^{\mathrm{T}_r^{k-s}} = 0. \end{aligned}$$

□

In the remainder of the section we state some lemmas that are proved in [7].

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Lemma A.1. *Let X be a Hilbert space, and let $g\colon X \to \mathbb{R}$ be given by*

$$g(x) := (\mu^2 + |x|^2)^{p/2}. \tag{A 2}$$

For every $p > 1$, there exist two constants $\theta_p > 0$ and $\Theta_p > 0$ such that, for every $\mu \geqslant 0$, the function g defined in (A 2) satisfies the following inequalities:

$$\begin{aligned}\theta_p(\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y|^2 &\leqslant g(x+y) - g(x) - \nabla g(x)\cdot y \\ &\leqslant \Theta_p(\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y|^2\end{aligned}$$

for every $x, y \in X$.

Lemma A.2. *Let X, Y be Hilbert spaces and let $p > 1$, $\mu \geqslant 0$, $\beta \geqslant 0$. Let*

$$g_\beta : X \times Y \to \mathbb{R}$$

be given by

$$g_\beta(x, y) := (\mu^2 + |x|^2 + \beta^2|y|^2)^{p/2}. \tag{A 3}$$

Then

$$\begin{aligned}g_\beta(x+\xi, y+\eta) &- g_\beta(x,y) - \nabla_x g_\beta(x,y)\cdot\xi - \nabla_y g_\beta(x,y)\cdot\eta \\ &\geqslant \theta_p(\mu^2 + |x|^2 + |\xi|^2 + \beta^2|y|^2 + \beta^2|\eta|^2)^{(p-2)/2}(|\xi|^2 + \beta^2|\eta|^2)\end{aligned} \tag{A 4}$$

for every $x, \xi \in X$, $y, \eta \in Y$, where θ_p is the first constant in lemma A.1. Therefore, if $p \geqslant 2$, we have

$$\begin{aligned}g_\beta(x+\xi, y+\eta) - g_\beta(x,y) &- \nabla_x g_\beta(x,y)\cdot\xi - \nabla_y g_\beta(x,y)\cdot\eta \\ &\geqslant \theta_p(\mu^2 + |x|^2 + |\xi|^2)^{(p-2)/2}|\xi|^2 \\ &\quad + \tfrac{1}{2}\theta_p\beta^2(\mu^2 + |x|^2)^{(p-2)/2}|\eta|^2 + \tfrac{1}{2}\theta_p\beta^p|\eta|^p\end{aligned} \tag{A 5}$$

for every $x, \xi \in X$, $y, \eta \in Y$.

Lemma A.3. *Let X be a Hilbert space and let $1 < p \leqslant 2$. Then, for every $\tilde\mu \geqslant 0$ and every $0 < \delta < 1$, we have*

$$(\tilde\mu^2 + |x+y|^2)^{(p-2)/2}|x+y|^2 \leqslant 2(\tilde\mu^2 + |x|^2)^{(p-2)/2}|x|^2 + 2(\tilde\mu^2 + |y|^2)^{(p-2)/2}|y|^2,$$

$$\begin{aligned}\delta^{(2-p)/2}(\tilde\mu^2 + |y|^2)^{(p-2)/2}|y|^2 \leqslant (\tilde\mu^2 &+ |x|^2 + |y|^2)^{(p-2)/2}|y|^2 \\ &+ \delta(\tilde\mu^2 + |x|^2)^{(p-2)/2}|x|^2\end{aligned}$$

for every $x, y \in X$.

Lemma A.4. *Let $1 < p \leqslant 2$. Then*

$$b^p \leqslant 8\varepsilon^{(p-2)/p}(\mu^2 + a^2 + b^2)^{(p-2)/2}b^2 + \varepsilon a^p + \varepsilon\mu^p$$

for every $a \geqslant 0$, $b \geqslant 0$, $\mu \geqslant 0$ and $0 < \varepsilon < 1$.

Lemma A.5. *Let X be a Hilbert space, and let $f \in C^1(X) \cap C^2(X \setminus \{0\})$. Assume that there exist $p > 1$, $C > 0$ and $\mu \geqslant 0$ such that*

$$|\nabla^2 f(x)| \leqslant C(\mu^2 + |x|^2)^{(p-2)/2} \tag{A 6}$$

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for every $x \in X \setminus \{0\}$. Then

$$|\nabla f(x+y) - \nabla f(x)| \leqslant K_p C(\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y| \tag{A 7}$$

for every $x, y \in X$, where $K_p \geqslant 1$ is a constant depending only on p.

LEMMA A.6. *Let X be a Hilbert space and let $f \in C^1(X)$. Assume that there exist $p > 1$ and $\mu \geqslant 0$ such that*

$$|\nabla f(x+y) - \nabla f(x)| \leqslant (\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y|$$

for every $x, y \in X$. If $1 < p \leqslant 2$, then for every $\varepsilon > 0$ there exists a constant $c_1 = c_1(\varepsilon, p) > 0$ depending only on ε and p such that

$$\begin{aligned} &|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ &\qquad \leqslant \varepsilon(\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y|^2 + c_1(\mu^2 + |z|^2)^{(p-2)/2}|z|^2 \end{aligned}$$

for every $x, y, z \in X$.

If $p \geqslant 2$, then for every $\varepsilon > 0$ there exists a constant $c_2 = c_2(\varepsilon, p) > 0$, depending only on ε and p, such that

$$\begin{aligned} &|f(x+y+z) - f(x+y) - \nabla f(x) \cdot z| \\ &\qquad \leqslant \varepsilon(\mu^2 + |x|^2 + |y|^2)^{(p-2)/2}|y|^2 + c_2(\mu^2 + |x|^2)^{(p-2)/2}|z|^2 + c_2|z|^p \end{aligned}$$

for every $x, y, z \in X$.

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