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DIFFERENTIAL GRADED QUIVERS OF SMOOTH RATIONAL SURFACES

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Abstract Let X be a smooth rational surface. We calculate a differential graded (DG) quiver of a full exceptional collection of line bundles on X obtained by an augmentation from a strong exceptional collection on the minimal model of X. In particular, we calculate canonical DG algebras of smooth toric surfaces.

Keywords: derived categories; exceptional collections; DG category; toric varieties

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1. Introduction

Derived categories of coherent sheaves have become one of the main research areas in modern algebraic geometry. An important tool allowing one to work with such complicated categories is given by full exceptional collections. Let X be a smooth projective variety and let $D^{\mathrm{b}}(X)$ denote the bounded derived category of coherent sheaves on X. It was proved in [**2**] that a full strong exceptional collection σ yields an equivalence between $D^{\mathrm{b}}(X)$ and the bounded derived category of modules over a finite quiver with relations. By a result of Bondal and Kapranov (see [**3**]), if σ is not strong, then $D^{\mathrm{b}}(X)$ is equivalent to the derived category of modules over some differential graded (DG) category $\mathcal{C}_\sigma$. It was proved in [**1**] that in the latter case the DG category $\mathcal{C}_\sigma$ is a path algebra of a finite DG quiver with relations Q_σ.

Calculating the quiver of a strong exceptional collection is equivalent to understanding endomorphisms of some sheaf. On the other hand, in order to calculate the DG quiver *a priori* one has to use injective resolutions. In [**1**] more comprehensive methods were given for determining DG quivers for two types of exceptional collections. Firstly, if a collection σ can be mutated to a strong one τ, then the DG quiver Q_σ of σ can be calculated by means of the quiver of τ. On the other hand, if a collection $\sigma = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle$ is almost strong, i.e. $\mathrm{Ext}^i(\mathcal{E}_j, \mathcal{E}_k) = 0$ for $i \neq 0, 1$, then one can construct a tilting object $\mathcal{E}_\sigma$ using universal extensions and coextensions defined in [**7**]. In this case endomorphisms of $\mathcal{E}_\sigma$ allow one to calculate the DG quiver of σ.

Many examples of almost strong exceptional collections are given by exceptional collections of line bundles on rational surfaces. Recall that every rational surface X, not isomorphic to the projective plane $\mathbb{P}^2$, is obtained from some Hirzebruch surface $\mathbb{F}_a$ by a sequence of blow-ups:

$$X = X_n \xrightarrow{\pi_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = \mathbb{F}_a.$$

In [**6**] Hille and Perling described an augmentation process that allows one to construct full exceptional collections of line bundles on X starting from a full exceptional collection on X_0. Moreover, in [**7**] it was proved that collections obtained by augmentation are almost strong.

The main purpose of this paper is to calculate the DG quiver of a full exceptional collection σ on a smooth rational surface obtained via augmentation from a strong full exceptional collection on X_0. To do this we first present σ in the canonical form (see Proposition 3.1). Using this presentation we calculate the tilting object $\mathcal{E}_\sigma$ (see Proposition 3.4) and its endomorphisms. Then, using twisted complexes, we can calculate the DG quiver of σ and any of its mutations.

In § 4 we apply these methods to a smooth toric surface Y with T-invariant divisors $D_1, \ldots, D_n$, which correspond to the rays in the fan $\Sigma_Y \subset N \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$ of Y. If the order of D_i is induced by an orientation of $\mathbb{Q}^2$, then the collection

$$\langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1 + D_2), \ldots, \mathcal{O}_Y(D_1 + \cdots + D_{n-1})\rangle$$

is full and exceptional on Y. For any $k \in \{1, \ldots, n\}$ the same remains true for the collection

$$\langle \mathcal{O}_Y, \mathcal{O}_Y(D_k), \mathcal{O}_Y(D_k + D_{k+1}), \ldots, \mathcal{O}_Y(D_k + \cdots + D_{k+n-2})\rangle$$

if the indices are considered as elements of $\mathbb{Z}/n\mathbb{Z}$. We consider all collections of such a form at once. Namely, let $Z = \operatorname{Tot}\omega_Y$ be the total space of the canonical bundle on Y and let $p\colon Z \to Y$ denote the canonical projection. As the vector bundle

$$\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_1 + \cdots + D_{n-1})$$

is a generator of $D^{\mathrm{b}}(Y)$, the sheaf $p^*(\mathcal{E})$ is a generator of $D^{\mathrm{b}}(Z)$. Moreover,

$$\begin{aligned} \operatorname{Hom}_Z(p^*(\mathcal{E}), p^*(\mathcal{E})) &= \operatorname{Hom}_Y(\mathcal{E}, p_*p^*(\mathcal{E})) \\ &= \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E} \otimes p_*(\mathcal{O}_Z)) \\ &= \bigoplus_{n \geqslant 0} \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_Y(-nK_Y)). \end{aligned}$$

On Y we can consider an infinite sequence $(A_k)_{k=0}^{\infty}$ of line bundles defined by

$$A_k = \mathcal{O}_Y(sK_Y + D_1 + \cdots + D_r),$$

where $k = sn + r$ for $0 \leqslant r < n$. Denote by $\mathcal{A}_Y = \bigoplus A_k$ the sum of all elements in this sequence. We define the canonical algebra of Y to be the DG algebra of endomorphisms of

$p^*(\mathcal{E})$ or equivalently of $\mathcal{A}_Y$. Methods described in § 3 allow us to calculate the canonical DG algebra of any smooth toric surface.

The structure of the paper is as follows. Definitions of quivers and DG quivers, twisted complexes and exceptional collections together with mutations, universal extensions and coextensions are contained in § 2. We also recall basic facts about rational surfaces. In § 3 we recall after [**6**] the construction of full exceptional collections on smooth rational surfaces. We present any such exceptional collection in the canonical form and describe its Ext-quiver. Then, using universal coextensions, we calculate the associated tilting object and we describe its endomorphisms. These data allow us to calculate the DG quiver of the collection. In § 4 we apply these methods to smooth toric surfaces. We start by recalling basic facts about toric surfaces and full exceptional collections on them. Then we define the canonical DG algebra of a toric surface and we show how to use the results of § 3 to calculate it. We conclude with examples of the canonical DG algebras for the first and second Hirzebruch surfaces and for surfaces obtained from the first Hirzebruch surface by blowing up one point.

2. Background

2.1. Quivers

A quiver Q consists of finite sets Q_0, Q_1 and two maps $h, t\colon Q_1 \to Q_0$. Elements of Q_0 are vertices of Q and elements of Q_1 are arrows of Q. The maps h and t indicate the head and the tail of an arrow, respectively. A path in Q is a sequence $p = a_n \cdots a_1$ of arrows such that $h(a_i) = t(a_{i+1})$ for $1 \leqslant i \leqslant n-1$; we put $h(p) = h(a_n)$ and $t(p) = t(a_1)$. A path algebra $\mathbb{C}Q$ of a quiver Q is an algebra with a basis consisting of paths in Q; the product $p \circ p'$ of two basis elements is defined by means of concatenation of paths if $t(p) = h(p')$, and is set to zero otherwise. We also assume that for any vertex $i \in Q_0$ there is a trivial path $e_i \in Q_1$ with $t(e_i) = i = h(e_i)$. Then the element $\sum_{i\in Q_0} e_i$ is the identity of $\mathbb{C}Q$.

A quiver with relations (Q, S) is a quiver Q together with a set $S \subset \mathbb{C}Q$. Let $I = \langle S\rangle \subset \mathbb{C}Q$ be an ideal generated by S. Then the path algebra $\mathbb{C}(Q, S)$ of a quiver with relations is defined to be $\mathbb{C}Q/I$.

If arrows in Q are $\mathbb{Z}$-graded in such a way that $\deg(e_i) = 0$ for any $i \in Q_0$, the path algebra $\mathbb{C}Q$ becomes a graded algebra; for a path $p = a_n \cdots a_1$ we put $\deg(p) = \deg(a_1) + \cdots + \deg(a_n)$.

A DG quiver is a quiver Q together with a $\mathbb{Z}$-grading on Q_1 and a structure of a DG algebra on $\mathbb{C}Q$ such that $\partial(e_i) = 0$ for any $i \in Q_0$. The Leibniz rule guarantees that $h(\partial(p)) = h(p)$ and $t(\partial(p)) = t(p)$ as soon as $\partial(p) \neq 0$. If the set $S \subset \mathbb{C}Q$ consists of homogeneous elements, one can analogously define a DG quiver with relations (Q, S).

2.2. DG categories

A DG category is a preadditive category $\mathcal{C}$ in which abelian groups $\operatorname{Hom}_{\mathcal{C}}(A, B)$ are endowed with a $\mathbb{Z}$-grading and a differential ∂ of degree 1. Moreover, the composition of morphisms

$$\mathrm{Hom}_{\mathcal{C}}(A,B)\otimes \mathrm{Hom}_{\mathcal{C}}(B,C)\to \mathrm{Hom}_{\mathcal{C}}(A,C)$$

is a morphism of complexes and for any object $C \in \mathcal{C}$ the identity morphism id_C is a closed morphism of degree 0.

The homotopy category $H(\mathcal{C})$ of a DG category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{H(\mathcal{C})}(A,B) = H^0(\mathrm{Hom}_{\mathcal{C}}(A,B)).$$

For a DG category $\mathcal{C}$, Bondal and Kapranov [**3**] defined the category $\mathcal{C}^{\text{pre-tr}}$ of one-sided twisted complexes. It is the smallest DG category containing $\mathcal{C}$ such that $H(\mathcal{C}^{\text{pre-tr}})$ is triangulated. Objects of $\mathcal{C}^{\text{pre-tr}}$ are expressions of the form $(\bigoplus_{i=1}^{n} C_i[r_i], q_{i,j})$ for $C_i \in \mathcal{C}$ and $r_i \in \mathbb{Z}$. We refer the reader to the original paper [**3**] for further details.

2.3. Exceptional collections

Let $\mathcal{T}$ be a $\mathbb{C}$-linear Ext-finite triangulated category such that $\mathcal{T} \simeq H^0(\mathcal{C}^{\text{pre-tr}})$ for some DG category $\mathcal{C}$. Recall that an object $\mathcal{E} \in \mathcal{T}$ is *exceptional* if $\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C}$ and $\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = 0$ for $i \neq 0$. A sequence $\sigma = \langle \mathcal{E}_1,\dots,\mathcal{E}_n\rangle$ of exceptional objects is an *exceptional collection* if $\mathrm{Ext}^i(\mathcal{E}_j,\mathcal{E}_k) = 0$ for $j > k$ and any i. For an exceptional collection σ, let $\mathcal{T}_\sigma$ be the smallest strictly full subcategory of $\mathcal{T}$ containing elements of σ. We say that σ is *full* if $\mathcal{T}_\sigma$ is equivalent to $\mathcal{T}$. Finally, the collection σ is *strong* if $\mathrm{Ext}^i(\mathcal{E}_j,\mathcal{E}_k) = 0$ for $i \neq 0$ and any j, k.

In [**2**] it was proved that a strong exceptional collection σ leads to an equivalence between $\mathcal{T}_\sigma$ and $D^{\mathrm{b}}(\text{mod-}A_\sigma)$ for a finite-dimensional algebra A_σ. The algebra $\mathcal{A}_\sigma$ is a path algebra of a quiver with relations obtained from objects $\mathcal{E}_1,\dots,\mathcal{E}_n$.

When the collection σ is not strong the category $\mathcal{T}_\sigma$ is equivalent to $D^{\mathrm{b}}(C_\sigma)$ for some DG algebra C_σ (see [**3**]). It was proved in [**1**] that C_σ can be chosen to be a path algebra of a finite DG quiver with relations Q_σ. If the collection σ is strong, the DG algebra C_σ is quasi-isomorphic to A_σ.

In [**2**] mutations of exceptional collections were defined. If a pair $\langle \mathcal{E},\mathcal{F}\rangle$ is exceptional, then so are the pairs $\langle L_{\mathcal{E}}\mathcal{F},\mathcal{E}\rangle$ and $\langle \mathcal{F}, R_{\mathcal{F}}\mathcal{E}\rangle$ for $L_{\mathcal{E}}\mathcal{F}$ and $R_{\mathcal{F}}\mathcal{E}$ defined by distinguished triangles in $\mathcal{T}$

$$L_{\mathcal{E}}\mathcal{F} \to \mathcal{E}\otimes \mathrm{Hom}(\mathcal{E},\mathcal{F}) \to \mathcal{F} \to L_{\mathcal{E}}\mathcal{F}[1],$$
$$R_{\mathcal{F}}\mathcal{E}[-1] \to \mathcal{E} \to \mathrm{Hom}(\mathcal{E},\mathcal{F})^* \otimes \mathcal{F} \to R_{\mathcal{F}}\mathcal{E}.$$

For an exceptional collection $\sigma = \langle \mathcal{E}_1,\dots,\mathcal{E}_n\rangle$, the ith left *mutation* $L_i\sigma$ and the ith right mutation $R_i\sigma$ are exceptional collections defined by

$$L_i\sigma = \langle \mathcal{E}_1,\dots,\mathcal{E}_{i-1}, L_{\mathcal{E}_i}\mathcal{E}_{i+1}, \mathcal{E}_i, \mathcal{E}_{i+2},\dots,\mathcal{E}_n\rangle,$$
$$R_i\sigma = \langle \mathcal{E}_1,\dots,\mathcal{E}_{i-1}, \mathcal{E}_{i+1}, R_{\mathcal{E}_{i+1}}\mathcal{E}_i, \mathcal{E}_{i+2},\dots,\mathcal{E}_n\rangle.$$

Mutations of DG quivers can be defined in such a way that $Q_{L_i\sigma} = L_iQ_\sigma$ and $Q_{R_i\sigma} = R_iQ_\sigma$ (see [**1**]). In particular, it is relatively easy to calculate a DG quiver of a collection that can be mutated to a strong one.

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2.4. Universal extensions and coextensions

We say that σ is *almost strong* if $\mathrm{Ext}^i(\mathcal{E}_j, \mathcal{E}_k) = 0$ for $i \neq 0, 1$ and for all j, k.

In [**7**] Hille and Perling described how to construct a tilting object in $\mathcal{T}_\sigma$ from an almost strong exceptional collection. The main tools in their construction are universal extension and coextension. For a pair $(\mathcal{E}, \mathcal{F})$ a universal extension $\bar{\mathcal{E}}$ of $\mathcal{E}$ by $\mathcal{F}$ is defined by means of a distinguished triangle

$$\mathcal{E}[-1] \xrightarrow{\text{can}} \mathcal{F} \otimes \mathrm{Ext}^1(\mathcal{E}, \mathcal{F})^* \to \bar{\mathcal{E}} \to \mathcal{E}.$$

Dually, a universal coextension of $\mathcal{F}$ by $\mathcal{E}$ is an object $\bar{\mathcal{F}}$ defined by a distinguished triangle

$$\mathcal{F} \to \bar{\mathcal{F}} \to \mathcal{E} \otimes \mathrm{Ext}^1(\mathcal{E}, \mathcal{F}) \xrightarrow{\text{can}} \mathcal{F}[1].$$

If $\mathcal{F}$ is exceptional and $\mathrm{Ext}^i(\mathcal{F}, \mathcal{E}) = 0$ for all i, then $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})^*$ is naturally isomorphic to $\mathrm{Hom}(\mathcal{F}, \bar{\mathcal{E}})$, and thus $\mathcal{E}$ is the cone of the canonical map

$$\mathcal{F} \otimes \mathrm{Hom}(\mathcal{F}, \bar{\mathcal{E}}) \xrightarrow{\text{can}} \bar{\mathcal{E}} \to \mathcal{E}.$$

Dually, if $\mathcal{E}$ is exceptional and $\mathrm{Ext}^i(\mathcal{F}, \mathcal{E}) = 0$ for all i, then $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ is naturally isomorphic to $\mathrm{Hom}(\bar{\mathcal{F}}, \mathcal{E})$ and, up to a shift, $\mathcal{F}$ is the cone of the canonical map

$$\bar{\mathcal{F}} \xrightarrow{\text{can}} \mathcal{E} \otimes \mathrm{Hom}(\bar{\mathcal{F}}, \mathcal{E}) \to \mathcal{F}[1].$$

Universal extensions and coextensions allow us to calculate a DG quiver of any almost strong exceptional collection in a category $\mathcal{T} \simeq H^0(\mathcal{C}^{\text{pre-tr}})$ (see [**1**]).

2.5. Rational surfaces

Let X be a smooth rational surface. X is obtained by a sequence of blow-ups from the projective plane $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_a$. We have a sequence of maps

$$X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow X_1 \xrightarrow{\pi_1} X_0,$$

where $X_0 = \mathbb{P}^2$ or $\mathbb{F}_a$. We can also assume that every π_i is a blow-up of one point $x_{i-1} \in X_{i-1}$.

Let $F_i \subset X_i$ be the exceptional divisor of π_i. Denote by $E_i \subset X$ the strict transform of F_i, and by $R_i \subset X$ its pullback under $\pi_{i+1} \cdots \pi_n$.

The divisors R_i are mutually orthogonal and $R_i^2 = -1$. Hille and Perling [**6**] introduced a partial order on the set of indices $\{1, \ldots, n\}$; $i \succeq j$ if $i > j$ and $\pi_{j-1} \cdots \pi_{i-1}(x_{i-1}) = x_{j-1}$. Then

$$\left.\begin{aligned} \mathrm{Hom}(\mathcal{O}_X(R_i), \mathcal{O}_X(R_j)) &\simeq \begin{cases} \mathbb{C} & \text{if } i \succeq j, \\ 0, & \text{otherwise;} \end{cases} \\ \mathrm{Ext}^1(\mathcal{O}_X(R_i), \mathcal{O}_X(R_j)) &\simeq \begin{cases} \mathbb{C} & \text{if } i \succ j, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{2.1}$$

Moreover, $H^0(\mathcal{O}_X(R_i)) = \mathbb{C}$, $H^j(\mathcal{O}_X(R_i)) = 0$ for $j > 0$ and $H^k(\mathcal{O}_X(-R_i)) = 0$ for all k.

3. DG quivers of exceptional collections on rational surfaces

3.1. Exceptional collections on rational surfaces

Again, let X be a smooth rational surface. We recall the augmentation procedure given in [**6**] that allows us to construct full exceptional collections of line bundles on X from an exceptional collection on X_0. To simplify the notation we identify a line bundle $\mathcal{L}$ on X_i with its pullback via π_js and denote them by the same letter.

Let $\sigma = \langle \mathcal{L}_1, \dots, \mathcal{L}_s \rangle$ be a full exceptional collection of line bundles on X_i. The augmentation of σ is $\sigma' = \langle \mathcal{L}_1(R_{i+1}), \dots, \mathcal{L}_{k-1}(R_{i+1}), \mathcal{L}_k, \mathcal{L}_k(R_{i+1}), \mathcal{L}_{k+1}, \dots, \mathcal{L}_s \rangle$—an exceptional collection on X_{i+1}.

It follows from a result of Orlov [**8**] that collections obtained via augmentation are full. It was proved in [**7**] that they are almost strong.

Mutations allow one to present each of the above described collections in the following form.

Proposition 3.1. *Any exceptional collection of line bundles on X obtained via augmentation can be mutated to $\langle \mathcal{O}_{R_n}(R_n)[-1], \dots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \dots, \mathcal{N}_t \rangle$, where $\langle \mathcal{O}_{X_0} = \mathcal{N}_0, \mathcal{N}_1, \dots, \mathcal{N}_t \rangle$ is an exceptional collection on X_0.*

Proof. The collection on X obtained via augmentation is of the form

$$\langle \mathcal{L}_1(R_n), \dots, \mathcal{L}_{i-1}(R_n), \mathcal{L}_i, \mathcal{L}_i(R_n), \mathcal{L}_{i+1}, \dots, \mathcal{L}_s \rangle,$$

where the $\mathcal{L}_j$s are pull-backs of line bundles on X_{n-1}. In particular, $\mathcal{L}_j|_{R_n} \simeq \mathcal{O}_{R_n}$ for any j.

Isomorphisms

$$\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_i(R_n)) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(R_n)) \simeq \mathbb{C}$$

and the short exact sequence

$$0 \to \mathcal{L}_i \to \mathcal{L}_i(R_n) \to \mathcal{O}_{R_n}(R_n) \to 0 \tag{3.1}$$

show that this collection can be mutated to

$$\langle \mathcal{L}_1(R_n), \dots, \mathcal{L}_{i-1}(R_n), \mathcal{O}_{R_n}(R_n)[-1], \mathcal{L}_i, \mathcal{L}_{i+1}, \dots \mathcal{L}_s \rangle.$$

Then

$$\mathrm{Hom}(\mathcal{L}_k(R_n), \mathcal{O}_{R_n}(R_n)) \simeq \mathrm{Hom}(\mathcal{O}_X(R_n), \mathcal{O}_{R_n}(R_n)) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_{R_n}) \simeq \mathbb{C}$$

and the exact sequence (3.1) for $\mathcal{L}_{i-1}, \dots, \mathcal{L}_1$ provides further mutations to

$$\langle \mathcal{O}_{R_n}(R_n)[-1], \mathcal{L}_1, \dots, \mathcal{L}_s \rangle.$$

The collection $\langle \mathcal{L}_1, \dots, \mathcal{L}_s \rangle$ is a pull-back of a collection on X_{n-1} and it again has the form $\langle \mathcal{L}'_1(R_{n-1}), \dots, \mathcal{L}'_{k-1}(R_{n-1}), \mathcal{L}'_k, \mathcal{L}'_k(R_{n-1}), \mathcal{L}'_{k+1}, \dots, \mathcal{L}'_{s-1} \rangle$ for some k. As before, it can be mutated to $\langle \mathcal{L}'_1(R_{n-1}), \dots, \mathcal{L}'_{k-1}(R_{n-1}), \mathcal{O}_{R_{n-1}}(R_{n-1})[-1], \mathcal{L}'_k, \dots, \mathcal{L}'_{s-1} \rangle$ and then to $\langle \mathcal{O}_{R_{n-1}}(R_{n-1})[-1], \mathcal{L}'_1, \dots, \mathcal{L}'_{s-1} \rangle$. Continuing, we can mutate the collection on X to $\langle \mathcal{O}_{R_n}(R_n)[-1], \dots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \dots, \mathcal{N}_t \rangle$. □

From now on we will assume that the collection $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \dots, \mathcal{N}_t \rangle$ on X_0 is strong.

3.2. Ext-quiver of $\langle \mathcal{O}_{R_n}(R_n)[-1], \dots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \dots, \mathcal{N}_t \rangle$.

To compute the Ext-quiver of the above collection one needs to understand the compositions

$$\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_k}(R_k)) \otimes \mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_k}(R_k)),$$
$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_k}(R_k)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_k}(R_k))$$

for $i \succeq j \succeq k$.

Denote by $\mathcal{C}$ the subcategory of $D^{\mathrm{b}}(X)$ generated by objects $\mathcal{O}_{R_n}(R_n), \dots, \mathcal{O}_{R_1}(R_1)$, and denote by $\mathcal{C}'$ the subcategory of $D^{\mathrm{b}}(X)$ generated by $\mathcal{O}(R_n), \dots, \mathcal{O}(R_1)$. Then $\mathcal{C}$ is a mutation of $\mathcal{C}'$ over $\mathcal{O}_X$, and hence understanding morphisms between generators of $\mathcal{C}$ is equivalent to understanding morphisms between generators of $\mathcal{C}'$. In particular, it follows from (2.1) that $\mathrm{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j))$ is one dimensional if $i \succeq j$ and 0 otherwise. Similarly, $\dim_{\mathbb{C}} \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_j}(R_j))$ is 1 if $i \succ j$ and 0 otherwise.

Lemma 3.2. *Let $i \succ j \succ k$. The composition*

$$\mathrm{Ext}^1(\mathcal{O}_X(R_j), \mathcal{O}_X(R_k)) \otimes \mathrm{Hom}(\mathcal{O}_X(R_i), \mathcal{O}_X(R_j)) \to \mathrm{Ext}^1(\mathcal{O}_X(R_i), \mathcal{O}_X(R_k))$$

is an isomorphism.

Proof. The exact sequence

$$0 \to \mathcal{O}_X(R_i) \to \mathcal{O}_X(R_j) \to \mathcal{O}_{R_j - R_i}(R_j) \to 0$$

gives

$$\begin{aligned} 0 &\to \mathrm{Hom}(\mathcal{O}_{R_j - R_i}(R_j), \mathcal{O}_X(R_k)) \to \mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_k)) \xrightarrow{\alpha} \mathrm{Hom}(\mathcal{O}_X(R_i), \mathcal{O}_X(R_k)) \\ &\to \mathrm{Ext}^1(\mathcal{O}_{R_j - R_i}(R_j), \mathcal{O}_X(R_k)) \to \mathrm{Ext}^1(\mathcal{O}_X(R_j), \mathcal{O}_X(R_k)) \xrightarrow{\beta} \mathrm{Ext}^1(\mathcal{O}_X(R_i), \mathcal{O}_X(R_k)) \\ &\to \mathrm{Ext}^2(\mathcal{O}_{R_j - R_i}(R_j), \mathcal{O}_X(R_k)) \to 0. \end{aligned}$$

The morphism $\alpha\colon \mathrm{Hom}(\mathcal{O}_X(R_j), \mathcal{O}_X(R_k)) \to \mathrm{Hom}(\mathcal{O}_X(R_i), \mathcal{O}_X(R_k))$ is an isomorphism because its kernel is 0 and both spaces are one dimensional.

Since both $\mathrm{Ext}^1(\mathcal{O}_X(R_j), \mathcal{O}_X(R_k))$ and $\mathrm{Ext}^1(\mathcal{O}_X(R_i), \mathcal{O}_X(R_k))$ are of dimension 1, β is an isomorphism if and only if $\mathrm{Ext}^1(\mathcal{O}_{R_j - R_i}(R_j), \mathcal{O}_X(R_k))$ is 0.

Applying $\mathrm{Hom}(\cdot, \mathcal{O}_X(R_k))$ to the short exact sequence

$$0 \to \mathcal{O}_{R_j - R_i}(R_j) \to \mathcal{O}_{R_j}(R_j + R_i) \to \mathcal{O}_{R_i}(R_j + R_i) \simeq \mathcal{O}_{R_i}(R_i) \to 0$$

gives an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_X(R_k)) &\to \mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j + R_i), \mathcal{O}_X(R_k)) \\ &\to \mathrm{Ext}^1(\mathcal{O}_{R_j - R_i}(R_j), \mathcal{O}_X(R_k)) \\ &\to 0. \end{aligned}$$

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Indeed, $\mathrm{Hom}(\mathcal{O}_{R_j-R_i}(R_j),\mathcal{O}_X(R_k))=0$ because $\mathcal{O}_X(R_k)$ is a torsion-free sheaf and $\mathrm{Ext}^2(\mathcal{O}_{R_i}(R_i),\mathcal{O}_X(R_k))\simeq\mathrm{Ext}^2(\mathcal{O}_{R_i}(R_i),\mathcal{O}_X)\simeq\mathrm{Hom}(\mathcal{O}_X,\mathcal{O}_{R_i}(R_i+K_X))^\vee\simeq 0$, by Serre duality.

It follows from (2.1) that $\mathrm{Ext}^1(\mathcal{O}_{R_i}(R_i),\mathcal{O}_{R_k}(R_k))=\mathbb{C}$. Thus, β is an isomorphism if and only if $\mathrm{Ext}^1(\mathcal{O}_{R_j-R_i}(R_j),\mathcal{O}_X(R_k))\simeq 0$, if and only if $\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j+R_i),\mathcal{O}_X(R_k))\simeq\mathbb{C}$. Since $j\succ k$, we have an isomorphism $\mathcal{O}_{R_j}(-R_k)\simeq\mathcal{O}_{R_j}$. It follows that $\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j+R_i),\mathcal{O}_X(R_k))\simeq\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j+R_i),\mathcal{O}_X)$.

From short exact sequences

$$0\to\mathcal{O}_X(R_i)\to\mathcal{O}_X(R_i+R_j)\to\mathcal{O}_{R_j}(R_i+R_j)\to 0,$$
$$0\to\mathcal{O}_X(R_j)\to\mathcal{O}_X(R_i+R_j)\to\mathcal{O}_{R_i}(R_i+R_j)\simeq\mathcal{O}_{R_i}(R_i)\to 0$$

and vanishing of $\mathrm{Hom}(\mathcal{O}_X(R_i),\mathcal{O}_X)$ and $\mathrm{Ext}^1(\mathcal{O}_X(R_i),\mathcal{O}_X)$, we deduce that

$$\mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j+R_i),\mathcal{O}_X)\simeq\mathrm{Ext}^1(\mathcal{O}_X(R_j+R_i),\mathcal{O}_X)\simeq\mathrm{Ext}^1(\mathcal{O}_{R_i}(R_j),\mathcal{O}_X)\simeq\mathbb{C},$$

which proves that β is an isomorphism. □

Remark 3.3. If $i\succeq j\succeq k$, the composition

$$\mathrm{Hom}(\mathcal{O}_X(R_j),\mathcal{O}_X(R_k))\otimes\mathrm{Ext}^1(\mathcal{O}_X(R_i),\mathcal{O}_X(R_j))\to\mathrm{Ext}^1(\mathcal{O}_X(R_i),\mathcal{O}_X(R_k))$$

does not have to be an isomorphism. Indeed, consider a surface X obtained from its minimal model by three blow-ups such that $E_1^2=-3$, $E_2^2=-2$, $E_3^2=-1$, $E_1E_2=0$, $E_1E_3=1$ and $E_2E_3=1$. Then $R_1=E_1+E_2+2E_3$, $R_2=E_2+E_3$ and $R_3=E_3$. Let $\bar\alpha\in\mathrm{Ext}^1(\mathcal{O}_X(R_3),\mathcal{O}_X(R_2))$ and $\beta\in\mathrm{Hom}(\mathcal{O}_X(R_2),\mathcal{O}_X(R_1))$ be non-zero elements. We have a short exact sequence

$$0\to\mathcal{O}_X(E_2+E_3)\xrightarrow{\beta}\mathcal{O}_X(E_1+E_2+2E_3)\to\mathcal{O}_{E_1+E_3}(E_1+E_2+2E_3)\to 0.$$

As in the proof of the previous lemma, $\beta\circ\bar\alpha=0$ if and only if $\mathrm{Hom}(\mathcal{O}_X(E_3),\mathcal{O}_{E_1+E_3}(E_1+E_2+2E_3))\neq 0$. We have

$$\mathrm{Hom}(\mathcal{O}_X(E_3),\mathcal{O}_{E_1+E_3}(E_1+E_2+2E_3))=H^0(X,\mathcal{O}_{E_1+E_3}(E_1+E_2+E_3)).$$

The latter sheaf fits into a short exact sequence

$$0\to\mathcal{O}_{E_3}\simeq\mathcal{O}_{E_3}(E_2+E_3)\to\mathcal{O}_{E_1+E_3}(E_1+E_2+E_3)\to\mathcal{O}_{E_1}(E_1+E_2+E_3)\simeq\mathcal{O}_{E_1}(-2)\to 0$$

from which it follows that $H^0(X,\mathcal{O}_{E_1+E_3}(E_1+E_2+E_3))=\mathbb{C}$.

Thus, we know that between $\mathcal{O}_{R_i}(R_i)$ and $\mathcal{O}_{R_j}(R_j)$ there is either no arrow or two arrows, one in degree 0 and one in degree 1. Moreover, $\bar\beta\circ\alpha\neq 0$ and $\beta\circ\alpha\neq 0$ for

$$\mathcal{O}_{R_i}(R_i)\underset{\bar\alpha}{\overset{\alpha}{\rightrightarrows}}\mathcal{O}_{R_k}(R_k)\underset{\bar\beta}{\overset{\beta}{\rightrightarrows}}\mathcal{O}_{R_j}(R_j),$$

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where α, β are non-zero morphisms and $\bar{\alpha}$, $\bar{\beta}$ are non-zero elements of the first Ext groups.

It remains to understand the space of morphisms from $\mathcal{O}_{R_k}(R_k)$ to $\mathcal{N}_i$ and how these morphisms compose with morphisms $\mathcal{O}_{R_i}(R_i) \to \mathcal{O}_{R_k}(R_k)$ and $\mathcal{N}_i \to \mathcal{N}_j$.

As the $\mathcal{N}_i$ are torsion free, we know that $\mathrm{Hom}(\mathcal{O}_{R_k}(R_k), \mathcal{N}_i) = 0$. From the short exact sequence

$$0 \to \mathcal{N}_i \to \mathcal{N}_i \otimes \mathcal{O}_X(R_k) \to \mathcal{O}_{R_k}(R_k) \to 0 \tag{3.2}$$

we deduce that $\mathrm{Ext}^1(\mathcal{O}_{R_k}(R_k), \mathcal{N}_i) \simeq \mathrm{Hom}(\mathcal{O}_{R_k}(R_k), \mathcal{O}_{R_k}(R_k)) = \mathbb{C}$. Let us fix a non-zero element ζ_k^i of the group $\mathrm{Ext}^1(\mathcal{O}_{R_k}(R_k), \mathcal{N}_i)$.

The diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}_i \otimes \mathcal{O}_X(R_j) & \longrightarrow & \mathcal{O}_{R_j}(R_j) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}_i \otimes \mathcal{O}_X(R_k) & \longrightarrow & \mathcal{O}_{R_k}(R_k) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{R_k - R_j}(R_k) & \xrightarrow{=} & \mathcal{O}_{R_k - R_j}(R_k) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

shows that the composition

$$\mathrm{Hom}(\mathcal{O}_{R_j}(R_j), \mathcal{O}_{R_k}(R_k)) \otimes \mathrm{Ext}^1(\mathcal{O}_{R_k}(R_k), \mathcal{N}_i) \to \mathrm{Ext}^1(\mathcal{O}_{R_j}(R_j), \mathcal{N}_i)$$

is an isomorphism.

To understand the composition

$$\mathrm{Ext}^1(\mathcal{O}_{R_k}(R_k), \mathcal{N}_i) \otimes \mathrm{Hom}(\mathcal{N}_i, \mathcal{N}_l) \to \mathrm{Ext}^1(\mathcal{O}_{R_k}(R_k), \mathcal{N}_l)$$

we apply the functor $\mathrm{Hom}(\cdot, \mathcal{N}_l)$ to the short exact sequence (3.2). It follows that for $\phi \in \mathrm{Hom}(\mathcal{N}_i, \mathcal{N}_l)$ the composition $\phi \circ \zeta_k^i$ is zero if and only if ϕ factors through $\mathcal{N}_l(-R_k)$.

3.3. DG quiver of $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$

Now we will present calculations allowing one to determine the DG quiver of the collection $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$. Recall that we work under the assumption that the collection $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ on X_0 is strong.

To calculate the DG category of the collection

$$\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$$

we substitute some objects with universal coextensions.

3.3.1. *Tilting object*

Note that if $2 \succeq 1$, then we have a unique non-trivial extension

$$0 \to \mathcal{O}_{R_1}(R_1) \to \mathcal{O}_{R_1+R_2}(R_1 + R_2) \to \mathcal{O}_{R_2}(R_2) \to 0.$$

Hence, $\mathcal{O}_{R_1+R_2}(R_1 + R_2)$ is the universal coextension of $\mathcal{O}_{R_1}(R_1)$ by $\mathcal{O}_{R_2}(R_2)$.

We will show that for $i_k \succeq \cdots \succeq i_1 \succeq s$ the universal coextension of $\mathcal{O}_{R_{i_1}+\cdots+R_{i_k}}(R_{i_1} + \cdots + R_{i_k})$ by $\mathcal{O}_{R_s}(R_s)$ is $\mathcal{O}_{R_s+R_{i_1}+\cdots+R_{i_k}}(R_s + R_{i_1} + \cdots + R_{i_k})$.

Proposition 3.4. *Let $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \ldots, \mathcal{N}_t\rangle$ be an exceptional collection on X such that $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t\rangle$ is a strong exceptional collection on X_0. Then*

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \cdots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_t$$

is tilting on X, where the S_k are defined as

$$S_k = \sum_{j \succeq k} R_j.$$

To prove Proposition 3.4 we shall need the following lemma.

Lemma 3.5. *For $i \succeq k \succeq l$ we have*

$$\begin{aligned}
&\operatorname{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_l+\cdots+R_k}(R_l + \cdots + R_k)) \\
&\quad \simeq \operatorname{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_{k+1}}(R_{k+1})) \otimes \operatorname{Hom}(\mathcal{O}_{R_{k+1}}(R_{k+1}), \mathcal{O}_{R_l+\cdots+R_k}(R_l + \cdots + R_k)) = \mathbb{C}, \\
&\operatorname{Ext}^1(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_l+R_{l+1}+\cdots+R_k}(R_l + R_{l+1} + \cdots + R_k)) \\
&\quad \simeq \operatorname{Hom}(\mathcal{O}_{R_i}(R_i), \mathcal{O}_{R_{k+1}}(R_{k+1})) \otimes \operatorname{Ext}^1(\mathcal{O}_{R_{k+1}}(R_{k+1}), \mathcal{O}_{R_l+\cdots+R_k}(R_l + \cdots + R_k)) = \mathbb{C},
\end{aligned}$$

where the sum $R_l + \cdots + R_k$ is taken over all divisors R_j such that $k \succeq j \succeq l$.

Proof. We proceed by induction. The initial case, for $k = l$, follows from Lemma 3.2. The induction step follows from applying the functor $\operatorname{Hom}(\mathcal{O}_{R_i}(R_i), \cdot)$ to the short exact sequence

$$0 \to \mathcal{O}_{R_l+\cdots+R_{k-1}}(R_l+\cdots+R_{k-1}) \to \mathcal{O}_{R_l+\cdots+R_k}(R_l+\cdots+R_k) \to \mathcal{O}_{R_k}(R_k) \to 0. \quad (3.3)$$

□

Proof of Proposition 3.4. From the above lemma and the short exact sequence (3.3) it follows that if $i \succeq k \succeq l$, the sheaf $\mathcal{O}_{R_l+\cdots+R_k}(R_l + \cdots + R_k)$ is the universal coextension of $\mathcal{O}_{R_l+\cdots+R_{k-1}}(R_l + \cdots + R_{k-1})$ by $\mathcal{O}_{R_i}(R_i)$. Hence, by the construction described in [**7**], the object

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \cdots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_t$$

is tilting on X. □

Endomorphisms of the tilting object depend not only on the order of the divisors but also on the dual graph of the exceptional curves.

Consider the blow-up with the following dual graph of exceptional divisors:

$$E_3 \text{———} E_2 \text{———} E_1.$$

Then

$$\begin{aligned} &E_1^2 = -2, && E_2^2 = -2, && E_3^2 = -1, \\ &E_1E_2 = 1, && E_1E_3 = 0, && E_2E_3 = 1, \\ &R_1 = E_1 + E_2 + E_3, && R_2 = E_2 + E_3, && R_3 = E_3, \end{aligned}$$

and the order is

$$3 \succeq 2 \succeq 1.$$

The endomorphisms of $\mathcal{O}_{R_3}(R_3) \oplus \mathcal{O}_{R_2+R_3}(R_2 + R_3) \oplus \mathcal{O}_{R_1+R_2+R_3}(R_1 + R_2 + R_3)$ are

$$\mathcal{O}_{R_3}(R_3) \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \mathcal{O}_{R_2+R_3}(R_2 + R_3) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \mathcal{O}_{R_1+R_2+R_3}(R_1 + R_2 + R_3)$$

with

$$\beta_3 \circ \alpha_3 = 0, \qquad \alpha_3 \circ \beta_3 = \beta_2 \circ \alpha_2.$$

However, if the dual graph is

$$E_2 \text{———} E_3 \text{———} E_1,$$

then

$$\begin{aligned} &E_1^2 = -3, && E_2^2 = -2, && E_3^2 = -1, \\ &E_1E_2 = 0, && E_1E_3 = 1, && E_2E_3 = 1, \\ &R_1 = E_1 + E_2 + 2E_3, && R_2 = E_2 + E_3, && R_3 = E_3, \end{aligned}$$

the order is still

$$3 \succeq 2 \succeq 1.$$

and the endomorphisms of the tilting object are

$$\mathcal{O}_{R_3}(R_3) \underset{\beta_3}{\overset{\alpha_3}{\rightleftarrows}} \mathcal{O}_{R_2+R_3}(R_2 + R_3) \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \mathcal{O}_{R_1+R_2+R_3}(R_1 + R_2 + R_3)$$

with

$$\beta_3 \circ \alpha_3 = 0, \qquad \beta_2 \circ \alpha_2 = 0.$$

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3.3.2. $\mathrm{Ext}^1(\mathcal{O}_{S_k}(S_k), \mathcal{N}_i)$

Lemma 3.6. *Let $i_k \succeq i_{k-1} \succeq \cdots \succeq i_1$. Then*

$$\mathrm{Ext}^1(\mathcal{O}_{R_{i_1}+\cdots+R_{i_k}}(R_{i_1} + \cdots + R_{i_k}), \mathcal{N}_i) = \mathbb{C}^k$$

and the remaining Ext groups are zero.

Proof. We proceed by induction. The short exact sequence

$$0 \to \mathcal{O}_{R_{i_1}+\cdots+R_{i_{k-1}}}(R_{i_1} + \cdots + R_{i_{k-1}}) \to \mathcal{O}_{R_{i_1}+\cdots+R_{i\,k}}(R_{i_1} + \cdots + R_{i_k}) \to \mathcal{O}_{R_{i_k}}(R_{i_k}) \to 0$$

together with an equality

$$\mathrm{Ext}^i(\mathcal{O}_{R_{i_k}}(R_{i_k}), \mathcal{N}_i) = \mathrm{Ext}^i(\mathcal{O}_{E_{i_k}}(E_{i_k}), \mathcal{N}_i)$$

completes the proof. □

If we apply the functor $\mathrm{Hom}(\mathcal{O}_{S_k}(S_k), \cdot)$ to the short exact sequence

$$0 \to \mathcal{N}_i \to \mathcal{N}_i \otimes \mathcal{O}_X(S_k) \to \mathcal{O}_{S_k}(S_k) \to 0,$$

we get an isomorphism

$$\mathrm{Ext}^1(\mathcal{O}_{S_k}(S_k), \mathcal{N}_i) \simeq \mathrm{Hom}(\mathcal{O}_{S_k}(S_k), \mathcal{O}_{S_k}(S_k)). \tag{3.4}$$

The identity morphism in the latter space corresponds to an element

$$\zeta_k^i \in \mathrm{Ext}^1(\mathcal{O}_{S_k}(S_k), \mathcal{N}_i).$$

The diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}_i \otimes \mathcal{O}_X(S_k) & \longrightarrow & \mathcal{O}_{S_k}(S_k) & \longrightarrow & 0 \\
 & & \downarrow{=} & & \downarrow & & \downarrow{\iota} & & \\
0 & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}_i \otimes \mathcal{O}_X(S_l) & \longrightarrow & \mathcal{O}_{S_l}(S_l) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_{S_l - S_k}(S_l) & \xrightarrow{=} & \mathcal{O}_{S_l - S_k}(S_l) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

shows that for an inclusion $\iota\colon \mathcal{O}_{S_k}(S_k) \to \mathcal{O}_{S_l}(S_l)$ we have $\zeta_l^i \circ \iota = \zeta_k^i$.

The isomorphism (3.4) allows us, in addition, to calculate the Yoneda composition

$$\mathrm{Hom}(\mathcal{N}_i, \mathcal{N}_k) \otimes \mathrm{Ext}^1(\mathcal{O}_{S_k}(S_k), \mathcal{N}_i) \to \mathrm{Ext}^1(\mathcal{O}_{S_k}(S_k), \mathcal{N}_k).$$

Thus, if the collection $\langle \mathcal{O}_{X_0}, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ on X_0 is strong, we know the endomorphism algebra of the tilting object

$$\mathcal{O}_{S_n}(S_n)[-1] \oplus \mathcal{O}_{S_{n-1}}(S_{n-1})[-1] \oplus \cdots \oplus \mathcal{O}_{S_1}(S_1)[-1] \oplus \mathcal{O}_X \oplus \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_t.$$

Using one-sided twisted complexes one can then calculate the DG quiver of the collection $\langle \mathcal{O}_{R_n}(R_n)[-1], \ldots, \mathcal{O}_{R_1}(R_1)[-1], \mathcal{O}_X, \mathcal{N}_1, \ldots, \mathcal{N}_t \rangle$ and of any of its mutations (see [**1**]).

4. Canonical DG algebras of toric surfaces

4.1. Toric surfaces

We recall some information about toric surfaces. More details can be found, for example, in [**5**].

A smooth projective toric surface Y is determined by its fan, spanned by a collection of elements $\rho_1, \ldots, \rho_n$ in a lattice $N = \mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^2$, where $T = (\mathbb{C}^*)^2$ is a two-dimensional torus. We enumerate the ρ_is clockwise and consider their indices, is, to be elements of $\mathbb{Z}/n\mathbb{Z}$. Then, for every $i \in \mathbb{Z}/n\mathbb{Z}$, the vectors ρ_i and ρ_{i+1} form an oriented basis of N. Moreover, for every such pair there is no other ρ_k lying in the rational polyhedral cone generated by ρ_i and ρ_{i+1} in $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$.

There is a one-to-one correspondence between one-dimensional orbits of the T-action on Y and the rays in the fan generated by the ρ_is. For every i we denote by D_i the closure of this orbit. Then the D_is are T-invariant divisors on X. Every D_i is isomorphic to $\mathbb{P}^1$ and the intersection form is given by

$$D_i D_j = \begin{cases} a_i & \text{if } i = j, \\ 1 & \text{if } j \in \{i-1, i+1\}, \\ 0 & \text{otherwise}, \end{cases}$$

where $a_i \in \mathbb{Z}$ are such that $\rho_{i-1} + a_i \rho_i + \rho_{i+1} = 0$. Conversely, the numbers $(a_1, \ldots, a_n)$ determine the toric surface Y.

Divisors D_i and D_{i+1} intersect transversely in a T-fixed point p_i corresponding to the cone spanned by the vectors ρ_i and ρ_{i+1}.

A surface Y_1 obtained from Y by a blow-up of a torus-fixed point p_i is again a toric surface. The fan of Y_1 is determined by vectors $\rho_1, \ldots, \rho_i, \rho_i + \rho_{i+1}, \rho_{i+1}, \ldots, \rho_n$. Moreover, every toric surface different from $\mathbb{P}^2$ can be obtained from some Hirzebruch surface $\mathbb{F}_a$ by a finite sequence of blow-ups of T-fixed points.

A canonical divisor of a toric surface is given by $K_Y = -\sum_{i=1}^n D_i$. The Picard group of Y is $\mathrm{Pic}(Y) = \mathbb{Z}^{n-2}$.

4.2. Exceptional collections on toric surfaces

The ath Hirzebruch surface $\mathbb{F}_a$ has a fan with four vectors and we can assume that

$$w_1 = (1, 0), \quad w_2 = (0, -1), \quad w_3 = (-1, a) \quad \text{and} \quad w_4 = (0, 1).$$

The collection $\langle \mathcal{O}_{\mathbb{F}_a}, \mathcal{O}_{\mathbb{F}_a}(D_1), \mathcal{O}_{\mathbb{F}_a}(D_1 + D_2), \mathcal{O}_{\mathbb{F}_a}(D_1 + D_2 + D_3)\rangle$ is a full strong exceptional collection on $\mathbb{F}_a$.

If Y is obtained from $\mathbb{F}_a$ by a sequence of T-equivariant blow-ups, then we can assume that the vectors $\rho_1, \dots, \rho_n$ determining Y are numbered in such a way that $\rho_n = w_4 = (0, 1)$. Then the collection $\langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1+D_2), \dots, \mathcal{O}_Y(D_1+\cdots+D_{n-1})\rangle$ on Y is obtained by augmentation from the strong collection on $\mathbb{F}_a$; hence, it is full. The following lemma tells us that in fact the numeration of T-invariant divisors is not important.

Lemma 4.1 (cf. Bondal [2, Theorem 4.1]). *Let $\langle \mathcal{E}_1, \dots, \mathcal{E}_n\rangle$ be a full exceptional collection on a smooth projective variety Z of dimension m. Then the n-fold mutation of $\mathcal{E}_n$ to the left, $L^n\mathcal{E}_n$, is isomorphic to $\mathcal{E}_n \otimes \omega_Z[m - n]$, where ω_Z is the canonical line bundle on Z.*

Let $\sigma_1 = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1 + D_2), \dots, \mathcal{O}_Y(D_1 + \cdots + D_{n-1})\rangle$ be a full exceptional collection on Y. Then, by the above lemma,

$$L^n\mathcal{O}_Y(D_1 + \cdots + D_{n-1}) = \mathcal{O}_Y(-D_n)[2 - n].$$

Hence, σ_1 can be mutated to a collection

$$\langle \mathcal{O}_Y(-D_n)[2 - n], \mathcal{O}_Y, \mathcal{O}_Y(D_1), \mathcal{O}_Y(D_1 + D_2), \dots, \mathcal{O}_Y(D_1 + \cdots + D_{n-2})\rangle,$$

which, in turn, after a shift and a twist by $\mathcal{O}_Y(D_n)$, is equivalent to the collection

$$\sigma_n = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_n), \mathcal{O}_Y(D_n + D_1), \dots, \mathcal{O}_Y(D_n + D_1 + \cdots + D_{n-2})\rangle.$$

One can repeat this operation and obtain full exceptional collections

$$\sigma_i = \langle \mathcal{O}_Y, \mathcal{O}_Y(D_i), \dots, \mathcal{O}_Y(D_i + \cdots + D_{i+n-2})\rangle$$

for any $i \in \mathbb{Z}/n$.

4.3. Canonical DG algebra of a toric surface

Let $Z = \operatorname{Tot}\omega_Y$ be the total space of the canonical bundle on Y and let $p\colon Z \to Y$ denote the canonical projection. As the vector bundle

$$\mathcal{E} = \mathcal{O}_Y \oplus \mathcal{O}_Y(D_1) \oplus \cdots \oplus \mathcal{O}_Y(D_1 + \cdots + D_{n-1})$$

is a generator of $D^{\mathrm{b}}(Y)$, the sheaf $p^*(\mathcal{E})$ is a generator of $D^{\mathrm{b}}(Z)$ (see [**4**, Proposition 4.1]). Moreover,

$$\begin{aligned}
\operatorname{Hom}_Z(p^*(\mathcal{E}), p^*(\mathcal{E})) &= \operatorname{Hom}_Y(\mathcal{E}, p_*p^*(\mathcal{E})) \\
&= \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E} \otimes p_*(\mathcal{O}_Z)) \\
&= \bigoplus_{n\geqslant 0} \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E} \otimes \mathcal{O}_Y(-nK_Y)).
\end{aligned}$$

On Y we can consider an infinite sequence $(A_k)_{k=0}^{\infty}$ of line bundles

$$A_{sn+r} = \mathcal{O}(sK_Y + D_1 + \cdots + D_r) \quad \text{for } 0 \leqslant r < n.$$

Denote by $\mathcal{A}_Y = \bigoplus A_k$ the sum of all elements in this sequence. It was proved in [**9**] that the DG enhancement of $\mathrm{Hom}^{\bullet}(\mathcal{A}_Y, \mathcal{A}_Y)$ can be calculated via the Čech enhancement. It follows that the DG enhancement of $\mathrm{Hom}_Z(p^*(\mathcal{E}), p^*(\mathcal{E}))$ is the same as the DG enhancement of $\mathrm{Hom}_Y(\mathcal{A}_Y, \mathcal{A}_Y)$.

The sequence $\langle \mathcal{O}_Y, \mathcal{O}_Y(D_1), \ldots, \mathcal{O}_Y(D_1 + \cdots + D_{n-1})\rangle$ is an augmentation of a strong exceptional collection on a Hirzebruch surface and therefore the methods described in § 3 allow one to calculate the DG algebra of endomorphisms of $\bigoplus_{k=0}^{n-1} A_k$. Lemma 4.1 guarantees that up to shifts the remaining elements of the sequence (A_k) are obtained by mutations from $A_0, \ldots, A_{n-1}$. Therefore, twisted complexes allow one to calculate the DG endomorphism algebra of $\mathrm{Hom}(\mathcal{A}_Y, \mathcal{A}_Y)$: *the canonical DG algebra of* Y.

The composition provides a natural map

$$\mathrm{Hom}(A_{i_{k-1}}, A_{i_k}) \otimes \cdots \otimes \mathrm{Hom}(A_{i_1}, A_{i_2}) \xrightarrow{\Psi_{i_1,\ldots,i_k}} \mathrm{Hom}(A_{i_1}, A_{i_k})$$

and an analogous one for elements of $\mathrm{Ext}^1(A_{i_1}, A_{i_k})$. If there exists $K \in \mathbb{N}$ such that for any i, j any element of $\mathrm{Hom}(A_i, A_j)$ or $\mathrm{Ext}^1(A_i, A_j)$ is in the image of some $\Psi_{i_1,\ldots,i_k}$ such that $i_{s+1} - i_s < K$ for all $s \in \{1, \ldots, k-1\}$, then the canonical DG algebra of Y can be presented as a path algebra of a cyclic DG quiver with K vertices.

If one can choose K to be the number n of T-invariant divisors of Y, then the DG quivers Q_i of exceptional collections σ_i can be read from the canonical DG quiver Q of Y

$$(Q_i)_0 = (Q_Y)_0,$$
$$(Q_i)_1 = (Q_Y)_1 \setminus \{a \in (Q_Y)_1 \mid t(a) > i - 1 > h(a)\},$$

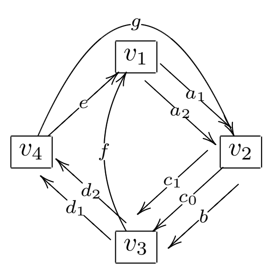
and the canonical DG quiver Q is obtained by gluing of the DG quivers Q_i.

Remark 4.2. The canonical DG algebra of $\mathbb{F}_3$ cannot be presented as a path algebra of such a quiver, i.e. in this case $K > 4$. If, as before, we consider the fan of $\mathbb{F}_3$ with $w_1 = (1, 0)$, $w_2 = (0, -1)$, $w_3 = (-1, 3)$ and $w_4 = (0, 1)$, then the map $\phi\colon \mathcal{O}_{\mathbb{F}_3}(D_1 + D_2) \to \mathcal{O}_{\mathbb{F}_3}(2D_1 + 2D_2 + 2D_3 + D_4)$ with zeroes along $2D_2$ is not a composition of any maps between line bundles.

4.4. Examples

We conclude with some examples of canonical DG quivers of toric surfaces.

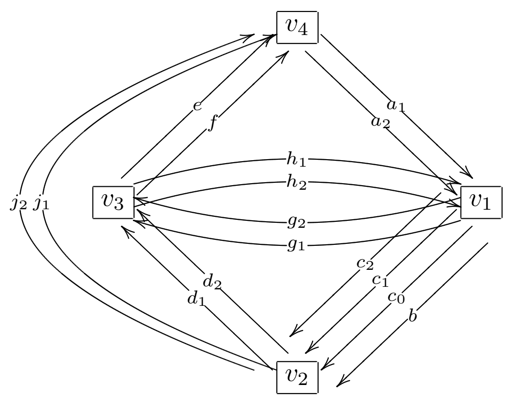
The canonical DG algebra of $\mathbb{F}_1$ is a path algebra of the quiver



with relations

$$\begin{aligned} c_0a_1 &= c_1a_2, & d_1c_0 &= d_2c_1, & d_1ba_2 &= d_2ba_1,\\ a_1ed_2 &= a_2ed_1, & a_1f &= gd_1, & a_2f &= gd_2,\\ ba_1e &= c_1g, & ba_2e &= c_0g, & fc_0 &= ed_2b,\\ fc_1 &= ed_1b. \end{aligned}$$

The canonical DG algebra of $\mathbb{F}_2$, with intersection numbers $(0,2,0,-2)$, is a path algebra of the DG quiver



with

$$\begin{aligned} &\deg(a_1)=0, && \deg(a_2)=0, && \deg(b)=0, && \deg(c_0)=0,\\ &\deg(c_1)=0, && \deg(c_2)=0, && \deg(d_1)=0, && \deg(d_2)=0,\\ &\deg(e)=0, && \deg(f)=1, && \deg(g_1)=-1, && \deg(g_2)=-1,\\ &\deg(h_1)=0, && \deg(h_2)=0, && \deg(j_1)=0, && \deg(j_2)=0, \end{aligned}$$

$$\begin{aligned} &\partial(g_1)=d_2c_1-d_1c_0, && \partial(g_2)=d_2c_2-d_1c_1, && \partial(h_1)=a_1f,\\ &\partial(h_2)=a_2f, && \partial(j_1)=fd_1, && \partial(j_2)=fd_2 \end{aligned}$$

and relations

$$\begin{aligned} &c_0a_1=c_1a_2, && c_1a_1=c_2a_2, && d_1ba_2=d_2ba_1, && c_1h_2=c_0h_1+ba_2e,\\ &c_2h_2=c_1h_1+ba_1e, && a_1j_2=a_2j_1, && h_1d_2=h_2d_1, && a_1ed_2=a_2ed_1,\\ &a_1fd_2=a_2fd_1, && fd_1c_0=fd_2c_1, && fd_1c_1=fd_2c_2, && fg_1=ed_2b,\\ &fg_2=ed_1b, && j_1c_0=j_2c_1, && j_1c_1=j_2c_1, && a_1j_1=0,\\ &a_2j_2=0, && h_1d_1=0, && h_2d_2=0. \end{aligned}$$

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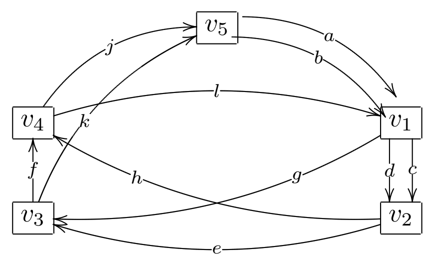
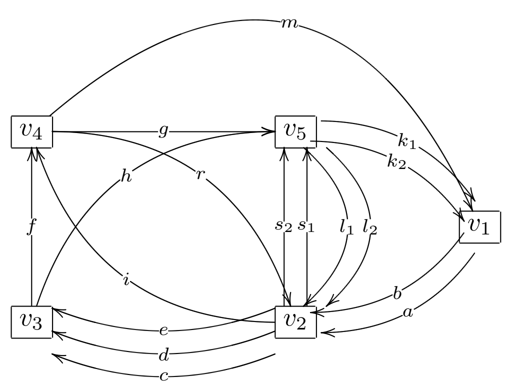
If we blow up $\mathbb{F}_1$ in such a way that the obtained toric surface Y_1 has intersection numbers $(-1,-1,0,0,-1)$, then the canonical algebra of Y_1 is a path algebra of the quiver



with relations

$$gb = eda, \quad hd = fg, \quad hcb = feca, \quad kg = jhc,$$
$$ked = jfec, \quad bk = lf, \quad bjh = ejfe, \quad lh = ake,$$
$$lfe = bke, \quad dl = cbj, \quad bk = lf, \quad dak = cajf,$$
$$gl = ecaj.$$

If we blow up $\mathbb{F}_1$ at another point, to obtain Y_2 with intersection numbers $(0,1,-1,-1,-2)$, then the canonical DG algebra is a path algebra of the DG quiver



with

$$\deg(a) = 0, \quad \deg(b) = 0, \quad \deg(c) = 0, \quad \deg(d) = 0,$$
$$\deg(e) = 0, \quad \deg(f) = 0, \quad \deg(g) = 0, \quad \deg(h) = 0,$$
$$\deg(i) = 0, \quad \deg(k_1) = 1, \quad \deg(k_2) = 0, \quad \deg(l_1) = 0,$$
$$\deg(l_2) = 0, \quad \deg(m) = 0, \quad \deg(r) = 0, \quad \deg(s_1) = -1,$$
$$\deg(s_2) = -1,$$

$$\partial(l_1) = bk_1, \quad \partial(l_2) = bk_2, \quad \partial(m) = k_1 g,$$
$$\partial(r) = k_1 h, \quad \partial(s_1) = he - gi, \quad \partial(s_2) = hd - gfe$$

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and relations

$$eb = da, \quad ib = fea, \quad gfca = hcb, \quad el_1 = cbk_2 + dl_2,$$
$$il_1 = fel_2 + fcak_2, \quad l_1g = bm, \quad l_2g = am, \quad ar = l_2h,$$
$$amf = br, \quad bk_2h = ak_2gf, \quad bk_1h = ak_1gf, \quad k_1s_1 = k_2hc,$$
$$k_1s_2 = k_2gfc.$$

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