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Stochastic representation of fluid equations

RYUTARO KANNO

National Institute for Fusion Science, Toki 509-5292, Japan
(kanno@nifs.ac.jp)

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Abstract. The stochastic representation of fluid equations describing a plasma transport in the peripheral region is discussed, which is based on the theorem of Yasue (1983 *J. Funct. Anal.* **51**, 133). The scheme solving the stochastic equations is given by employing a path integral method.

1. Introduction

In recent years, great effort has been put in to analyzing plasma transport in the peripheral region in order to understand the divertor physics [1]. In the studies of the transport, a simplified fluid model is frequently employed when the collisionality of plasma particles is strong. The model is based on simplified Braginskii's fluid equations [1,2] neglecting the electric field and plasma current, i.e. a simple neutral plasma is assumed. The fluid equation for a fluid property f is given generally as

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla-\nu^{(f)}\nabla^2\right\}f=S^{(f)}, \tag{1.1}$$

where $\nu^{(f)}=$ constant is a diffusion coefficient of f, $S^{(f)}$ is a source/sink of f, and an incompressible fluid is assumed, i.e. $\nabla\cdot\boldsymbol{v}=0$. If the fluid property f is a density $\rho(\boldsymbol{x},t)$, then the fluid equation (1.1) represents the Fokker–Planck (FP) equation. As is well known, the FP equation corresponds to the Langevin equation [3]:

$$d\boldsymbol{X}_t(\boldsymbol{x}_a)=\boldsymbol{v}(\boldsymbol{X}_t(\boldsymbol{x}_a),t)\,dt+\sqrt{2\nu^{(f)}}\,d\boldsymbol{W}_t, \tag{1.2}$$

where $d\boldsymbol{W}_t/dt$ is the Gaussian white noise and start and end points of a stochastic process $\boldsymbol{X}_t$ are given as $\boldsymbol{X}_{t_a}(\boldsymbol{x}_a)=\boldsymbol{x}_a$ at $t=t_a$ and $\boldsymbol{X}_{t_b}(\boldsymbol{x}_a)=\boldsymbol{x}_b$ at $t=t_b$, respectively. For the case of $f=\rho(\boldsymbol{x},t)$, the source/sink term is treated as creation/annihilation of random particles described by (1.2). If a fluid property f is a velocity field $\boldsymbol{v}(\boldsymbol{x},t)$, then the fluid equation represents the Navier–Stokes (NS) equation:

$$\left\{\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla-\nu\nabla^2\right\}\boldsymbol{v}(\boldsymbol{x},t)=\boldsymbol{S}^{(\boldsymbol{v})}, \tag{1.3}$$

where the function $\boldsymbol{S}^{(\boldsymbol{v})}$ is given as $\boldsymbol{S}^{(\boldsymbol{v})}=\boldsymbol{S}_{\rm p}+\boldsymbol{S}_{\rm m}$. Here $\boldsymbol{S}_{\rm p}=-(1/\rho)\nabla p(\boldsymbol{x},t)$, p is a pressure and $\boldsymbol{S}_{\rm m}$ is a source/sink of momentum (or an external force). From an analogy with the FP equation, one may intuit that the Langevin equation can also be used to solve the NS equation, instead of directly solving it. Actually, in the studies of [4–6] a Monte Carlo method was proposed to solve these fluid equations. The method in previous studies [4–6] is summarized quite briefly as follows. The

Monte Carlo particles described as the Langevin equation (1.2), which corresponds to each fluid equation, are traced and scored in each cell to obtain the particle weight. The evolution of a fluid property is determined by using the weight. In order to obtain solutions of the fluid equations in a steady state, the computation continues until the desired accuracy of statistics. However, unfortunately, it is not clear whether the method in the previous studies is justified mathematically. The present article is devoted to clarifying the mathematical background of the stochastic treatment and proposes a scheme of treatment with a Monte Carlo technique by employing a path integral method [7], which differs from previous studies.

2. Stochastic representation of fluid equations

In this section, we focus on one of the fluid equations, i.e. the NS equation in d-dimensional space, by assuming the incompressibility $\nabla \cdot \boldsymbol{v} = 0$,

$$\left\{ \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla - \nu \nabla^2 \right\} \boldsymbol{v}(\boldsymbol{x}, t) = \boldsymbol{S}_{\mathrm{p}} + \boldsymbol{S}_{\mathrm{m}} = -\frac{1}{\rho} \nabla p(\boldsymbol{x}, t) - \nabla \varphi(\boldsymbol{x}, t),$$

where a region $\mathscr{M}$ in which the fluid exists is fixed, the source/sink of momentum is assumed to be given as $\boldsymbol{S}_{\mathrm{m}} = -\nabla \varphi(\boldsymbol{x}, t)$, φ is a scalar function and $d \leq 3$. From the theorem of Yasue [8], a variation of the following functional $J[\boldsymbol{X}]$ gives the NS equation (1.3);

$$J[\boldsymbol{X}] = \int_{t_a}^{t_b} dt \, \mathrm{E}\left[\tfrac{1}{2} |D\boldsymbol{X}_t|^2 - \varphi(\boldsymbol{X}_t, t) \right], \tag{2.1}$$

where $\mathrm{E}[\cdot]$ denotes a mathematical expectation, and the volume-preserving diffusion process $\boldsymbol{X}_t$ with fixed start and end points $\boldsymbol{X}_{t_a} = \boldsymbol{x}_a$ and $\boldsymbol{X}_{t_b} = \boldsymbol{x}_b$ is mean forward differentiable:

$$D\boldsymbol{X}_t \equiv \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \mathrm{E}[\boldsymbol{X}_{t+\varepsilon} - \boldsymbol{X}_t | \boldsymbol{X}_t] = \boldsymbol{v}(\boldsymbol{X}_t, t). \tag{2.2}$$

Here $\mathrm{E}[\cdot|\mathscr{B}]$ denotes a conditional expectation with respect to a σ-algebra $\mathscr{B}$, the stochastic process $\boldsymbol{X}_t$ satisfies the Langevin equation (1.2), i.e.

$$d\boldsymbol{X}_t = \boldsymbol{v}(\boldsymbol{X}_t, t) \, dt + \sqrt{2\nu} \, d\boldsymbol{W}_t.$$

A variation of $J[\boldsymbol{X}]$ leads to the NS equation (1.3) as follows [8]:

$$\begin{aligned} 0 = \delta J[\boldsymbol{X}] &= -\int_{t_a}^{t_b} dt \, \mathrm{E}[\delta \boldsymbol{X}_t \cdot \{ D_* \boldsymbol{v}(\boldsymbol{X}_t, t) + \nabla \varphi(\boldsymbol{X}_t, t) \}] \\ &= -\int_{t_a}^{t_b} dt \int_{\mathscr{M}} dV(\boldsymbol{x}) \, \mu(\boldsymbol{x}, t) [\boldsymbol{h}(\boldsymbol{x}, t) \cdot \{ A_* \boldsymbol{v}(\boldsymbol{x}, t) - \boldsymbol{S}_{\mathrm{m}} \}], \end{aligned} \tag{2.3}$$

where $dV(\boldsymbol{x})$ is the volume element, $\mu(\boldsymbol{x}, t)$ is a probability density of the process $\boldsymbol{X}_t$, $\boldsymbol{h}(\boldsymbol{x}, t) = \delta \boldsymbol{X}_t$ is an arbitrary function satisfying $\boldsymbol{h}(\,\cdot\,, t_a) = \boldsymbol{h}(\,\cdot\,, t_b) = 0$ and A_* is a differential operator defined as

$$A_* \equiv \frac{\partial}{\partial t} + \boldsymbol{v}_*(\boldsymbol{x}, t) \cdot \nabla - \nu \nabla^2. \tag{2.4}$$

Here $\boldsymbol{v}_*(\boldsymbol{x}, t)$ is a backward local velocity field, $\boldsymbol{v}_*(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{x}, t) - 2\nu \nabla \ln \mu(\boldsymbol{x}, t)$,

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and a mean backward derivative is defined as

$$D_*\boldsymbol{X}_t \equiv \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}[\boldsymbol{X}_t - \boldsymbol{X}_{t-\varepsilon}|\boldsymbol{X}_t] = \boldsymbol{v}_*(\boldsymbol{X}_t, t), \tag{2.5}$$

and the following relation is used; for arbitrary differentiable functions g and s,

$$\int_{t_a}^{t_b} dt\,\mathrm{E}[g(\boldsymbol{X}_t,t)Ds(\boldsymbol{X}_t,t)] = \mathrm{E}\big[g(\boldsymbol{X}_t,t)s(\boldsymbol{X}_t,t)|_{t_a}^{t_b}\big] - \int_{t_a}^{t_b} dt\,\mathrm{E}[s(\boldsymbol{X}_t,t)D_*g(\boldsymbol{X}_t,t)]. \tag{2.6}$$

It is helpful to give the following equations in order to understand the relation between the mean forward or the mean backward derivative and differential operators, e.g. A_*,

$$Dg(\boldsymbol{X}_t,t) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla + \nu\nabla^2\right) g(\boldsymbol{X}_t,t), \tag{2.7}$$

$$D_*g(\boldsymbol{X}_t,t) = \left(\frac{\partial}{\partial t} + \boldsymbol{v}_*\cdot\nabla - \nu_*\nabla^2\right) g(\boldsymbol{X}_t,t), \tag{2.8}$$

where ν_* is a diffusion coefficient of a Wiener process with time reversed, $\boldsymbol{W}_{*t}$: $d\boldsymbol{X}_t = \boldsymbol{v}_*(\boldsymbol{X}_t,t)\,dt + \sqrt{2\nu_*}\,d\boldsymbol{W}_{*t}$, and it is easily seen that by using (2.6) and the FP equation, $\boldsymbol{v}_* = \boldsymbol{v} - 2\nu\nabla\ln\mu$ and $\nu_* = \nu$. Note that the incompressibility of the fluid implies $\mu(\boldsymbol{x},t) = \text{constant} = 1/V_{\mathscr{M}}$, thus $\boldsymbol{v}_*(\boldsymbol{x},t) = \boldsymbol{v}(\boldsymbol{x},t)$, where $V_{\mathscr{M}}$ is the volume of a region $\mathscr{M}$. Therefore, the variational principle (2.3) gives

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}(\boldsymbol{x},t)\cdot\nabla - \nu\nabla^2\right\}\boldsymbol{v}(\boldsymbol{x},t) = \boldsymbol{S}_{\mathrm{p}} + \boldsymbol{S}_{\mathrm{m}}.$$

Using the Hodge decomposition theorem [9], we have $\boldsymbol{S}_{\mathrm{p}} = -(1/\mu)\nabla p(\boldsymbol{x},t)$. Note that the probability density $\mu(\boldsymbol{x},t)$ must be equivalent to the density $\rho(\boldsymbol{x},t)$ in the framework presented here. The stochastic process having the probability density $\mu(\boldsymbol{x},t)$ is expressed as the equation of continuity $\nabla\cdot\boldsymbol{v} = 0$, i.e. only $\mu(\boldsymbol{x},t) = \rho(\boldsymbol{x},t) = \text{constant}$ is permitted. If not, the variational principle (2.3) is not equivalent to the NS equation, because of the relation $\boldsymbol{v}_* = \boldsymbol{v} - 2\nu\nabla\ln\mu$. Due to the same reason, a compressible fluid is unacceptable in the framework and the source/sink of momentum $\boldsymbol{S}_{\mathrm{m}}$ cannot be treated as creation/annihilation of random particles described by the Langevin equation (1.2).

In general, it is not easy to generate the stochastic process $\boldsymbol{X}_t$ with the fixed start and end points, $\boldsymbol{X}_{t_a} = \boldsymbol{x}_a$ and $\boldsymbol{X}_{t_b} = \boldsymbol{x}_b$, by directly solving the Langevin equation (1.2). For the simplicity of the boundary condition, we consider the simplified fluid with the periodic boundary in the d-dimensional Cartesian coordinates $\boldsymbol{x} = (x, y, \ldots)$. The region $\mathscr{M}$ is assumed to be simply-connected. For instance, the ABC flow is known as one of steady solutions in this situation [10]. The mathematical expectation of an arbitrary function $g(\boldsymbol{X}_t,t)$ can be given by using a path integral method [7, 11, 12] as

$$\mathrm{E}[g(\boldsymbol{X}_t,t)] = \frac{1}{V_{\mathscr{M}}}\int_{\mathscr{M}} dV(\boldsymbol{x}_a)\,\mathrm{E}\big[g(\boldsymbol{X}_t,t)|\boldsymbol{X}_{t_a} = \boldsymbol{x}_a; \boldsymbol{X}_{t_b} = \boldsymbol{x}_b\big], \tag{2.9}$$

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and

$$\mathrm{E}\big[g(\boldsymbol{X}_t,t)|\boldsymbol{X}_{t_a}=\boldsymbol{x}_a;\boldsymbol{X}_{t_b}=\boldsymbol{x}_b\big]$$
$$=\lim_{N\to\infty}\frac{1}{C_N}\int\int\cdots\int dx_{N-1}\,dx_{N-2}\ldots dx_1\,dy_{N-1}\,dy_{N-2}\ldots dy_1\ldots g(\boldsymbol{x}_k,t_k)$$
$$\times\exp\left\{-\frac{\varepsilon}{4\nu}\sum_{m=0}^{N-1}\left[\frac{\boldsymbol{x}_{m+1}-\boldsymbol{x}_m}{\varepsilon}-\boldsymbol{v}(\boldsymbol{x}_m,t_m)\right]^2\right\},\tag{2.10}$$

where $V_{\mathscr{M}}=$ constant is volume of the periodic region $\mathscr{M}$, $t_0=t_a$, $t_N=t_b$, $t=t_k$, $t_{k+1}=t_k+\varepsilon$, $\varepsilon=(t_b-t_a)/N$, $\boldsymbol{x}_0=\boldsymbol{x}_a\in\mathscr{M}$, $\boldsymbol{x}_N=\boldsymbol{x}_b\in\mathscr{M}$, $\boldsymbol{x}_m=\boldsymbol{x}(t_m)$, $\boldsymbol{x}_k=\boldsymbol{x}(t_k)=\boldsymbol{X}_t$, and

$$C_N=\lim_{N\to\infty}\int\cdots\int dx_{N-1}\ldots dx_1\,dy_{N-1}\ldots dy_1\ldots$$
$$\times\exp\left\{-\frac{\varepsilon}{4\nu}\sum_{m=0}^{N-1}\left[\frac{\boldsymbol{x}_{m+1}-\boldsymbol{x}_m}{\varepsilon}-\boldsymbol{v}(\boldsymbol{x}_m,t_m)\right]^2\right\}.$$

Here a set of start points $\{\boldsymbol{x}_a\}$ is in one-to-one onto correspondence with a set of end points $\{\boldsymbol{x}_b\}$. The Lévy's interpolation formula [13] is useful when expressing a Wiener path $\boldsymbol{x}^{(\ell)}(t)$:

$$\boldsymbol{x}^{(\ell)}(t)=\frac{(t_b-t)\boldsymbol{x}_a+(t-t_a)\boldsymbol{x}_b}{t_b-t_a}+\sqrt{\frac{2\nu(t_b-t)(t-t_a)}{t_b-t_a}}\boldsymbol{\xi}^{(\ell)}(t),\tag{2.11}$$

where $\boldsymbol{\xi}^{(\ell)}=(\xi_x^{(\ell)},\xi_y^{(\ell)},\ldots)$, $\xi_i^{(\ell)}$ is a Gaussianly distributed random variable with mean 0 and variance 1 for $i=x,y,\ldots$. Using the Wiener path (2.11), a mean forward derivative is represented as

$$D\boldsymbol{X}_t=\lim_{\substack{N'\to\infty\\N_{\mathrm{p}}\to\infty}}\frac{1}{C_{N',N_{\mathrm{p}}}}\sum_{\ell=1}^{Np}\left(\frac{\boldsymbol{x}_{k+1}^{(\ell)}-\boldsymbol{x}_k^{(\ell)}}{\varepsilon'}\right)$$
$$\times\exp\left\{\frac{\varepsilon'}{2\nu}\sum_{m=k}^{N'-1}\left[\frac{\boldsymbol{x}_{m+1}^{(\ell)}-\boldsymbol{x}_m^{(\ell)}}{\varepsilon'}\cdot\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)},t_m\big)-\frac{1}{2}\big|\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)},t_m\big)\big|^2\right]\right\}\Bigg|_{\boldsymbol{x}_k^{(\ell)}=\boldsymbol{X}_t}$$
$$=\boldsymbol{v}(\boldsymbol{X}_t,t),\tag{2.12}$$

and (2.10) is expressed for $g(\boldsymbol{X}_t,t)=D\boldsymbol{X}_t$ as

$$\mathrm{E}[D\boldsymbol{X}_t|\boldsymbol{X}_{t_a}=\boldsymbol{x}_a;\boldsymbol{X}_{t_b}=\boldsymbol{x}_b]=\mathrm{E}[\boldsymbol{v}(\boldsymbol{X}_t,t)|\boldsymbol{X}_{t_a}=\boldsymbol{x}_a;\boldsymbol{X}_{t_b}=\boldsymbol{x}_b]$$
$$=\lim_{\substack{N\to\infty\\N_{\mathrm{p}}\to\infty}}\frac{1}{C_{N,N_{\mathrm{p}}}}\sum_{\ell=1}^{Np}\boldsymbol{v}\big(\boldsymbol{x}_k^{(\ell)},t_k\big)$$
$$\times\exp\left\{\frac{\varepsilon}{2\nu}\sum_{m=0}^{N-1}\left[\frac{\boldsymbol{x}_{m+1}^{(\ell)}-\boldsymbol{x}_m^{(\ell)}}{\varepsilon}\cdot\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)},t_m\big)-\frac{1}{2}\big|\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)},t_m\big)\big|^2\right]\right\},\tag{2.13}$$

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where $t_0 = t_a$, $t_N = t_{N'} = t_b$, $t = t_k$, $t_{k+1} = t_k + \varepsilon = t_k + \varepsilon'$, $\varepsilon = (t_b - t_a)/N$, $\varepsilon' = (t_b - t_k)/N'$, $\boldsymbol{x}_0^{(\ell)} = \boldsymbol{x}_a$, $\boldsymbol{x}_N^{(\ell)} = \boldsymbol{x}_{N'}^{(\ell)} = \boldsymbol{x}_b$, $\boldsymbol{x}_m^{(\ell)} = \boldsymbol{x}^{(\ell)}(t_m)$, $\boldsymbol{x}_k^{(\ell)} = \boldsymbol{x}^{(\ell)}(t_k) = \boldsymbol{X}_t$,

$$C_{N',N_{\mathrm{p}}} = \sum_{\ell=1}^{Np} \exp\left\{ \frac{\varepsilon'}{2\nu} \sum_{m=k}^{N'-1} \left[\frac{\boldsymbol{x}_{m+1}^{(\ell)} - \boldsymbol{x}_m^{(\ell)}}{\varepsilon'} \cdot \boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big) - \frac{1}{2}\big|\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big)\big|^2 \right] \right\},$$

and

$$C_{N,N_{\mathrm{p}}} = \sum_{\ell=1}^{Np} \exp\left\{ \frac{\varepsilon}{2\nu} \sum_{m=0}^{N-1} \left[\frac{\boldsymbol{x}_{m+1}^{(\ell)} - \boldsymbol{x}_m^{(\ell)}}{\varepsilon} \cdot \boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big) - \frac{1}{2}\big|\boldsymbol{v}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big)\big|^2 \right] \right\}.$$

Therefore, an approximate solution of the NS equation, $\boldsymbol{v}$, is obtained as $\boldsymbol{v}^{(n)} \to \boldsymbol{v}$ with $n \to \infty$, where the nth velocity field $\boldsymbol{v}^{(n)}$ is given, for example, by the Ritz method [14] with $\boldsymbol{v}^{(n)} = \sum_i c_i^{(n)} \boldsymbol{u}_i$, $c_i^{(n)}$ is a coefficient, $\boldsymbol{u}_i$ is a given function satisfying $\nabla \cdot \boldsymbol{u}_i = 0$ and the periodic boundary condition. $D\boldsymbol{Y}_t^{(i)} = \boldsymbol{u}_i$, $d\boldsymbol{Y}_t^{(i)} = D\boldsymbol{Y}_t^{(i)}\, dt + (\sqrt{2\nu}/c_i)\, d\boldsymbol{W}_t$ and $d\boldsymbol{X}_t = \sum_i c_i d\boldsymbol{Y}_t^{(i)}$. Then,

$$0 = \frac{\partial J^{(n)}}{\partial c_i^{(n)}} \quad \text{with} \quad J^{(n)}\big[\{c_i^{(n)}\}\big] = \int_{t_a}^{t_b} dt\, \mathrm{E}^{(n-1)} \left[\frac{1}{2}\big|\boldsymbol{v}^{(n)}(\boldsymbol{X}_t, t)\big|^2 - \varphi^{(n)}(\boldsymbol{X}_t, t) \right], \tag{2.14}$$

and the expectation $\mathrm{E}^{(n-1)}[\cdot]$ is calculated by using the $(n-1)$th velocity field $\boldsymbol{v}^{(n-1)}$ as

$$\begin{aligned} &\mathrm{E}^{(n-1)} \left[\frac{1}{2}\big\{\boldsymbol{v}^{(n)}(\boldsymbol{X}_t, t)\big\}^2 - \varphi^{(n)}(\boldsymbol{X}_t, t) | \boldsymbol{X}_{t_a} = \boldsymbol{x}_a; \boldsymbol{X}_{t_b} = \boldsymbol{x}_b \right] \\ &\quad = \lim_{\substack{N\to\infty \\ N_{\mathrm{p}}\to\infty}} \frac{1}{C_{N,N_{\mathrm{p}}}} \sum_{\ell=1}^{Np} \left[\frac{1}{2} \left\{ \sum_i c_i^{(n)} \boldsymbol{u}_i\big(\boldsymbol{x}_k^{(\ell)}, t_k\big) \right\}^2 - \varphi\big(\boldsymbol{x}_k^{(\ell)}, t_k, \{c_i^{(n)}\}, \{c_i^{(n-1)}\}\big) \right] \\ &\qquad \times \exp\left\{ \frac{\varepsilon}{2\nu} \sum_{m=0}^{N-1} \left[\frac{\boldsymbol{x}_{m+1}^{(\ell)} - \boldsymbol{x}_m^{(\ell)}}{\varepsilon} \cdot \boldsymbol{v}^{(n-1)}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big) - \frac{1}{2}\big|\boldsymbol{v}^{(n-1)}\big(\boldsymbol{x}_m^{(\ell)}, t_m\big)\big|^2 \right] \right\}, \end{aligned} \tag{2.15}$$

where $\varphi^{(n)}(\boldsymbol{X}_t, t) = \varphi(\sum_i c_i^{(n)} \boldsymbol{Y}_t^{(i)}, \sum_i c_i^{(n-1)} \boldsymbol{Y}_t^{(i)}, t)$, $\boldsymbol{v}^{(n-1)} = \sum_i c_i^{(n-1)} \boldsymbol{u}_i$ and $\boldsymbol{v}^{(0)}$ is an appropriate initial trial function based on $\{\boldsymbol{u}_i\}$. The basic Monte Carlo techniques used to numerically obtain the probability density have been given in [15], which discusses the transport in the ABC flow. The techniques are useful when carrying out the integral (2.15).

When some other boundary condition is considered, it may be necessary to modify the generation of a Wiener path (2.11); e.g. a path across the boundary is restricted by the reflection. Note that the emission/absorption of a path at the boundary cannot be permitted in (2.11). It is also possible to consider a case where the diffusion coefficient ν is expressed as a function of $\boldsymbol{x}$, i.e. $\nu = \nu(\boldsymbol{x})$, and the region $\mathscr{M}$ has no boundary. In this case, the path integral method (2.10) cannot be used and the Langevin equation, $d\boldsymbol{X}_t = \boldsymbol{v}^{(n-1)}(\boldsymbol{X}_t, t)\, dt + \sqrt{2\nu(\boldsymbol{X}_t)}\, d\boldsymbol{W}_t$, should be directly solved instead; however, it is not convenient because of the fixed start and end points.

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3. Stochasticity of turbulence

In the previous section, we have seen that the NS equation can be represented using the Langevin equation, i.e. the equation of motion of a fluid particle is given as the Langevin equation. This result implies the fact that the stochasticity of turbulence can be understood through the stochastic representation presented in the previous section. The equation of motion of a fluid particle is expressed as

$$d\boldsymbol{X}_t = \boldsymbol{v}(\boldsymbol{X}_t, t)\, dt + \sqrt{2\nu}\, d\boldsymbol{W}_t, \tag{3.1}$$

where the velocity $\boldsymbol{v}(\boldsymbol{X}_t, t) = D\boldsymbol{X}_t$ is assumed to satisfy (1.3). Frequently, the effect of the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ itself in strong turbulence is interpreted as a white noise [16]. However, in this section we see that this hypothesis is an *ad hoc* assumption, by the same manner as [7].

Using the path integral method of [7], we have the following probability density of the motion (3.1) when the initial distribution is given as the delta function $P(\boldsymbol{x}_0, t_0) = \delta(\boldsymbol{x}_0 - \boldsymbol{x}_a)$,

$$P(\boldsymbol{x}_b, t_b) = P(x_b, y_b, \ldots, t_b) = \frac{1}{C} \prod_{i=x,y,\ldots} \exp\left\{ -\frac{(r_{i,b} - r_{i,a})^2}{4\nu(t_b - t_a)} \right\}$$

$$\times \lim_{N_\mathrm{p}\to\infty} \frac{1}{N_\mathrm{p}} \sum_{\ell=1}^{N_\mathrm{p}} \underbrace{\lim_{N\to\infty} \prod_{i=x,y,\ldots} \exp\left\{ \frac{\varepsilon}{2\nu} \sum_{j=0}^{N-1} \left[v_{i,j}^{(\ell)} \frac{r_{i,j+1}^{(\ell)} - r_{i,j}^{(\ell)}}{\varepsilon} - \frac{1}{2} \big(v_{i,j}^{(\ell)}\big)^2 \right] \right\}}_{G^{(\ell)}}, \tag{3.2}$$

where C is a normalizing factor, the coordinates of the d-dimensional configuration space are expressed as $\boldsymbol{x} = (r_x, r_y, \ldots) = (x, y, \ldots)$, N_p is the number of Wiener paths $\{\boldsymbol{x}^{(\ell)}(t)\}$ starting from $(x_a, y_a, \ldots)$ and arriving at $(x_b, y_b, \ldots)$, $r_{i,k}^{(\ell)}$ is the kth element of the ℓth Wiener path $r_i^{(\ell)}(t)$, and $r_{x,j}^{(\ell)} = x_j^{(\ell)} = x^{(\ell)}(t_j)$, $r_{x,0}^{(\ell)} = r_{x,a} = x_a$, $r_{x,N}^{(\ell)} = r_{x,b} = x_b$, $v_{x,j}^{(\ell)} = v_x(x_j^{(\ell)}, y_j^{(\ell)}, \ldots, t_j)$, etc. An average of the weight $G^{(\ell)}$ in (3.2) is carried out by using the Wiener paths (2.11).

If the process described by the Langevin equation (3.1) is a Brownian process with an effective diffusion coefficient caused by a velocity field in turbulence, then the final distribution $P(\boldsymbol{x}_b, t_b)$ becomes the Gaussian distribution with the effective diffusion coefficient, i.e. it is necessary that an average of the weight $G^{(\ell)}$ given in (3.2) on an end point $(\boldsymbol{x}_b, t_b)$ is Gaussian. Consequently, the following relation for a Wiener path $\boldsymbol{x}^{(\ell)}(t)$ is required:

$$\left\langle \sum_{i=x,y,\ldots} \frac{1}{2\nu} \int_{t_a}^{t_b} dt \left\{ v_i\big(\boldsymbol{x}^{(\ell)}(t), t\big) \frac{dr_i^{(\ell)}}{dt} - \frac{1}{2} \big[v_i\big(\boldsymbol{x}^{(\ell)}(t), t\big) \big]^2 \right\} \right\rangle_\mathrm{w}$$

$$= \sum_{i=x,y,\ldots} \left\{ -\frac{(r_{i,b} - r_{i,a})^2}{c_i(t_b - t_a)} + \cdots \right\}, \tag{3.3}$$

where c_i is a coefficient given by the function v_i. An average on the left-hand side of (3.3) is defined as

$$\ln \left\{ \lim_{N_\mathrm{p}\to\infty} \frac{1}{N_\mathrm{p}} \sum_{\ell=1}^{N_\mathrm{p}} G^{(\ell)} \right\}, \tag{3.4}$$

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where

$$G^{(\ell)} = \exp\left[\sum_{i=x,y,\dots} \frac{1}{2\nu}\int_{t_a}^{t_b} dt\left\{v_i\big(\boldsymbol{x}^{(\ell)}(t),t\big)\frac{dr_i^{(\ell)}}{dt} - \frac{1}{2}\big[v_i\big(\boldsymbol{x}^{(\ell)}(t),t\big)\big]^2\right\}\right]. \tag{3.5}$$

The second term on the right-hand side of (3.3) means the remainder terms in the average. It is natural to consider that the function $\boldsymbol{v}(\boldsymbol{x},t)$ is continuous almost everywhere and it is bounded, i.e. there is a positive constant M satisfying a condition $|v_i(\boldsymbol{x},t)| \le M$ for all $\boldsymbol{x}$ and for all t. A Wiener path $\boldsymbol{x}^{(\ell)}(t)$ is continuous almost everywhere [17], thus $v_i(\boldsymbol{x}^{(\ell)}(t),t)$ is also continuous almost everywhere. The first term on the left-hand side of (3.3) is estimated as

$$\begin{aligned} I_{i,1}^{(\ell)} &\equiv \int_{t_a}^{t_b} dt\left\{v_i\big(\boldsymbol{x}^{(\ell)}(t),t\big)\frac{dr_i^{(\ell)}}{dt}\right\} \\ &= \lim_{N\to\infty}\sum_{j=1}^{N} v_i\big(x_{j-1}^{(\ell)}, y_{j-1}^{(\ell)},\dots,t_{j-1}\big)\big\{r_{i,j}^{(\ell)} - r_{i,j-1}^{(\ell)}\big\}. \end{aligned} \tag{3.6}$$

From the interpolation formula (2.11), the displacement of a path, $(r_{i,j}^{(\ell)} - r_{i,j-1}^{(\ell)})$, is estimated as

$$r_{i,j}^{(\ell)} - r_{i,j-1}^{(\ell)} = \frac{\varepsilon(r_{i,b}-r_{i,a})}{t_b - t_a} + F\big(t_a,t_{j-1},t_j,t_b;\xi_i^{(\ell)}(t_{j-1}),\xi_i^{(\ell)}(t_j)\big), \tag{3.7}$$

where F is not a function of $r_{i,a}$ and $r_{i,b}$. Since the function $v_i(\boldsymbol{x}^{(\ell)}(t),t)$ is bounded, the function $v_i(\boldsymbol{x}^{(\ell)}(t),t)$ must include higher-order terms of $r_{i,a}$ and $r_{i,b}$, except for the case where v_i is a function of t only, or is a constant. Here the Weierstrass approximation theorem [18] is used. Thus the first term $(1/2\nu)I_{i,1}^{(\ell)}$ is not given as $-(r_{i,b}-r_{i,a})^2/c_i(t_b-t_a)$. The second term on the left-hand side of (3.3) is estimated as

$$I_{i,2}^{(\ell)} \equiv \int_{t_a}^{t_b} dt\ \frac{1}{2}\big[v_i\big(\boldsymbol{x}^{(\ell)}(t),t\big)\big]^2 \le \frac{M^2}{2}(t_b - t_a). \tag{3.8}$$

Therefore, in general, the relation (3.3) is not obtained for a well-behaved function $\boldsymbol{v}(\boldsymbol{x},t)$ such that it is continuous almost everywhere and is bounded; hence the average of the weight $G^{(\ell)} = \prod_{i=x,y,\dots}\exp\{(1/2\nu)[I_{i,1}^{(\ell)} - I_{i,2}^{(\ell)}]\}$ on an end point is not Gaussian. Furthermore, even if the probability density $P(\boldsymbol{x}_b,t_b)$ is Gaussian, its diffusion coefficient must be the same coefficient as the original ν. Note that this result is independent of the value of ν.

4. Discussion and conclusion

Finally, we will discuss a topic which is relevant to the Euler equation of the ideal fluid. For the simplicity of a physical situation, hereafter, the scalar function $\varphi(\boldsymbol{x},t)$ is neglected. If the diffusion coefficient $\nu \to 0$, then the conditional expectation (2.10) is expressed as

$$\begin{aligned} \mathrm{E}\big[\tfrac{1}{2}\{D\boldsymbol{X}_t\}^2 | \boldsymbol{X}_{t_a} = \boldsymbol{x}_a; \boldsymbol{X}_{t_b} = \boldsymbol{x}_b\big] &\to \lim_{N\to\infty}\int\cdots\int dx_{N-1}\dots dx_1 dy_{N-1}\dots dy_1\dots \\ &\times \frac{1}{2}\left\{\frac{\boldsymbol{x}_{k+1}-\boldsymbol{x}_k}{\varepsilon}\right\}^2 \prod_{m=0}^{N-1}\delta\left[\frac{\boldsymbol{x}_{m+1}-\boldsymbol{x}_m}{\varepsilon} - \boldsymbol{v}(\boldsymbol{x}_m,t_m)\right]. \end{aligned} \tag{4.1}$$

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Thus, the functional $J[\boldsymbol{X}]$ becomes $J_0[\boldsymbol{X}]$ with $\nu \to 0$:

$$J[\boldsymbol{X}] \to J_0[\boldsymbol{X}] = \int_{t_a}^{t_b} dt \int_{\mathscr{M}} dV(\boldsymbol{x})\, \frac{1}{2} \left| \frac{d\boldsymbol{X}_t(\boldsymbol{x})}{dt} \right|^2, \tag{4.2}$$

where $\nabla \cdot \boldsymbol{v} = 0$, $d/dt = \partial/\partial t + \boldsymbol{v} \cdot \nabla$, and the time derivative $d\boldsymbol{X}_t/dt$ is described as

$$\frac{d\boldsymbol{X}_t(\boldsymbol{x})}{dt} = \boldsymbol{v}(\boldsymbol{X}_t(\boldsymbol{x}), t), \quad \text{and} \quad \boldsymbol{X}_{t_a}(\boldsymbol{x}) = \boldsymbol{x} \quad \text{at} \quad t = t_a. \tag{4.3}$$

Note that $\boldsymbol{X}_t$ is not stochastic any longer. A variation of $J_0[\boldsymbol{X}]$ leads to the Euler equation of the ideal fluid as follows [19]:

$$\begin{aligned} 0 = \delta J_0[\boldsymbol{X}] &= -\int_{t_a}^{t_b} dt \int_{\mathscr{M}} dV(\boldsymbol{x})\, \delta \boldsymbol{X}_t \cdot \frac{d\boldsymbol{v}(\boldsymbol{X}_t, t)}{dt} \\ &= -\int_{t_a}^{t_b} dt \int_{\mathscr{M}} dV(\boldsymbol{x})\, \boldsymbol{h}(\boldsymbol{x}, t) \cdot \left\{ \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right\} \boldsymbol{v}(\boldsymbol{x}, t), \end{aligned} \tag{4.4}$$

thus we obtain

$$\left\{ \frac{\partial}{\partial t} + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \nabla \right\} \boldsymbol{v}(\boldsymbol{x}, t) = -\frac{1}{\rho} \nabla p(\boldsymbol{x}, t). \tag{4.5}$$

Although the probability density is given as $\mu(\boldsymbol{x}, t) = \text{constant}$, the functional $J[\boldsymbol{X}]$ is not equivalent to $J_0[\boldsymbol{X}]$ in general. It should be noted that the equation of motion of a fluid particle in $J[\boldsymbol{X}]$ is given as (2.2), i.e. $D\boldsymbol{X}_t = \boldsymbol{v}(\boldsymbol{X}_t, t)$; on the other hand the equation in $J_0[\boldsymbol{X}]$ is expressed as (4.3), i.e. $d\boldsymbol{X}_t/dt = \boldsymbol{v}(\boldsymbol{X}_t, t)$. The equation of motion (2.2) takes the diffusion term $-\nu\nabla^2\boldsymbol{v}$ into account, but this term is neglected in (4.3). In contrast to (2.7) and (2.8), we should note that $dg(\boldsymbol{X}_t, t)/dt = (\partial/\partial t + \boldsymbol{v} \cdot \nabla) g(\boldsymbol{X}_t, t)$. When the variation of (4.2) determines the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, the path integral method in Sec. 2 may also be useful in solving the equation of motion $d\boldsymbol{X}_t/dt = \boldsymbol{v}(\boldsymbol{X}_t, t)$ under the conditions $\boldsymbol{X}_{t_a} = \boldsymbol{x}_a$ and $\boldsymbol{X}_{t_b} = \boldsymbol{x}_b$ [11, 20].

We have discussed the mathematical background of the Monte Carlo method solving the fluid equations. We have seen that the Langevin equation (1.2) can be used to solve the fluid equations and the velocity field, which is the solution of the NS equation, satisfies (2.3). The scheme treating the stochastic method has been proposed by employing the path integral. The results may be useful when laying the foundations of the Monte Carlo method, but the method is restricted by severe conditions: an incompressible fluid (i.e. $\nabla \cdot \boldsymbol{v} = 0$ and $\rho(\boldsymbol{x}, t) = \text{constant}$) with a constant diffusion coefficient is considered, an external force $-\nabla\varphi(\boldsymbol{x}, t)$ describing the momentum source/sink is assumed and a region $\mathscr{M}$ in which the fluid exists is fixed. In the present article, we have not discussed the equation of energy. In order to treat the equation of energy by using a Monte Carlo technique, a functional leading the equation is needed. This will be investigated in future work.

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