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On Euler's function

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By giving some new treatments we can improve a classical result of Walfisz (1963) on the asymptotic formula of Euler's function.

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1. Introduction

Let $\phi(n)$ be Euler's totient function. The following asymptotic formula is well known (and can be found in many books on number theory):

$$\sum_{n \leqslant x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x). \tag{1.1}$$

In 1963 [7, ch. 4] Walfisz improved the error term of (1.1) to

$$O(x(\log x)^{2/3}(\log\log x)^{4/3}) \tag{1.2}$$

by using both Vinogradov's and van der Corput's methods of exponential sums. His method turns out to be rather complicated (occupying 32 pp.; see [7, pp. 114–145]), for example, he had to use Vinogradov's method to firstly estimate sums of the shape (see [7, Hilfssatz 8, p. 136])

$$\sum_{p \leqslant N} e\left(\frac{x}{p}\right).$$

(From now on, $e(t) = \exp(2\pi i t)$.) We know that the error term of (1.1) is

$$-x \sum_{d \leqslant x} \frac{\mu(d)}{d} \psi\left(\frac{x}{d}\right) + O(x), \quad \psi(t) = t - [t] - 1/2, \tag{1.3}$$

where $\mu(\cdot)$ is the Möbius function. In this paper we instead use the method of [4] for treating sums involving $\mu(\cdot)$, and we suitably combine van der Corput's and Vinogradov's methods. Thus, we can improve (1.2) as follows.

THEOREM 1.1. *In (1.1) we have the error term* $O(x(\log x)^{2/3}(\log\log x)^{1/3})$.

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2. Lemmas

Lemma 2.1. *Let $X^{1/2}(\log X)^{-1} > 2D \geqslant D_1 > D$. Then*

$$S := \sum_{D<n\leqslant D_1} e\biggl(\frac{X}{n}\biggr) \ll D\exp\biggl(-\frac{\gamma(\log D)^3}{(\log X)^2}\biggr).$$

for some positive constant γ.

Proof. For $\exp((\log X)^{2/3}) < D < X^{1/24}$ the required result follows from [7, Satz 1, p. 47]. For $X^{1/24} < D < X^{1/2}(\log X)^{-1}$ let $n = 23$, and we use the exponent pair

$$(p, q) = \biggl(\frac{1}{2^{n+1}-2}, 1 - \frac{1}{2^{n+1}-2}\biggr)$$

of [5] to obtain (using $X/D^2 \gg \log X$ to verify the condition of using an exponent pair in the strict sense of [5])

$$S \ll \biggl(\frac{X}{D^2}\biggr)^p D^q \ll D\biggl(\frac{X}{D^{n+2}}\biggr)^p \ll D^{1-p}.$$

Finally, for $D \leqslant \exp((\log X)^{2/3})$ the required estimate holds trivially. Therefore, lemma 2.1 holds. □

Lemma 2.2. *Let $\psi(t) = t - [t] - 1/2$. Then for $H > 2$ we have*

$$\psi(t) = -\sum_{0<|h|\leqslant H} \frac{e(ht)}{2\pi \mathrm{i} h} + O\biggl(\min\biggl(1, \frac{1}{H\|t\|}\biggr)\biggr),$$

$$\min\biggl(1, \frac{1}{H\|t\|}\biggr) = \sum_{-\infty<h<\infty} a_h e(ht)$$

$$a_0 \ll \frac{\log H}{H}, \qquad a_h \ll \min\biggl(\frac{\log(2H)}{H}, Hh^{-2}\biggr) \quad \text{for } h \neq 0,$$

where $\|t\| = \min(1 - \{t\}, \{t\})$, $\{t\} = t - [t]$.

Proof. See [4, p. 254]. Actually we need to explain the estimate

$$a_h \ll \frac{\log H}{H},$$

for in [4] Montgomery and Vaughan already mentioned that $a_h \ll \min(1/|h|, Hh^{-2})$ for $h \neq 0$. In fact, for

$$\varPhi(x) = \min\biggl(1, \frac{1}{H\|x\|}\biggr)$$

we have (for $h \neq 0$)

$$a_h = \int_0^1 \varPhi(x) e(-hx)\,\mathrm{d}x,$$

and thus, obviously,

$$|a_h| \leqslant a_0 = 2H^{-1}(1 + \log(\tfrac{1}{2}H)) \ll \frac{\log(2+H)}{H}.$$

Thus, by combining with the known bound $a_h \ll Hh^{-2}$ (this can be obtained by means of the integration by parts in the integral representation of a_h of [4, (18)]), we see that lemma 2.2 follows. □

LEMMA 2.3. *Let $Z > 1$, $U = Z^{1/3}$. Then for any arithmetic function $f(n)$ there exist real coefficients b_u, g_u, c_u such that $|b_u| + |g_u| + |c_u| = O(d(u))$; here $d(u)$ is the divisor function, which is the number of distinct divisors of u, and*

$$\sum_{Z<n\leqslant 2Z} \mu(n) f(n) = S_1 + S_2,$$

$$S_1 = \sum_{\substack{u\leqslant U,\\ Z/u<v\leqslant 2Z/u}} g_u f(uv), \qquad S_2 = \sum_{\substack{u>U,\, v>U,\\ Z<uv\leqslant 2Z}} b_u c_v f(uv),$$

where $b_u, g_u \ll d(u)$, $c_v \ll d(v)$ (as usual $d(n)$ is the number of positive divisors of n).

Proof. A similar formula is derived in [4, p. 251], which actually implies our version, because we can split the sum as

$$\sum_{\substack{u\leqslant U^2,\\ Z/u<v\leqslant 2Z/u}} g_u f(uv) = \left(\sum_{\substack{u\leqslant U,\\ Z/u<v\leqslant 2Z/u}} + \sum_{\substack{U<u\leqslant U^2,\\ Z/u<v\leqslant 2Z/u}} \right) g_u f(uv),$$

and we find that the second part can be included in S_2. □

LEMMA 2.4. *For positive numbers A_m, B_n, u_m, v_n $(1 \leqslant m \leqslant M,\ 1 \leqslant n \leqslant N)$, $0 < Q_1 < Q_2$, there is a number $q \in [Q_1, Q_2]$ such that*

$$\begin{aligned}\sum_{1\leqslant m\leqslant M} A_m q^{u_m} &+ \sum_{1\leqslant n\leqslant N} B_n q^{-v_n} \\ &\leqslant (M+N)\biggl(\sum_{1\leqslant m\leqslant M} A_m Q_1^{u_m} + \sum_{1\leqslant n\leqslant N} B_n Q_2^{-v_n} \\ &\qquad + \sum_{1\leqslant m\leqslant M} \sum_{1\leqslant n\leqslant N} (A_m^{v_n} B_n^{u_m})^{1/(u_m+v_n)} \biggr).\end{aligned}$$

Proof. See [3, lemma 6]. □

3. Proof of theorem 1.1

Note that $t \sim T$ means that $T < t \leqslant 2T$, and $t \approx T$ means $t/T \in [\omega, \rho]$ for some absolute and positive constants ω and ρ. The meaning of Vinogradov's symbols $\ll$ or $\gg$ is standard.

Let $N = \exp(AL^{2/3}(\log L)^{1/3})$. Here $L = \log x$, A is some sufficiently large constant that will be specified later. In view of (1.3), by using the familiar estimate $(t \geqslant 2)$

$$\sum_{n\leqslant t} \frac{1}{n} = \log t + O(1),$$

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to derive theorem 1.1 we only need to treat

$$\sum_{N<d<xN^{-1}} \frac{\mu(d)}{d}\psi\left(\frac{x}{d}\right).$$

We split the sum into less than 10 subsums that have ranges of summation as $D < d \leqslant D'$ ($D' \leqslant 2D$, $N < D < xN^{-1}$), and, for each of them, by the partial summation we only need to establish (for any $D_1 \leqslant 2D$) that

$$S := \sum_{D<d\leqslant D_1} \mu(d)\psi\left(\frac{x}{d}\right) \ll DL^{-3}. \tag{3.1}$$

Let $H = L^5$. By lemma 2.2 we have

$$S = -\sum_{1\leqslant|h|\leqslant H} \frac{1}{2\pi \mathrm{i}h} \sum_{D<d\leqslant D_1} \mu(d)e\left(\frac{hx}{d}\right) + O\left(\sum_{D<d\leqslant D_1} \min\left(1, \frac{1}{H\|x/d\|}\right)\right) \tag{3.2}$$

and

$$\sum_{D<d\leqslant D_1} \min\left(1, \frac{1}{H\|x/d\|}\right) \ll DL^{-4} + \log L \sum_{1\leqslant|h|\leqslant H^2} \min\left(\frac{1}{H}, \frac{H}{h^2}\right)\left|\sum_{D<d\leqslant D_1} e\left(\frac{hx}{d}\right)\right|. \tag{3.3}$$

We first treat the contribution of (3.3). If $D > x^{5/12}$, we use the exponent pair (1/2,1/2) according to the manner of Heath-Brown [2] to obtain (note that, by the inclusion of an extra term in the final upper bound, and the use of auxiliary tools of Titchmarsh [6], Heath-Brown [2] was able to relax the strict conditions of Phillips [5] regarding the use of exponent pairs; see also [1, (3.3.4)])

$$\sum_{D<d\leqslant D_1} e\left(\frac{hx}{d}\right) \ll \sqrt{|h|xD^{-1}} + D^2(|h|x)^{-1},$$

which gives

$$\sum_{D<d\leqslant D_1} \min\left(1, \frac{1}{H\|x/d\|}\right) \ll DL^{-4} + \log L(\sqrt{HxD^{-1}} + D^2x^{-1}) \ll DL^{-4}.$$

If $D \leqslant x^{5/12}$, by lemma 2.1 we have (by taking $A^3\gamma > 10$)

$$\sum_{D<d\leqslant D_1} e\left(\frac{hx}{d}\right) \ll DL^{-10},$$

and thus the above estimate also holds. In the following we estimate the first summation of the right-hand side of (3.2). We will give the following estimate:

$$\sum_{D<d\leqslant D_1} \mu(d)e\left(\frac{hx}{d}\right) \ll DL^{-4}.$$

By lemma 2.3, it suffices to establish both

$$T_1 := \sum_{\substack{u\sim U,\, v\sim V,\\ D<uv\leqslant D_1}} b_u c_v e\left(\frac{hx}{uv}\right) \ll DL^{-10} \tag{3.4}$$

and

$$T_2 := \sum_{\substack{u\sim U,\, v\sim V,\\ D<uv\leqslant D_1}} g_u\left(\frac{hx}{uv}\right) \ll DL^{-10}, \tag{3.5}$$

where in (3.4) $D^{1/3} \ll U,\ V \ll D^{2/3}$, $UV \approx D$, and in (3.5) $1 \ll U \ll D^{1/3}$, $UV \approx D$, and $b_u, g_u \ll d(u)$, $c_v \ll d(v)$. Obviously, in the following we can suppose that $h > 0$.

(a) Let $D > x^{2/7}N$. Due to the symmetric positions of u and v, we can suppose that $D^{1/3} \ll V \ll D^{1/2}$. Using

$$\sum_{n\leqslant t} d^2(n) \ll t\log^3 t$$

(see [7, Hilfssatz 1, p. 126]), Cauchy's inequality and Weyl's inequality (a strict proof of Weyl's inequality was given in [2], which we use here), after exchanging the order of summations, we obtain

$$T_1^2 \ll L^6D^2Q^{-1} + L^3DQ^{-1}\sum_{v,q}|c_vc_{v+q}|\left|\sum_u e\left(\frac{hx}{u}\left(\frac{1}{v}-\frac{1}{v+q}\right)\right)\right|, \tag{3.6}$$

where $Q \in [10, VL^{-1}]$ is a parameter to be specified later, and $1 \leqslant |q| \leqslant Q$, $v \sim V$, $(v+q) \sim V$, $D < uv$, $u(v+q) \leqslant D_1$ and $u \sim U$. We treat only $q > 0$. Applying the exponent pair (1/14,11/14) to the innermost sum of (3.6), we get

$$\sum_u e\left(\frac{hx}{u}\left(\frac{1}{v}-\frac{1}{v+q}\right)\right) \ll \left(\frac{hxq}{D^2}\right)^{1/14} U^{11/14} + D^2(hxq)^{-1}.$$

For a fixed q we have

$$\sum_{v,q}|c_vc_{v+q}| \ll \sum_v d(v)d(v+q) \ll \sum_v (d^2(v)+d^2(v+q)) \ll VL^3.$$

Thus, we get from (3.6)

$$L^{-6}T_1^2 \ll D^2Q^{-1} + \sqrt[14]{hxQD^{23}V^3} + LD^3V(hx)^{-1}Q^{-1}. \tag{3.7}$$

Obviously, (3.7) holds for all $Q \in [0, VL^{-1}]$. Using lemma 2.4 to choose an optimal parameter Q in this range, we get (using $D^{1/3} \ll V \ll D^{1/2}$)

$$\begin{aligned} L^{-4}T_1 &\ll \sqrt{D^2V^{-1}} + \sqrt[30]{hxD^{25}V^3} + \sqrt{D^3(hx)^{-1}} + \sqrt[30]{D^{26}V^4} \\ &\ll D(\sqrt[30]{hxD^{-3.5}} + D^{-1/15} + \sqrt{Dx^{-1}}), \end{aligned}$$

and thus (3.4) holds for $D > x^{2/7}N$. Using directly the exponent pair $(1/14, 11/14)$ to estimate the innermost sum of (3.5), we get the required bound (3.5) as follows:

$$\begin{aligned} T_2 &\ll \sum_{u\sim U} d(u)\left(\left(\frac{hx}{DV}\right)^{1/14} V^{11/14} + VD(hx)^{-1}\right) \\ &\ll LD(\sqrt[14]{hxD^{-11/3}} + D(hx)^{-1}) \\ &\ll DL^{-9}; \end{aligned}$$

in the last step we have used $U \ll D^{1/3}$ and $D > x^{2/7}N$.

(b) Let $N < D \leqslant x^{2/7}N$. Here we just use Cauchy's inequality to obtain (similarly to (3.6))

$$T_1^2 \ll L^6D^2V^{-1} + L^3U \sum_{v_1\neq v_2} d(v_1)d(v_2)\left|\sum_u e\left(\frac{hx}{u}\left(\frac{1}{v_1} - \frac{1}{v_2}\right)\right)\right|. \tag{3.8}$$

Here the variables satisfy a similar restriction condition to those of (3.6). Let $X = hx|v_1 - v_2|(v_1v_2)^{-1}$. It is easy to verify that $U < X^{1/2}(\log X)^{-1}$ follows from $D^{1/3} \ll U$, $V \ll D^{2/3}$ and $X \gg hxV^{-2} \geqslant xV^{-2}$. Thus by lemma 2.1 the innermost sum of (3.8) is

$$\sum_u e\left(\frac{hx}{u}\left(\frac{1}{v_1} - \frac{1}{v_2}\right)\right) \ll U\exp\left(-\frac{\gamma\log^3 U}{\log^2 X}\right) \ll U\exp\left(-\gamma_1\frac{\log^3 D}{\log^2 X}\right)$$

(for suitable positive constants γ and γ_1), and thus we can obtain

$$T_1^2 \ll L^6D^{5/3} + L^5D^2\exp\left(-\gamma_1\frac{\log^3 U}{\log^2 X}\right),$$

and (3.4) follows by noting that $D > N$, and A is large enough such that $A^3\gamma_1 > 25$. Similarly, we can use lemma 2.1 to bound the innermost sum of T_2 and obtain (3.5).

The proof of theorem 1.1 is finished.

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References

1 S. W. Graham and G. Kolesnik. *Van der Corput's method of exponent pairs* (Cambridge University Press, 1991).

2 D. R. Heath-Brown. The Pjateckii-Sapiro prime number theorem. *J. Number Theory* **16** (1983), 242–266.

3 H.-Q. Liu. On the estimates for double exponential sums. *Acta Arith.* **129** (2007), 203–247.

4 H. L. Montgomery and R. C. Vaughan. The distribution of squarefree numbers. In *Recent progress in analytic number theory* (ed. H. Halberstam and C. Hooley), vol. I, pp. 247–256 (Academic Press, 1981).

5 E. Phillips. The zeta-function of Riemann: further developments of van der Corput's method. *Q. J. Math.* **4** (1933), 209–225.

6 E. C. Titchmarsh. *The theory of the Riemann zeta-function*, 2nd edn (Oxford University Press, 1986).

7 A. Walfisz. *Weylsche Exponentialsummen in der neueren Zahlentheorie*, Mathematische Forschungsberichte, vol. 15 (Berlin: Deutscher Verlag der Wissenschaften, 1963).