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THE SLIMMING NUMBER AND GENUS OF GRAPHS

BY

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For G. Ringel and the late J. W. T. Youngs,
on settling the Heawood conjecture

J. Ch. Boland[2] suggested, and Mrs. Sheehan named, the idea of the *slimming number* of a graph G, i.e. the minimum number, $s(G)$, of edges, $e_1, e_2, \ldots, e_s$, which must be removed from G in order that $G-\bigcup e_i$ be planar.

For the complete graph, K_n $(n \geq 3)$, it may be seen by Euler's formula that a planar subgraph contains at most $3n-6$ edges; moreover one may construct such a subgraph inductively, starting from K_3, and adding points successively, joining them to the three vertices of the region in which they lie, so

$$s(K_n) = \binom{n}{2}-(3n-6) = \tfrac{1}{2}(n-3)(n-4), \quad (n \geq 3). \tag{1}$$

For the complete bipartite graph, $K_{m,n}$ $(m, n \geq 2)$, all circuits are even, so a planar subgraph with $m+n$ points and e edges contains $e+2-(m+n)$ regions, which are at least 4-sided, so $4(e+2-m-n) \leq 2e$, $e \leq 2m+2n-4$. Again we may construct a subgraph with the maximal number of edges (Fig. 1), so that

$$s(K_{m,n}) = mn-(2m+2n-4) = (m-2)(n-2), \quad (m, n \geq 2). \tag{2}$$

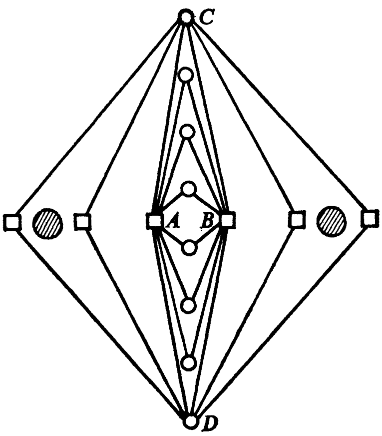

FIGURE 1.

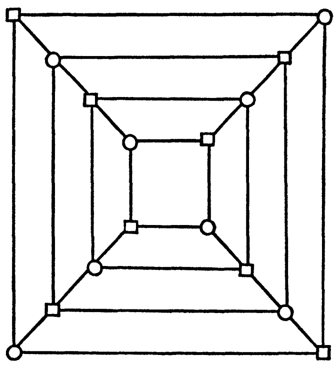

FIGURE 2.

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The n-cube, Q_n, i.e. the 1-skeleton of the n-dimensional cube, is also bipartite, so we may construct a subgraph with $2^{n+1}-4$ edges, though since the valence must not now exceed n, it is necessary to use an arrangement such as that shown in Fig. 2. Hence

$$s(Q_n) = n\cdot 2^{n-1}-(2^{n+1}-4) = (n-4)2^{n-1}+4, \quad (n \geq 2). \tag{3}$$

More generally, we may define the *generic slimming number*, $s_p(G)$, of a graph G as the minimum number of edges to be removed from G in order that $G-\bigcup e_i$ be embeddable in an orientable surface of genus p. (A further generalization, due to W. Vollmerhaus([3]), is the *characteristic slimming number*, $s_\chi(G)$, defined correspondingly for embeddings in non-orientable surfaces of Euler characteristic χ.) As before, we may show by an appeal to Euler's formula, that, for sufficiently large n (and m),

$$s_p(K_n) \geq \tfrac{1}{2}(n-3)(n-4)-6p, \tag{4}$$

$$s_p(K_{m,n}) \geq (m-2)(n-2)-4p, \tag{5}$$

$$s_p(Q_n) \geq (n-4)2^{n-1}+4(1-p). \tag{6}$$

It is natural to conjecture that equality holds in each case, for sufficiently large n (and m). In fact we give constructions to show equality in (6), and in (5) provided m, n are both even. This also confirms the results of Ringel ([5], [6]) on the genus of Q_n and of $K_{m,n}$ (the genus, $\gamma(G)$, of a graph G is the least possible genus of an orientable surface in which G is embeddable):

$$\gamma(Q_n) = (n-4)2^{n-3}+1, \quad (n \geq 2), \tag{7}$$

$$\gamma(K_{m,n}) = \left\{\frac{(m-2)(n-2)}{4}\right\}, \quad (m, n \geq 2), \tag{8}$$

where braces denote the "post office function", least integer not less than, and are not required in the case we consider.

To prove (7) and show equality in (6), we prove by induction on r that Q_r is embeddable in an orientable closed surface of genus $(r-4)2^{r-3}+1$ $(r\geq 2)$. Observe that Q_2 and Q_3 may be embedded in the plane, and Q_4 in the torus (Fig. 3). Take two copies of an embedding of Q_r $(r\geq 3)$. Let the 2^r vertices of each of these be labelled with the 2^r r-tuples of zeros and ones so that edges join vertices if and only if their r-tuples differ in exactly one coordinate. Two corresponding sets of four vertices (for example, those whose labels agree in all coordinates except possibly the last two) are joined by a handle containing four edges. $2^r/4$ such handles are needed and the genus of the resulting surface is

$$2((r-4)2^{r-3}+1)+2^{r-2}-1 = (r-3)2^{r-2}+1.$$

([3]) Tagung über Graphentheorie, Oberwolfach, July, 1969.

Hence, by induction

$$\gamma(Q_n) \leq (n-4)2^{n-3}+1. \tag{9}$$

Since each of the 2^{n-3} handles used in the last step contains four edges, we may remove $(n-4)2^{n-3}+1-p$ of them, provided $0 \leq (n-4)2^{n-3}+1-p < 2^{n-3}$, and thus obtain

$$s_p(Q_n) \leq (n-4)2^{n-1}+4(1-p),$$

giving equality in (6), which in turn implies (7). If $2^{n-3} \leq (n-4)2^{n-3}+1-p$, so that $0 \leq p \leq (n-5)2^{n-3}+1$, we assume inductively that

$$s_p(Q_{n-1}) = (n-5)2^{n-2}+4(1-p)$$

and write

$$p = h+k \quad \text{where } 0 \leq h \leq k \leq (n-5)2^{n-4}+1.$$

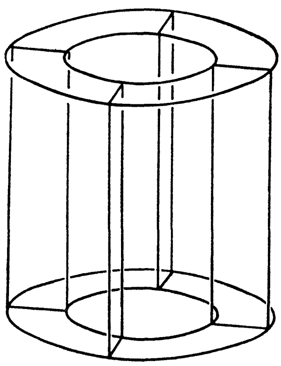

FIGURE 3.

By the inductive hypothesis we can embed Q_{n-1}, with $4(\gamma(Q_{n-1})-h)$ appropriate edges removed, in a surface of genus h. Make a similar embedding with h replaced by k and connect the two surfaces by a handle carrying four edges joining corresponding vertices of the $(n-1)$-cubes. This gives an embedding of Q_n, with

$$8\gamma(Q_{n-1})-4(h+k)+(2^{n-1}-4) = (n-4)2^{n-1}+4(1-p)$$

edges removed, in a surface of genus $h+k=p$, so

$$s_p(Q_n) \leq (n-4)2^{n-1}+4(1-p)$$

and equality is established in (6).

To confirm (8) in case m and n are both even, observe that of the m $(=6)$ points in Fig. 1, A and B are connected to all the n $(=8)$ points, and of the n points, C and D are connected to each of the m. To connect the remaining points we plant $\frac{1}{2}m-1$ "trees" of finite girth in the holes indicated in Fig. 1 and form $\frac{1}{2}n-1$

branches on each (Fig. 4). These trees each carry $\frac{1}{2}n-1$ edges from the $\frac{1}{2}m-1$ pairs of the m points other than A and B, one edge from each of 2 points on each branch. The $(\frac{1}{2}m-1)(\frac{1}{2}n-1)$ branches are connected to $\frac{1}{2}m-1$ branches on each of $\frac{1}{2}n-1$ other trees planted between pairs of the n points other than C and D. The genus of the surface thus produced is $(m-2)(n-2)/4$. Since removal of $(\frac{1}{2}m-1)(\frac{1}{2}n-1)-p$ of the $(\frac{1}{2}m-1)(\frac{1}{2}n-1)$ branches and of the $(m-2)(n-2)-4p$ edges carried by these branches yields an embedding of the rest of $K_{m,n}$ in a surface of genus p $(0\le p \le(\frac{1}{2}m-1)(1\frac{1}{2}-1))$, our construction also establishes equality in (5), if m and n are even.

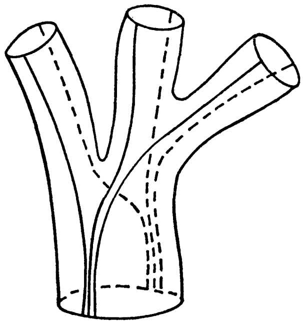

FIGURE 4.

Not surprisingly, we do not produce a proof of the Heawood conjecture, since, starting with a triangulation of the plane, it is not clear how to add triangular handles and be sure one is not duplicating edges which already exist. In fact, even if one assumes the Heawood conjecture (proved by Ringel and Youngs [7–14], [16–18]) this gives no immediate guarantee of equality in (4), which must remain an open question; one which, however, should yield to the methods of Ringel and Youngs.

We illustrate the case $\gamma(K_7)=1$ by starting from the plane triangulation shown by continuous lines in Fig. 5, in which the points labelled 2 and 6 are identified, and

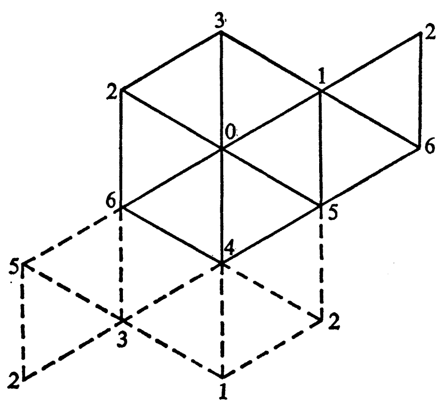

FIGURE 5.

adding a handle in the form of a triangular antiprism (broken lines in the figure) to connect the triangles 123 and 456. Note that Fig. 5 is part of the infinite tesselation, 3^6, of the plane, with its vertices labelled with arithmetic progressions from the additive group Z_7, the common differences depending (only) on the directions of the edges. The triangulation is the dual of Heawood's map of seven contiguous countries on the torus and can be realized geometrically in Euclidean 3-space. It is connected with "cyclic" and "neighborly" polytopes [3].

Another generalization of slimming number, not named or investigated here, may be defined as the minimum number, $s(G; t)$, of edges to be removed from G, that the remaining graph have thickness t. The *thickness* of G is the minimum number of planar graphs whose union is G. P. Erdős (oral communication) and others ask for the least number of edges, $m(t)$, in a graph of thickness t. It follows from Kuratowski's theorem [4] that $m(2)=9$, the unique critical graph being $K_{3,3}$. The results of Beineke *et al.* [2] on the thickness of the complete bipartite graph show that $m(t) \le (4t-5)^2$. The example K_9 shows that $m(3) \le 36$, but there are grounds ([1], [15]) for believing this to be exceptional.

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REFERENCES

1. J. Battle, F. Harary, and Y. Kodama, *Every planar graph with nine points has a non-planar complement*, Bull. Amer. Math. Soc., **68** (1962), 569–571; M.R. **27** (1964), #5248.

2. L. W. Beineke, F. Harary, and J. W. Moon, *On the thickness of the complete bipartite graph*. Proc. Cambridge Philos. Soc., **60** (1964), 1–5; M.R. **28** (1964), #1611.

3. B. Grünbaum, *Convex Polytopes*, Interscience, New York, (1967), 61, 122.

4. K. Kuratowski, *Sur le problème des courbes gauches en topologie*, Fund. Math., **15** (1930), 271–283.

5. G. Ringel, *Über drei kombinatorische Probleme am n-dimensionalen Würfel und Würfelgitter*, Abh. Math. Sem. Univ. Hamburg, **20** (1955), 10–19; M.R. **17** (1956), #772.

6. ———, *Das Geschlecht des vollständigen paaren Graphen*, Abh. Math. Sem. Univ. Hamburg, **28** (1965), 139–150; M.R. **32** (1966), #6439.

7. ———, *Färbungsprobleme auf Flächen und Graphen*, VEB Deutscher Verlag, Berlin, 1959; M.R. **22** (1961), #A235.

8. ———, *The genus of graphs*, in R. K. Guy, *et al.* (editors), Combinatorial Structures and their Applications (Proc. Calgary Internat. Conference, 1969), Gordon and Breach, New York, (1970), 361–366.

9. G. Ringel and J. W. T. Youngs, *Lösung des problems der Nachbargebiete auf orientierbaren Flächen*, Arch. Math. (Basel), **20** (1969), 190–201; M.R. **39** (1970), #4049.

10. ———, *Remarks on the Heawood conjecture*, in F. Harary, (editor), Proof Techniques in Graph Theory, Academic Press, New York, (1969), 133–138.

11. ———, *Solution of the Heawood map-coloring problem—case 11*, J. Combinatorial Theory, **7** (1969), 71–93; M.R. **39** (1970), #1360. —*case 2*, ibid., 342–352. —*case 8*, ibid., 353–363.

12. C. M. Terry, L. R. Welch, and J. W. T. Youngs, *The Genus of K_n, $n=12(2^m)$*, Bull. Amer. Math. Soc., **71** (1965), 653–656. Ditto, $n=12s$, ibid., 657–660; M.R. **31** (1966), #4022a, b.

13. ———, *The genus of* K_{12s}, J. Combinatorial Theory, **2** (1967), 43–60; M.R. **34** (1967), #6755.

14. ———, *Solution of the Heawood map-coloring problem, case 4*, J. Combinatorial Theory, **8** (1970), 170–174.

15. W. T. Tutte, *On the non-biplanar character of the complete 9-graph*, Canad. Math. Bull., **6** (1963), 319–330; M.R. **28** (1964), #2535.

16. J. W. T. Youngs, *Minimal imbeddings and the genus of a graph*, J. Math. Mech., **12** (1963), 303–315; M.R. **26** (1963), #3043.

17. ———, *The Heawood map-coloring conjecture*, in F. Harary (editor), Graph Theory and Theoretical Physics, Academic Press, London, 1967, 313–354; M.R. **38** (1969), #4357.

18. ———, *Solution of the Heawood map-coloring problem, cases 3, 5, 6 and 9*. J. Combinatorial Theory, **8** (1970), 175–219. *cases 1, 7 and 10*, ibid., 220–231.

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