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Straight-sided solutions to classical and modified plume flux equations

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The classical plume model due to Morton, Taylor & Turner (*Proc. R. Soc. Lond.* A, vol. 234, 1956, pp. 1–23) is re-cast in terms of the non-dimensional plume radius, the plume 'laziness' defined as the squared ratio of the source radius and the jet length, and the buoyancy flux. It is shown that many of the key results of this classical model can then be read straight from the equations without recourse to solving them. Based on this observation, derivative models that consider plumes propagating through stratified environments or undergoing chemical reactions are similarly re-cast. We show again that key results can be read straight from the governing equations and results that have previously only been demonstrated numerically can be found analytically. In particular, we unify two previously distinct models that consider plumes propagating through stable and unstable stratified environments whose stratification has a power-law dependence on height. We present analytical solutions for the range of stratification power-law decay rates for which straight-sided plumes are possible. This result unifies the sets of solutions by Batchelor (*Q. J. R. Meteorol. Soc.*, vol. 80, 1954, pp. 339–358) and Caulfield & Woods (*J. Fluid Mech.*, vol. 360, 1998, pp. 229–248). We are able to explain the unstable behaviour previously found when the power lies in the range $(-4, -8/3)$. Finally we show that this method also has limited advantages when applied to plumes with unsteady source conditions.

Key words: plumes/thermals, stratified flows

1. Introduction

We focus on various steady or quasi-steady straight-sided solutions to the plume equations of Morton, Taylor & Turner (1956) (MTT). The original MTT model consisted of a set of coupled first-order ordinary differential equations for the fluxes of the plume volume, momentum and buoyancy. The system of equations was closed using the entrainment assumption in which the mean horizontal inflow velocity into the plume due to entrainment is proportional to the mean vertical velocity in the plume at that height. This approach differed from the closure scheme of Priestly & Ball (1955) who assumed a conical shape for the plume at all heights with a universal radial growth with height.

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The MTT model has been used to predict the volume flow rate in a plume as a function of height (Baines 1983), the rise height in a linearly stratified environment (Batchelor 1954; MTT), virtual origin corrections for non-ideal source conditions (Morton 1959; Caulfield & Woods 1995; Hunt & Kaye 2001; Scase, Caulfield & Dalziel 2008), the rise height of turbulent fountains (Baines, Turner & Campbell 1990; Kaye & Hunt 2006), and plumes in nonlinearly stratified environments (Caulfield & Woods 1998). More recently, modified versions of the MTT equations have been used to model chemically reacting plumes, in which there is a change in buoyancy flux due to the heat of reaction (Diez & Dahm 2007; Conroy, Smith & Caulfield 2005; Conroy & Llewellyn Smith 2008; Campbell & Cardoso 2010), and plumes with unsteady source conditions (Scase, Caulfield & Dalziel 2006*a*; Scase *et al.* 2006*b*).

We revisit these models and show that the steady straight-sided solutions (i.e. plumes for which the radii of the mean profiles of buoyancy and velocity increase linearly with height) found under certain source conditions can be easily calculated by re-casting the governing equations in terms of the plume local radius, flux balance parameter Γ (see Morton 1959; Morton & Middleton 1973), and buoyancy flux, which is constant. This approach was introduced by Hunt & Kaye (2005) who showed that the classical 'pure plume' solution, $\Gamma = 1$, and the 'pure jet' solution, $\Gamma = 0$, can be read directly from the re-cast equations, and the steady power-law equations for the volume flux can be found without having to solve the governing differential equations. The model was extended in Kaye & Hunt (2006) to predict the rise height of a turbulent fountain for a broad range of source conditions.

The remainder of the paper is structured as follows. In the next section, we review the modelling approach of Hunt & Kaye (2005) and re-state their results for jets and plumes. We then apply this modelling approach to the work of Caulfield & Woods (1998), on plume propagation through non-uniformly stratified environments, and Diez & Dahm (2007), Conroy *et al.* (2005), Conroy & Llewellyn Smith (2008) and Campbell & Cardoso (2010) on chemically reacting plumes. Finally we examine certain properties of the straight-sided solutions of the unsteady plume equations of Scase *et al.* (2006*a*,*b*).

2. Pure jets and plumes

We begin with a brief review of the work of Hunt & Kaye (2005) to demonstrate the analysis technique that will later be used to analyse plumes in stratified environments, chemically reacting plumes and plumes with time varying source conditions. Following MTT, we define a volume flux, momentum flux and buoyancy flux, respectively, as

$$\pi Q = \int_0^{2\pi}\int_0^b wr\,\mathrm{d}r\,\mathrm{d}\theta,\ \ \pi M = \int_0^{2\pi}\int_0^b w^2 r\,\mathrm{d}r\,\mathrm{d}\theta,\ \ \pi F = \int_0^{2\pi}\int_0^b wg'r\,\mathrm{d}r\,\mathrm{d}\theta, \quad (2.1\,a\text{–}c)$$

where r is the radial coordinate and θ is the azimuthal coordinate. We assume top hat profiles where b is the radius of the profile, w is the vertical velocity and g' is the reduced gravity. It should be noted that different authors have used slightly different definitions of Q, M and F. The governing equations describing the evolution of the three fluxes, Q, M and F with height for an unstratified ambient were shown by MTT to be

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\alpha M^{1/2}, \quad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{QF}{M} \quad \text{and} \quad \frac{\mathrm{d}F}{\mathrm{d}z} = 0. \qquad (2.2\,a\text{–}c)$$

The general source conditions are

$$Q|_{z=0} = Q_0, \quad M|_{z=0} = M_0, \quad F|_{z=0} = F_0. \tag{2.3}$$

From these source conditions, two length scales, based upon the source radius and the source jet length, can be defined and are given by

$$L_{Q0} = \frac{5}{6\alpha}\frac{Q_0}{M_0^{1/2}} \quad \text{and} \quad L_{M0} = \left(\frac{10}{9\alpha}\frac{M_0^{3/2}}{F_0}\right)^{1/2}, \tag{2.4}$$

respectively. The ratio of the square of these length scales is the source flux balance parameter, referred to as the source 'laziness',

$$\Gamma_0 = \frac{L_{Q0}^2}{L_{M0}^2} = \frac{5}{8\alpha}\frac{Q_0^2 F_0}{M_0^{5/2}}, \tag{2.5}$$

where $\Gamma_0 = 0$ for a 'pure jet' and $\Gamma_0 = 1$ for a 'pure plume' (see Morton & Middleton 1973). Although Γ has typically been used to characterize the source fluxes (e.g. 'lazy' for $\Gamma_0 > 1$, or 'forced' for $\Gamma_0 < 1$), it can be defined at any height

$$\Gamma(z) = \frac{5}{8\alpha}\frac{Q(z)^2 F(z)}{M(z)^{5/2}}. \tag{2.6}$$

If the ambient fluid is homogeneous, it can be shown that for $\Gamma_0 \neq 0$, as $z \to \infty$, $\Gamma \to 1$, i.e. provided the source conditions are not those of a pure jet, the plume fluxes tend towards those of a pure plume as $z \to \infty$. Far from the source, it is as if a pure plume has originated from a point source at a 'virtual origin' (Hunt & Kaye 2005; Scase *et al.* 2008).

Scaling the fluxes on their source value and height on L_{Q0}, we get the non-dimensional variables

$$q = \frac{Q}{Q_0}, \quad m = \frac{M}{M_0}, \quad f = \frac{F}{F_0} \quad \text{and} \quad \zeta = \frac{z}{L_{Q0}}. \tag{2.7}$$

The radius $b(z)$ and vertical velocity $W(z)$ are also scaled on their source values,

$$\beta = \frac{b}{b_0} = \frac{q}{m^{1/2}} \quad \text{and} \quad w = \frac{W}{W_0} = \frac{m}{q}. \tag{2.8 a, b}$$

Note that the radius scaling is not b/L_{Q0} but $5b/(6\alpha L_{Q0})$. The non-dimensional version of (2.2) is, therefore,

$$\frac{dq}{d\zeta} = \frac{5}{3}m^{1/2}, \quad \frac{dm}{d\zeta} = \frac{4\Gamma_0}{3}\frac{qf}{m} \quad \text{and} \quad \frac{df}{d\zeta} = 0. \tag{2.9 a–c}$$

The local value of Γ may be expressed, relative to the source value as

$$\frac{\Gamma}{\Gamma_0} = \frac{q^2 f}{m^{5/2}}. \tag{2.10}$$

Rewriting (2.9 a, b) in terms of Γ and β rather than q and m, we have

$$\frac{d\beta}{d\zeta} = \frac{1}{3}(5 - 2\Gamma), \quad \frac{d\Gamma}{d\zeta} = \frac{10\Gamma}{3\beta}(1 - \Gamma). \tag{2.11 a, b}$$

By inspection, we see that there are two possible constant Γ solutions to (2.11), namely the pure jet ($\Gamma = 0$, $b = 2\alpha z$), and the pure plume ($\Gamma = 1$, $b = 6\alpha z/5$). To demonstrate the value of this approach in analysing plume equations, we present analysis of the governing equations applied to plumes in a stratified environment, chemically reacting plumes and plumes with unsteady source fluxes, in which we re-cast the equations in terms of Γ, β and any additional dimensionless parameters required to close the problem.

3. Plumes in a stratified ambient

The solution for the finite rise height of a plume in a linearly stratified (constant buoyancy frequency) ambient environment was solved numerically by MTT and analytically by Scase *et al.* (2006*a*). However, it was shown by Caulfield & Woods (1998) that it is possible to attain straight-sided solutions to the governing equations for plumes propagating through stratified ambient environments in which there is a power-law decay in the buoyancy frequency of the ambient with height. Further, Batchelor (1954) showed that solutions also exist for an unstable power-law stratification and a zero initial buoyancy flux. In both cases, the governing equations become

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\alpha M^{1/2}, \quad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{QF}{M} \quad \text{and} \quad \frac{\mathrm{d}F}{\mathrm{d}z} = -QN_0^2\left(\frac{z}{z_s}\right)^{\kappa}, \tag{3.1 a–c}$$

where z_s is the height of the plume source and N_0 is the buoyancy frequency at the source height. Caulfield & Woods (1998) considered the case of $N_0^2 > 0$ and Batchelor (1954) considered $N_0^2 < 0$.

We define a new dimensionless source parameter

$$\Pi_0 = \frac{5}{6\alpha}\frac{Q_0^2 N_0^2}{F_0 M_0^{1/2}}\left(\frac{L_{Q0}}{z_s}\right)^{\kappa} \tag{3.2}$$

that represents the rate at which the stratification decays relative to the source volume flux length scale. Again, this parameter can be defined at any height as

$$\frac{\Pi}{\Pi_0} = \frac{q^2}{fm^{1/2}}. \tag{3.3}$$

The buoyancy flux equation (3.1*c*) may, therefore, be rewritten as

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = -\Pi_0 q\zeta^{\kappa}. \tag{3.4}$$

Following the same analysis technique as the previous section, we re-cast (3.1) in terms of Γ, β and Π to get

$$\frac{\mathrm{d}\beta}{\mathrm{d}\zeta} = \frac{1}{3}(5-2\Gamma), \quad \frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \frac{\Gamma}{3\beta}[10(1-\Gamma) - 3\Pi\zeta^{\kappa}], \quad \frac{\mathrm{d}\Pi}{\mathrm{d}\zeta} = \frac{\Pi}{3\beta}[10 - 2\Gamma + 3\Pi\zeta^{\kappa}]. \tag{3.5 a–c}$$

For all modifications to the steady plume equation systems that we consider, it is only the buoyancy flux equation that changes between systems. Therefore, the equation for the plume radius (3.5*a*) is unchanged as $\beta = q/m^{1/2}$ and the volume flux and momentum flux equations are unchanged. This implies that any straight-sided solution to modified versions of the MTT equations considered herein will have a constant Γ value. Therefore, any solution to $\mathrm{d}\Gamma/\mathrm{d}\zeta = 0$ will be straight sided.

We, therefore, seek solutions that tend to constant values of Γ as $\zeta \to \infty$. That is $\Gamma \to \Gamma_s$, say, and $\mathrm{d}\Gamma/\mathrm{d}\zeta \to 0$ as $\zeta \to \infty$. It is not necessarily true that $\Gamma_s = \Gamma_0$. Hence, for large ζ

$$3\Pi\zeta^{\kappa} = 10(1-\Gamma_s) \quad \text{and} \quad \beta = \beta_s + \tfrac{1}{3}(5-2\Gamma_s)\zeta, \tag{3.6 a, b}$$

therefore substituting into (3.5c) and integrating yields

$$\frac{\Pi}{\Pi_s} = \left(1 + \frac{5-2\Gamma}{3\beta_s}\zeta\right)^{(20-12\Gamma_s)/(5-2\Gamma_s)}. \tag{3.7}$$

In order for (3.7) to be consistent with (3.6) as $\zeta \to \infty$, substitution of (3.7) into (3.5b) requires

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \frac{\Gamma_s}{3\beta_s}\left\{10\,(1-\Gamma_s) - 3\Pi_s\left(1 + \frac{5-2\Gamma_s}{3\beta_s}\zeta\right)^{(20-12\Gamma_s)/(5-2\Gamma_s)}\zeta^{\kappa}\right\}. \tag{3.8}$$

For consistency, we require the right-hand side of (3.8) to tend to zero as $\zeta \to \infty$, such that Γ tends to a constant value. This can only occur when

$$\kappa \leqslant -\frac{20-12\Gamma_s}{5-2\Gamma_s} = \kappa_c. \tag{3.9}$$

Two possible solution sets exist, $\kappa = \kappa_c$ and $\kappa < \kappa_c$. When $\kappa < \kappa_c$ the second term in the curly brackets on the right-hand side of (3.8) tends to zero as $\zeta \to \infty$, $\Gamma \to \Gamma_s = 1$, and $\kappa_c = -8/3$. For $\kappa = \kappa_c$, we get the condition that

$$\Pi_s = \frac{10}{3}(1-\Gamma_s)\left(\frac{3\beta_s}{5-2\Gamma_s}\right)^{\kappa_c}. \tag{3.10}$$

Straight-sided solutions can only occur for $0 \leqslant \Gamma_s < 5/2$ as negative Γ yields fountains with finite rise heights, while for $\Gamma \geqslant 5/2$ it follows that $\mathrm{d}\beta/\mathrm{d}\zeta \leqslant 0$ and the plume is not conical in shape. Using '$\sim$' to denote 'scales as', given $\Pi \sim \zeta^{-\kappa_c}$ from (3.6a), and $\beta = q/m^{1/2} \sim \zeta$ from (2.8a) it follows from (3.3) that we can write

$$\Pi \propto \frac{q^2}{f\,m^{1/2}} \sim \zeta^{-\kappa_c} \quad \text{or} \quad f \sim q\,\zeta^{1+\kappa_c}. \tag{3.11}$$

We seek solutions to the fluxes of volume, momentum and buoyancy that are power-law functions of height. That is,

$$q \sim \zeta^{\phi_q}, \quad m \sim \zeta^{\phi_m} \quad \text{and} \quad f \sim \zeta^{\phi_f}. \tag{3.12}$$

Solution of (3.11) and (3.12) leads to

$$\phi_q = \frac{\kappa_c + 6}{2}, \quad \phi_m = \kappa_c + 4 \quad \text{and} \quad \phi_f = \frac{3\kappa_c + 8}{2} \tag{3.13}$$

(cf. (4.3) in Caulfield & Woods 1998). A plot of κ_c and the flux power-law exponents ϕ_q, ϕ_m and ϕ_f, as a function of Γ_s are shown in figure 1.

The restriction that $0 \leqslant \Gamma_s < 5/2$ results in the full range of possible κ_c values found by Caulfield & Woods (1998) and Batchelor (1954), namely $-4 \leqslant \kappa_c$. For $\kappa_c < -4$ we have that $\phi_m < 0$ and, therefore, the momentum flux decreases with height so the plume must eventually stall. For $\Gamma = 1$, $\kappa_c = -8/3$, f is constant with height ($\phi_f = 0$) and the pure plume solution is attained. However, for $\Gamma_s < 1$ ($\phi_f < 0$), the buoyancy flux decreases with height and, therefore, $\Pi_0 > 0$ as considered by Caulfield & Woods (1998). For $\Gamma_s > 1$, the buoyancy flux increases with height ($\phi_f > 0$) and $\Pi_0 < 0$ as considered by Batchelor (1954). It is interesting to note that Caulfield & Woods (1998) and Batchelor (1954) are entirely consistent and that there is a smooth transition in all parameters over the range $0 \leqslant \Gamma_s < 5/2$ with only a change in the sign of Π_0 at $\Gamma_s = 1$ as shown in figure 1.

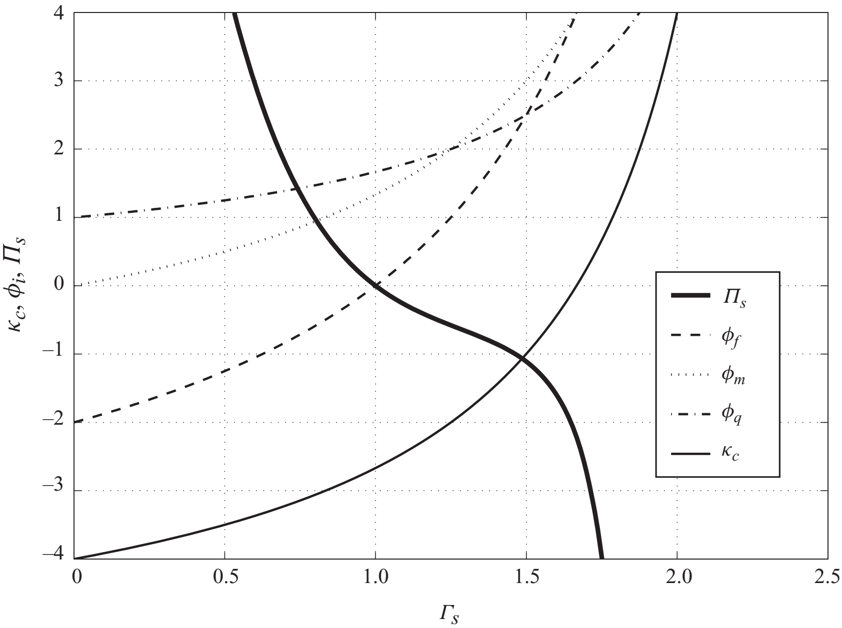

FIGURE 1. Plot of the power-law exponents for the stratification, fluxes of momentum, buoyancy and volume and the critical initial value of Π_0 as a function of Γ.

Specific results from Caulfield & Woods (1998) and Batchelor (1954) can also be established. When $\kappa_c = -4$, we recover the $\Gamma_s = 0$ jet solution. If $\kappa_c = -8/3$ the pure plume $\Gamma_s = 1$ solution is attained and Π_0 can be either positive or negative. Further, taking the example given in Batchelor (1954), if $\kappa_c = -1$ then the exponents for q and f are equal ($\phi_q = \phi_f = 5/2$) implying that the buoyancy $g' = f/q$ is constant with height.

Finally, Caulfield & Woods (1998) showed numerically that the $\kappa_c = (20-12\Gamma_s)/(5-2\Gamma_s)$ solution was unstable. Slight variations in source conditions lead the solution to diverge. They also established numerically the condition under which the solution exists. The solution approach presented herein leads to an analytical solution for the stability condition, specifically (3.10).

4. Chemically reacting plumes

We now consider plumes with internally generated buoyancy. This problem has been addressed in the context of buoyancy generated by phase changes (Bhat & Narasimha 1996; Basu & Narasimha 1999) and buoyancy generated by chemical reactions (Conroy *et al.* 2005; Diez & Dahm 2007; Conroy & Llewellyn Smith 2008; Campbell & Cardoso 2010). Buoyancy generated by latent heat release in clouds was analysed experimentally by Bhat & Narasimha (1996) who ran experiments in which the buoyancy flux of a plume was increased linearly with height. The steady solution to the MTT equations for a linear increase in buoyancy flux with height was solved by Hunt & Kaye (2005) who showed that the resulting plume is narrower than a pure plume ($\Gamma = 5/4$) and has a constant velocity.

Various models have been proposed for chemically reacting plumes in which one of the reactants is supplied in the plume and a second is in the surrounding ambient (Conroy *et al.* 2005; Diez & Dahm 2007; Conroy & Llewellyn Smith 2008). In this case, additional equations are required to keep track of the two reactants and the reaction product. Buoyancy flux terms are added to account for heat release and changes in fluid density due to the species change in the reaction (Conroy & Llewellyn Smith 2008). Various flows can develop depending on the speed of the reaction relative to the rate at which reactants are mixed across the entire plume width and the rate at which fluid is transported vertically within the plume. If the reaction rate is fast compared to the time taken to mix the reactants across the whole width of the plume then the integral modelling approach is invalid (see Cardoso & McHugh 2010). If the reaction rate is slower than this then the change in buoyancy flux is controlled by the rate of mixing of the reactants, assumed equivalent to the rate of entrainment $\mathrm{d}Q/\mathrm{d}z$. Both Diez & Dahm (2007) and Conroy & Llewellyn Smith (2008) present the buoyancy flux equation to be of the form

$$\frac{\mathrm{d}F}{\mathrm{d}z} = h\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\alpha h M^{1/2}, \tag{4.1}$$

where h is a constant and a property of the reaction. In general, h can be either positive (exothermic) or negative (endothermic). Straight-sided solutions are possible for upward flowing plumes with $h > 0$ and downward flowing plumes with $h < 0$. Equation (4.1) results in another dimensionless source parameter

$$\varPhi_0 = h\frac{Q_0}{F_0}, \tag{4.2}$$

where the local value of $\varPhi$ is given by $\varPhi = \varPhi_0 q/f$.

The plume equations written in terms of $\varGamma$, β and $\varPhi$ are

$$\frac{\mathrm{d}\beta}{\mathrm{d}\zeta} = \frac{1}{3}(5 - 2\varGamma), \quad \frac{\mathrm{d}\varGamma}{\mathrm{d}\zeta} = \frac{5\varGamma}{3\beta}(2\,[1-\varGamma] + \varPhi), \quad \frac{\mathrm{d}\varPhi}{\mathrm{d}\zeta} = \frac{5\varPhi}{3\beta}(1-\varPhi). \tag{4.3 a–c}$$

A straight-sided plume will develop such that $\varPhi = 1$ and $\varGamma = 3/2$ provided there is sufficient reactant available and the density change due to species change during the reaction is small compared to the density change due to heat release. These constraints also allow us to ignore the additional equations that track individual species concentrations. The $\varGamma = 3/2$ flow is an accelerating plume that is narrower than a pure plume ($\varGamma = 1$). Conroy & Llewellyn Smith (2008) presented an analytical solution for MTT equations for a high reaction rate chemically reacting plume in terms of gamma functions and found the same result, that is a steady plume with $\varGamma = 3/2$. This result is different to the model of Diez & Dahm (2007) as they assumed that the radial growth rate of the plume is constant and equal to that of a pure plume, as opposed to making the entrainment assumption. The approach presented allows the solution to be established without integration but simply by solution of a pair of linear equations in $\varPhi$ and $\varGamma$.

Cardoso & McHugh (2010) also consider plumes that undergo chemical reactions as they propagate. As above, they consider a plume whose buoyancy flux may vary as a result of a reaction between the plume fluid and the ambient. The reaction they consider is controlled by a parameter, G, which describes the generation of buoyancy within the plume. Their system has similar governing volume and momentum flux equations to the systems described above, but their buoyancy flux equation (see Cardoso & McHugh 2010, (2.17)) is

$$\frac{\mathrm{d}F}{\mathrm{d}z} = -N_0^2 Q + G F_0 \frac{Q}{M} \tag{4.4}$$

(their factor γ may be scaled out) where N_0 is the constant buoyancy frequency of the ambient fluid.

We introduce the non-dimensional quantity Λ, a ratio of the reaction rate to the local plume velocity, defined by $\Lambda = L_{Q0} G Q/M = \Lambda_0 q/m$. The governing system may then be written as

$$\frac{\mathrm{d}\beta}{\mathrm{d}\zeta} = \frac{1}{3}(5-2\Gamma), \quad \frac{\mathrm{d}\Lambda}{\mathrm{d}\zeta} = \frac{\Lambda}{3\beta}(5-4\Gamma), \quad \frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \frac{10\Gamma}{3\beta}(1-\Gamma)+\frac{\Gamma_0}{\Lambda_0^3}\frac{\Lambda^2}{\beta}(\Lambda^2-\Lambda_0\Pi_0\beta^2). \tag{4.5 a–c}$$

For straight-sided solutions we again require constant Γ, and so it follows that the right-hand side of (4.5c) must vanish. It can be seen that in the case of a homogeneous ambient, no stratification $\Pi_0 = 0$, it must also be necessary to have constant Λ. The solution is given by $\Gamma = 5/4$, $\Lambda = 5/6$, and $\beta = \beta_s + (5/6)\zeta$. This corresponds to the Bhat & Narasimha (1996) plume, with constant velocity, and a linear increase in buoyancy flux with height. As above, as $\zeta \to \infty$ the plume tends towards a straight-sided solution and $\Gamma \to 5/4$, $\Lambda \to 5/6$. This would have been observed in figure 1 of Cardoso & McHugh (2010) had they continued their calculation out to $\hat{Z} \approx 25$ (their notation), although this was far out of the range of their experiments for practical reasons. In the case of a stratified ambient, straight-sided point source solutions are only possible when N_0^2 has the exact $z^{-4/3}$ dependence required.

5. Plumes with time varying buoyancy flux

Recently, plumes whose source conditions may change in time have received attention. In many real-life applications the strength, or indeed the size, of a plume's source may vary significantly in time. For example, a catastrophic failure or plugging of a volcanic vent (see e.g. Houghton *et al.* 2004) or volcanic eruption, or the ignition of fuel storage containers (see e.g. Johnson *et al.* 1991; Mather *et al.* 2007). In order to model such unsteady plumes Scase *et al.* (2006b) rederived the steady model of MTT, using the vertical Euler equation as the starting point, preserving the terms involving temporal gradients. In so doing, they derived a system of three partial differential equations for the mass, momentum and buoyancy fluxes. Specifically, it was shown that

$$\frac{\partial}{\partial t}\left(\frac{Q^2}{M}\right) + \frac{\partial Q}{\partial z} = 2\alpha M^{1/2}, \quad \frac{\partial Q}{\partial t} + \frac{\partial M}{\partial z} = \frac{QF}{M}, \quad \frac{\partial}{\partial t}\left(\frac{QF}{M}\right) + \frac{\partial F}{\partial z} = 0. \tag{5.1 a–c}$$

These equations may be non-dimensionalized based upon a reference source buoyancy flux, as described in Scase *et al.* (2008), yielding identical equations except that the coefficient of $M^{1/2}$ in the right-hand side of (5.1a) is replaced by 1. As in the previous sections, we re-cast the equations in terms of a non-dimensional plume radius, β, vertical velocity, ω, and the plume laziness parameter Γ. The same non-dimensionalization as used earlier is employed, but a scale is required for time and we take $T_{Q0} = 5Q_0^2 M_0^{-3/2}/(6\alpha)$. This particular non-dimensionalization is only valid when $Q_0 \neq 0$ and $M_0 \neq 0$. Writing $\boldsymbol{\beta} = (\beta, \omega, \Gamma)^{\mathrm{T}}$, we find

$$\frac{\partial \boldsymbol{\beta}}{\partial \tau} + \begin{pmatrix} \omega & \beta/2 & 0 \\ 0 & \omega & 0 \\ 0 & \Gamma/2 & \omega \end{pmatrix} \frac{\partial \boldsymbol{\beta}}{\partial \zeta} = \frac{\omega}{6\beta}\begin{pmatrix} 5\beta \\ 2\omega(4\Gamma-5) \\ -\Gamma(16\Gamma-15) \end{pmatrix}. \tag{5.2}$$

The advantage of casting the system (5.1) in this form is that it is straightforward to show that there are only three sets of solutions for straight-sided plumes with

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constant laziness. This is demonstrated by substituting $\Gamma(z,t)=\Gamma$ is constant and $\beta \propto \zeta$ into (5.2). The first and third rows immediately yield that either $\Gamma=0$, the jet solution, or the constant of proportionality for β is $(8\Gamma-5)/3$. It then follows from the third row that $\omega=f(\tau)\zeta^{-(16\Gamma-15)/(8\Gamma-5)}$. Substitution back into the second row reveals

$$\frac{\mathrm{d}f}{\mathrm{d}\tau}=\frac{20(\Gamma-1)}{8\Gamma-5}\zeta^{-4(6\Gamma-5)/(8\Gamma-5)}f(\tau)^2. \tag{5.3}$$

Solutions to (5.3) only exist when there is no ζ dependence on the right-hand side, i.e. for $\Gamma=1$, the classical steady plume, or $\Gamma=5/6$, the neck.

Alternatively, following the approach of Scase *et al.* (2006*b*), we seek a straight-sided solution

$$\Gamma=\Gamma_s, \quad \beta=C\zeta \quad \text{and} \quad w=A\zeta^a\tau^b \tag{5.4}$$

subject to the condition $b\neq 0$. Substituting into the second row of (5.2) yields the two exponents $a=1$ and $b=-1$. The constants can then be evaluated explicitly using, in order, rows 1, 3 and then 2 to yield

$$\beta=\frac{5}{9}\zeta, \quad \Gamma=\frac{5}{6} \quad \text{and} \quad w=\frac{\zeta}{2\tau} \tag{5.5}$$

as found by Scase *et al.* (2006*b*). While the analysis approach is similar to that of Scase *et al.* (2006*b*), the re-cast equations only require the evaluation of two exponents and three pre-factors, rather than six exponents and three pre-factors. This is because we are able to take advantage of the fact that the solution will have a constant Γ value and be straight sided. No such simplifying assumptions can be made when directly seeking solutions to the fluxes of volume, momentum and buoyancy.

6. Conclusions

One motivation for expressing the governing plume equations in terms of a volume, momentum and buoyancy flux is that these quantities lend themselves to accurate measurement in both the laboratory and the field. These quantities can be accurately measured not only at the source of the plume, but also throughout the vertical extent of the plume. However, as first demonstrated by Hunt & Kaye (2005), these are not necessarily the most helpful quantities to work with analytically. By re-casting the equations in terms of plume radius, vertical velocity and laziness, strong statements and significant physical insight, about the behaviour of the plume can be made without any solution of the governing equations being required.

We have presented analytical solutions for the range of background stratification power-law decay rates for which straight-sided plumes are possible. This demonstrates the usefulness of the technique presented here, as this result has only previously been fully attained by numerical search.

It has been shown that the models of Batchelor (1954) and Caulfield & Woods (1998) are two halves of the same continuum of solutions, and there is a smooth transition between the two solution sets across the critical decay rate of $\kappa=-8/3$. For $\kappa<-8/3$ the buoyancy flux of a straight-sided plume decays with height, and so for the plume to continue rising, the stratification must be stable. Conversely, for $\kappa>-8/3$ the buoyancy flux of a straight-sided plume increases with height, and hence an unstable stratification is required.

Similarly, previous results for chemically reacting plumes are recovered straightforwardly. However, for the case of the unsteady plume model, the method of re-casting the equations into plume radius, vertical velocity and plume laziness results in only modest simplifications.

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REFERENCES

BAINES, W. D. 1983 A technique for the direct measurement of volume flux of a plume. *J. Fluid Mech.* **132**, 247–256.

BAINES, W. D., TURNER, J. S. & CAMPBELL, I. H. 1990 Turbulent fountains in an open chamber. *J. Fluid Mech.* **212**, 557–592.

BASU, A. J. & NARASIMHA, R. 1999 Direct numerical simulation of turbulent flows with cloud-like off-source heating. *J. Fluid Mech.* **385**, 199–228.

BATCHELOR, G. K. 1954 Heat convection and buoyancy effects in fluids. *Q. J. R. Meteorol. Soc.* **80**, 339–358.

BHAT, G. S. & NARASIMHA, R. 1996 A volumetrically heated jet: large-eddy structure and entrainment characteristics. *J. Fluid Mech.* **325**, 303–330.

CAMPBELL, A. & CARDOSO, S. 2010 Turbulent plumes with internal generation of buoyancy by chemical reaction. *J. Fluid Mech.* **655**, 122–151.

CARDOSO, S. S. & MCHUGH, S. T. 2010 Turbulent plumes with heterogeneous chemical reaction on the surface of small buoyant droplets. *J. Fluid Mech.* **642**, 49–77.

CAULFIELD, C. P. & WOODS, A. W. 1995 Plumes with non-monotonic mixing behaviour. *Geophys. Astrophys. Fluid Dyn.* **79**, 173–199.

CAULFIELD, C. P. & WOODS, A. W. 1998 Turbulent gravitational convection from a point source in a non-uniformly stratified environment. *J. Fluid Mech.* **360**, 229–248.

CONROY, D. T. & LLEWELLYN SMITH, S. G. 2008 Endothermic and exothermic chemically reacting plumes. *J. Fluid Mech.* **612**, 291–310.

CONROY, D. T., SMITH, S. G. L. & CAULFIELD, C. P. 2005 Evolution of a chemically reacting plume in a ventilated room. *J. Fluid Mech.* **537**, 221–253.

DIEZ, F. J. & DAHM, W. J. A. 2007 Effects of heat release on turbulent shear flows. Part 3. Buoyancy effects due to heat release in jets and plumes. *J. Fluid Mech.* **575**, 221–255.

HUNT, G. R. & KAYE, N. B. 2005 Lazy plumes. *J. Fluid Mech.* **533**, 329–338.

HUNT, G. R. & KAYE, N. G. 2001 Virtual origin correction for lazy turbulent plumes. *J. Fluid Mech.* **435**, 377–396.

KAYE, N. B. & HUNT, G. R. 2006 Weak fountains. *J. Fluid Mech.* **558**, 319–328.

MATHER, T. A., HARRISON, R. G., TSANEV, V. I., PYLE, D. M., KARUMUDI, M. L., BENNETT, A. J., SAWYER, G. M. & HIGHWOOD, E. J. 2007 Observations of the plume generated by the December 2005 oil depot explosions and prolonged fire at Buncefield (Hertfordshire, UK) and associated atmospheric changes. *Proc. R. Soc. Lond.* A **463** (2081), 1153–1177.

MORTON, B. R. 1959 Forced plumes. *J. Fluid Mech.* **5**, 151–163.

MORTON, B. R. & MIDDLETON, J. 1973 Scale diagrams for forced plumes. *J. Fluid Mech.* **58**, 165–176.

MORTON, B. R., TAYLOR, G. I. & TURNER, J. S. 1956 Turbulent gravitational convection from maintained and instantaneous sources. *Proc. R. Soc. Lond.* A **234**, 1–23.

PRIESTLY, C. H. B. & BALL, F. K. 1955 Continuous convection from an isolated source of heat. *Q. J. R. Meteorol. Soc.* **81**, 144–157.

SCASE, M. M., CAULFIELD, C. P. & DALZIEL, S. B. 2006*a* Boussinesq plumes and jets with decreasing source strengths in stratified environments. *J. Fluid Mech.* **563**, 463–472.

SCASE, M. M., CAULFIELD, C. P. & DALZIEL, S. B. 2008 Temporal variation of non-ideal plumes with sudden reductions in buoyancy flux. *J. Fluid Mech.* **600**, 181–199.

SCASE, M. M., CAULFIELD, C. P., DALZIEL, S. B. & HUNT, J. C. R. 2006*b* Time-dependent plumes and jets with decreasing source strengths. *J. Fluid Mech.* **563**, 443–461.