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PUBLIC DEBT AND AGGREGATE RISK

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In this paper, we investigate the importance of aggregate fluctuations for the assessment of the optimal level of public debt in an incomplete-markets economy. We start by building a steady state model in which households are only subject to uninsurable idiosyncratic risk and evaluate the optimal level of public debt. We then augment the model to allow for aggregate risk and measure the impact on the optimal level. We show that the cyclical behavior of the economy has a quantitative impact on this level that can be decomposed into the effects of the aggregate productivity shock and the cyclicality of unemployment. Moreover, we find that matching wealth distribution statistics substantially changes the optimal level of public debt.

Keywords: Public Debt, Aggregate Risk, Precautionary Saving, Credit Constraints

1. INTRODUCTION

In this paper, we investigate the importance of aggregate fluctuations for the assessment of the optimal level of public debt. The setting used to conduct this analysis is an incomplete-markets economy where agents are subject to both uninsurable idiosyncratic risk and aggregate risk. Incomplete-markets models in which agents face uninsurable income risk have been used to assess to what extent the government should issue public debt [e.g., Aiyagari and McGrattan (1998) or Floden (2001)]. However, to the best of our knowledge, there have been surprisingly few attempts so far to analyze the impact of aggregate risk on the optimal level of public debt in such a non-Ricardian setting. In this paper, we quantify the optimal level of public debt in the presence of aggregate fluctuations and analyze the impact of matching wealth distribution characteristics on our quantification. We also conduct distributional and business cycle effects analysis in this economy.

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We use a minimal setup to address the issues raised above. Following the Bewley (1986)–Huggett (1993)–Aiyagari (1994) class of models, we first build a steady state production economy with capital market imperfections where a large number of ex ante identical infinitely-lived agents face idiosyncratic income shocks. Households are borrowing-constrained, and their saving behavior is influenced by a precautionary saving motive that helps to smooth consumption. Public debt and private capital assets can be claimed to insure against future risk. This steady state model is then augmented to exhibit aggregate fluctuations, introduced following the approach of Den Haan (1996) and Krusell and Smith (1998).

A key feature of our model is the cyclical behavior of the economy and its correlation with the idiosyncratic labor market risk. The literature that has reexamined the welfare cost of business cycles in models that feature both aggregate and idiosyncratic risk reveals that aggregate risk can have asymmetric welfare effects across the population.[1] In such a setup, the welfare cost of aggregate risk is much higher than in the seminal paper of Lucas (1987). Moreover, Storesletten et al. (2004) report that idiosyncratic risk is strongly countercyclical: the variance of idiosyncratic labor income risk is much greater in recessions than in expansions. As aggregate risk exacerbates uninsurable idiosyncratic income risk, the motive for precautionary saving is likely to increase. Moreover, because saving decisions depend on prices, the cost of precautionary saving changes as prices fluctuate along the cycle. Both the cost and the motive for precautionary saving impact the optimal level of public debt, and their interplay with the business cycle might have been underestimated by the previous literature.

The main findings of the paper are that the cyclical behavior of the economy has a quantitative impact on the optimal level of public debt that can be decomposed and that matching wealth distribution characteristics significantly changes that optimal level. Our baseline steady state model displays a negative optimal level of public debt that contrasts with the standard result found in Aiyagari and McGrattan (1998).[2] We identify the source of this discrepancy to be the exogenous growth assumption in that paper. Next, we show that introducing aggregate risk into the economy increases the optimal level of public debt. This level is still significantly negative. We then decompose and quantify the direct impact of the aggregate technology shock and contrast it with the impact of the cyclicality of the unemployment process. We argue that the number of wealth-rich and wealth-poor households in the economy is an important element to consider when quantifying the optimal level of public debt that has not been assessed by the previous literature. We therefore introduce a preference heterogeneity setting into the model in order to match wealth distribution characteristics. We compute that the optimal level of public debt in a model with aggregate fluctuations matching the U.S. wealth distribution is now positive and amounts to 5% of annual gross domestic product (GDP). The welfare gain of being at this optimal level instead of the calibration benchmark of 67% is 0.25% of consumption. This average gain contrasts with large variations across the population when a distributional decomposition is conducted. Moreover, we show that the reported optimal level can change substantially when the characteristics of the business cycle, particularly the unemployment duration and the unemployment rate, are altered. Finally, we also document the impact on the optimal level of public debt of adding an exogenous growth factor as in Aiyagari and McGrattan (1998).

A vast literature explores the implications of public debt, starting from papers studying optimal taxation issues. Barro (1979) builds a public debt theory in which there are aggregate risk and a deadweight cost to tax. Lucas and Stokey (1983) characterize optimal fiscal policy in a Ramsey model where government expenditures follow a stochastic process and public debt is state-contingent. Chari et al. (1994) recast the optimal taxation problem in the Lucas and Stokey (1983) model extended to allow for capital accumulation and business cycles. Aiyagari et al. (2002) revisit the result of Lucas and Stokey (1983) in a setting with only risk-free one-period debt. Finally, Farhi (2010) reconsiders the results of Aiyagari et al. (2002) in a model with capital accumulation and capital taxation. In this class of models, either households are fully insured against idiosyncratic income risk or no such risk is considered. Therefore, these models do not take precautionary saving into account. Precautionary saving is likely to be an important source of aggregate wealth accumulation, as has been documented by Gourinchas and Parker (2002), Cagetti (2003), and Fuchs-Schundeln and Schundeln (2005). Our paper is more closely related to the literature analyzing the impact of public debt in models that depart from the representative agent assumption. In a liquidity-constrained economy, Woodford (1990) shows that raising the level of public debt may be welfare-enhancing. Aiyagari and McGrattan (1998) address the question of the optimal level of public debt in an incomplete-markets economy where agents face idiosyncratic labor productivity risk. They find the optimal level of public debt to be positive, even though welfare gains appear to be very small. Floden (2001) extends the latter model to take public transfers into account and conducts a detailed risk-sharing and welfare analysis. Desbonnet and Weitzenblum (2012) emphasize the short-run effects of an increase in public debt. In this strand of the literature, there is, however, no aggregate risk. Shin (2006) can be set apart from the preceding two strands of the literature, as the author attempts to bridge the gap between them. His paper studies optimal fiscal policy in an incomplete-markets model. However, our analysis departs from Shin (2006), as the nature of aggregate uncertainty is different in our model. Shin (2006) depicts an aggregate tax-smoothing motive due to a government expenditure shock. In such an environment, the government may find it optimal to save for a precautionary motive in order to smooth tax distortions. We emphasize a different source for aggregate fluctuations in the economy: a productivity shock that exacerbates individual unemployment risk. When the variability of the government expenditures shock increases in Shin (2006), the government is willing to accumulate assets, whereas in our setting, the government is willing to increase public debt to help households smooth consumption. Finally, Heathcote (2005) studies an economy with heterogeneous agents, idiosyncratic income risk, and aggregate risk coming from the tax rate. Nevertheless, the scope of his analysis differs: he investigates the effect of a temporary tax cut financed by debt, and there are no business cycles in the way defined in our paper.

The rest of the paper is organized as follows. In the next section, we describe a baseline steady state economy that we use as a building block. The following section describes the aggregate risk model. The last section concludes.

2. STEADY STATE MODEL

In this section, we build a steady state model to explain the basic mechanisms we want to outline when public debt is introduced. There is no aggregate risk in this model, and we describe the stationary equilibrium associated with it. This economy is a Bewley (1986)–Huggett (1993)–Aiyagari (1994)–type dynamic stochastic general equilibrium model with public debt. Markets are incomplete. Agents face idiosyncratic risk and are borrowing-constrained. These assumptions lead agents into precautionary saving [Aiyagari (1994)].

2.1. Model Specification

Firms. We assume that there is a continuum of firms with a neoclassical production technology that behave competitively in product and factor markets. The output is given by

$$Y_t = F(K_t, N_t),$$

where K is aggregate capital, N aggregate labor, and F the technology. The function F exhibits constant returns to scale with respect to K and N, has positive and strictly diminishing marginal products, and satisfies the Inada conditions. Capital depreciates at a constant rate δ.

Because input markets are competitive, the wage w and the interest rate r satisfy

$$r_t + \delta = F_K(K_t, N_t), \tag{1}$$

$$w_t = F_N(K_t, N_t). \tag{2}$$

The government. The government issues public debt and levies taxes to finance public expenses. Both capital and labor income are taxed proportionally at an identical rate τ. The government's budget constraint satisfies

$$G_t + r_t B_t = B_{t+1} - B_t + T_t,$$

and total taxes can be computed as follows:

$$T_t = \tau_t (w_t N_t + r_t A_t).$$

G_t is the level of public expenses, B_t the level of public debt, and T_t tax revenues. A_t accounts for total average wealth in the economy. It is the sum of average physical capital K and public debt B, such that $A_t = K_t + B_t$.

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This budget constraint has the following simpler representation in the steady state:

$$G + rB = T.$$

Households. This economy is populated by a continuum of ex ante identical, infinitely lived households of unit mass. Their preferences are summarized by the function V:

$$V = \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\}. \tag{3}$$

c_t is household consumption and β is the discount factor. Agents are subject to idiosyncratic unemployment shocks. Let s be the household's labor market status. A household can either be unemployed ($s = u$) or employed ($s = e$). When agents are in the employed state, they receive the wage w. However, when agents are unemployed, their income corresponds to their home production, which we denote as θ. Markets are incomplete, so agents can only partially self-insure against idiosyncratic risk. Following Aiyagari and McGrattan (1998), no borrowing is allowed. The only way for households to self-insure against idiosyncratic risk is to accumulate physical capital and government bonds, both yielding the same return r. We do not distinguish at the household level holdings of each separate asset, and we denote overall holdings as a. Therefore the recursive problem of a household is

$$v(a, s) = \max_{c,\, a'} \{u(c) + \beta E\left[v(a', s')|s\right]\}, \tag{4}$$

subject to

$$c + a' = (1 + r(1 - \tau))a + \chi(s), \tag{5}$$

$$c \geq 0, \tag{6}$$

$$a' \geq 0, \tag{7}$$

with

$$\chi(s) = \begin{cases} \theta \text{ if } s = u, \\ (1 - \tau)w \text{ if } s = e. \end{cases}$$

Equilibrium. The recursive equilibrium consists of a set of decision rules for consumption and asset holding $\left\{c(a, s), a'(a, s)\right\}$, aggregate capital and labor $\{K, N\}$, and factor prices $\{r, w\}$ and a tax rate τ satisfying the following conditions:

1. Given the prices $\{r, w\}$ and the tax rate τ, the decision rules $\{c(a, s), a'(a, s)\}$ solve the dynamic programming problem (4) subject to the constraints (5), (6), and (7).
2. Market price arrangements are as follows:

$$r = F_K(K_t, N_t) - \delta,$$

$$w = F_N(K_t, N_t).$$

3. The government budget is balanced.

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4. The capital market clears when

$$K + B = \int a'(a, s) d\Gamma(a, s),$$

with $\Gamma(a, s)$ the distribution of agents over asset holdings and employment status.

Calibration. The model economy is calibrated to match certain observations in the U.S. data. We let one period in the model be one quarter in the data.

Technology and preferences: We choose the production function to be Cobb–Douglas:

$$Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}, \quad 0 < \alpha < 1.$$

Technology parameters are standard. The capital share of output α is set to 0.36 and the capital depreciation rate δ is 0.025.

We assume that utility is log, so that

$$u(c_t) = \log(c_t).$$

We set the discount rate β to target a quarterly capital–output ratio of 10.6.[3] The associated discount factor is 0.99396.

Labor market process: The logic behind the calibration of the labor market process is relative to the aggregate risk model detailed in a later section. But we provide the intuition behind our approach here. It is important for comparison purposes that the unemployment rate and unemployment duration in the steady state model be equal to their unconditional mean with respect to their aggregate risk counterparts, as described in Imrohoroglu (1989). Thus, the unemployment rate u is set to 7%, which is the average between the unemployment rate in recessions (10%) and in expansions (4%). We follow the same approach for the unemployment duration duru. It is set to two quarters, which corresponds to the average of the duration of an unemployment spell in recession (2.5 quarters) and in expansion (1.5 quarters). These two assumptions define the transition matrix π:

$$\begin{cases} \pi_{uu} + \pi_{ue} = 1 \\ \pi_{eu} + \pi_{ee} = 1 \\ \pi_{ue} = \frac{1}{\text{duru}} \\ \pi_{ee} = 1 - \frac{u\pi_{ue}}{1-u} \end{cases} \Longrightarrow \pi = \begin{pmatrix} 0.5 & 0.5 \\ 0.0376 & 0.9624 \end{pmatrix}.$$

Note that we assume that unemployed agents receive income too and fix the home production income θ to be 0.10.[4]

Government: We set the (long-run) ratio of government purchases to GDP, γ, to be 0.217. The debt-to-GDP ratio, denoted b, is set to an annual value of 67%. Those values are the observed ratios in the United States as reported by Aiyagari and McGrattan (1998).

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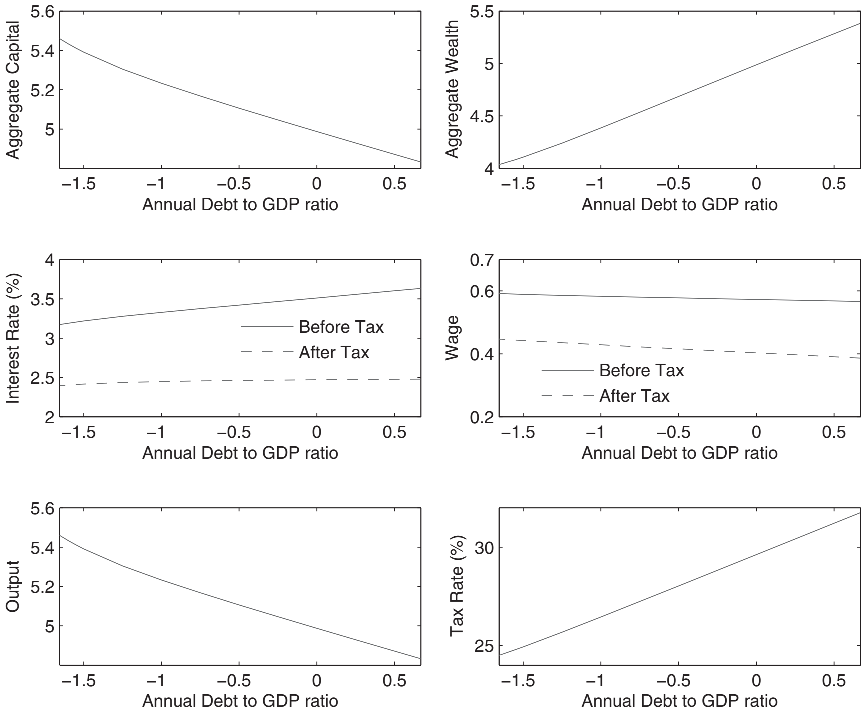

FIGURE 1. Macroeconomic behavior of the steady state economy.

2.2. Results

Macroeconomic behavior. We start by discussing the macroeconomic behavior of the steady state model when our policy experiment is carried out. This experiment consists in changing the debt-to-GDP ratio in the economy. Our steady state computations are reported in Figure 1.

Increasing the level of public debt raises the supply of assets in the economy. Consequently, the before-tax interest rate increases. Because the repayment of debt interest is higher, the income tax rate increases. Nevertheless, the after-tax interest rate unambiguously increases. In turn, public debt has a crowding-out effect on private capital: higher levels of debt decrease the aggregate amount of private capital in the economy. The crowding out of capital induces a decline in output. However, the decline in physical capital is smaller than the increase in public debt. The increase in the after-tax interest rate reduces the gap between the after-tax interest rate and the rate of time preference. The cost of postponing consumption to build up a buffer stock of saving is then reduced. Households choose to hold more assets at the steady state equilibrium. That is why the overall private wealth level, which is the combination of private capital and public debt, is higher. These mechanisms mimic those described in Aiyagari and McGrattan (1998).

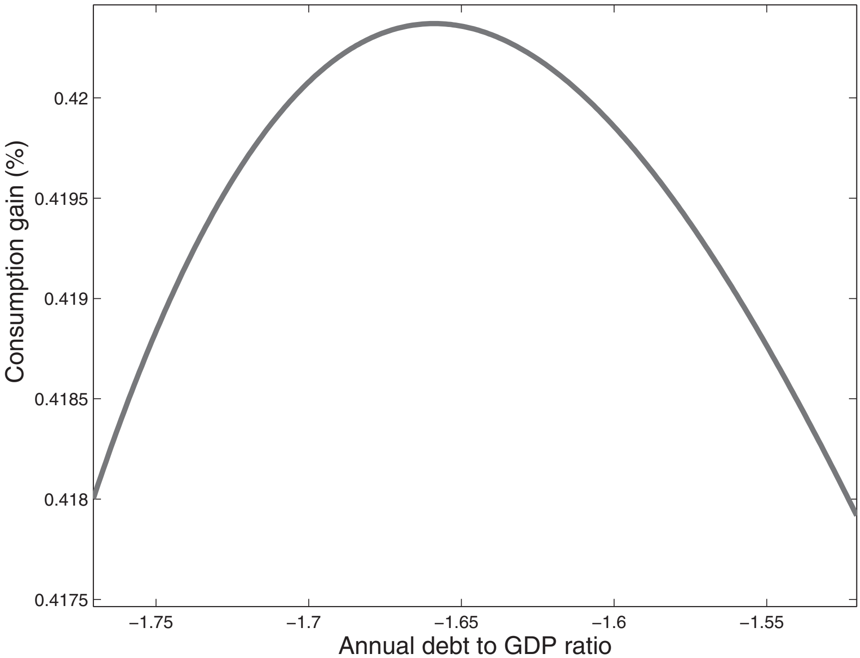

FIGURE 2. Welfare gains in percent consumption in the steady state economy.

Optimal level of public debt. We define the optimal level of public debt as the debt-to-GDP ratio that maximizes the traditional utilitarian welfare criterion. We follow Lucas (1987) in measuring the amount of consumption that one would have to remove or add in order to make the agent indifferent between the benchmark debt-to-GDP ratio and some other ratio of public debt.

The main mechanisms at stake in determining an optimal level of debt in this type of economy can be explained as follows. The result of the introduction of public debt on welfare is unknown a priori because of opposing effects. The first of these is a *crowding-out effect*: A higher level of public debt crowds out capital and reduces per capita consumption and thus welfare. Moreover, the increase in the income tax rate tends to amplify the negative impact of public debt on welfare. Another effect is related to the cost of precautionary saving: the increase in the after-tax interest rate makes it less costly to accumulate precautionary saving in order to smooth consumption as the interest rate gets closer to the time preference rate. This latter effect is welfare-enhancing.[5] These effects take place whether the setting has aggregate fluctuations or not and explain why the level of debt is not infinitely positive or negative.

Figure 2 depicts the welfare profile we find in the steady state model. The optimal level of public debt computed in this economy is -165.5% of output on an annual basis. In this basic model it is not optimal for the government to

TABLE 1. Steady state experiments

Model	Performed change	Parameter(s) changed	Optimal level of debt (annual)
A	Baseline steady state model		−165.5%
B	Adjusting labor market calibration		−166.5%
C	Adjusting technology parameters	α: 0.3 and δ: 0.075	Negligible
D	Adding transfers to the model	Transfers over GDP: 8.2%	−200%
E	Increasing the risk aversion	Risk aversion parameter σ : 1.5	−180%
F	Targeting a lower capital-output ratio	Target K/Y: 2.5	−220%
G	Adding exogenous growth	Growth rate: 1.85%	80%
H	Adding endogenous labor supply	Labor elasticity: 0.328	66.667%

sustain a positive level of debt, and it should instead accumulate a significant level of assets. As in other models in this literature, the welfare effects of changing the level of public debt are small: for instance, the gain of being at the optimal level of −165.5% instead of our calibration benchmark of 67% is only 0.42% of consumption. The equivalent result in Aiyagari and McGrattan (1998) is that the optimal level of debt is 67%. In the next section we explain step by step this difference in optimal levels. Thus we will use this model both as a comparison to Aiyagari and McGrattan (1998) and as a benchmark when aggregate risk is introduced.

Steady state experiments. In this part, we perform an experiment with the steady state model to explain the discrepancy between our optimal level and the level reported by Aiyagari and McGrattan (1998) and to better understand the impact of our calibration on the optimal level of public debt. This experiment consists in changing various model specifications and parameters in order to make it virtually equivalent to the model in Aiyagari and McGrattan (1998). This experiment is reported in Table 1.

We consider seven intermediary changes from our model to the one described in Aiyagari and McGrattan (1998) and each time adjust the discount factor accordingly. We start with our baseline steady state model, which we call (A), and first change the labor market process to reflect the process in Aiyagari and McGrattan (1998). In their paper, the underlining AR(1) process governing idiosyncratic income risk is discretized to a seven-state Markov chain to obtain model (B). We use the same approach and the same calibration of the process here and therefore switch from an unemployment risk process to an income risk process. The outcome of this change on the optimal level of public debt is very small. We then change technology parameters to match those in Aiyagari and McGrattan (1998): we set the capital share α to 0.3 and the depreciation rate δ to 0.075. A byproduct of

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this change is to make the model virtually annual instead of quarterly. We obtain model (C), and there is very small quantitative change on the optimal level of public debt. Next we add transfers to the model. We introduce them exactly in the way depicted by Aiyagari and McGrattan (1998): they are lump-sum and exogenously set to be 8.2% of GDP. We adjust government and household budget to reflect this change. As explained by Floden (2001), transfers when chosen optimally reduce the need for public debt. But as this is not the object here, we do not perform such a computation. We simply observe that, even at a suboptimal transfer level, there is incidentally less need for debt, as the optimal level of debt is now -200% in model (D). As explained by Floden (2001), transfers will reduce the need for precautionary saving at the cost of higher distortionary taxes. Next we adjust the risk aversion parameter and set it to a higher value. In our initial setup, we have log utility, with the risk aversion parameter being unity at the limit. We call the risk aversion parameter σ and change its value to 1.5 as in Aiyagari and McGrattan (1998). Model (E) exhibits an optimal debt of -180%; this is the combination of a higher risk aversion and the adjustment of the discount factor. In model (F), we target a lower capital–output ratio of 2.5, the same as in Aiyagari and McGrattan (1998). This yields a lower discount factor and a lower level of public debt of -220%. The next change is the single most important one, as the impact on the optimal level of public debt is substantial. We introduce in model (G) an exogenous growth parameter equal to $g = 1.85\%$ as in Aiyagari and McGrattan (1998). The optimal level of public debt is now positive and amounts to 80%. In the stationary detrended model that follows, the effective discount factor is now $\tilde{\beta} = \beta(1+g)^{1-\sigma}$. Consequently, there is an effect of adjusting the discount rate. At the same time, this growth factor appears in the budget constraint of the government and alleviate the interest payment cost. This plays an important role in reducing the cost of public debt. The final change we make is adding an endogenous labor supply. This yields model (H), which is identical to the model in Aiyagari and McGrattan (1998). The optimal level of public debt we find is at 67%, the same as reported by the former paper. Adding elastic labor introduces a new channel for tax distortion through adjustment of the labor supply. Higher public debt, by increasing the income tax rate, reduces labor supply. This effect decreases the importance of public debt as insurance.

This experiment emphasizes the key differences between our model and the setup in Aiyagari and McGrattan (1998), and the impact on these differences on the optimal level of public debt. Most of the specifications in our model are made to generate a simple setup, easily compatible with a model with aggregate fluctuations à la Krusell and Smith (1998).

3. AGGREGATE RISK MODEL

In this section we augment the steady state model of the last section to allow for aggregate fluctuations à la Den Haan (1996) or Krusell and Smith (1998). On top of idiosyncratic risk in the labor market, agents are also subject to an aggregate risk correlated with their idiosyncratic risk. We outline only key differences with the steady state model.

3.1. Model Specification

Firms. The production function is now the following:

$$Y_t = z_t F(K_t, N_t).$$

The economy is subject to an exogenous aggregate shock that we denote as z. There are two possible aggregate states: a good state where $z = z_g$ and a bad state where $z = z_b$. The aggregate shock follows a first-order Markov process with transition probability $\eta_{z|z'} = \Pr(z_{t+1} = z'|z_t = z)$. Thus, $\eta_{z|z'}$ is the probability that the aggregate state tomorrow is z' given that it is z today. We denote as η the matrix that describes the transition from one aggregate state to another such that

$$\eta = \begin{pmatrix} \eta_{gg} & \eta_{gb} \\ \eta_{bg} & \eta_{bb} \end{pmatrix}.$$

Prices are modified accordingly:

$$r_t + \delta = z_t F_K(K_t, N_t), \tag{8}$$

$$w_t = z_t F_N(K_t, N_t). \tag{9}$$

The government. We introduce the simplest setup for the behavior of the government with aggregate fluctuations and public debt. Public debt is assumed to be a one-period bond. One complication—which did not arise in the steady state model—is the fact that the tax base is random and that it is, in general, not possible to exclude a nonstationary path for the newly contracted public debt. To overcome this difficulty, we assume that the budget constraint of the government holds in the long-run and impose a constant level of public debt—corresponding to a specific debt-to-GDP ratio—and a constant tax rate. But then the period-by-period budget constraint of the government might not be balanced. Our strategy is thus to offset the effect of the random tax base by adjusting government expenditures. The outcome of this strategy is that, on average, government expenditures are equal to a fixed and positive expenditures over GDP ratio denoted γ, but they fluctuate slightly around this ratio over the business cycle. This fluctuation is quantitatively small and government expenditures play no role in this model. so that this strategy has no particular consequences.[6] Moreover, we retain some realistic features of fiscal policy and public debt—mainly the facts that governments will try to smooth taxation across periods and should not in general be able to trade state-contingent debt. With these specifications, the period-by-period government budget constraint simplifies to the following:

$$G_t + r_t B = T_t,$$

with

$$T_t = \tau(w_t N_t + r_t A_t).$$

Households. Now households are subject to both idiosyncratic and aggregate risk. Aggregate shocks exacerbate idiosyncratic unemployment risk. The unemployment rate and the unemployment duration are higher in recessions than in expansions. Therefore, transitions in the labor market are correlated with the aggregate state. We denote as $\Pi_{zz'|ss'}$ the joint transition probability to a state (s', z') conditional on a state (s, z). The matrix that jointly describes the transition from a state (s, z) to a state (s', z') is the following:

$$\Pi = \begin{pmatrix} \Pi_{bbuu} & \Pi_{bbue} & \Pi_{bguu} & \Pi_{bgue} \\ \Pi_{bbeu} & \Pi_{bbee} & \Pi_{bgeu} & \Pi_{bgee} \\ \Pi_{gbuu} & \Pi_{gbue} & \Pi_{gguu} & \Pi_{ggue} \\ \Pi_{gbeu} & \Pi_{gbee} & \Pi_{ggeu} & \Pi_{ggee} \end{pmatrix},$$

where $\Pi_{ggee} = \Pr(z_{t+1} = z_g, s_{t+1} = e | z_t = z_g, s_t = e)$.

The recursive problem of a household is

$$v(a, s; z, \Gamma) = \max_{c,\, a'} \{u(c) + \beta \mathbf{E}\left[v(a', s'; z', \Gamma') | (s; z, \Gamma)\right]\}, \qquad \textbf{(10)}$$

subject to

$$c + a' = (1 + r(z, \Gamma)(1 - \tau))a + \chi(s),$$

$$\chi(s) = \begin{cases} \theta \text{ if } s = u, \\ (1 - \tau)w(z, \Gamma) \text{ if } s = e, \end{cases}$$

$$\Gamma' = H(\Gamma, z, z'),$$

$$c \geq 0,$$

$$a' \geq 0.$$

The existence of aggregate risk leads us to distinguish between individual state variables and aggregate state variables. The individual state variables are given by the vector (a, s). The aggregate state variables are summarized by the vector (z, Γ), where $\Gamma(a, s)$ is a distribution of agents over asset holdings, employment status, and preferences. This distribution is needed because agents use this information to predict futures prices for each possible aggregate state. Predicting the prices imposes that agents use the current wealth distribution to forecast the next period's aggregate capital. This is cumbersome because the wealth distribution is an infinite-dimensional object. In the computational Appendix, we explain how we avoid manipulating the wealth distribution by approximating it with some of its moments using the methodology developed in Den Haan (1996) and Krusell and Smith (1998). Finally, we detail the computational strategy that we use to solve the model in Appendix A.1.

Equilibrium. The recursive equilibrium consists of a set of decision rules for consumption and asset holding $\{c(a, s; z, \Gamma), a'(a, s; z, \Gamma)\}$, aggregate capital and labor $\{K(z, \Gamma), N(z, \Gamma)\}$, factor prices $\{r(z, \Gamma), w(z, \Gamma)\}$, and tax rate τ and a law of motion for the distribution $\Gamma' = H(\Gamma, z, z')$, which satisfy these conditions:

(i) Given the prices $\{r(z, \Gamma), w(z, \Gamma)\}$, the tax rate τ, and the law of motion for the distribution $\Gamma' = H(\Gamma, z, z')$, the decision rules $\{c(a, s; z, \Gamma), a'(a, s; z, \Gamma)\}$ solve the dynamic programming problem (10).

(ii) Market price arrangements are

$$r(z, \Gamma) = zF_K(K, N) - \delta,$$
$$w(z, \Gamma) = zF_N(K, N).$$

(iii) The capital market satisfies

$$K + B = \int a'(a, s, \beta; \Gamma, z)d\Gamma.$$

(iv) The law of motion H is consistent with individual behavior.

Calibration. To remain simple and allow comparisons, we closely follow Krusell and Smith (1998, 1999) and Krusell et al. (2009) when calibrating the characteristics of the labor market and the aggregate risk. We only outline differences with the steady state model.

As in Krusell and Smith (1998), we assume that the value of the aggregate shock z is equal to 0.99 in recessions (z_b) and 1.01 in expansions (z_g). The process for z is set so that the average duration of good and bad times is eight quarters. Therefore, the transition matrix η for aggregate state changes is defined by

$$\eta = \begin{pmatrix} 0.8750 & 0.1250 \\ 0.1250 & 0.8750 \end{pmatrix}.$$

The average duration of an unemployment spell is 1.5 quarters in good times and 2.5 quarters in bad times. We also set the unemployment rate accordingly: in good periods it is 4% and in bad periods it is 10%. These assumptions let us define the transition matrices for labor market status for each aggregate state change: Π^{gg} for a transition from a good period to a good period, Π^{bb} for a transition from a bad period to a bad period, Π^{gb} for a transition from a good period to a bad period, and Π^{bg} for a transition from a bad period to a good period:[7]

$$\Pi^{bb} = \begin{pmatrix} 0.6000 & 0.4000 \\ 0.0445 & 0.9555 \end{pmatrix} \quad \Pi^{bg} = \begin{pmatrix} 0.2500 & 0.7500 \\ 0.0167 & 0.9833 \end{pmatrix}$$

$$\Pi^{gb} = \begin{pmatrix} 0.7500 & 0.2500 \\ 0.0729 & 0.9271 \end{pmatrix} \quad \Pi^{gg} = \begin{pmatrix} 0.3333 & 0.6667 \\ 0.0278 & 0.9722 \end{pmatrix}.$$

Finally, the joint transition matrix Π for labor market statuses and aggregate states can be defined as

$$\Pi = \begin{pmatrix} \eta_{bb}\Pi^{bb} & \eta_{bg}\Pi^{bg} \\ \eta_{gb}\Pi^{gb} & \eta_{gg}\Pi^{gg} \end{pmatrix} = \begin{pmatrix} 0.5250 & 0.3500 & 0.0313 & 0.0938 \\ 0.0388 & 0.8361 & 0.0021 & 0.1229 \\ 0.0938 & 0.0313 & 0.2916 & 0.5833 \\ 0.0911 & 0.1158 & 0.0243 & 0.8507 \end{pmatrix}.$$

Again we pinpoint precisely the capital–output ratio and the associated discount factor is now 0.99386. It is lower than in the steady state model: there is more risk in this model, so that agents save more and need to be a little less patient to attain the same capital–output ratio in the economy.

3.2. Results

Welfare analysis and optimal level of debt. The welfare analysis we conduct below applies to the long-run optimal level of public debt with aggregate fluctuations. Because the aggregate variables are not constant, not even in the limit, we consider for our welfare analysis the values of aggregate variables averaged over long periods of time for a given debt-to-GDP ratio.

Figure 3 depicts the long-run optimal level of public debt in the aggregate risk economy. The optimal level found is -152.75% of output and the gain of being at the optimal level instead of the calibration benchmark of 67% is 0.31% of consumption. We note that the optimal level of debt with aggregate risk is still largely negative, and this is to be expected with regard to the results and experiments with the steady state model. However, the introduction of aggregate risk has made the optimal level of public debt higher than in the equivalent steady state case.

Aggregate risk can have several effects in this type of model, and in turn they can impact the optimal level of public debt. The prime effects that we want to underline are (a) aggregate productivity shocks and (b) employment fluctuations. We argue that these effects alter the cost and the motive for precautionary saving. During a recession, both the unemployment rate and the unemployment duration increase. Thus for an employed agent, the risk of losing its job is higher, and for an unemployed agent, the opportunity to find a job is lower. As a result, the motive for precautionary saving is higher. Moreover, during a recession, the interest rate is lower because of the aggregate productivity shock. A lower interest rate is costly in an incomplete-markets setting because it is more difficult to smooth consumption: an agent would need to save more in order to sustain the same level of consumption. Thus the cost of precautionary saving is higher in a recession. The aggregate fluctuations setting is such that, in recessions, when the need for precautionary saving is high, the cost of precautionary saving is also high. Next we implement a protocol to disentangle these effects.

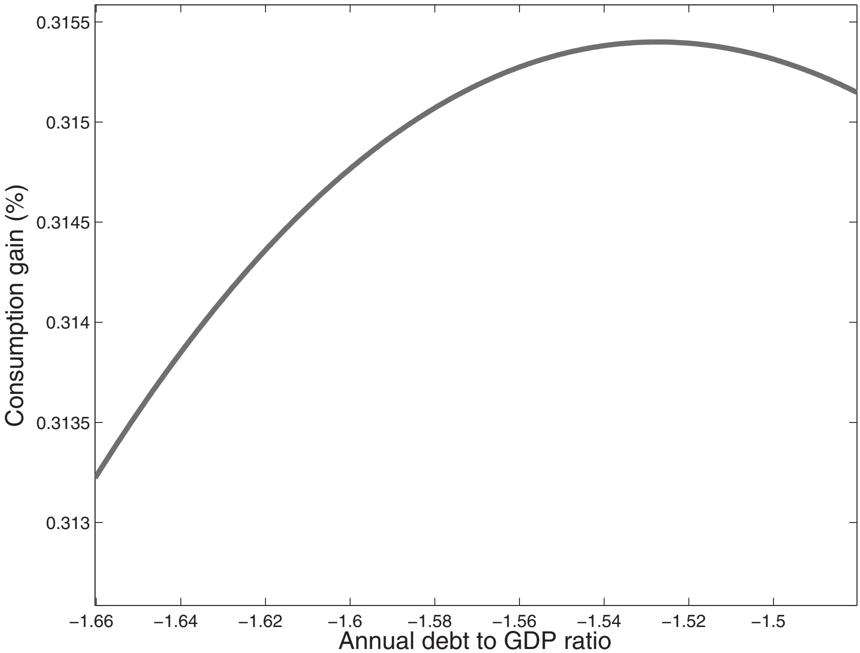

FIGURE 3. Welfare gains in percent consumption in the aggregate risk economy.

TABLE 2. Disentangling the effects of aggregate fluctuations

Model	Performed change	Optimal level of debt	Welfare gain at the optimal level
I	Baseline aggregate risk model	−152.75%	0.31%
II	Technology shock to uncond. mean	−157.25%	0.35%
III	Labor market process to uncond. mean	−161.75%	0.37%
IV	Idiosyncratic risk model	−165.50%	0.42%

Disentangling the effects of aggregate fluctuations. To disentangle the effects of aggregate productivity shocks and employment fluctuations on the optimal level of public debt, we turn to alternative models that each concentrate on a specific effect. Our results are reported in Table 2.

The direct effect of aggregate productivity shocks can be isolated by considering a model where this aggregate shock is set to its unconditional mean. In our calibration, $z_t = z_b = 0.99$ in recessions and $z_t = z_g = 1.01$ in expansions. Thus we build model II by setting $z = E(z_t) = 1$. We leave the model unchanged otherwise except for the fact that we adjust the discount factor accordingly.[8] The optimal level of public debt in this model is -157.25%, which is lower than in I

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but higher than in the steady state model IV. The welfare gain from being at the optimal level instead of the calibration level of 67% is 0.35% of consumption. This difference in optimal levels captures the impact of the aggregate productivity shock on the optimal level of public debt. This shock has a direct impact on the production function and on prices, most notably the interest rate. In model II, the aggregate productivity shock does not decrease (resp. increase) the interest rate in recession (resp. expansion). This has an impact on the cost of accumulating precautionary savings, as this cost is lower in model II than in model I. In turn, the trade-off between the welfare-improving and welfare-decreasing effects of public debt is also affected. Public debt is less necessary in model II, as households do not experience a decrease in the interest rate in recessions. They can be insured in the same way as before without needing a higher interest rate coming from a higher level of public debt.

To isolate the effect of employment fluctuations, we consider model III. In this model, the technology shock remains the same as in the baseline model I but we alter the labor market process in the following manner. The variance of the labor market process transition matrix does not change with the cycle, and we set this matrix to be identical to the transition matrix in the steady state case in all states of nature.[9] As before, to get the joint transition matrix Π, the individual transition probabilities have to be multiplied by the probability of being in a recession or an expansion as computed in matrix η. We leave the model unchanged otherwise, but we do adjust the discount factor accordingly.[10] This specification neutralizes the cyclicality of the unemployment process. The optimal level of public debt in this case is -161.75% and is lower than in model I, although remaining higher than in the steady state model IV. The welfare gain of being at the optimal level instead of being at the calibration benchmark is 0.42%. This difference in optimal levels captures the impact of employment fluctuations on the optimal level of public debt. In model III, households do not face higher (resp. lower) unemployment rate and duration in recessions (resp. expansions) as in model I. Therefore the motive for precautionary saving is lower, as there is less need for insurance against adverse states of nature. Again, this has an impact on the welfare-improving and welfare-decreasing effects of public debt, and less public debt is necessary.

We note that the isolated effect of the cyclicality of the unemployment process on the optimal level of public debt is greater here than the isolated effect of the aggregate productivity shock. Moreover, when the two effects are neutralized, we revert to the steady state model IV, and as both the cost and the motive for precautionary saving are lower in that case, there is a lesser need for public debt.

3.3. Wealth Distribution and Business Cycle Considerations

Preference heterogeneity setting. In the preceding sections, we have abstracted from one element that we believe is important for the determination of the optimal level of public debt: the exact specification of the wealth distribution in the economy. We argue that producing a realistic wealth distribution should be a calibration

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TABLE 3. Comparison of wealth distribution for United States and models

Model	W. Gini	Percentage wealth held by top 1%	5%	10%	20%
Aiyagari and McGrattan (1998)	0.41	4	15	26	44
Floden (2001)	0.61	7	25	41	62
Baseline aggregate risk model	0.29	3	11	21	37
Preference heterogeneity model	0.82	22	53	71	89
U.S. data (SCF 1992)	0.78	29.5	53.5	66.1	79.5

Model	Percentage wealth held by bottom 20%	40%	60%	80%
Aiyagari and McGrattan (1998)	3	13	30	56
Floden (2001)	0	4	15	38
Baseline aggregate risk model	8	21	39	63
Preference heterogeneity model	0	2	4	11
U.S. data (SCF 1992)	0	1	6	19

target in this family of models if distributional effects matter. It is primarily important to have an empirically plausible wealth distribution here because the optimal level of public debt is influenced by the proportion of wealth-rich and wealth-poor individuals in the economy. This result is suggested in Ball and Mankiw (1996) and Floden (2001): wealth-poor individuals tend to favor lower levels of public debt and in contrast, wealth-rich individuals favor higher levels of public debt. The basic mechanism is the following: the income of wealth-rich households is biased toward capital income, whereas the income of wealth-poor households is biased toward labor income. When the level of public debt rises, wealth-poor households suffer from reduction in output, higher taxes, lower wages, and crowding-out of capital. On the other hand, wealth-rich households are better off because their capital income benefits from the increase in the interest rate. This mechanism applies to this whole family of models, from Aiyagari and McGrattan (1998) to Floden (2001) and then to this model. However, there has been no attempt in the previous papers to match the wealth distribution when computing the optimal level of public debt. This is especially apparent from the computations we report in Table 3. In this table, we detail our computations of statistics of the wealth distribution in several models in the literature.

The first model we report on is Aiyagari and McGrattan (1998). The authors do not explore wealth distribution considerations in this paper, and unsurprisingly, their wealth statistics are far from empirical data. The Gini index is considerably lower than that of the U.S. wealth distribution. Moreover, the model does not generate enough wealth-rich households on the right tail of the distribution, and in

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the bottom part of the distribution agents are too wealthy. Floden (2001) improves on the Gini index, the right tail, and the bottom of the distribution, but the result is still unsatisfactory with respect to empirics. Next we comment on the wealth statistics of the model with aggregate risk that we built in the preceding section, called the baseline aggregate risk model in the table. Again, wealth statistics are poor, but like Aiyagari and McGrattan (1998) and Floden (2001), we did not try to match those statistics. Thus even though those models are very helpful in producing a series of results, they are also lacking with respect to wealth statistics when it comes to determining the optimal level of public debt.

There are several ways of generating a plausible wealth distribution in this literature. Introducing entrepreneurship [Quadrini (2000)], adding a bequest motive [De Nardi (2004)], and a preference heterogeneity setting [Krusell and Smith (1998)] are the prime candidates. But as Cagetti and De Nardi (2008) suggest, the current literature still has to understand the quantitative importance of each of the preceding factors. We follow Krusell and Smith (1998) and match wealth statistics with a preference heterogeneity setting. Cagetti (2003) builds an incomplete-market life-cycle model and estimates discount factor and risk aversion in order to match the median wealth profiles constructed from the Panel Study on Income Dynamics and the Survey of Consumer Finances (SCF). He shows that heterogeneity in patience levels is needed to generate a wide dispersion of wealth. This supports our choice of such a setting. The idea is to allow households to discount time differently in order to reproduce a natural behavior: some households are more patient and save more, others are impatient and save less. The fact that we allow unemployed agents to receive a positive income also helps in generating a large group of poor agents. The exact specification and the equilibrium of this preference heterogeneity setting are reported in Appendix B. This setting is quite successful in generating an empirically plausible wealth distribution, as can be seen in Table 3. The line denoted "Preference heterogeneity model" reports the wealth Gini and the top and bottom percentiles of the distribution. The Gini index matches its empirical counterpart, and the setting is successful in reproducing the thick right tail of the distribution and generating wealth-rich and wealth-poor households.

We renew the welfare computations we did previously with the aggregate risk model but applied to this preference-heterogeneity setting.[11] This setting generates a positive optimal level of public debt of 5%, and the welfare gain from being at this level instead of the calibration benchmark of 67% is 0.25% of consumption. A better reproduction of the wealth distribution has a direct implication on the proportions of wealth-rich and wealth-poor individuals in the economy and in turn has a substantial effect on the optimal level of public debt. We also perform the equivalent computation in the steady state model by adding preference heterogeneity. We find that the optimal level of public debt also increases substantially with respect to the original nonstochastic preference setting. The optimal level of public debt reaches 2.5% with a consumption gain of 0.29%. This level is below the corresponding aggregate risk model with preference heterogeneity; however, the difference in optimal levels between these two models is smaller than the

TABLE 4. Welfare gains in percent consumption for being at the optimal level of public debt for bottom and top percentiles of the population and quintiles

	Consumption gains in (%) held by						
Population	1%	5%	10%	20%	30%	40%	50%
Bottom percentiles	2.6793	2.6689	2.5287	3.4858	3.4383	3.4070	3.3871
Top percentiles	−18.1823	−11.6597	−8.7367	−5.7483	−3.7966	−2.2727	−1.5595

	Consumption gains in (%) for				
	$Q1$	$Q2$	$Q3$	$Q4$	$Q5$
Quintile	3.4858	3.3457	3.3146	2.6033	−5.7483

one we found between the aggregate and steady state models without preference heterogeneity. Our computations show that this fact can be explained in the following way. In the original nonstochastic preference setting, there is much more change in the distribution of wealth than in the current setting when switching from the steady state model to the aggregate risk model. Notably, the aggregate risk model displays more inequality and more difference in wealth percentiles, when compared with the corresponding steady state model in the original setting, than in the preference heterogeneity models. Obviously, in the original setting as only the capital–output ratio is targeted, the wealth distribution is somewhat unconstrained. In the preference heterogeneity setting, in contrast, we target in the same way the wealth statistics in both models. From this we conclude that even when the wealth distribution is constrained, there is a quantitative impact of aggregate risk. At the same time, to have a quantitatively accurate picture of the optimal level of public debt, monitoring the changes in the wealth distribution is crucial.

Next we emphasize that the overall consumption gains or losses of a change in the level of public debt in the aggregate risk setting with preference heterogeneity are shared very differently across the population. We decompose consumption gains from changing the level of public debt for a number of percentiles in the population. This decomposition shows that the consumption gain for many percentiles in the population is orders of magnitude higher than the average gain reported earlier. Moreover, gains of rich and poor percentiles are not symmetrical. Table 4 documents that a change in public debt from the calibration benchmark of 67% to the optimal level leads to an increase in consumption for the poorer percentiles of the population.[12] The bottom 1% of the population gains as much as 2.679% of consumption, whereas the bottom 10% of the population gains 2.529%. Most of the agents in the lower percentiles are unemployed and impatient. There is a jump in the consumption gain when the bottom 20% are considered, as they gain as much as 3.486%. This is explained by the fact that the new agents in the bottom 20%, when compared with those in the bottom 10%, are mostly employed. This effect also applies to the bottom 30%, 40%, and 50%, although the consumption

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gain slowly decreases: as agents become richer, the adverse effect of a lower interest rate has a greater impact. A quintile decomposition of consumption gains support our previous comment. Higher quintiles describe the consumption gains of richer households. We see that these gains steadily decrease until they become negative for the highest quintile.

Contrastingly, the richest percentiles in the population suffer when there is a change in the level of public debt from the calibration benchmark to the optimal level, and wealth-rich households favor higher levels of public debt. The magnitude of their losses is substantially higher than that of the gains of the poorest percentiles. For the top 1% of the population, this amounts to a loss of 18.182% of consumption. The richer the agent is, the greater the loss. This is explained by the fact that the income of rich agents depends mainly on capital income, whereas poor agents rely more on labor income. When the level of public debt is lower, poor agents benefit from the decrease in taxes, higher wages, and smaller crowding-out of capital. But as the interest rate decreases, rich agents become worse off. This particular analysis underlines the importance of having a realistic reproduction of the wealth distribution and especially of its fat right tail.

Higher unemployment rate and longer unemployment spells. In this section, we consider alternative calibrations of the labor market in the cycle that can serve as robustness exercises. We take as a benchmark for these changes the aggregate risk economy with preference heterogeneity of the preceding section. We modify only the labor market features and leave the rest of the calibration unchanged except for the stochastic discount factors. We adjust the discount factors to closely match the U.S. wealth distribution as in the preceding section and we keep targeting the same capital–output ratio.

In our experiments, we increase either the duration of an unemployment spell or the unemployment rate in the business cycle with respect to their values in the benchmark case. For instance, if we consider a symmetric 10% (percent change) increase in the duration of unemployment with respect to our initial calibration of the benchmark model, we would set the duration of an unemployment spell to be 2.75 quarters in recessions and 1.65 quarters in expansions. Similarly, for a 10% (percent change) increase in the unemployment rate, we would set the unemployment rate at 11% in recessions and 4.4% in expansions. Figure 4 reports the results of this investigation. The left-hand side represents the symmetric increase in the unemployment duration whereas the right-hand side represents the symmetric increase in the unemployment rate. Raising either the unemployment duration or the unemployment rate increases the optimal level of public debt. In our simulations, after a 40% (percent change) increase in the unemployment duration, the optimal level of debt is 10.625% instead of 5% in the benchmark economy, which corresponds to a 112.5% (percent) change. With a 40% (percent change) increase in the unemployment rate, the optimal level of public debt is 11.25%, which corresponds to a 125% (percent) change. Changing those parameters affects the precautionary saving behavior of agents as the labor market

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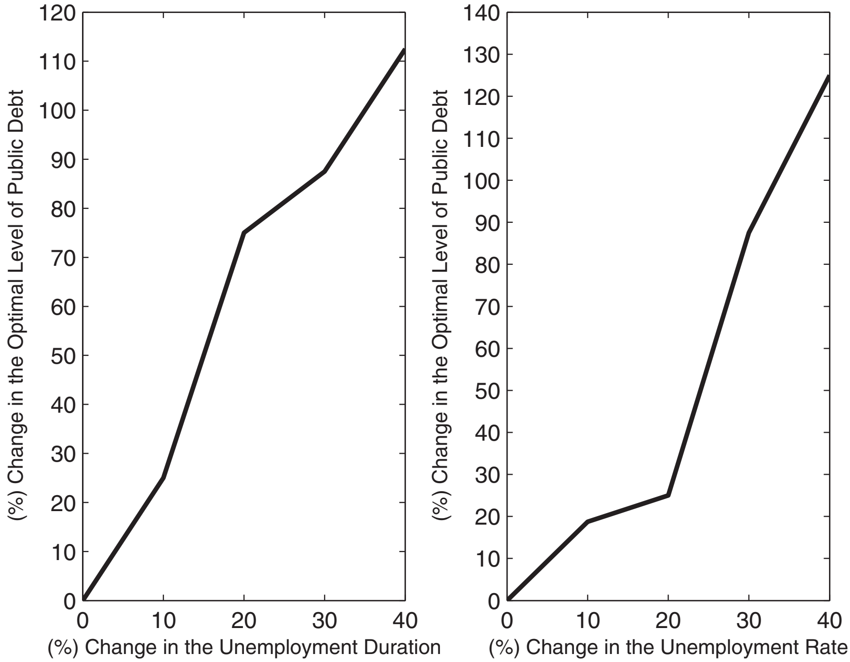

FIGURE 4. Change in the optimal level of public debt after a change in business cycle variables.

becomes more risky in the cycle. Therefore, a higher level of public debt becomes optimal.

We also consider an economy where both the unemployment rate and the unemployment duration are changed. We investigate a symmetric (percent change) increase of 40% in these parameters: the unemployment rate is 14% in recessions and 5.6% in expansions, and the unemployment duration is 3.5 quarters in recessions and 2.1 quarters in expansions. We now find the optimal level of debt to be 20%, which corresponds to a 300% (percent change) increase with respect to the optimal level in the benchmark case. As we could have expected after the previous experiment, longer unemployment spells and higher unemployment rates raise the optimal level of debt. This alternative calibration purposely strengthens the effect of employment fluctuation along the cycle: unemployed agents have a harder time finding a job in this economy as compared to the benchmark economy and employed agents face a higher risk of losing their jobs. In recessions, this is amplified. The precautionary motive is stronger here than in the benchmark model with preference heterogeneity, and thus the costs of reducing the level of public debt are higher. Because of this, agents settle for a higher level of public debt than in the benchmark case. Even though the changes in the unemployment rate and duration with respect to the benchmark economy are not comparable, the decomposition in the previous experiment helps in disentangling the contribution of each change.

These experiments underline the importance of the cyclical elements of the labor market in understanding the optimal level of public debt as these elements only appear in a model with aggregate risk. Also, this can illustrate potential outcomes in economies with a more volatile labor market, such as some European economies. But we do not strongly argue in this direction, as many other aspects of such economies are not captured by our exercise.

3.4. Sensitivity to Exogenous Growth and Capital–Output Ratio Adjustments

Adding an exogenous growth factor. We have shown in the steady state section that changing the exogenous growth factor has an important impact on the optimal level of public debt. In this part we quantitatively evaluate the impact of an exogenous growth factor on the optimal level of public debt in both the aggregate risk model and its extension with stochastic discount factors. We introduce exogenous growth as in Aiyagari and McGrattan (1998) or Floden (2001): we assume that labor productivity grows at the exogenous rate g and initial productivity is normalized to unity.[13] We use the same growth rate of 1.85% as Aiyagari and McGrattan (1998) but compute the quarterly equivalent of 0.4593% to match the periodicity of our model. To simplify our computations with stochastic discounting and exogenous growth, we also use in this section a CRRA utility with $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and a value for σ sufficiently close to unity to emulate the log function.

We first report the addition of the exogenous growth factor in the aggregate risk model without stochastic discounting. We first adjust the discount factor to match the target capital–output ratio of 2.65. Our calibration procedure yields $\beta = 0.99827$, whereas it was 0.99386 without the exogenous growth factor. The discount factor is now higher, and agents base their decision on the *perceived discount factor*, $\tilde{\beta} = \beta(1+g)^{1-\sigma}$. Our simulations indicate that the optimal level is now 530% of annual GDP. This is significantly higher than the result in the model without exogenous growth, where the optimal level of public debt was -152.75%. Our explanation of this result is the same as in the steady-state case. The growth factor appears in the government budget constraint and alleviate interest payment costs. At the same time, there is an effect of adjusting the discount rate.

We next introduce the exogenous growth factor into the model with aggregate fluctuations and stochastic discount factors. In this case we need to adjust the three discount factors and the proportions of individuals in each impatience state to both find the targeted capital–output ratio of 2.65 and match the wealth distribution statistics. Our calibration procedure yields that $\beta_l = 0.9870$, $\beta_m = 0.9965$, and $\beta_h = 1.0024$.[14] We also have that the invariant distribution of discount factors has 10% of agents at the lowest discount rate, 80% at the medium discount rate, and 10% at the highest discount rate. The corresponding wealth distribution statistics are reported in Table 5, and the wealth statistics are very close to what we have

TABLE 5. Comparison of wealth distribution for United States and model with exogenous growth

Model	W. Gini	Percentage wealth held by top 1%	5%	10%	20%
Preference heterogeneity model	0.82	22	53	71	89
Pref. heterogeneity and exogenous growth	0.80	20	53	73	89
U.S. data (SCF 1992)	0.78	29.5	53.5	66.1	79.5

Model	Percentage wealth held by bottom 20%	40%	60%	80%
Preference heterogeneity model	0	2	4	11
Pref. heterogeneity and exogenous growth	1	3	6	11
U.S. data (SCF 1992)	0	1	6	19

in the model without exogenous growth. The optimal level found after simulating this model is 665% of annual GDP, whereas the optimal level in the model without exogenous growth was 5%. As shown in both the model without aggregate risk and the previous model with aggregate risk but without preference heterogeneity, adding an exogenous growth component has a substantial impact on the optimal level of public debt. By reproducing the wealth distribution statistics in the latter model, we generate a large group of patient individuals and a small group of very patient individuals. Both these groups will substantially engage in precautionary saving and create an economy with wealth-rich individuals that will be favorable to higher levels of public debt. In this context, the exogenous growth component is able to reduce the burden of public debt repayment (and change the perceived discount factor), changing the trade-off between the costs and gains of higher public debt toward a higher level of public debt.

Targeting a lower capital–output ratio. Depending on the definition of capital and empirical computations, the value of the capital–output ratio can vary in the range from 2.5 to 3.0. We have adopted the quarterly value of 10.6 (equivalent to an annual ratio of 2.65) elsewhere in the paper, following Quadrini (2000). As a robustness exercise, we also consider the case of a quarterly target capital–output of 10 (equivalent to an annual ratio of 2.5) as in Aiyagari and McGrattan (1998). There is a simple relation between the optimal level of public debt and the capital–output ratio: with everything else constant, the higher the targeted capital–output ratio, the higher the optimal level of public debt. This is a Woodford (1990)-type mechanism: given a capital–output ratio we have to reach, in a liquidity-constrained economy, the closer the interest rate is to the time preference rate, the higher the welfare is. A higher capital-output ratio has to be reached through setting a higher discount factor and conversely for a lower capital-output ratio. We conduct this experiment in the aggregate risk model with stochastic discount factors and exogenous growth, detailed in the preceding section. We have adjusted the values of the discount factors to reduce the capital–output ratio while maintaining the necessary wealth distribution statistics.[15] The optimal level of debt found is indeed lower than in the previous case, as it is now 542.5% of annual GDP.

4. CONCLUSIONS

This paper investigates how aggregate fluctuations impact the optimal level of public debt in an incomplete-markets economy where households face both idiosyncratic and aggregate risk. We show that taking aggregate risk into account increases the optimal level of public debt relative to an economy where there are no such fluctuations. A decomposition shows that this is due to the effects of the aggregate productivity shock itself and to the cyclicality of the unemployment process. Furthermore, we show that the optimal level of public debt changes substantially when the model matches the characteristics of the U.S. wealth distribution. Finally, exogenous growth is an important factor that can substantially change the value of the optimal level of public debt in this family of models.

These results call for several remarks. First, the optimal level of public debt we report in the aggregate risk model that reproduces the U.S. wealth distribution can be regarded as a lower bound on the optimal level of public debt. Aggregate risk in our study has been introduced in the same way as in Krusell and Smith (1998, 1999).[16] The latter compute the welfare costs of business cycles in an incomplete-markets economy where households face aggregate risk and uninsurable idiosyncratic unemployment risk. Their result has been, since then, commented on and has given rise to several other contributions that can shed some light on our result. Mukoyama and Sahin (2006) point out that Krusell and Smith (1999) do not take skill heterogeneity into account. Unskilled workers are more likely to become unemployed in recessions than skilled workers [Mincer (1991)]. Moreover, the unemployment rate of unskilled workers is higher and much more volatile than that of skilled workers [Topel (1993)]. Additionally, Krusell and Smith (1999) focus on unemployment risk, whereas fluctuations can contribute to earnings risk beyond just unemployment risk. Storesletten et al. (2001) use microeconomic data from the Panel Study on Income Dynamics to calibrate the process of idiosyncratic earnings risk. In recessions, the idiosyncratic earnings shock tends to be more volatile than in Krusell and Smith (1999). When we alter business cycle properties in our paper, we conduct a robustness exercise in this direction. Finally, these works abstract from the permanent component of earnings risk: a decrease in income today may become permanent in the future. Moreover, the income loss the worker may experience is likely to be higher in downturns. Krebs (2003) introduces both these characteristics and concludes that the costs of business cycles can be higher than in Krusell and Smith (1999) or Storesletten et al. (2001). These results support our lower bound argument on the optimal level of public debt.

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Second, we do not consider optimal taxation issues and do not model stochastic government expenditures here. However, as Shin (2006) points out, the government may find it optimal to save for its own precautionary saving motive in order to smooth tax distortions over time. This behavior would certainly mitigate our result, as the government would face a trade-off between increasing public debt to help the heightened desire for consumption smoothing of households subject to both idiosyncratic and aggregate risk and accumulating assets to smooth tax distortions.

Third, our contribution abstracts from transitional dynamics. Desbonnet and Weitzenblum (2012) extend Floden (2001) by modeling the transition from one steady state to another to capture the short-run effects of public debt. They point out that at the date of the increase in public debt, the resources of the government are raised, and in turn taxes can be reduced or transfers increased. They show that the consumption gains associated with a higher level of public debt are much more important when short-run effects of public debt are taken into account and that this could create an incentive to increase the level of public debt beyond its long-run value. Following this result, it is reasonable to think that the optimal level of public debt in the steady state model would be higher with transitional dynamics. It is also reasonable to think that this result would extend to the aggregate risk model. However, as prices and employment fluctuate along the cycle in that case, there are an exceedingly large number of paths for the prices and employment when the economy is switching from the postwar U.S. average level of public debt to another.

Finally, another limitation is the exogenous nature of the policies imposed on the economy, in the sense that agents do not expect policy changes. An active literature on the time-consistent determination of public policies[17] sheds a very promising light on this issue.

NOTES

1. See, for instance, Storesletten et al. (2001) or Krusell et al. (2009).

2. Aiyagari and McGrattan (1998) report that the average debt-to-GDP ratio in the United States over the postwar period is 67% on a yearly basis. That level is precisely the optimal level of public debt obtained by their model.

3. The value of the capital–output ratio can change with the definition of capital. Here we adopt the definition in Quadrini (2000). Thus, aggregate capital results from the aggregation of plant and equipment, inventories, land at market value, and residential structures. This definition is close to the findings of Prescott (1986) and is also used, for instance, in Floden and Linde (2001). This yields a capital–output ratio of 2.65 on an annual basis that we convert to its quarterly equivalent of 10.6.

4. This corresponds to about 13% of the quarterly wage. This is roughly in the range of the value used by Krusell et al. (2009).

5. Woodford (1990) shows in a liquidity-constrained economy that the closer the interest rate is to the time preference rate, the higher welfare is. We have the same effect here.

6. It is also possible to offset the effect of the random tax base with a lump-sum transfer to the households. In our simulations, both assumptions lead to the same optimal level of public debt and quantitative results of the same order for the aggregate risk economy. More details on this can be found in Appendix A.2.

7. Further details on this calibration can be found in Appendix C.

8. We keep targeting the capital–output ratio defined in the calibration section and find a discount factor of 0.99388.

9. In other words, $\Pi^{bb} = \Pi^{bg} = \Pi^{gb} = \Pi^{gg} = \pi$.

10. We find a discount factor of 0.99389.

11. Computing the welfare gains and costs is somewhat more complex in this setting. We detail our methodology in Appendix B.3.

12. When we report consumption gains or losses across the population, we have to take a stand on how households are sorted. In our case, we sort agents from poorest to richest based on their wealth. Thus, for instance, the consumption gain of the bottom 1% of the population refer to the consumption gain of the poorest 1% of households in terms of wealth in the economy.

13. We consider only the balanced growth path and detrend the model appropriately. Details about the detrending procedure can be found in Appendix D.

14. Note that even in the case where $\beta_h > 1$, the perceived discount factor $\tilde{\beta}_h = \beta_h(1+g)^{1-\sigma} < 1$.

15. Those values are now $\beta_l = 0.9870$, $\beta_m = 0.9963$, and $\beta_h = 1.0022$.

16. The contributions of this paper are revised in Krusell et al. (2009). Our remarks also apply to the former paper.

17. See, for instance, Krusell (2002); Hassler et al. (2003); Klein and Rios-Rull (2003); Hassler et al. (2005); Klein et al. (2005, 2008); or Grechyna (2016).

18. Further details about this are given in Section A.2.

19. This number of simulations ensures that there are as many expansion as recession periods in the simulation sample with the particular random sequence used here.

20. For further details on the calibration of this matrix, see appendix C.

21. The data we report on the U.S. distribution come from Krusell and Smith (1998) and Budria-Rodriguez et al. (2002).

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APPENDIX A: COMPUTATIONAL STRATEGY

A.1. SOLVING THE MODEL

We only describe how we solve the most complex model: the model with aggregate risk and preference heterogeneity. We apply similar principles to solve the other models.

We solve the model by following strategies described in Den Haan (1996), Den Haan (1997), and Krusell and Smith (1997, 1998). Agents need only a restrictive set of statistics about the wealth distribution to determine prices, and limiting this set to the mean of the wealth distribution might be sufficient: a linear prediction rule based only on the average level of capital provides an accurate prediction. This result comes from the near linearity of the decision rule $a'(a, s; z, \Gamma)$. As the aggregate capital stock is mainly held by rich people who have approximately the same propensity to save, the next period's aggregate capital is accurately predicted by the current period's aggregate capital. In our model, we use the following prediction rules:

$$\log(\bar{K}') = a_0 + a_1 \log(\bar{K}), \text{ if } z = z_g,$$
$$\log(\bar{K}') = c_0 + c_1 \log(\bar{K}), \text{ if } z = z_b,$$

where $\bar{K}'$ and $\bar{K}$ denote respectively the average stock of capital of the next period and of the current period. Thus, the strategy is the following:

1. As the aggregate variables are not constant, even in the limit, in an economy with aggregate fluctuations, we approximate the steady state of this economy by averaging aggregate variables over long periods of time. We first make a guess at the average long-run interest rate.[18] From this, we can deduce the long-run GDP, and using the debt-over-GDP ratio and the public expenses ratio, we can derive the level of public debt and public expenses in this economy, given our guess of the long-run interest rate. From the long-run budget constraint of the government, we then derive the long-run tax rate for the economy. Given this tax rate, we execute steps (2), (3) and (4) below.
2. Given a set of parameter values (a_0, a_1, c_0, c_1) for the law of motion, we solve the individual problem. To solve the individual problem, we iterate on the Euler equation,

$$U'(c) = E\left\{\beta' U'(c')[1 + r'(z', \Gamma')(1-\tau)]|s; z, \Gamma\right\},$$

on a discrete grid until a fixed point is found. When the borrowing constraint binds, the solution can be deduced from the budget constraint.

3. We perform simulations to derive new values for the coefficients of the law of motion.
4. If the parameters (a_0, a_1, c_0, c_1) found are close to the parameter values used to solve step (2), the algorithm has converged. Otherwise steps (2), (3), and (4) are repeated until convergence. At this step, the convergence criterion has a precision of 10^{-6}.
5. When steps (2), (3), and (4) have converged, we update our guess of the long-run interest rate until a fixed point is found. At this step, the convergence criterion has a precision of 10^{-8}.

When all steps are completed, the long-run average interest rate coincides with the average interest rate in the actual economy with aggregate fluctuations, the government budget constraint is balanced.

All results in the model are derived from simulated data. The simulated sample consists of 12, 858 periods,[19] and the first 1, 000 are discarded. The distribution is approximated by a sample of 30, 000 households in each period. The algorithm is implemented in the C++ language.

A.2. TAX AND PUBLIC DEBT DEFINITION

We describe here our strategy regarding the tax rate and the level of public debt in the aggregate risk model. We impose a constant tax rate and a constant non-state-contingent public debt issuance. Because the tax base is random, with the preceding assumptions, the period-by-period budget constraint of the government may not be balanced.

In Aiyagari and McGrattan (1998), the budget constraint of the government is written as follows:

$$\widehat{G}_t + r\widehat{B}_t + \widehat{\mathrm{Tr}}_t = \widehat{B}_{t+1} - \widehat{B}_t + \widehat{T}_t,$$

with

$$\widehat{T}_t = \tau[wN_t + r(K_t + \widehat{B}_t)],$$

with $\widehat{G}_t$ the level of public expenses, $\widehat{B}_t$ the level of public debt, $\widehat{T}_t$ tax revenues, $\widehat{\mathrm{Tr}}_t$ transfers, and r and w the prices. Because the next period's level of public debt $\widehat{B}_{t+1}$ is unknown, this expression is cumbersome in an aggregate fluctuations setting. Aiyagari and McGrattan (1998) use the long-run steady state government budget constraint, which can be written as follows:

$$\widehat{G} + r\widehat{B} = \widehat{T},$$

and derive the tax rate from this last equation.

In our aggregate risk model, we cannot directly use such a long-run budget constraint because of the aggregate fluctuations property. Current prices, employment level, and aggregate capital fluctuate in such a way that the tax base and the amount of interest paid on the contracted public debt fluctuate.

We adopt the following approach. We guess a value $\bar{r}$ for the interest rate that the fluctuating economy would reach on average in the long run. We derive a constant level of public debt from $\bar{r}$ such that $B = b\bar{Y}(\bar{r})$, where B is the constant value of public debt, b is the targeted debt-to-GDP ratio, and $\bar{Y}$ is the long-run GDP. Similarly, we define

$\bar{G} = \gamma \bar{Y}(\bar{r})$, where γ is the ratio of government expenses to GDP, $\bar{G}$ the long-run level of public expenses, $\bar{K} = \bar{K}(\bar{r})$ the long-run level of aggregate capital, $\bar{w} = \bar{W}(\bar{r})$ the long-run value for wages, and $\bar{N}$ the average employment level in the economy. We can then define the constant tax rate in the following manner:

$$\tau = \frac{(\bar{G} + \bar{r}B)}{\bar{w}\bar{N} + \bar{r}(B + \bar{K})}.$$

In each period of the aggregate risk economy, a constant level of public debt B is issued by the government and households are taxed at the long-run rate τ. As prices, aggregate capital, and employment fluctuate, the period-by-period government budget constraint may not be balanced at each date along the cycle. Thus, we impose that a short-term adjustment in government expenditures balances the budget in each period. This adjustment can be viewed as the usual compromise the government makes in balancing its budget constraint. We define this adjustment $\widetilde{G}_t$ in the following manner:

$$\widetilde{G}_t = \tau[(K_t + B)r_t + N_t w_t] - (\bar{G} + r_t B),$$

and we have

$$G_t = \bar{G} + \widetilde{G}_t.$$

Depending on the tax base, this adjustment can be positive or negative and is small. On the average it amounts to zero. This adjustment has no particular incidence, as government expenditures play no particular role in the model.

On the computational side, as long as the average interest rate in the simulated economy is far from the guessed long-run rate $\bar{r}$, $\widetilde{G}_t$ may be large. But when the fixed point on the average interest rate has been found, it is small and averages out to zero, so that the government budget constraint is balanced over the cycle.

Alternatively, we could assume that instead of government expenditures, a lump-sum transfer to households, Tr_t, balances the period-by-period budget constraint so that

$$\mathrm{Tr}_t = \tau[(K_t + B)r_t + N_t w_t] - (\bar{G} + r_t B).$$

In all our simulations of the aggregate risk economy with preference heterogeneity, this alternative specification does not change any qualitative results of this paper and the quantitative implications are very small. Most importantly, both specifications lead to the same optimal level of public debt.

APPENDIX B: PREFERENCE HETEROGENEITY SETTING

B.1. GENERATING AN EMPIRICALLY PLAUSIBLE WEALTH DISTRIBUTION

To reproduce the shape of the U.S. wealth distribution, we first assume that unemployed agents receive income too and fix the home production income θ to be 0.10. This assumption produces a large group of poor agents. Next, we use the preference heterogeneity setting discussed in Krusell and Smith (1998) to generate a long thick right tail. We impose that the discount factor β takes on three values, $\{\beta_l, \beta_m, \beta_h\}$, where $\beta_l < \beta_m < \beta_h$:

$$\begin{pmatrix} \beta_l \\ \beta_m \\ \beta_h \end{pmatrix} = \begin{pmatrix} 0.9750 \\ 0.9880 \\ 0.9985 \end{pmatrix}.$$

Thus, an agent with a discount factor β_m is more patient than an agent with a discount factor β_l. To calibrate the transition matrix, we impose that the invariant distribution for discount factors has 10% of the population at the lowest discount rate β_l, 70% at the medium discount factor β_m, and 20% at the highest discount factor β_h. Like Krusell and Smith (1998), we assume that there is no immediate transition between extreme values of the discount factors. Finally, we set the average duration of the lowest discount factor and the highest discount factor to be 50 years (200 quarters) to roughly match the length of a generation. These assumptions yield the following transition matrix:[20]

$$\Upsilon = \begin{pmatrix} 0.9950 & 0.0050 & 0.0000 \\ 0.0007 & 0.9979 & 0.0014 \\ 0.0000 & 0.0050 & 0.9950 \end{pmatrix}.$$

Household preferences are summarized by the function V:

$$V = \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) u(c_t) \right\}. \tag{B.1}$$

The discount factor satisfies

$$\begin{cases} \beta_0 = 1 \\ \beta_j \in]0; 1[, \quad j \geq 1 \end{cases}.$$

The results of this calibration are shown in Table 3. The preference heterogeneity setting helps to reproduce the shape and the skewness of the U.S. wealth distribution[21] and yields a Gini index of 0.82. The quarterly capital–output ratio we target here is still 10.6.

B.2. EQUILIBRIUM

The recursive equilibrium in the preference heterogeneity setting consists of a set of decision rules for consumption and asset holding $\{c(a, s, \beta; z, \Gamma), a'(a, s, \beta; z, \Gamma)\}$, aggregate capital and labor $\{K(z, \Gamma), N(z, \Gamma)\}$, factor prices $\{r(z, \Gamma), w(z, \Gamma)\}$, tax rate τ, and a law of motion for the distribution $\Gamma' = H(\Gamma, z, z')$, which satisfy the following conditions:

(i) Given the prices $\{r(z, \Gamma), w(z, \Gamma)\}$ and the law of motion for the distribution $\Gamma' = H(\Gamma, z, z')$, the decision rules $\{c(a, s, \beta; z, \Gamma), a'(a, s, \beta; z, \Gamma)\}$ solve the dynamic programming problem (10).

(ii) Market price arrangements are

$$r(z, \Gamma) = zF_K(K, N) - \delta,$$

$$w(z, \Gamma) = zF_N(K, N).$$

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(iii) The capital market satisfies

$$K + B = \int a'(a, s, \beta; \Gamma, z) d\Gamma.$$

(iv) The law of motion H is consistent with individual behavior.

B.3. WELFARE COMPUTATION

In this section of the Appendix, we explain how the welfare computations are performed in the preference heterogeneity setting. As in Lucas (1987) and Mukoyama and Sahin (2006), we compute μ, the amount of consumption that one would have to remove or add in order to make the utilitarian welfare criterion equal between a benchmark debt over GDP ratio and some other level of public debt. It satisfies

$$\int \mathbf{E}_0 \left\{ \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \ln \left[(1+\mu) c_t^{\text{bench.}} \right] \right\} d\Gamma^{\text{bench.}} = \int \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \ln(c_t) \right] d\Gamma,$$

with $\left\{ c_t^{\text{bench.}} \right\}_{t=0}^{\infty}$ the consumption stream in the benchmark model when the debt-over-GDP ratio is equal to $8/3$. $\{c_t\}_{t=0}^{\infty}$ is the consumption stream when the debt-over-GDP ratio is set to some other level than the benchmark level. For logarithmic utility we can show that

$$\mu = \exp\left[\left(W - W^{\text{bench.}} \right) / S \right] - 1,$$

where

$$W^{\text{bench.}} = \int \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \ln(c_t^{\text{bench.}}) \right] d\Gamma^{\text{bench.}},$$

$$W = \int \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \ln(c_t) \right] d\Gamma,$$

$$S = \int \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \right] d\Gamma.$$

Next, we explain how $f = \mathbf{E}_0 \left[\sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \right]$ is computed. By assumption, we have that β_0 is equal to 1 and β_1 is given by the initial condition. If we denote f_i the value of f when $\beta_1 = \beta^i$ for $i = l, m, h$, we have

$$\begin{pmatrix} f_l \\ f_m \\ f_h \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \beta_l & 0 & 0 \\ 0 & \beta_m & 0 \\ 0 & 0 & \beta_h \end{pmatrix} \begin{pmatrix} \Upsilon_{ll} & \Upsilon_{lm} & \Upsilon_{lh} \\ \Upsilon_{ml} & \Upsilon_{mm} & \Upsilon_{mh} \\ \Upsilon_{hl} & \Upsilon_{hm} & \Upsilon_{hh} \end{pmatrix} \begin{pmatrix} f_l \\ f_m \\ f_h \end{pmatrix}.$$

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We deduce that

$$\begin{pmatrix} f_l \\ f_m \\ f_h \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \beta_l & 0 & 0 \\ 0 & \beta_m & 0 \\ 0 & 0 & \beta_h \end{pmatrix} \begin{pmatrix} \Upsilon_{ll} & \Upsilon_{lm} & \Upsilon_{lh} \\ \Upsilon_{ml} & \Upsilon_{mm} & \Upsilon_{mh} \\ \Upsilon_{hl} & \Upsilon_{hm} & \Upsilon_{hh} \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

APPENDIX C: CALIBRATION DETAILS

We now show how we derived the transition matrices for aggregate state changes (η), for joint transition between aggregate states and labor market statuses (Π), and for discount factor changes (Υ).

C.1. AGGREGATE STATE TRANSITION MATRIX

To deduce the aggregate state change transition matrix η, we solve the following system:

$$\left.\begin{array}{r} \eta_{gg} = \eta_{bb} \\ \eta_{bg} = \eta_{gb} \\ \eta_{gg} + \eta_{gb} = 1 \\ \eta_{bb} + \eta_{bg} = 1 \\ \eta_{bg} = \dfrac{1}{8} \end{array}\right\} \Longrightarrow \begin{pmatrix} 0.875 & 0.125 \\ 0.125 & 0.875 \end{pmatrix}.$$

As we assume that the duration of an expansion or a recession is the same, we deduce the two first equations. Moreover, the duration of a cycle is set to 8 quarters. It follows that $\eta_{bg} = \Pr(z_{t+1} = g/z_t = b) = \frac{1}{8}$ and $\eta_{gb} = \Pr(z_{t+1} = b/z_t = g) = \frac{1}{8}$.

C.2. MATRIX FOR JOINT TRANSITION BETWEEN AGGREGATE STATES AND LABOR MARKET STATUSES

The determination of the matrix Π that describes the transition between unemployment and employment requires the identification of the aggregate shock (whether we are in a recession or in an expansion). The transition matrix Π is built thanks to the matrix η and to the transition matrixes Π^{gg}, Π^{bb} ,Π^{gb}, and Π^{bg}. Π satisfies

$$\Pi = \begin{pmatrix} \eta_{bb}\Pi^{bb} & \eta_{bg}\Pi^{bg} \\ \eta_{gb}\Pi^{gb} & \eta_{gg}\Pi^{gg} \end{pmatrix}.$$

We assume that in recessions the duration of the unemployment, which we denote durub, amounts to 2.5 quarters and the unemployment rate u_b is set to 10%. In expansions, the duration of unemployment, durug, is equal to 1.5 quarters and the unemployment rate, u_g, is set to 4%. From this information, we can deduce the matrices Π^{gg} and Π^{bb}.

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The transition matrix Π^{gg} corresponds to the case $(z, z') = (g, g)$. It satisfies

$$\Pi^{gg} = \begin{pmatrix} \Pi^{gg}_{uu} & \Pi^{gg}_{ue} \\ \Pi^{gg}_{eu} & \Pi^{gg}_{ee} \end{pmatrix}.$$

Solving the following system gives the values of Π^{gg}_{ee}, Π^{gg}_{eu}, Π^{gg}_{ue}, and Π^{gg}_{uu}:

$$\begin{cases} \Pi^{gg}_{ee} + \Pi^{gg}_{eu} = 1 \\ \Pi^{gg}_{ue} + \Pi^{gg}_{uu} = 1 \\ \Pi^{gg}_{ue} = \dfrac{1}{\text{durug}} \\ \Pi^{gg}_{ee} = 1 - \dfrac{u_g \Pi^{gg}_{ue}}{1 - u_g} \end{cases} \Longrightarrow \Pi^{gg} = \begin{pmatrix} 0.3333 & 0.6667 \\ 0.0278 & 0.9722 \end{pmatrix}.$$

The transition matrix Π^{bb} corresponds to the case $(z, z') = (b, b)$. It satisfies

$$\Pi^{bb} = \begin{pmatrix} \Pi^{bb}_{uu} & \Pi^{bb}_{ue} \\ \Pi^{bb}_{eu} & \Pi^{bb}_{ee} \end{pmatrix}.$$

Solving the following system gives the values of Π^{bb}_{ee}, Π^{bb}_{eu}, Π^{bb}_{ue}, and Π^{bb}_{uu}:

$$\begin{cases} \Pi^{bb}_{ee} + \Pi^{bb}_{eu} = 1 \\ \Pi^{bb}_{ue} + \Pi^{bb}_{uu} = 1 \\ \Pi^{bb}_{ue} = \dfrac{1}{\text{durub}} \\ \Pi^{bb}_{ee} = 1 - \dfrac{u_b \Pi^{bb}_{ue}}{1 - u_b} \end{cases} \Longrightarrow \Pi^{bb} = \begin{pmatrix} 0.6 & 0.4 \\ 0.0445 & 0.9555 \end{pmatrix}.$$

When the cycle changes, the unemployment rate changes. The transitions between unemployment and employment are modified. We make the same assumptions as Krusell and Smith (1998):

$$\begin{cases} \Pi^{bg}_{uu} = \Pr(s_{t+1} = u^g / s_t = u^b) = 0.75\Pi^{gg}_{uu} \\ \Pi^{gb}_{uu} = \Pr(s_{t+1} = u^b / s_t = u^g) = 1.25\Pi^{bb}_{uu}. \end{cases}$$

The probability of remaining unemployed when the next period is a recession (resp. expansion) increases (resp. decreases), because by assumption the unemployment rate is higher in recession than in expansion.

The transition matrix Π^{bg} corresponds to the case $(z, z') = (b, g)$. It satisfies

$$\Pi^{bg} = \begin{pmatrix} \Pi^{bg}_{uu} & \Pi^{bg}_{ue} \\ \Pi^{bg}_{eu} & \Pi^{bg}_{ee} \end{pmatrix}.$$

The following system gives us Π^{bg}_{ee}, Π^{bg}_{eu}, Π^{bg}_{ue}, and Π^{bg}_{uu}:

$$\begin{cases} \Pi^{bg}_{ee} + \Pi^{bg}_{eu} = 1 \\ \Pi^{bg}_{ue} + \Pi^{bg}_{uu} = 1 \\ \Pi^{bg}_{uu} = 0.75\Pi^{gg}_{uu} \\ \Pi^{bg}_{ee} = \dfrac{[(1 - u_g) - u_b \Pi^{bg}_{ue}]}{1 - u_b} \end{cases} \Longrightarrow \Pi^{bg} = \begin{pmatrix} 0.25 & 0.75 \\ 0.0167 & 0.9833 \end{pmatrix}.$$

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The transition matrix Π^{gb} corresponds to the case $(z, z') = (g, b)$:

$$\Pi^{gb} = \begin{pmatrix} \Pi^{gb}_{ee} & \Pi^{gb}_{eu} \\ \Pi^{gb}_{ue} & \Pi^{gb}_{uu} \end{pmatrix}.$$

The following system gives us Π^{gb}_{ee}, Π^{gb}_{eu}, Π^{gb}_{ue}, and Π^{gb}_{uu}:

$$\begin{cases} \Pi^{gb}_{ee} + \Pi^{gb}_{eu} = 1 \\ \Pi^{gb}_{ue} + \Pi^{gb}_{uu} = 1 \\ \Pi^{gb}_{uu} = 1.25\Pi^{bb}_{uu} \\ \Pi^{gb}_{ee} = \dfrac{[(1 - u_b) - u_g \Pi^{gb}_{ue}]}{1 - u_g} \end{cases} \Longrightarrow \Pi^{gb} = \begin{pmatrix} 0.75 & 0.25 \\ 0.0729 & 0.9271 \end{pmatrix}.$$

C.3. DISCOUNT FACTOR TRANSITION MATRIX

We assume that discount factors follow a three-states first-order Markov process. Therefore, the matrix describing the transition from the discount factor β_i to the discount factor β_j is the following:

$$\Upsilon = \begin{pmatrix} \Upsilon_{ll} & \Upsilon_{lm} & \Upsilon_{lh} \\ \Upsilon_{ml} & \Upsilon_{mm} & \Upsilon_{mh} \\ \Upsilon_{hl} & \Upsilon_{hm} & \Upsilon_{hh} \end{pmatrix}.$$

As we assume that there is no immediate transition between β_l and β_h as in Krusell and Smith (1998), $\Upsilon_{lh} = \Upsilon_{hl} = 0$. Moreover, as we set the duration of the extreme states (β_l and β_h) to 50 years, or 200 quarters, we have $\Upsilon_{lm} = \frac{1}{200} = \Upsilon_{hm}$. Solving the following system gives us the transition matrix Υ:

$$\left.\begin{array}{c} \Upsilon_{ll} + \Upsilon_{lm} + \Upsilon_{lh} = 1 \\ \Upsilon_{ml} + \Upsilon_{mm} + \Upsilon_{mh} = 1 \\ \Upsilon_{hl} + \Upsilon_{hm} + \Upsilon_{hh} = 1 \\ \Upsilon_{lh} = \Upsilon_{hl} = 0 \\ \Upsilon_{lm} = \frac{1}{200} = \Upsilon_{hm} \\ \Upsilon_{ml} = \frac{\Pr(\beta_t = \beta_l)\Upsilon_{lm}}{\Pr(\beta_t = \beta_m)} \\ \Upsilon_{mh} = \frac{\Pr(\beta_t = \beta_h)\Upsilon_{hm}}{\Pr(\beta_t = \beta_m)} \end{array}\right\} \Longrightarrow \Upsilon = \begin{pmatrix} 0.995 & 0.005 & 0 \\ 0.0007 & 0.9979 & 0.0014 \\ 0 & 0.005 & 0.995 \end{pmatrix}.$$

APPENDIX D: AGGREGATE RISK MODEL WITH EXOGENOUS GROWTH

We here describe the aggregate risk model with exogenous growth and stochastic discounting. We explain in the following how we detrend the model. In the general case, we denote the nondetrended variable as X, so that $\tilde{X}_t = \frac{X_t}{(1+g)^t}$, with $\tilde{X}_t$ the detrended variable.

We can detrend the production function as follows (in the Cobb–Douglas case): $\tilde{Y}_t = z_t \tilde{K}_t^{\alpha} N_t^{1-\alpha}$. Factor prices are then

$$r_t + \delta = \frac{\partial \tilde{Y}_t}{\partial \tilde{K}_t} = \alpha z_t \tilde{K}_t^{\alpha-1} N_t^{1-\alpha},$$

$$\tilde{w}_t = \frac{\partial \tilde{Y}_t}{\partial N_t} = (1-\alpha) z_t \tilde{K}_t^{\alpha} N_t^{-\alpha}.$$

The detrended household problem is the following (we abstract from the nonnegativity and transversality conditions):

$$\max_{\tilde{c}_t, \tilde{a}_{t+1}} \mathbf{E}_0 \sum_{t=0}^{\infty} \left(\prod_{j=0}^{t} \beta_j \right) \frac{[(1+g)^t \tilde{c}_t]^{1-\sigma}}{1-\sigma}$$
$$\text{s.t. } \tilde{c}_t + (1+g)\tilde{a}_{t+1} = [1 + r_t(z_t, \Gamma_t)(1-\tau)]\tilde{a}_t + \tilde{\chi}(s).$$

This problem will yield the following Euler equation:

$$\tilde{c}_t^{-\sigma} = \mathbf{E}_t \{ \beta_{t+1} (1+g)^{-\sigma} [1 + r_{t+1}(z_{t+1}, \Gamma_{t+1})(1-\tau)] \tilde{c}_{t+1}^{-\sigma} \}.$$

The associated value function is the following:

$$V_t = \max \left[\frac{\tilde{c}_t^{1-\sigma}}{1-\sigma} + \beta^{t \to t+1} (1+g)^{1-\sigma} \mathbf{E}_t V_{t+1} \right],$$

with $\beta^{t \to t+1}$ the appropriate discount factor to discount period $t+1$ when the current period is t.

For the government, our approach is the following. We assume that the government issues in the long run a level of debt relative to an average long-run level of GDP. We guess a value $\bar{r}$ for the interest rate that the fluctuating economy would reach on the average in the long run. Together with the average value of aggregate labor between good and bad periods, denoted $\overline{N}$, we can deduce long-run production $\overline{\tilde{Y}} = \overline{\tilde{K}}^{\alpha} \overline{N}^{1-\alpha}$ and wage $\overline{\tilde{w}} = (1-\alpha)\overline{\tilde{K}}^{\alpha} \overline{N}^{-\alpha}$. The amount of debt issued by the government depends on the long-run production and the debt-to-GDP ratio $\tilde{B} = b\overline{\tilde{Y}}$, and there is the following relation between two consecutive debt levels: $\tilde{B}' = (1+g)\tilde{B}$. Similarly, we define the long-run value of government expenditures $\tilde{G} = \gamma \overline{\tilde{Y}}$, and taxes in the long run are equal to $\tilde{T} = \bar{\tau}(\overline{\tilde{w}}\overline{N} + \bar{r}(B + \overline{\tilde{K}}))$. Thus the long-run (detrended) budget constraint in our case is

$$\tilde{G} + \bar{r}\tilde{B} = (1+g)\tilde{B} - \tilde{B} + \tilde{T}.$$

From this we can derive a constant tax rate relative to this long-run economy:

$$\bar{\tau} = \frac{\tilde{G} + (\bar{r} - g)\tilde{B}}{\overline{\tilde{w}}\overline{N} + \bar{r}(\tilde{B} + \overline{\tilde{K}})}.$$

Now, outside of the long-run case, in each period of the fluctuating economy, the government issues a level of debt $\tilde{B}$ derived from the long-run case and taxes agents at the long-run rate

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$\bar{\tau}$. The budget constraint can be written as follows:

$$\tilde{G} + (r_t - g)\tilde{B} = \bar{\tau}((\tilde{K}_t + \tilde{B})r_t + N_t\tilde{w}_t).$$

As in the aggregate risk model without exogenous growth, we impose that a short-term adjustment in government expenditures balances the budget in each period in the cycle:

$$\hat{G}_t = \bar{\tau}((\tilde{K}_t + \tilde{B})r_t + N_t\tilde{w}_t) - \tilde{G} - (r_t - g)\tilde{B},$$

such that government expenditures in the cycle are $G_t = \tilde{G} + \hat{G}_t$. In the end, this specification is very similar to that for the case without exogenous growth.