ANZIAM J. **64** (2022), 250–263
doi:10.1017/S144618112200013X

VOLATILITY SWAPS VALUATION UNDER A MODIFIED RISK-NEUTRALIZED HESTON MODEL WITH A STOCHASTIC LONG-RUN VARIANCE LEVEL

XIN-JIANG HE [1] and SHA LIN [✉2]

(Received 3 March, 2022; accepted 2 August, 2022; first published online 26 September, 2022)

Abstract

We consider the pricing of discretely sampled volatility swaps under a modified Heston model, whose risk-neutralized volatility process contains a stochastic long-run variance level. We derive an analytical forward characteristic function under this model, which has never been presented in the literature before. Based on this, we further obtain an analytical pricing formula for volatility swaps which can guarantee the computational accuracy and efficiency. We also demonstrate the significant impact of the introduced stochastic long-run variance level on volatility swap prices with synthetic as well as calibrated parameters.

2020 *Mathematics subject classification*: 91G20.

Keywords and phrases: volatility swaps, modified Heston model, risk-neutralized volatility, stochastic long-run variance, analytical solution.

1. Introduction

Managing financial risk attracts much attention from researchers and market practitioners, which contributes to the high volume of trading volatility derivatives in real markets, since it is an easy way to trade volatility and hedge risk. Volatility swaps, as one important volatility derivative, have received considerable attention. A number of authors have worked on the pricing of a volatility swap.

Volatility swaps can be mainly classified into two categories according to the sampling method, that is, continuously sampled and discretely sampled ones. In the first category, general model independent results are presented by Carr and Lee [7, 8], while other authors focus on pricing volatility swaps under different stochastic

[1] School of Economics, Zhejiang University of Technology, Hangzhou, China;
e-mail: xinjiang@zjut.edu.cn
[2] School of Finance, Zhejiang Gongshang University, Hangzhou, China;
e-mail: linsha@mail.zjgsu.edu.cn

volatility models [12, 14, 18, 20]. Despite their appealing results, the underlying assumption is not consistent with practice, as volatility swaps traded in real markets are usually discretely sampled. This can cause mis-valuation problems if one makes use of these results, as suggested by a number of authors [11, 24].

To properly reflect the discrete sampling effect and be closer to financial reality, it is very popular to consider the valuation of discretely sampled volatility swaps. In particular, Zhu and Lian [29] presented an analytical pricing formula for volatility swaps under the well-known Heston model [19]. Despite their appealing results, the study of pricing volatility swaps should not be stopped, since the Heston model is not perfect for modelling the underlying dynamics. For example, the square root specification for the so-called volatility of volatility is generally rejected as a model of stock index returns [1, 26], while evidence of the substantial nonlinear mean-reverting property for a volatility process has been provided by a number of authors (for example, Bakshi et al. [2]). All of these drawbacks have led to the development of different modifications to the Heston model, trying to incorporate more stochastic factors. These variations have also been applied in volatility derivative pricing, including a regime switching Heston model [11], Heston model with stochastic interest rate [4, 6, 16] and Heston model with stochastic interest rate as well as regime switching [5]. A hybrid constant elasticity of variance (CEV) and stochastic volatility model were adopted by Cao et al. [3], while stochastic volatility was combined with the Hawkes jump-diffusion process by Liu and Zhu [25]. A general framework for variance swap pricing under stochastic volatility models with jumps was established by Cui et al. [10].

Recently, multi-factor stochastic volatility models have started to gain attention, because they have been shown to provide a better fit to market data [9]. In fact, there have already been various results on the pricing of volatility derivatives under multi-factor stochastic volatility models. For example, for variance swap pricing, Pun et al. [27] considered a combination of multi-factor stochastic volatility and jumps, while Wu et al. [28] introduced the stochastic interest rate into a double Heston stochastic volatility model. A double exponential Ornstein–Uhlenbeck stochastic volatility was adopted by Kim and Kim [22]. Both variance and volatility swaps were valued under a two-factor Heston model with an additional regime switching factor [17] and a multi- factor Heston stochastic volatility model [21]. Being quite similar to these multi-factor stochastic volatility models, another trend for introducing additional stochastic factors is to make the parameters of stochastic volatility models as random variables to increase the flexibility of the model. Belonging to this category, Lee et al. [23] considered multiscale stochastic volatility of volatility, while He and Chen [15] introduced a stochastic long-run variance level into the risk-neutralized Heston model and obtained a closed-form solution for European option prices.

In this paper, we focus on pricing volatility swaps under the model proposed by He and Chen [15], which assumes a stochastic long-run variance level under the risk-neutralized Heston model. Although the considered model is much more complicated than the original Heston model due to the involvement of an additional stochastic source, we have still successfully obtained an analytical pricing formula for volatility swaps, based on the forward characteristic function of the underlying price derived in closed form. The contribution of this paper can be summarized from two aspects. On the one hand, we present an analytical formulation of the forward characteristic function under the considered model, which has not been presented before. This leads to an analytical solution being available for volatility swap prices. In this case, computational time can be significantly reduced and computational accuracy can be greatly improved, compared with the case to which numerical methods have to be resorted. On the other hand, we demonstrate the significant impact of the introduced stochastic long-run variance level under the risk-neutralized Heston model with synthetic and calibrated model parameters.

The rest of the paper is organized as follows. In Section 2, the adopted model is briefly introduced, and the forward characteristic function of the underlying price is derived, followed by the closed-form pricing formula for volatility swaps. In Section 3, numerical experiments are conducted to show various properties of the newly derived formula. Concluding remarks are given in Section 4.

2. Closed form solution

In this section, the modified Heston model proposed by He and Chen [15] will be briefly introduced, after which the price of discretely sampled volatility swaps will be worked out based on the derived forward characteristic function of the underlying price.

2.1. The modified Heston model We start with a filtered probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_{t\in[0,T]})$, which describes the uncertainty of the economy, with P representing a probability measure (typically a risk-neutral measure considered in this paper) and T denoting a finite time horizon. All stochastic processes involved are assumed to be $\mathcal{F}_{t\in[0,T]}$ adapted. Let $\{S_t, t \geq 0\}$ and $\{v_t, t \geq 0\}$ denote the underlying price and the volatility process, respectively. The modified Heston model under the risk-neutral measure is characterized as

$$\begin{aligned} \frac{dS_t}{S_t} &= r\,dt + \sqrt{v_t}\,dW_t^1, \\ dv_t &= k(\bar{v} + \theta_t - v_t)dt + \sigma_1\sqrt{v_t}\,dW_t^2, \\ d\theta_t &= \lambda\,dt + \sigma_2\,dW_t^3, \end{aligned} \tag{2.1}$$

where W_t^1, W_t^2 and W_t^3 are standard Brownian motions [15]. We further assume that W_t^3 is independent of W_t^1 and W_t^2, with $dW_t^1 dW_t^2 = \rho\,dt$. We remark that following a number of different authors including Heston [19], we analyse the model in terms of the risk-neutralized volatility process instead of the "true" process under the physical measure throughout the paper, since the risk-neutralized process exclusively determines prices. Therefore, we know that k, $\bar{v}$, θ_t and σ respectively represent the mean reversion speed, constant part of the long-run variance level, stochastic part

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

of the long-run variance level at time t (stochastic long-run variance level for short hereafter) and volatility of volatility, associated with the risk-neutralized volatility process. The stochastic part θ_t can actually be viewed as corrections to the constant part due to outside information, which can explain the independence between W_t^3 and the other two Brownian motions. We also note that this model will degenerate to the Heston model if both λ and σ_2 take the value of zero, in which case θ_t becomes a constant.

2.2. Volatility swaps For the completeness of this paper, we first sketch the derivation of a general formula for the delivery price in a volatility swap contract, while the full details can be found in the existing literature [17].

One of the most popular measures of the realized volatility σ_R can be specified as

$$\sigma_R = 100\sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N}\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right|,$$

where t_i, $i = 0, \ldots, N$, represents the ith observation time of the realized volatility with $t_i = iT/N$, $i = 0, \ldots, N$. According to the risk-neutral pricing rule, as well as the fact that the value of volatility swaps should equal to zero when it is entered, we obtain

$$K = E(\sigma_R) = \sqrt{\frac{\pi}{2NT}}\sum_{i=1}^{N} E\left(\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right| \middle| S_0, v_0, \theta_0\right).$$

If we assume that $x_{t,T} = \ln(S_T) - \ln(S_t)$, $t < T$ with the current time being 0, and let $p(x_{t_{i-1},t_i})$ be the probability density function of the stochastic variable x_{t_{i-1},t_i}, the target expectation can be calculated as

$$\begin{aligned} E\left(\left|\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right| \middle| S_0, v_0, \theta_0\right) &= \int_0^{\infty} (e^{x_{t_{i-1},t_i}} - 1)p(x_{t_{i-1},t_i})\,dx_{t_{i-1},t_i} \\ &\quad + \int_{-\infty}^{0} (-e^{x_{t_{i-1},t_i}} + 1)p(x_{t_{i-1},t_i})\,dx_{t_{i-1},t_i}. \end{aligned} \tag{2.2}$$

We further define $f(\phi; t, T, v_0, \theta_0)$ as the conditional forward characteristic function of $x_{t,T}$. By making use of the Gil–Pelaez theorem [13] that relates the characteristic function and the cumulative function of a random variable, we obtain

$$P_{1,i} \triangleq \int_0^{\infty} p(x_{t_{i-1},t_i})\,dx_{t_{i-1},t_i} = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{f(\phi; t_{i-1}, t_i, v_0, \theta_0)}{j\phi}\right] d\phi$$

and

$$\begin{aligned} P_{2,i} &\triangleq \int_0^{\infty} e^{x_{t_{i-1},t_i}} p(x_{t_{i-1},t_i})\,dx_{t_{i-1},t_i} \\ &= f(-j; t_{i-1}, t_i, v_0, \theta_0)\left\{\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \mathrm{Re}\left[\frac{f(\phi - j; t_{i-1}, t_i, v_0, \theta_0)}{j\phi f(-j; t_{i-1}, t_i, v_0, \theta_0)}\right] d\phi\right\}, \end{aligned}$$

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

where Re(·) denotes the real part of the argument. In this case, equation (2.2) can be further simplified as

$$E\left(\left|\frac{S_{t_i}-S_{t_{i-1}}}{S_{t_{i-1}}}\right| | S_0, v_0, \theta_0\right) = (1-2P_{1,i}) - f(-j; t_{i-1}, t_i, v_0, \theta_0)(2P_{2,i}-1)$$
$$= \frac{2}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{f(\phi-j; t_{i-1}, t_i, v_0, \theta_0) - f(\phi; t_{i-1}, t_i, v_0, \theta_0)}{j\phi}\right] d\phi.$$

Therefore, it is clear that our task is converted into finding the conditional forward characteristic function $f(\phi; t, T, v_0, \theta_0)$, the solution to which is presented in the following proposition.

PROPOSITION 2.1. *If the underlying asset price S_t follows the dynamics (2.1), then the conditional forward characteristic function can be derived as*

$$f(\phi; t, T, v_0, \theta_0) = e^{C(\phi;\tau)+\bar{C}(\phi;t)+\bar{D}(\phi;t)v_0+\bar{E}(\phi;t)\theta_0}$$

with $\tau = T - t$, and

$$\bar{D}(\phi; t) = \frac{2k}{\sigma_1^2}\frac{1}{1-(1-2k/\sigma_1^2 D(\phi;\tau))e^{kt}},$$

$$\bar{E}(\phi; t) = E(\phi;\tau) + \frac{2k}{\sigma_1^2}\left\{kt - \ln\left(1-\left(1-\frac{2k}{\sigma_1^2 D(\phi;\tau)}\right)e^{kt}\right) + \ln\left(\frac{2k}{\sigma_1^2 D(\phi;\tau)}\right)\right\},$$

$$\bar{C}(\phi; t) = \frac{1}{2}\sigma_2^2\int_0^t \bar{E}^2(\phi; s)\,ds + \lambda\int_0^t \bar{E}(\phi; s)\,ds + \bar{v}[\bar{E}(\phi;t) - E(\phi;\tau)],$$

$$C(\phi;\tau) = \bar{v}E + jr\phi\tau + \frac{1}{2}\sigma_2^2\int_0^\tau E^2(\phi; s)\,ds + \lambda\int_0^\tau E(\phi; s)\,ds,$$

$$D(\phi;\tau) = \frac{d-(\rho\sigma_1 j\phi - k)}{\sigma_1^2}\frac{1-e^{d\tau}}{1-ge^{d\tau}},$$

$$E(\phi;\tau) = \frac{k}{\sigma_1^2}\left\{[d-(\rho\sigma_1 j\phi - k)]\tau - 2\ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right\},$$

$$d = \sqrt{(\rho\sigma_1 j\phi - k)^2 + \sigma_1^2(j\phi + \phi^2)},$$

$$g = \frac{(\rho\sigma_1 j\phi - k) - d}{(\rho\sigma_1 j\phi - k) + d}.$$

The proof of this proposition is left in the Appendix.

Having worked out the conditional forward characteristic function, the final solution of the delivery price K can be expressed as

$$K = 100\sqrt{\frac{\pi}{2NT}}\int_0^\infty \sum_{i=1}^N \mathrm{Re}\left[\frac{f(\phi-j; t_{i-1}, t_i, v_0, \theta_0) - f(\phi; t_{i-1}, t_i, v_0, \theta_0)}{j\phi}\right] d\phi.$$

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

By now, we have derived the closed-form pricing formula for volatility swaps under the modified Heston model. In the next section, the accuracy of our newly derived formula will be verified by comparing numerical results obtained from our formula and those through Monte Carlo simulation [17]. Also, we will show the difference caused by the introduction of the stochastic long-run variance level through the comparison of our results and those under the Heston model [29].

3. Numerical experiments and examples

In this section, numerical experiments are carried out to study the properties of volatility swap prices under the modified Heston model. In particular, we first show the accuracy of our formula by comparing numerical results obtained with our formula and those from Monte Carlo simulation. With confidence in our formula, we further show the influence of introducing the stochastic long-run variance level into the volatility process by comparing volatility swap prices under our adopted model and the original Heston model. As mentioned earlier, we focus on analysing the model with risk-neutralized parameters instead of the "true" ones, since the risk-neutralized process actually determines prices, as pointed out by Heston [19]. In addition, all of our calculations in this paper are done on a laptop with the following specifications: Intel(R) Core(TM), i5-1135G7 CPU@2.40 GHz and 16.0 GB of RAM.

What is presented in Table 1 is the comparison of volatility swap prices obtained through our formula (Ours) with those from Monte Carlo (MC) simulation. The MC simulation is implemented with 500 000 sample paths, and it is accompanied by a 98% confidence interval provided in the parentheses. One can clearly observe from this table that our results are quite close to those obtained through MC simulation. We also provide the absolute relative error (RE) between the two prices to demonstrate the accuracy of our formula. It is not difficult to find that the maximum absolute relative error in this test case is only 0.06%, which implies that our formula is accurate. However, the CPU time cost by our formula (t_1) is far less than that consumed by MC simulation (t_2). It should be remarked that the CUP time cost by our formula reported here measures the computational time when the involved integrals are computed using the trapezoidal rule. It can be highly reduced if one uses some software built-in functions, such as *integral* in MATLAB.

Once we are confident of our formula, the pricing performance of our model is compared with that of the Heston model, the dynamics of which are specified as

$$\frac{dS}{S} = r\,dt + \sqrt{v}\,dW_t^1,$$
$$dv = k(\widetilde{v} - v)\,dt + \sigma\sqrt{v}\,dW_t^2.$$

Note that if we make $\widetilde{v} = \bar{v} + \theta_0$, and let λ and σ_2 be equal to zero, our model would become exactly the same as the Heston model. Thus, it is not difficult to deduce that with all the other corresponding parameters being the same, our results will approach the results under the Heston model when the values of λ and σ_2 approach zero.

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

TABLE 1. Our prices versus Monte Carlo prices.

N	4	6	8	10	12	14	16	18
Ours	28.3079	28.2010	28.1470	28.1140	28.0934	28.0761	28.0651	28.0580
MC	28.3087	28.2152	28.1422	28.1167	28.0844	28.0642	28.0764	28.0568
	(±0.036)	(±0.029)	(±0.025)	(±0.023)	(±0.021)	(±0.019)	(±0.018)	(±0.017)
RE(%)	2.48×10^{-3}	5.05×10^{-2}	1.70×10^{-2}	9.65×10^{-3}	3.19×10^{-2}	4.21×10^{-2}	4.03×10^{-2}	4.35×10^{-3}
t_1	4.37	7.44	9.31	13.55	16.36	17.02	19.91	22.39
t_2	27.46	40.53	60.98	80.36	94.79	112.47	123.12	144.40
N	20	22	24	26	28	30	32	34
Ours	28.0501	28.0435	28.0408	28.0372	28.0327	28.0282	28.0261	28.0217
MC	28.0537	28.0363	28.0352	28.0370	28.0363	28.0390	28.0339	28.0274
	(±0.016)	(±0.015)	(±0.015)	(±0.014)	(±0.014)	(±0.013)	(±0.013)	(±0.012)
RE(%)	1.29×10^{-2}	2.59×10^{-2}	1.97×10^{-2}	7.08×10^{-4}	1.29×10^{-2}	3.88×10^{-2}	2.78×10^{-2}	2.05×10^{-2}
t_1	26.87	31.47	33.93	35.02	37.32	40.51	43.36	45.81
t_2	158.52	181.90	187.78	208.25	224.97	266.56	298.72	326.28
N	36	38	40	42	44	46	48	50
Ours	28.0199	28.0182	28.0164	28.0121	28.0103	28.0084	28.0066	28.0042
MC	28.0202	28.0200	28.0095	28.0294	28.0213	28.0153	28.0084	28.0064
	(±0.012)	(±0.012)	(±0.012)	(±0.011)	(±0.011)	(±0.011)	(±0.011)	(±0.010)
RE(%)	1.21×10^{-3}	6.51×10^{-3}	2.47×10^{-2}	6.17×10^{-2}	3.90×10^{-2}	2.47×10^{-2}	6.51×10^{-3}	7.80×10^{-3}
t_1	49.38	51.15	53.92	55.88	57.98	59.99	61.56	63.66
t_2	362.41	407.71	446.61	482.32	518.14	552.93	583.66	618.89

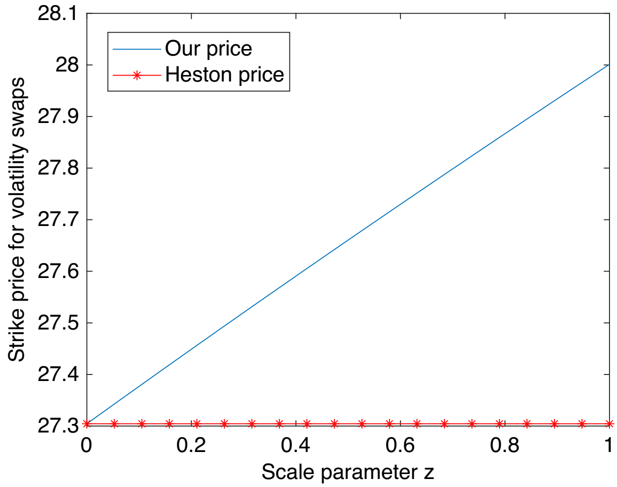

FIGURE 1. Comparison of our prices and Heston prices with different value of the scale parameter. Parameters are $T = 1, \sigma_1 = 0.1, \sigma_2 = 0.01, \lambda = 0.01, k = 10, \rho = -0.5, \bar{v} = 0.05, r = 0.05, v_0 = 0.03, \theta_0 = 0.03, S_0 = 10$.

To smoothly show this phenomenon, we introduce a scale parameter z, which varies within [0,1]. We then assume that $\lambda = \bar{\lambda}z$ and $\sigma_2 = \bar{\sigma}_2 z$, so that the two prices could be depicted with respect to z, which is shown in Figure 1. As expected, our price will be the same as the Heston price (the star line that is very closed to the x-axis) when $z = 0$, while they can become quite different when z takes large values.

With the time to expiry being unchanged and the sampling frequency being altered, a similar phenomenon could be observed in Figure 2 that our price is always lower than the Heston price under the current parameter settings. This can be explained by the negative value of λ, which contributes to the decrease in the long-run variance level of the risk-neutralized volatility and thus the lower volatility swap prices. We also observe that both prices are the decreasing function of the sampling frequency. In other words, the delivery price of a volatility swap would decrease if sampling times per year are increased.

As shown in Figure 3, the delivery price of volatility swaps under the Heston model is a monotonic increasing function of the time to expiry, while that under our model shows an increasing trend before it starts to decrease. This is in fact reasonable, as the negative value of λ typically leads to a smaller long-run variance level of the risk-neutralized volatility, and this can result in the decrease of the realized volatility as well as the delivery price when the time to expiry is large.

All the above sensitivity analysis was carried out by setting the corresponding parameters of both models to be the same. One may also be interested in whether the two models would behave differently when the parameters are calibrated to real market data. Therefore, we make use of the parameters calibrated to European

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

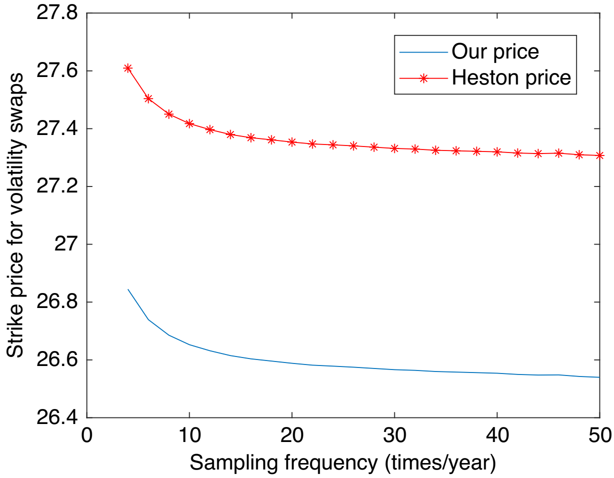

FIGURE 2. Comparison of our prices and Heston prices with different value of the time to expiry. Parameters are $N = 52, T = 1, \sigma_1 = 0.1, \sigma_2 = 0.01, \lambda = -0.01, k = 10, \rho = -0.5, \bar{v} = 0.05, r = 0.05, v_0 = 0.03, \theta_0 = 0.03$.

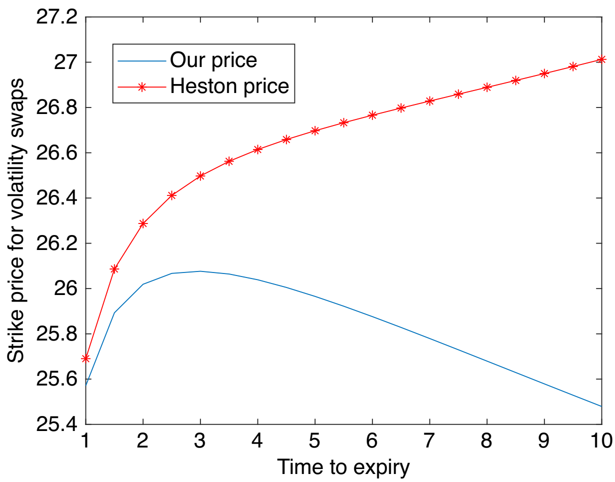

FIGURE 3. Comparison of our prices and Heston prices with different value of the sampling frequency. Parameters are $N = 52, \sigma_1 = 0.1, \sigma_2 = 0.01, \lambda = -0.001, k = 10, \rho = -0.5, \bar{v} = 0.05, r = 0.05, v_0 = 0.03, \theta_0 = 0.03$.

options written on the S&P 500 index from He and Chen [15] for calculating the delivery prices of volatility swaps under both models. With the calibrated parameters for our model being $k = 5.4897, \theta_0 = 0.0523, \sigma_1 = 0.7751, \lambda = 0.0822, \sigma_2 = 0.0074, \rho = -0.7439, v_0 = 0.0342$, and for the Heston model being $k = 4.4766, \widetilde{v} = 0.0702, \sigma = 1.0371, \rho = -0.4230, v_0 = 0.0356$, the results with respect to different sampling

https://doi.org/10.1017/S144618112200013X Published online by Cambridge University Press

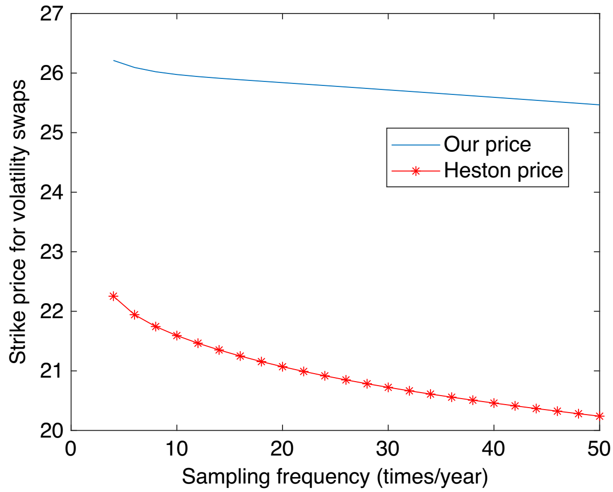

FIGURE 4. Market test with different sampling frequency.

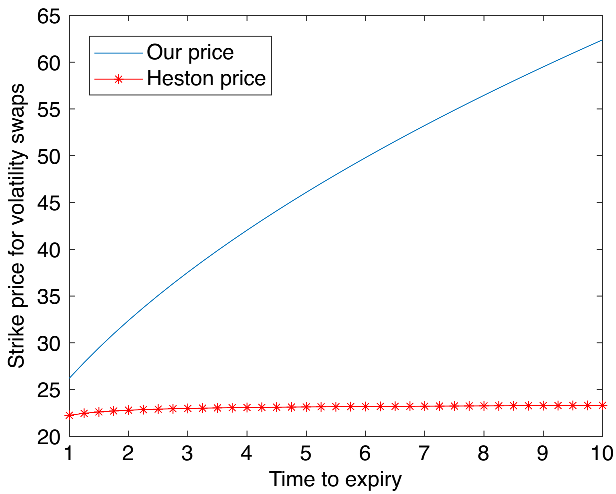

FIGURE 5. Market test with different time to expiry.

frequency are presented in Figure 4. We observe that there is a large difference between the two models, and such difference is further widened when the sampling frequency increases. A similar phenomenon is shown in Figure 5, where the delivery prices are plotted against different time to expiry. With the lifetime of the contract being larger, a greater gap between the two model prices is generated. We then conclude that the inclusion of a stochastic long-run variance level in the risk-neutralized volatility process can make a significant difference in volatility swap prices. Thus, the adopted model can serve as an alternative to the Heston model in practice when pricing volatility swaps.

4. Conclusion

In this paper, we present a closed-form pricing formula for discretely sampled volatility swaps under the modified Heston model, after successfully working out the forward characteristic function of the underlying price. The newly derived formula is shown to be accurate through numerical comparison with the results from the Monte Carlo simulation. The influence of introducing the stochastic long-run variance level into the risk-neutralized Heston model on volatility swap prices is also shown to be significant, implying that the modified Heston model may serve as a competitor to the Heston model for volatility swap pricing.

Acknowledgements

This work was supported by Zhejiang Provincial Natural Science Foundation of China (No. LQ22A010010), the National Natural Science Foundation of China (No. 12101554), the Fundamental Research Funds for Zhejiang Provincial Universities (No. GB202103001) and A Project Supported by Scientific Research Fund of Zhejiang Provincial Education Department (No. Y202147703). The authors would also like to gratefully acknowledge the anonymous referees' constructive comments and suggestions, which greatly improved the quality and presentation of the manuscript.

Appendix

Following is the proof of Proposition 2.1.

PROOF. According to the tower rule of expectation, the conditional forward characteristic function $f(\phi; t, T, v_0, \theta_0)$ can be calculated as

$$\begin{aligned} f(\phi; t, T, v_0, \theta_0) &= E(e^{j\phi x_{t,T}} | v_0, \theta_0) \\ &= E[E(e^{j\phi x_{t,T}} | v_t, \theta_t) | v_0, \theta_0]. \end{aligned}$$

As a result, the calculation of $f(\phi; t, T, v_0, \theta_0)$ can be divided into two steps, that is, the inner expectation and outer expectation. If we define

$$h(\phi; \tau, v_t, \theta_t) = E(e^{j\phi x_{t,T}} | v_t, \theta_t),$$

as the inner expectation with $\tau = T - t$, then h can be formulated as

$$h(\phi; \tau, v_t, \theta_t) = e^{C(\phi;\tau)+D(\phi;\tau)v_t+E(\phi;\tau)\theta_t+j\phi x_{t,t}},$$

according to the results in [15]. With the expressions of $C(\phi; \tau)$, $D(\phi; \tau)$ and $E(\phi; \tau)$, the forward characteristic function $f(\phi; t, T, v_0, \theta_0)$ can be expressed as

$$\begin{aligned} f(\phi; t, T, v_0, \theta_0) &= E(e^{C(\phi;\tau)+D(\phi;\tau)v_t+E(\phi;\tau)+\theta_t+j\phi x_{t,t}} | v_0, \theta_0), \\ &= e^{C(\phi;\tau)} E(e^{D(\phi;\tau)v_t+E(\phi;\tau)\theta_t} | v_0, \theta_0), \end{aligned} \tag{A.1}$$

by noticing the fact that $x_{t,t} = 0$. Hence, what remains is to work out the expectation shown in equation (A.1). If we define

$$m(\phi; 0, t, v_0, \theta_0) = E(e^{D(\phi;\tau)v_t + E(\phi;\tau)\theta_t} | v_0, \theta_0),$$

the Feynman–Kac theorem shows that m should satisfy

$$\begin{cases} \dfrac{\partial m}{\partial s} + \dfrac{1}{2}\sigma_1^2 v \dfrac{\partial^2 m}{\partial v^2} + \dfrac{1}{2}\sigma_2^2 \dfrac{\partial^2 m}{\partial \theta^2} + k(\bar{v} + \theta - v)\dfrac{\partial m}{\partial v} + \lambda \dfrac{\partial m}{\partial \theta}, \\ m(\phi; 0, t, v_0, \theta_0) = e^{D(\phi;\tau)v + E(\phi;\tau)\theta}. \end{cases} \tag{A.2}$$

With careful observation of the expression for the terminal condition, we further assume that

$$m(\phi; s, t, v_s, \theta_s) = e^{\bar{C}(\phi;t) + \bar{D}(\phi;t)v + \bar{E}(\phi;t)\theta},$$

and substitute it into the partial differential equation (A.2). In this case, we could also obtain three ordinary differential equations (ODEs):

$$\begin{aligned} \frac{\partial \bar{D}}{\partial \tau} &= \frac{1}{2}\sigma_1^2 \bar{D}^2 - k\bar{D}, \\ \frac{\partial \bar{E}}{\partial \tau} &= k\bar{D}, \\ \frac{\partial \bar{C}}{\partial \tau} &= \frac{1}{2}\sigma_2^2 \bar{E}^2 + \lambda \bar{E} + k\bar{v}\bar{D}, \end{aligned}$$

with the terminal condition $\bar{C}(\phi; t) = 0, \bar{D}(\phi; t) = D(\phi; \tau), \bar{E}(\phi; t) = E(\phi; \tau)$. We could reach the final solution by solving these three ODEs. This completes the proof of the proposition. □

References

[1] T. G. Andersen, L. Benzoni and J. Lund, "An empirical investigation of continuous-time equity return models", *J. Finance* **57** (2002) 1239–1284; doi:10.1111/1540-6261.00460.

[2] G. Bakshi, N. Ju and H. Ou-Yang, "Estimation of continuous-time models with an application to equity volatility dynamics", *J. Financ. Econ.* **82** (2006) 227–249; doi:10.1016/j.jfineco.2005.09.005.

[3] J. Cao, J.-H. Kim and W. Zhang, "Pricing variance swaps under hybrid CEV and stochastic volatility", *J. Comput. Appl. Math.* **386** (2021) Article Id 113220; doi:10.1016/j.cam.2020.113220.

[4] J. Cao, G. Lian and T. R. N. Roslan, "Pricing variance swaps under stochastic volatility and stochastic interest rate", *Appl. Math. Comput.* **277** (2016) 72–81; doi:10.1016/j.amc.2015.12.027.

[5] J. Cao, T. R. N. Roslan and W. Zhang, "Pricing variance swaps in a hybrid model of stochastic volatility and interest rate with regime-switching", *Methodol. Comput. Appl. Probab.* **20** (2018) 1359–1379; doi:10.1007/s11009-018-9624-5.

[6] J. Cao, T. R. N. Roslan and W. Zhang, "The valuation of variance swaps under stochastic volatility, stochastic interest rate and full correlation structure", *J. Korean Math. Soc.* **57** (2020) 1167–1186; doi:10.4134/JKMS.j190616.

[7] P. Carr and R. Lee, "Realized volatility and variance: options via swaps", *Risk* **20** (2007) 76–83; https://www.risk.net/derivatives/equity-derivatives/1500285/realised-volatility-and-variance-options-swaps.

[8] P. Carr and R. Lee, "Robust replication of volatility derivatives", In PRMIA award for Best Paper in Derivatives, MFA 2008 Annual Meeting, http://faculty.baruch.cuny.edu/lwu/890/CarrLee2004.pdf.

[9] P. Christoffersen, S. Heston and K. Jacobs, "The shape and term structure of the index option smirk: why multifactor stochastic volatility models work so well", *Manage. Sci.* **55** (2009) 1914–1932; doi:10.1287/mnsc.1090.1065.

[10] Z. Cui, J. L. Kirkby and D. Nguyen, "A general framework for discretely sampled realized variance derivatives in stochastic volatility models with jumps", *European J. Oper. Res.* **262** (2017) 381–400; doi:10.1016/j.ejor.2017.04.007.

[11] R. J. Elliott and G.-H. Lian, "Pricing variance and volatility swaps in a stochastic volatility model with regime switching: discrete observations case", *Quant. Finance* **13** (2013) 687–698; doi:10.1080/14697688.2012.676208.

[12] R. J. Elliott, T. K. Siu and L. Chan, "Pricing volatility swaps under Heston's stochastic volatility model with regime switching", *Appl. Math. Finance* **14** (2007) 41–62; doi:10.1080/13504860600659222.

[13] J. Gil-Pelaez, "Note on the inversion theorem", *Biometrika* **38** (1951) 481–482; doi:10.2307/2332598.

[14] A. Grünbichler and F. A. Longstaff, "Valuing futures and options on volatility", *J. Bank. Financ.* **20** (1996) 985–1001; doi:10.1016/0378-4266(95)00034-8.

[15] X.-J. He and W. Chen, "A closed-form pricing formula for European options under a new stochastic volatility model with a stochastic long-term mean", *Math. Financ. Econ.* **15** (2021) 381–396; doi:10.1007/s11579-020-00281-y.

[16] X.-J. He and S.-P. Zhu, "A series-form solution for pricing variance and volatility swaps with stochastic volatility and stochastic interest rate", *Comput. Math. Appl.* **76** (2018) 2223–2234; doi:10.1016/j.camwa.2018.08.022.

[17] X.-J. He and S.-P. Zhu, "Variance and volatility swaps under a two-factor stochastic volatility model with regime switching", *Int. J. Theor. Appl. Finance* **22** (2019) Article Id 1950009; doi:10.1142/S0219024919500092.

[18] S. Heston and S. Nandi, "Derivatives on volatility: some simple solutions based on observables" (Federal Reserve Bank of Atlanta WP No. 2000-20, November 2000); doi:10.2139/ssrn.249173.

[19] S. L. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Rev. Financ. Stud.* **6** (1993) 327–343; doi:10.1093/rfs/6.2.327.

[20] S. Howison, A. Rafailidis and H. Rasmussen, "On the pricing and hedging of volatility derivatives", *Appl. Math. Finance* **11** (2004) 317–346; doi:10.1080/1350486042000254024.

[21] A. Issaka, "Variance swaps, volatility swaps, hedging and bounds under multi-factor Heston stochastic volatility model", *Stoch. Anal. Appl.* **38** (2020) 856–874; doi:10.1080/07362994.2020.1730903.

[22] S.-W. Kim and J.-H. Kim, "Variance swaps with double exponential Ornstein–Uhlenbeck stochastic volatility", *North Am. J. Econ. Finance* **48** (2019) 149–169; doi:10.1016/j.najef.2019.01.018.

[23] M.-K. Lee, S.-W. Kim and J.-H. Kim, "Variance swaps under multiscale stochastic volatility of volatility", *Methodol. Comput. Appl. Probab.* **24** (2022) 39–64; doi:10.1007/s11009-020-09834-6.

[24] T. Little and V. Pant, "A finite difference method for the valuation of variance swaps", *J. Comput. Finance* **5** (2001) 81–101; doi:10.21314/JCF.2001.057.

[25] W. Liu and S.-P. Zhu, "Pricing variance swaps under the Hawkes jump-diffusion process", *J. Futures Markets* **39** (2019) 635–655; doi:10.1002/fut.21997.

[26] J. Pan, "The jump-risk premia implicit in options: evidence from an integrated time-series study", *J. Financ. Econ.* **63** (2002) 3–50; doi:10.1016/S0304-405X(01)00088-5.

[27] C. S. Pun, S. F. Chung and H. Y. Wong, "Variance swap with mean reversion, multifactor stochastic volatility and jumps", *European J. Oper. Res.* **245** (2015) 571–580; doi:10.1016/j.ejor.2015.03.026.

[28] H. Wu, Z. Jia, S. Yang and C. Liu, "Pricing variance swaps under double Heston stochastic volatility model with stochastic interest rate", *Probab. Engrg. Inform. Sci.* **36** (2022) 564–580; doi:10.1017/S0269964820000662.

[29] S.-P. Zhu and G.-H. Lian, "Analytically pricing volatility swaps under stochastic volatility", *J. Comput. Appl. Math.* **288** (2015) 332–340; doi:10.1016/j.cam.2015.04.036.