

MODULI OF CURVES WITH NONSPECIAL DIVISORS AND RELATIVE MODULI OF A_∞ -STRUCTURES

ALEXANDER POLISHCHUK^{1,2}

¹*University of Oregon, USA* (apolish@uoregon.edu)

²*National Research University Higher School of Economics, Russia*

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Abstract In this paper, for each $n \geq g \geq 0$ we consider the moduli stack $\tilde{\mathcal{U}}_{g,n}^{ns}$ of curves $(C, p_1, \dots, p_n, v_1, \dots, v_n)$ of arithmetic genus g with n smooth marked points p_i and nonzero tangent vectors v_i at them, such that the divisor $p_1 + \dots + p_n$ is nonspecial (has $h^1 = 0$) and ample. With some mild restrictions on the characteristic we show that it is a scheme, affine over the Grassmannian $G(n-g, n)$. We also construct an isomorphism of $\tilde{\mathcal{U}}_{g,n}^{ns}$ with a certain relative moduli of A_∞ -structures (up to an equivalence) over a family of graded associative algebras parametrized by $G(n-g, n)$.

Keywords: A_∞ -algebra; moduli of curves; Sato Grassmannian

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Introduction

This paper continues the study of connections between moduli spaces of curves and A_∞ -algebras, started in [10]. Recall that in [10] we gave an interpretation of the moduli of curves of arithmetic genus g with g (distinct smooth) marked points forming a nonspecial divisor, as a certain moduli space of A_∞ -algebras. In the present paper, we consider a generalization of this picture to the case of curves with n marked points, where $n \geq g$.

The second motivation for this work is the relation, pointed out in [6], between the moduli space of curves (C, p_1, \dots, p_n) of arithmetic genus 1 such that $H^1(C, \mathcal{O}(p_i)) = 0$ for every i and one of the moduli spaces studied by Smyth in [14] and [15]. For each $1 \leq m < n$ he constructed an alternate compactification $\overline{\mathcal{M}}_{1,n}(m)$ of the moduli space of n -pointed curves of genus 1 consisting of m -stable curves. The moduli space that shows up in [6] is $\overline{\mathcal{M}}_{1,n}(n-1)$. In the present paper, we consider a bigger moduli stack of curves of genus 1, so that Smyth's moduli spaces for all $m \geq (n-1)/2$ admit natural regular morphisms to our moduli stack (see Section 1.5).

Let us fix $n \geq g$. The main object of study of this paper is the moduli stack $\mathcal{U}_{g,n}^{ns}$ of (C, p_1, \dots, p_n) , where C is a reduced connected projective curve of arithmetic genus

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g, p_1, \dots, p_n are distinct smooth points on C such that $H^1(C, \mathcal{O}(p_1 + \dots + p_n)) = 0$ and $\mathcal{O}(p_1 + \dots + p_n)$ is ample. We also consider the \mathbb{G}_m^n -torsor over this stack, $\tilde{\mathcal{U}}_{g,n}^{ns}$, corresponding to choices of nonzero tangent vectors v_1, \dots, v_n at the marked points.

Note that the vanishing of $H^1(C, \mathcal{O}(p_1 + \dots + p_n))$ is equivalent to the surjectivity of the map

$$H^0(C, \mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C) \rightarrow H^1(C, \mathcal{O}_C). \tag{0.0.1}$$

Hence, its kernel is $(n - g)$ -dimensional. Thus, we have a natural morphism

$$\pi : \tilde{\mathcal{U}}_{g,n}^{ns} \rightarrow G(n - g, n), \tag{0.0.2}$$

where $G(n - g, n)$ is the Grassmannian of $(n - g)$ -dimensional subspaces in the n -dimensional space, associating with $(C, p_1, \dots, p_n, v_1, \dots, v_n)$ the kernel of the map (0.0.1), where $H^0(C, \mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C)$ is trivialized using the basis v_1, \dots, v_n .

Note that some closely related moduli stacks were considered in [11]. Namely, the preimages of the standard cells in $G(n - g, n)$ under π are the open substacks $\tilde{\mathcal{U}}_{g,n}^{ns}(S) \subset \tilde{\mathcal{U}}_{g,n}^{ns}$, for subsets $S \subset \{1, \dots, n\}$ such that $|S| = g$, given by the condition $H^1(C, \mathcal{O}_C(\sum_{i \in S} p_i)) = 0$. These stacks are precisely the stacks $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a})$, defined in [11] for collections $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geq 0}^n$ such that $\sum_i a_i = g$, in the case when each a_i is either 0 or 1.

Working over \mathbb{Q} we proved in [11] that each $\tilde{\mathcal{U}}_{g,n}^{ns}(S)$ is in fact an affine scheme of finite type, and identified it with the quotient of a certain locally closed subset of the Sato Grassmannian of subspaces in $\mathcal{H} = \bigoplus_{i=1}^n k((t_i))$ by the free action of the group of changes of variables. The first result of this paper, Theorem A below, gives analogous statements for $\tilde{\mathcal{U}}_{g,n}^{ns}$ (in this case the morphism π is affine of finite type).

As in [11] we consider the closed subset ASG of the Sato Grassmannian consisting of W that are subalgebras of \mathcal{H} . Let $ASG^{ns} \subset ASG$ be the open subset consisting of W such that

$$W \cap \bigoplus_{i=1}^n k[[t_i]] = k, \quad \dim\left(\mathcal{H}/\left(W + \bigoplus_{i=1}^n k[[t_i]]\right)\right) = g, \quad \text{and} \quad \mathcal{H} = W + \bigoplus_{i=1}^n t_i^{-1}k[[t_i]].$$

There is a natural action on ASG^{ns} of the group \mathfrak{G} of changes of variables of the form $t_i \mapsto t_i + c_{1i}t_i^2 + c_{2i}t_i^3 + \dots, i = 1, \dots, n$.

Theorem A (see Theorem 1.2.2). (i) *Assume that either*

- $n \geq g \geq 1, n \geq 2$ and the base is $\text{Spec}(\mathbb{Z}[1/2])$, or
- $n = g = 1$ and the base is $\text{Spec}(\mathbb{Z}[1/6])$, or
- $g = 0, n \geq 2$ and the base is $\text{Spec}(\mathbb{Z})$.

Then the stack $\tilde{\mathcal{U}}_{g,n}^{ns}$ is isomorphic to a scheme, affine of finite type over the Grassmannian $G(n - g, n)$, so that the preimages of the standard open cells $U_S \subset G(n - g, n)$, for $S \subset \{1, \dots, n\}, |S| = g$, are the moduli schemes $\tilde{\mathcal{U}}_{g,n}^{ns}(\mathbf{a}_S)$, where \mathbf{a}_S has 1's at the places corresponding to S and 0's elsewhere.

(ii) *Now let us work over $\text{Spec}(\mathbb{Q})$. Then the action of \mathfrak{G} on ASG^{ns} is free, and the Krichever map induces an isomorphism*

$$\tilde{\mathcal{U}}_{g,n}^{ns} \simeq ASG^{ns}/\mathfrak{G}.$$

Note that part (i) of this Theorem is an improvement of [10, Theorem 1.2.4] (where the case $n = g$ was considered, but over $\mathbb{Z}[1/6]$) and of the special case of [11, Theorem A(i)] when \mathbf{a} is a collection of 0's and 1's.

Next, generalizing the work [10] (corresponding to the case $n = g$), we consider A_∞ -algebras associated with curves $(C, p_\bullet, v_\bullet) \in \tilde{\mathcal{U}}_{g,n}^{ns}$. Namely, we consider the object

$$G = \mathcal{O}_C \oplus \mathcal{O}_{p_1} \oplus \cdots \oplus \mathcal{O}_{p_n} \tag{0.0.3}$$

in the perfect derived category of C and consider the natural minimal A_∞ -structure on the corresponding algebra $\text{Ext}^*(G, G)$ (which arises from a dg-model of this Ext-algebra and is defined uniquely up to a gauge equivalence). The key observation is that the associative algebra structure on $\text{Ext}^*(G, G)$ depends only on the corresponding $(n - g)$ -dimensional subspace in k^n .

More precisely, let Q_n be the quiver with $n + 1$ vertices marked as $\mathcal{O}, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n}$ and with the arrows

$$A_i : \mathcal{O}_{p_i} \rightarrow \mathcal{O}, \quad B_i : \mathcal{O} \rightarrow \mathcal{O}_{p_i}, \quad i = 1, \dots, n.$$

Let J_0 be the two-sided ideal in the path algebra $k[Q_n]$ of Q_n , generated by the elements

$$A_i B_i A_i, B_i A_i B_i, A_i B_j,$$

where $i \neq j$. For an $(n - g)$ -dimensional subspace $W \subset k^n$ we define $J_W \subset k[Q_n]$ to be the ideal generated by J_0 together with the additional relations

$$\sum x_i B_i A_i = 0 \quad \text{for every} \quad \sum x_i e_i \in W,$$

and consider the corresponding quotient algebra

$$E_W = k[Q_n]/J_W. \tag{0.0.4}$$

We equip E_W with the \mathbb{Z} -grading by $\text{deg}(A_i) = 0, \text{deg}(B_i) = 1$.

Now for a curve $(C, p_\bullet, v_\bullet) \in \tilde{\mathcal{U}}_{g,n}^{ns}$ there is a canonical isomorphism of associative algebras

$$\text{Ext}^*(G, G) \simeq E_W,$$

for $W = \pi(C, p_\bullet, v_\bullet)$. Thus, from such a curve we get an A_∞ -structure on the algebra E_W .

The family of associative algebras E_W defines a sheaf of \mathcal{O} -algebras $\mathcal{E}_{g,n}$ over $G(n - g, n)$. Extending the techniques developed in [10] we consider the relative moduli space \mathcal{M}_∞ over $G(n - g, n)$, classifying minimal A_∞ -structures on the fibers of $\mathcal{E}_{g,n}$ (for a precise definition, see Definition 2.2.5). We prove that in fact \mathcal{M}_∞ is an affine scheme over $G(n - g, n)$ (over $\mathbb{Z}[1/6]$), and the above construction of A_∞ -structures associated with curves gives an isomorphism of the moduli spaces.

Theorem B (see Theorem 2.3.11). *Under the assumptions of Theorem A(i) we have a natural isomorphism*

$$\tilde{\mathcal{U}}_{g,n}^{ns} \xrightarrow{\sim} \mathcal{M}_\infty$$

of affine schemes over $G(n - g, n)$, compatible with the \mathbb{G}_m^n -action, where $(\lambda_i) \in \mathbb{G}_m^n$ acts on $\tilde{\mathcal{U}}_{g,n}^{ns}$ by rescaling the tangent vectors at the marked points and on \mathcal{M}_∞ by the rescalings

$$A_i \mapsto A_i, \quad B_i \mapsto \lambda_i B_i.$$

Note that in the case $n = g \geq 2$ (respectively, $n \geq 3, g = 0$) this result is a strengthening of a similar isomorphism in [10, Theorem A] (respectively, [10, Theorem 5.2.1]), since we now work over $\mathbb{Z}[1/2]$ (respectively, \mathbb{Z}), not over a field.

The construction of the scheme \mathcal{M}_∞ of minimal A_∞ -structures on $\mathcal{E}_{g,n}$ is a particular case of a more general construction (see Theorem 2.2.6) of the affine scheme classifying equivalence classes of minimal A_∞ -structures on a sheaf of \mathcal{O} -algebras \mathcal{E} over a scheme S , such that \mathcal{E} is locally free of finite rank as an \mathcal{O} -module, and the associative algebras given by the fibers \mathcal{E}_s , for $s \in S$, satisfy the following vanishing condition:

$$HH^i(\mathcal{E}_s)_{<0} = 0 \quad \text{for } i \leq 1, \text{ for every point } s \in S. \tag{0.0.5}$$

As in [10], the important part of the proof of Theorem B is identifying the curves in $\tilde{\mathcal{U}}_{g,n}^{ns}$ such that the corresponding A_∞ -algebras are homotopically trivial, i.e., have $m_i = 0$ for $i \geq 3$. In the case $n = g$, considered in [10], there is only one such curve, C_g^{cusp} , which is the union of g cuspidal curves of genus 1, glued at the cusp. In general, there is a family of such curves, parametrized by $G(n - g, n)$. Namely, these are precisely the invariant points for the action of the diagonal $\mathbb{G}_m \subset \mathbb{G}_m^n$ on $\tilde{\mathcal{U}}_{g,n}^{ns}$. We refer to curves in this family as *special curves*. In Theorem 1.2.2 we prove that special curves form a section of the projection (0.0.2). Special curves are used in the proof of Theorem B as follows. Since the \mathbb{G}_m -action contracts both spaces, $\tilde{\mathcal{U}}_{g,n}^{ns}$ and \mathcal{M}_∞ , to the \mathbb{G}_m -invariant locus (which is $G(n - g, n)$ in both cases), it is enough to study deformations of each special curve and show that they precisely correspond to deformations of E_W as an A_∞ -algebra. This is done using the same ideas as in the case $n = g$, although there are some new features that appear because we now work with a family of associative algebras (see Proposition 2.3.10).

Note that our moduli scheme of A_∞ -structures \mathcal{M}_∞ has a natural extension to a derived stack that can be constructed as in [2, (3.2)]. It would be interesting to find an interpretation of this derived extension in terms of moduli of curves.

The paper is organized as follows. Section 1 is devoted to geometric aspects of the moduli stacks $\tilde{\mathcal{U}}_{g,n}^{ns}$. In particular, in § 1.1 we describe a family of special curves in $\tilde{\mathcal{U}}_{g,n}^{ns}$. Then in § 1.2 we prove Theorem A. In § 1.3 we describe the natural gluing morphism that associates with a pair of curves from the moduli spaces $\tilde{\mathcal{U}}_{g_1,n_1}^{ns}$ and $\tilde{\mathcal{U}}_{g_2,n_2}^{ns}$, each equipped with an additional point different from all the marked points, a glued curve in $\tilde{\mathcal{U}}_{g_1+g_2,n_1+n_2}^{ns}$. In §§ 1.4 and 1.5 we study the case $g = 1$: we describe explicitly the space $\tilde{\mathcal{U}}_{1,2}^{ns}$, as well as construct regular morphisms from the Smyth’s moduli spaces of m -stable curves to $\mathcal{U}_{1,n}^{ns}$ for $m \geq (n - 1)/2$. Section 2 is devoted to the relative moduli of A_∞ -structures. After proving some technical results in § 2.1, we give in § 2.2 a general construction of the affine scheme parametrizing A_∞ -structures over a given family of associative algebras (under the assumption (0.0.5)). Finally, in § 2.3 we prove Theorem B.

1. Moduli of curves with nonspecial divisors

1.1. Some special curves

First, we are going to construct some special curves that will play an important role in our study of the moduli spaces $\tilde{\mathcal{U}}_{g,n}^{ns}$.

Definition 1.1.1. (i) Let x_1, \dots, x_n be independent variables, and let R be a commutative ring. For a subset $S \subset \{1, \dots, n\}$ with $|S| = g$ let us consider the subalgebra in $\bigoplus_{i=1}^n R[x_i]$ given by

$$B(S) := R \cdot 1 + \bigoplus_{i \in S} x_i^2 R[x_i].$$

Next, let $(\bar{h}_j)_{1 \leq j \leq n, j \notin S}$ be a collection of linear combinations of $(x_i)_{i \in S}$ with coefficients in R . We define the R -algebra $A(\bar{h}_\bullet)$ as the $B(S)$ -subalgebra in $\bigoplus_{i=1}^n R[x_i]$ generated by the elements $h_j = x_j + \bar{h}_j$, $j \notin S$. We view $A(\bar{h}_\bullet)$ as a graded R -algebra, where $\deg(x_i) = 1$.

(ii) We define two relative curves, one affine and another projective over R , by

$$\begin{aligned} C^{\text{aff}}(\bar{h}_\bullet) &= \text{Spec}(A(\bar{h}_\bullet)), \\ C(\bar{h}_\bullet) &= \text{Proj}(\mathcal{R}(A(\bar{h}_\bullet))), \end{aligned}$$

where $\mathcal{R}(A(\bar{h}_\bullet)) = \bigoplus_{m \geq 0} F_m$ is the Rees algebra associated with the increasing filtration (F_m) on $A(\bar{h}_\bullet)$ coming from the grading. Note that we have an action of \mathbb{G}_m on $C(\bar{h}_\bullet)$ associated with the grading on $A(\bar{h}_\bullet)$, and that $C^{\text{aff}}(\bar{h}_\bullet)$ is a \mathbb{G}_m -invariant affine open in $C(\bar{h}_\bullet)$.

Proposition 1.1.2. (i) For any matrix $(a_{ij})_{i \in S, j \notin S}$ with entries in R , let A be the graded R -algebra defined by the generators $(f_i, h_i, h_{S,j})_{i \in S, j \notin S}$ subject to the equations

$$\begin{aligned} f_i f_{i'} &= 0, \quad f_i h_{i'} = 0, \quad h_i h_{i'} = 0, \quad h_i^2 = f_i^3, \\ h_{S,j} h_{S,j'} &= \sum_{i \in S} a_{ij} a_{ij'} f_i, \\ f_i h_{S,j} &= a_{ij} h_i, \\ h_i h_{S,j} &= a_{ij} f_i^2, \end{aligned} \tag{1.1.1}$$

where $i, i' \in S$, $i \neq i'$, $j, j' \notin S$, $j \neq j'$, and the grading is given by

$$\deg(h_{S,j}) = 1, \quad \deg(f_i) = 2, \quad \deg(h_i) = 3.$$

Then there is an injective homomorphism of graded R -algebras

$$\rho : A \rightarrow \bigoplus_{i=1}^n R[x_i],$$

such that

$$\begin{aligned} \rho(f_i) &= x_i^2, \quad \rho(h_i) = x_i^3, \quad i \in S, \\ \rho(h_{S,j}) &= x_j + \sum_{i \in S} a_{ij} x_i, \quad j \notin S, \end{aligned}$$

inducing an isomorphism $A \simeq A(\bar{h}_\bullet)$, where

$$\bar{h}_j = \sum_{i \in S} a_{ij} x_i.$$

The elements

$$(f_i^n, f_i^n h_i, h_{S,j}^m), \quad i \in S, j \notin S, m \geq 1, n \geq 0, \tag{1.1.2}$$

form a basis of A as an R -module.

- (ii) Assume now $R = k$, where k is a field. Let $C = C(\bar{h}_\bullet)$, $C^{\text{aff}} = C^{\text{aff}}(\bar{h}_\bullet)$ be as in Definition 1.1.1. Then $C \setminus C^{\text{aff}}$ consists of n smooth points p_1, \dots, p_n on C . Furthermore, C is a union of n components C_i , joined in a single point q , which is the only singular point of C (with $p_i \in C_i \setminus \{q\}$). Each component C_i is either \mathbb{P}^1 , or the cuspidal curve of arithmetic genus 1.

Proof. (i) It is easy to see that ρ is well defined, and that the elements (1.1.2) span A over R . On the other hand, one immediately checks that their images under ρ are linearly independent over R and generate the subalgebra $A(\bar{h}_\bullet)$. This implies our assertions.

(ii) The complement $C \setminus C^{\text{aff}}$ is naturally identified with

$$\text{Proj } A(\bar{h}_\bullet) \simeq \text{Proj} \left(\bigoplus_{i=1}^n k[x_i] \right) = \bigsqcup_{i=1}^n p_i.$$

The ring extension $B(S) \subset A(\bar{h}_\bullet)$ induces a finite morphism

$$\phi : C^{\text{aff}} \rightarrow \text{Spec } B(S).$$

Note that $\text{Spec } B(S)$ is the transversal union of g affine cuspidal curves glued at the cusp (numbered by $i \in S$), and our morphism is given by the functions $(f_i, h_i)_{i \in S}$. The fiber over the cusp is the union of the coordinate axes in \mathbb{A}^{n-g} . We claim that over the complement to the cusp ϕ is an isomorphism. Say, $h_i = f_i = 0$ for all $i \neq i_0$, and h_{i_0}, f_{i_0} are invertible. Then the equations

$$h_{S,j} h_{S,l} = a_{i_0 j} a_{i_0 l} f_{i_0}, \quad f_{i_0} h_{S,j} = a_{i_0 j} h_{i_0}, \quad h_{i_0} h_{S,j} = a_{i_0 j} f_{i_0}^2$$

are equivalent to

$$h_{S,j} = a_{i_0 j} \frac{h_{i_0}}{f_{i_0}},$$

for $j \notin S$.

Thus, C^{aff} has n irreducible rational components C_i^{aff} , all joined at one point, where $h_i = f_i = h_{S,j} = 0$. More precisely, for $j \notin S$, we have $C_j^{\text{aff}} \simeq \mathbb{A}^1$: C_j^{aff} is given by the equations $f_i = h_i = 0, h_{S,l} = 0$ for $l \neq j$, and $x_j = h_{S,j}$ is the coordinate on it.

For $i \in S$ there are two cases:

Case 1. There exists $j \notin S$ such that $a_{ij} \neq 0$. In this case $C_i^{\text{aff}} \simeq \mathbb{A}^1$ with the coordinate $x_i = h_{S,j}/a_{ij}$, and the restriction of ϕ to C_i^{aff} is the normalization of the i th component of $\text{Spec } B(S)$. Note that if $a_{il} \neq 0$ for some other $l \notin S$ then $h_{S,j}/a_{ij} = h_{S,l}/a_{il}$ on C_i^{aff} .

Case 2. $a_{ij} = 0$ for all $j \notin S$. Then all $h_{S,j} = 0$ on C_i^{aff} , so C_i^{aff} is the cuspidal curve with the coordinate x_i on the normalization of C_i^{aff} such that $x_i^2 = f_i, x_i^3 = h_i$. The restriction of ϕ to C_i^{aff} maps it isomorphically to the i th component of $\text{Spec } B(S)$.

Note the map ρ is precisely the map associating with a function $f \in A$ its pull-backs to normalizations of the irreducible components $C_1^{\text{aff}}, \dots, C_n^{\text{aff}}$, where we choose

coordinates x_i as above. Let C_i be the closure of C_i^{aff} . The point at infinity $p_i \in C_i$ corresponds either to the point on \mathbb{P}^1 where $x_i = \infty$, or to the infinite point on the projective closure of the cuspidal curve (corresponding to $x_i = \infty$ on its normalization). In particular, all p_i are smooth. □

Note that the curve $C(\bar{h})$ is determined by the corresponding subspace

$$W := \langle x_j + \bar{h}_j \mid j \notin S \rangle \subset \bigoplus_{i=1}^n k \cdot x_i \simeq k^n,$$

which can be any point of the open cell in $G(n - g, n)$ where $W + k^S = k^n$. Later we will show that each curve $C(\bar{h})$ with the marked points p_i and the natural tangent vectors at them induced by x_i^{-1} (viewed as elements of the local ring of $C(\bar{h})$ at p_i) defines a point in $\tilde{\mathcal{U}}_{g,n}^{ns}$, which we denote simply by C_W . We refer to the curves of the form C_W as *special curves*.

Remark 1.1.3. The special curves above are particular cases of the curves considered in [11, § 2.1]. Note that in the case $g = 0$ there is a unique special curve for each n : the union of n projective lines joined in a rational n -fold point.

1.2. Moduli spaces

Recall that for each collection $a_1, \dots, a_n \geq 0$ such that $a_1 + \dots + a_n = g$, we considered in [11] the stack $\tilde{\mathcal{U}}_{g,n}^{ns}(a_1, \dots, a_n)$ of curves (C, p_1, \dots, p_n) of arithmetic genus g such that $H^1(C, \mathcal{O}(\sum a_i p_i)) = 0$ and $\mathcal{O}(p_1 + \dots + p_n)$ is ample, equipped with nonzero tangent vectors at the marked point. Working over $\mathbb{Z}[1/N]$ for sufficiently divisible N , we proved that all of these are affine schemes of finite type, and related them (working over \mathbb{Q}) to certain subschemes of the Sato Grassmannian via the Krichever map.

The moduli spaces considered below are glued from various $\tilde{\mathcal{U}}_{g,n}^{ns}(a_1, \dots, a_n)$, where a_i 's are all 0's and 1's.

Definition 1.2.1. Let us denote by $\mathcal{U}_{g,n}^{ns}$ the moduli stack of (reduced connected projective) curves (C, p_1, \dots, p_n) of arithmetic genus g and n smooth distinct marked points, such that $H^1(C, \mathcal{O}(p_1 + \dots + p_n)) = 0$ and $\mathcal{O}_C(p_1 + \dots + p_n)$ is ample. Let $\tilde{\mathcal{U}}_{g,n}^{ns}$ denote the \mathbb{G}_m^n -torsor over $\mathcal{U}_{g,n}^{ns}$ corresponding to choices of nonzero tangent vectors at all the marked points.

For each subset $S \subset \{1, \dots, n\}$ such that $|S| = g$, we consider the open substack $\tilde{\mathcal{U}}_{g,n}(S) \subset \tilde{\mathcal{U}}_{g,n}^{ns}$ corresponding to curves (C, p_1, \dots, p_n) for which $H^1(C, \mathcal{O}_C(\sum_{i \in S} p_i)) = 0$. This is equivalent to requiring that $H^0(C, \mathcal{O}_C(\sum_{i \in S} p_i)/\mathcal{O}_C)$ surjects onto $H^1(C, \mathcal{O}_C)$. Thus, we have

$$\tilde{\mathcal{U}}_{g,n}(S) = \pi^{-1}(U_S)$$

where $U_S \subset G(n - g, n)$ is the open cell in the Grassmannian corresponding to subspaces $W \subset k^n$ such that $W + k^S = k^n$.

It follows that $\tilde{\mathcal{U}}_{g,n}^{ns}$ is the union of open substacks $\tilde{\mathcal{U}}_{g,n}(S)$, where S ranges over subsets of $\{1, \dots, n\}$ such that $|S| = g$. On the other hand, by definition, we have

$$\tilde{\mathcal{U}}_{g,n}(S) = \tilde{\mathcal{U}}_{g,n}^{ns}(a_1, \dots, a_n),$$

where $a_i = 1$ for $i \in S$ and $a_j = 0$ for $j \notin S$.

We have a natural action of \mathbb{G}_m^n on $\tilde{\mathcal{U}}_{g,n}^{ns}$, so that for $\lambda = (\lambda_1, \dots, \lambda_n)$,

$$\lambda \cdot (C, p_1, \dots, p_n, v_1, \dots, v_n) = (C, p_1, \dots, p_n, \lambda_1^{-1}v_1, \dots, \lambda_n^{-1}v_n).$$

Note that the map $\pi : \tilde{\mathcal{U}}_{g,n}^{ns} \rightarrow G(n - g, n)$ (see (0.0.2)) is \mathbb{G}_m^n -equivariant, where \mathbb{G}_m^n acts on $G(n - g, n)$ via the embedding $\mathbb{G}_m^n \subset GL(n)$ and the natural action of $GL(n)$ on the Grassmannian.

As in [11], we consider the locally closed subset $SG_1(g)$ in the Sato Grassmannian of subspaces of $\mathcal{H} = \bigoplus_{i=1}^n k((t_i))$, consisting of W such that $1 \in W$,

- $W \cap \mathcal{H}_{\geq 0} = \langle 1 \rangle$ and
- $\dim \mathcal{H}/(W + \mathcal{H}_{\geq 0}) = g$,

where $\mathcal{H}_{\geq 0} = \bigoplus_{i=1}^n k[[t_i]]$. We denote by $ASG \subset SG_1(g)$ the closed subscheme consisting of W which are subalgebras in \mathcal{H} . We denote by $SG^{ns} \subset SG_1(g)$ the open subset consisting of W such that

$$W + \bigoplus_{i=1}^n t_i^{-1}k[[t_i]] = \mathcal{H},$$

and we set $ASG^{ns} = ASG \cap SG^{ns}$. All of these schemes can be defined over \mathbb{Z} (see [11, §1.1] for details).

Let $\mathcal{U}_{g,n}^{ns,(\infty)}$ be the torsor over $\mathcal{U}_{g,n}^{ns}$ corresponding to choices of formal parameters (t_1, \dots, t_n) at the marked points. Using [11, Proposition 1.1.5] we see that there is a natural morphism (*Krichever map*)

$$\text{Kr} : \mathcal{U}_{g,n}^{ns,(\infty)} \rightarrow ASG^{ns} : (C, p_\bullet, t_\bullet) \mapsto H^0(C \setminus \{p_1, \dots, p_n\}, \mathcal{O}) \subset \mathcal{H}, \tag{1.2.1}$$

where the embedding into \mathcal{H} is given by the Laurent expansions with respect to the formal parameters (t_\bullet) . This morphism is \mathfrak{G} -equivariant, where $\mathfrak{G} = \prod_{i=1}^n \mathfrak{G}_i$, and \mathfrak{G}_i is the group of changes of t_i of the form $t_i \mapsto t_i + c_1t_i^2 + c_2t_i^3 + \dots$.

Theorem 1.2.2. *Assume that either $n \geq g \geq 1$, $n \geq 2$ and the base is $\text{Spec}(\mathbb{Z}[1/2])$, or $n = g = 1$ and the base is $\text{Spec}(\mathbb{Z}[1/6])$, or $g = 0$, $n \geq 2$ and the base is $\text{Spec}(\mathbb{Z})$.*

- (i) *The stack $\tilde{\mathcal{U}}_{g,n}^{ns}$ is a scheme, the morphism $\pi : \tilde{\mathcal{U}}_{g,n}^{ns} \rightarrow G(n - g, n)$ is affine of finite type and \mathbb{G}_m^n -equivariant.*
- (ii) *The diagonal subgroup $\mathbb{G}_m \subset \mathbb{G}_m^n$ acts on the ring of functions on each open affine $\tilde{\mathcal{U}}_{g,n}(S)$, where $S \subset \{1, \dots, n\}$, $|S| = g$, with nonnegative weights. There is a \mathbb{G}_m^n -equivariant section*

$$\sigma : G(n - g, n) \rightarrow \tilde{\mathcal{U}}_{g,n}^{ns}$$

of the morphism π , such that the locus of \mathbb{G}_m -invariant points in $\tilde{\mathcal{U}}_{g,n}^{ns}$ coincides with the image of σ . These \mathbb{G}_m -invariant points correspond to the special curves considered in Proposition 1.1.2.

(iii) Now let us work over \mathbb{Q} . Then the Krichever map (1.2.1) induces an isomorphism

$$\overline{\text{Kr}} : \tilde{\mathcal{U}}_{g,n}^{ns} \simeq \text{ASG}^{ns} / \mathfrak{G},$$

where the action of \mathfrak{G} on ASG^{ns} is free.

Proof. (i) The case $n = g = 1$ is well known (see e.g., [10, Theorem 1.2.4] or [6, Theorem A]), while the case $n \geq 3, g = 0$ is [10, Theorem 5.1.4]. The case $n = 2, g = 0$ is elementary (and can be worked out as in [10, Lemma 5.1.2]): one has $\tilde{\mathcal{U}}_{0,2}^{ns} \simeq \mathbb{A}^1$, with the universal affine curve $C \setminus \{p_1, p_2\}$ given by the equation $x_1x_2 = t$ (where t is the coordinate on \mathbb{A}^1). Thus, we assume that $n \geq g \geq 1$ and $n \geq 2$.

Since $G(n - g, n)$ is covered by affine open cells U_S , it is enough to prove that each $\tilde{\mathcal{U}}_{g,n}(S)$ is an affine scheme of finite type over the base. As in [10, Theorem 1.2.4] and [11, Theorem A], the main point is to find a canonical basis of the algebra $H^0(C \setminus \{p_1, \dots, p_n\}, \mathcal{O})$ for a family of curves in $\tilde{\mathcal{U}}_{g,n}(S)$ with any affine base $\text{Spec}(R)$.

Set $D_S = \sum_{i \in S} p_i$. We start by constructing an R -basis of the algebra $H^0(C \setminus D_S, \mathcal{O})$. Similarly to [10, Theorem 1.2.4], using the condition $H^1(C, \mathcal{O}(D_S)) = 0$ we choose elements

$$f_i \in H^0(C, \mathcal{O}(p_i + D_S)), \quad h_i \in H^0(C, \mathcal{O}(2p_i + D_S)), \quad \text{for } i \in S,$$

with $f_i \equiv \frac{1}{t_i} \pmod{\frac{1}{t_i} R[[t_i]]}, h_i \equiv \frac{1}{t_i^2} \pmod{\frac{1}{t_i^2} R[[t_i]]}$, where t_i are some formal parameters at p_i , compatible with the chosen tangent vectors. The elements (f_i, h_i) are defined uniquely up to the following transformations

$$(h_i, f_i) \mapsto (h_i + a_i f_i + b_i, f_i + c_i), \tag{1.2.2}$$

for some $a_i, b_i, c_i \in R$. The assumption that $H^1(C, \mathcal{O}(D_S)) = 0$ implies that $H^0(C, \mathcal{O}(D_S)) = R$, hence, the monomials $(f_i^m, f_i^m h_i)_{i \in S, m \geq 0}$ form an R -basis of $H^0(C \setminus D_S, \mathcal{O})$ (cf. [10, Lemma 1.2.1(ii)]). By considering the polar parts, as in [10, Lemma 1.2.1], we see that the generators (f_i, h_i) should satisfy relations of the following form:

$$\begin{aligned} f_i f_j &= \alpha_{ji} h_i + \alpha_{ij} h_j + \gamma_{ji} f_i + \gamma_{ij} f_j + \sum_{k \neq i, j} c_{ij}^k f_k + a_{ij}, \\ f_i h_j &= d_{ij} f_j^2 + t_{ji} h_i + v_{ij} h_j + r_{ji} f_i + \delta_{ij} f_j + \sum_{k \neq i, j} e_{ij}^k f_k + b_{ij}, \\ h_i h_j &= \beta_{ji} f_i^2 + \beta_{ij} f_j^2 + \varepsilon_{ji} h_i + \varepsilon_{ij} h_j + \psi_{ji} f_i + \psi_{ij} f_j + \sum_{k \neq i, j} l_{ij}^k f_k + u_{ij}, \\ h_i^2 &= f_i^3 + q_i h_i f_i + r_i f_i^2 + u_i h_i + \sum_{j \neq i} g_i^j h_j + \pi_i f_i + \sum_{j \neq i} k_i^j f_j + s_i, \end{aligned}$$

where $i \neq j$ (the coefficients are some elements of R). Since we assume that 2 is invertible, choosing a_i and b_i in (1.2.2) appropriately we can make the coefficients q_i and u_i in the last equation to be zero. This fixes the ambiguity in a choice of h_i . Assume now that $g \geq 2$. Then to fix the ambiguity in a choice of f_i we observe that by making appropriate changes $f_i \mapsto f_i + c_i$ we can make $\gamma_{i0} = 0$ for $i \neq 0, \gamma_{i0i_1} = 0$ for fixed $i_0, i_1 \in S (i_0 \neq i_1)$. In the case $g = 1$ we leave the ambiguity in choosing f_i for now and fix it later.

Next, for each $j \notin S$ we have $h^0(p_j + D_S) = 2$, so we can choose $h_{S,j} \in H^0(C, \mathcal{O}(p_j + D_S))$ with the polar part $1/t_j$ at p_j , uniquely up to an additive constant. Let us set $D := \sum_{i=1}^n p_i$. Then

$$(f_i^m, f_i^m h_i, h_{S,j}^{m+1}), \quad i \in S, j \notin S, m \geq 0 \tag{1.2.3}$$

is an R -basis of $\mathcal{O}(C \setminus D)$. Indeed, $(h_{S,j}^m)_{m \geq 1, j \notin S}$ generate arbitrary polar parts at points $p_j, j \notin S$, while $(f_i^m, f_i^m h_i)_{i \in S, m \geq 0}$ form a basis of $H^0(C \setminus D_S, \mathcal{O})$.

Let us define $a_{ij}(S) \in R$, where $i \in S, j \notin S$, by

$$h_{S,j} \equiv \frac{a_{ij}(S)}{t_i} \pmod{R[[t_i]]}$$

at p_i . Then using the basis (1.2.3) we see that in addition to the relations satisfied by $(f_i, h_i)_{i \in S}$ we should have relations of the following form:

$$\begin{aligned} h_{S,j} h_{S,j'} &= c_{j'j}(S) h_{S,j} + c_{jj'}(S) h_{S,j'} + \sum_{i \in S} a_{ij}(S) a_{ij'}(S) f_i + \text{const} \\ f_i h_{S,j} &= b_{ij}(S) h_{S,j} + a_{ij}(S) h_i + \sum_{l \in S} d_{ij}^l(S) f_l + \text{const} \\ h_i h_{S,j} &= e_{ij}(S) h_{S,j} + a_{ij}(S) f_i^2 + r_{ij}(S) h_i + \sum_{l \in S} s_{ij}^l(S) f_l + \text{const} \end{aligned}$$

where $i \in S, j, j' \notin S$. Note that $c_{jj'}(S) = h_{S,j}(p_{j'})$, $b_{ij}(S) = f_i(p_j)$, $e_{ij}(S) = h_i(p_j)$. Using these relations we can get rid of the ambiguity in adding a constant to each $h_{S,j}$ by requiring that $d_{i_0 j}^l(S) = 0$ for fixed $i_0 \in S$. Also, in the case $g = 1$ we can now fix the ambiguity in adding a constant to f_i by requiring $b_{i j_0}(S) = 0$ for a fixed $j_0 \notin S$ (which exists since $|S| = 1$ and $n \geq 2$).

As in [10, Lemma 1.2.2(i)], Buchberger’s algorithm gives a system of equations on the constants in the relations between the generators $f_i, h_i, h_{S,j}$, which is equivalent to (1.2.3) being a basis of $H^0(C \setminus D_S, \mathcal{O})$ as an R -module. This system of equations defines an affine scheme S_{GB} of finite type over $\mathbb{Z}[1/2]$. Thus, we get a morphism $\tilde{\mathcal{U}}_{g,n}(S) \rightarrow S_{GB}$. The remainder of the proof is similar to that of [10, Theorem 1.2.4]: starting from Groebner relations of the above form we construct a family of curves with required properties, which gives an inverse morphism $S_{GB} \rightarrow \tilde{\mathcal{U}}_{g,n}(S)$ (to see that this is indeed the inverse morphism one uses the ampleness of $\mathcal{O}(p_1 + \dots + p_n)$).

(ii) The action of $\lambda \in \mathbb{G}_m^n$ on the coordinates on $\tilde{\mathcal{U}}_{g,n}(S)$ is induced by the rescalings

$$f_i \mapsto \lambda_i^2 f_i, \quad h_i \mapsto \lambda_i^3 h_i, \quad h_{S,j} \mapsto \lambda_j h_{S,j}.$$

From this we see that the diagonal action of \mathbb{G}_m does not change (a_{ij}) and acts with positive weights on all the other coordinates. In particular, the \mathbb{G}_m -invariant points correspond to the $(C, p_\bullet, v_\bullet)$ such that the affine curve $C \setminus \{p_1, \dots, p_n\}$ is given by equations (1.1.1) (where we think of f_i and h_i as independent variables). So these are exactly the equations of the special curves considered in Proposition 1.1.2.

Conversely, from part (i), we see that $\tilde{\mathcal{U}}_{g,n}(S)$ can be identified with the affine scheme S_{GB} parametrizing commutative algebras with generators $(f_i, h_i, h_{S,j})$ and relations of

the form specified in (i), such that the elements (1.2.3) form a basis. Thus, by Proposition 1.1.2, the family of special curves over U_S gives a morphism

$$\sigma : U_S \rightarrow \tilde{U}_{g,n}(S).$$

It remains to check that this morphism is a section of the projection $\pi : \tilde{U}_{g,n}(S) \rightarrow U_S$. By definition, we have to prove that for $C = C(a_{ij})$ as in Proposition 1.1.2, where (a_{ij}) is the matrix with $i \in S, j \notin S$, the kernel of the map

$$H^0(C, \mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C) \rightarrow H^1(C, \mathcal{O}_C)$$

gets identified with the subspace $W \subset k^n$ spanned by $(\mathbf{e}_j + \sum_{i \in S} a_{ij} \mathbf{e}_i)_{j \notin S}$, where $H^0(C, \mathcal{O}(p_i)/\mathcal{O})$ is trivialized using the rational function x_i that has a pole of order 1 at p_i . But this follows from the fact that $x_j + \sum_{i \in S} a_{ij} x_i = h_{S,j}$ defines a global section of $\mathcal{O}_C(p_1 + \dots + p_n)$.

(iii) We have seen that $\tilde{U}_{g,n}^{ns}$ is the union of $\tilde{U}_{g,n}^{ns}(\mathbf{a})$, over \mathbf{a} consisting of 0's and 1's. Similarly, ASG^{ns} is the union of $ASG^{\mathbf{a}}$ (where $ASG^{\mathbf{a}} = ASG \cap SG^{\mathbf{a}}$ with $SG^{\mathbf{a}}$ being the open cell of the Sato Grassmannian defined in [11, § 1.3]), over the same set of \mathbf{a} , and the Krichever map is compatible with these open coverings. Hence, the assertion follows from [11, Theorem B]. □

Note that Theorem A is a part of Theorem 1.2.2.

Remarks 1.2.3. (1) If C has arithmetic genus 0 then the vanishing of $H^1(C, \mathcal{O}(p_1 + \dots + p_n))$ is automatic, so the moduli scheme $\tilde{U}_{0,n}^{ns}$ is exactly the space $\tilde{U}_{0,n}[\psi]$ of ψ -prestable curves (with tangent vectors at the marked points) considered in [10, § 5.1]. This case of Theorem 1.2.2(i)(ii) was considered in [10, Theorem 5.1.4]. Note that one of the GIT quotients of $\tilde{U}_{0,n}^{ns}$ by \mathbb{G}_m^n is the space of Boggi-stable (or ψ -stable) curves studied in [1], [3, § 4.2.1] and [4, § 7.2].

(2) The schemes $\tilde{U}_{g,n}^{ns}$ can be reducible. For example, by [9, Theorem (11.10)], if C is the union of n generic lines through one point in \mathbb{P}^{n-3} , then C is not smoothable for $n \geq 15$. It is easy to see that equipping each component of C with a marked point we get a curve (C, p_1, \dots, p_n) satisfying $H^1(C, \mathcal{O}(p_1 + \dots + p_n)) = 0$. Thus, we deduce that $\tilde{U}_{3,n}^{ns}$, for $n \geq 15$, has a component with nonsmoothable curves.

Definition 1.2.4. Let us denote by $U_{g,n}^{ns, '}$ (respectively, $\tilde{U}_{g,n}^{ns, '}$) the moduli stack, defined exactly like $U_{g,n}^{ns}$ (respectively, $\tilde{U}_{g,n}^{ns}$), but without the condition of ampleness of $\mathcal{O}(p_1 + \dots + p_n)$.

Proposition 1.2.5. *There is a natural morphism $U_{g,n}^{ns, '} \rightarrow U_{g,n}^{ns}$ sending a curve (C, p_1, \dots, p_n) to the curve $(\bar{C}, \bar{p}_1, \dots, \bar{p}_n)$, where \bar{C} is the image of C under the morphism to a projective space induced by the linear system $|\mathcal{O}_C(N(p_1 + \dots + p_n))|$ for $N \gg 0$.*

Proof. We will construct a \mathbb{G}_m^n -equivariant morphism

$$\tilde{U}_{g,n}^{ns, '} \rightarrow \tilde{U}_{g,n}^{ns}, \tag{1.2.4}$$

as sketched in [11, Remark 1.7.2.2] (where $\tilde{\mathcal{U}}_{g,n}^{ns,\prime} \rightarrow \mathcal{U}_{g,n}^{ns,\prime}$ is the \mathbb{G}_m^n -torsor corresponding to choices of nonzero tangent vectors at marked points). Let us restrict to the open substack $\tilde{\mathcal{U}}_{g,n}^{ns,\prime}(S)$, defined by the condition $h^1(\mathcal{O}(\sum_{i \in S} p_i)) = 0$, for a fixed subset $S \subset \{1, \dots, n\}$, $|S| = g$ (it will be clear that our morphisms are the same on the intersections).

Let R be a commutative ring, and let $(C, p_\bullet, v_\bullet)$ be a family in $\tilde{\mathcal{U}}_{g,n}^{ns,\prime}(S)(R)$. Now repeating literally the first part of the proof of Theorem 1.2.2(i), we construct a canonical basis of the R -algebra

$$A = H^0(C \setminus \{p_1, \dots, p_n\}, \mathcal{O})$$

and use it to get an R -point in the scheme S_{GB} from that proof (note that this part of the proof does not require the ampleness of $\mathcal{O}(p_1 + \dots + p_n)$). Since the construction is functorial in R , we get a morphism

$$\tilde{\mathcal{U}}_{g,n}^{ns,\prime}(S) \rightarrow S_{GB}.$$

It remains to recall that by the proof of Theorem 1.2.2(i), we have a natural isomorphism $\tilde{\mathcal{U}}_{g,n}^{ns} \simeq S_{GB}$. Thus, we get the required morphism (1.2.4). By the definition, to a curve (C, p_1, \dots, p_n) our morphism associates the curve $\bar{C} := \text{Proj}(\mathcal{R}(A))$ with the induced marked points, where $\mathcal{R}(A) = \bigoplus_m F_m$ is the Rees algebra associated with the filtration on A by the order of pole along $p_1 + \dots + p_n$. This curve can be identified with the image of C in the projective embedding given by $|\mathcal{O}_C(N(p_1 + \dots + p_n))|$ for $N \gg 0$. □

1.3. Gluing morphism

Let $\tilde{\mathcal{C}}_{g,n}^{ns} \rightarrow \tilde{\mathcal{U}}_{g,n}^{ns}$ denote the universal affine curve, i.e., the complement to the sections p_1, \dots, p_n in the universal curve. By definition, the stack $\tilde{\mathcal{C}}_{g,n}^{ns}$ classifies the data $(C, p_1, \dots, p_n, v_1, \dots, v_n; q)$, where $(C, p_\bullet, v_\bullet)$ is in $\tilde{\mathcal{U}}_{g,n}^{ns}$ and q is a point of C , different from the marked points p_1, \dots, p_n (where C can be singular at q). In the case when the marked points are in bijection with a finite set I we use the notation $\tilde{\mathcal{U}}_{g,I}^{ns}$ and $\tilde{\mathcal{C}}_{g,I}^{ns}$ for these moduli stacks.

Example 1.3.1. In the case $n = g = 1$ we need to invert 6 to ensure that the stack $\tilde{\mathcal{U}}_{1,1}^{ns}$ is a scheme. However, it is easy to see that already the stack $\tilde{\mathcal{C}}_{1,1}^{ns} \times \text{Spec}(\mathbb{Z}[1/2])$ is a scheme. Indeed, starting with a family $(C, p, v; q)$ in $\tilde{\mathcal{C}}_{1,1}^{ns}(R)$, where R is a commutative ring, we can normalize functions $f \in H^0(C, \mathcal{O}(2p))$, $h \in H^0(C, \mathcal{O}(3p))$, with the Laurent expansions $f = \frac{1}{t^2} + \dots$, $g = \frac{1}{t^3} + \dots$ at p (where the local parameter t is compatible with the tangent vector v) by the conditions $f(q) = h(q) = 0$, so that the only remaining ambiguity is $h \mapsto h + cf$. We can fix this ambiguity by requiring that

$$h^2 - f^3 = \alpha f^2 + \beta h + \gamma f$$

for uniquely defined constants $\alpha, \beta, \gamma \in R$ (note that there is no fh term on the right-hand side). This gives an isomorphism $\tilde{\mathcal{C}}_{1,1}^{ns} \simeq \mathbb{A}^3$ over $\mathbb{Z}[1/2]$.

Proposition 1.3.2. *Assume that either $n \geq g \geq 1$, $n \geq 2$, and we work over $\mathbb{Z}[1/2]$, or $g = 0$, $n \geq 2$, and we work over \mathbb{Z} .*

- (i) For every partition $[1, n] = I \sqcup J$ into nonempty subsets and a pair of numbers $h \leq |I|, k \leq |J|$, such that $h + k = g$, there is a natural gluing morphism

$$\rho_{I,J}^{h,k} : \tilde{\mathcal{C}}_{h,I}^{ns} \times \tilde{\mathcal{C}}_{k,J}^{ns} \rightarrow \tilde{\mathcal{U}}_{g,n}^{ns}, \tag{1.3.1}$$

sending a pair of curves C_I, C_J (equipped with the marked points, tangent vectors and with the extra points q_I, q_J) to the curve $C = C_I \cup C_J$, where C_I and C_J are glued transversally, so that the points q_I and q_J are identified. The curve C is equipped with n marked points (and tangent vectors at them), so that the points indexed by I come from the marked points on C_I , while those indexed by J come from C_J .

- (ii) The morphism $\rho_{I,J}^{h,k}$ is compatible with the projections to the Grassmannians and with the morphism

$$G(|I| - h, |I|) \times G(|J| - k, |J|) \rightarrow G(n - g, n)$$

sending a pair of subspaces (V, V') to $V \oplus V'$.

- (iii) The morphism $\rho_{I,J}^{h,k}$ is a closed embedding that factors through the closed subscheme $Z_{I,J} \subset \tilde{\mathcal{U}}_{g,n}^{ns}$, given by the conditions

- (1) $H^1(C, \mathcal{O}(\sum_{i \in S} p_i)) \neq 0$ for all $S \subset [1, n]$, such that $|S| = g$ and either $|S \cap I| < h$ or $|S \cap J| < k$;
- (2) for every $S \subset [1, n]$ and $s, t \in [1, n]$, such that $|S| = g, |S \cap I| = h, |S \cap J| = k$, and either $s \in S \cap I, t \in S \cap J$ or $s \in S \cap J, t \in S \cap I$, the morphism

$$H^0\left(C, \mathcal{O}\left(2p_s + \sum_{i=1}^n p_i\right)\right) \rightarrow H^0(C, \mathcal{O}(p_t)/\mathcal{O}) \oplus \bigoplus_{i \notin S} H^0(C, \mathcal{O}(p_i)/\mathcal{O})$$

has rank $\leq n - g$.

Note that the nonvanishing of H^1 in (1) can also be expressed as a degeneracy locus of the morphism of vector bundles over $\tilde{\mathcal{U}}_{g,n}^{ns}$, so we have a natural subscheme structure on $Z_{I,J}$. Furthermore, there exists a retraction morphism from $Z_{I,J}$ onto the image of $\rho_{I,J}^{h,k}$.

Proof. (i,ii) The fact that C_I and C_J are glued transversally means that there is an exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{O}_{C_I} \oplus \mathcal{O}_{C_J} \rightarrow \mathcal{O}_q \rightarrow 0.$$

This leads to an isomorphism

$$H^1(\mathcal{O}_C) \simeq H^1(\mathcal{O}_{C_I}) \oplus H^1(\mathcal{O}_{C_J}), \tag{1.3.2}$$

so the arithmetic genus of C is $h + k = g$. Similarly, the exact sequence

$$0 \rightarrow \mathcal{O}_C(p_1 + \dots + p_n) \rightarrow \mathcal{O}_{C_I}\left(\sum_{i \in I} p_i\right) \oplus \mathcal{O}_{C_J}\left(\sum_{j \in J} p_j\right) \rightarrow \mathcal{O}_q \rightarrow 0$$

shows that

$$H^1(\mathcal{O}_C(p_1 + \dots + p_n)) = H^1\left(\mathcal{O}_{C_I}\left(\sum_{i \in I} p_i\right)\right) \oplus H^1\left(\mathcal{O}_{C_J}\left(\sum_{j \in J} p_j\right)\right)$$

(since the constants in $H^0(\mathcal{O}_{C_I}(\sum_{i \in I} p_i))$ surject onto $H^0(\mathcal{O}_q)$). Similar argument works in families, so our morphism is well defined. The compatibility with the morphism of Grassmannians follows from the decomposition (1.3.2), which is compatible with the similar decomposition of $H^1(\mathcal{O}_C(p_1 + \dots + p_n)/\mathcal{O}_C)$.

(iii) For any subset $S \subset [1, n]$ we have an exact sequence

$$0 \rightarrow \mathcal{O}_C\left(\sum_{i \in S} p_i\right) \rightarrow \mathcal{O}_{C_I}\left(\sum_{i \in S \cap I} p_i\right) \oplus \mathcal{O}_{C_J}\left(\sum_{j \in S \cap J} p_j\right) \rightarrow \mathcal{O}_q \rightarrow 0$$

which gives an isomorphism

$$H^1\left(C, \mathcal{O}_C\left(\sum_{i \in S} p_i\right)\right) \simeq H^1\left(C_I, \mathcal{O}\left(\sum_{i \in S \cap I} p_i\right)\right) \oplus H^1\left(C_J, \mathcal{O}\left(\sum_{j \in S \cap J} p_j\right)\right).$$

Thus, if $|S \cap I| < h$ then $H^1(C_I, \mathcal{O}(\sum_{i \in S \cap I} p_i)) \neq 0$, so that $H^1(C, \mathcal{O}_C(\sum_{i \in S} p_i)) \neq 0$. Similarly, we get the nonvanishing of $H^1(C, \mathcal{O}_C(\sum_{i \in S} p_i))$ if $|S \cap J| < k$. Assume now that S is as in condition (2), $s \in S \cap I$ and $t \in S \cap J$. Using the exact sequence

$$0 \rightarrow \mathcal{O}_C\left(2p_s + \sum_{i=1}^n p_i\right) \rightarrow \mathcal{O}_{C_I}\left(2p_s + \sum_{i \in I} p_i\right) \oplus \mathcal{O}_{C_J}\left(\sum_{j \in J} p_j\right) \rightarrow \mathcal{O}_q \rightarrow 0$$

we see that the degeneracy required in (2) is equivalent to the condition that the morphism

$$\begin{aligned} &H^0\left(C_I, \mathcal{O}\left(2p_s + \sum_{i \in I} p_i\right)\right) \oplus H^0\left(C_J, \mathcal{O}\left(\sum_{j \in J} p_j\right)\right) \\ &\rightarrow H^0(\mathcal{O}_q) \oplus H^0(\mathcal{O}(p_t)/\mathcal{O}) \oplus \bigoplus_{i \notin S} H^0(\mathcal{O}(p_i)/\mathcal{O}) \end{aligned}$$

has rank $\leq n - g + 1$. But this follows from the fact that the composition of this morphism with the projection to $H^0(\mathcal{O}(p_t)/\mathcal{O}) \oplus \bigoplus_{j \in J \setminus S} H^0(\mathcal{O}(p_j)/\mathcal{O})$ has rank $\leq |J| - k$. Indeed, this composition factors through a map

$$H^0\left(C_J, \mathcal{O}\left(\sum_{j \in J} p_j\right)\right) \rightarrow H^0(\mathcal{O}(p_t)/\mathcal{O}) \oplus \bigoplus_{j \in J \setminus S} H^0(\mathcal{O}(p_j)/\mathcal{O})$$

whose cokernel is $H^1(C_J, \mathcal{O}(\sum_{j \in S \cap J \setminus \{t\}} p_j)) \neq 0$. This shows that our morphism factors through the subscheme $Z_{I,J}$.

Next, we are going to construct a retraction

$$r : Z_{I,J} \rightarrow \tilde{\mathcal{C}}_{h,I}^{ns} \times \tilde{\mathcal{C}}_{k,J}^{ns}. \tag{1.3.3}$$

It is enough to construct compatible morphisms on all the affine opens $Z_{I,J} \cap \tilde{\mathcal{U}}_{g,n}(S)$, where $S \subset [1, n]$, $|S| = g$. By condition (1), the intersection is nonempty only when $|S \cap I| = h$ and $|S \cap J| = k$. Let $(C, p_\bullet, v_\bullet)$ be the restriction of the universal family to $Z_{I,J} \cap \tilde{\mathcal{U}}_{g,n}(S)$.

Recall that over $\tilde{U}_{g,n}(S)$ we have generators $f_i, h_i, h_{S,j}$ of $\mathcal{O}(C \setminus D)$, where $D = p_1 + \dots + p_n$ (see the proof of Theorem 1.2.2). Let us consider their restrictions over $Z_{I,J}$ (denoted in the same way). We claim that for every $i \in S \cap I$ and $i' \in I \setminus S$ the functions f_i, h_i and $h_{S,i'}$ are regular at any p_j , where $j \in S \cap J$. Indeed, for $h_{S,i'}$ this follows from the exact sequence

$$H^0\left(C, \mathcal{O}\left(p_{i'} + \sum_{i \in S} p_i\right)\right) \rightarrow H^0(C, \mathcal{O}(p_{i'})/\mathcal{O}) \rightarrow H^1\left(C, \mathcal{O}\left(p_{i'} + \sum_{i \in S \setminus \{j\}} p_i\right)\right) \rightarrow 0$$

since by condition (1) the first arrow is zero (due to the way we represent the nonvanishing of H^1 as a degeneracy locus). Since over $\tilde{U}_{g,n}(S)$, for $i \in S \cap I$, we have an exact sequence

$$0 \rightarrow H^0\left(C, \mathcal{O}\left(2p_i + \sum_{i' \in S} p_{i'}\right)\right) \rightarrow H^0\left(C, \mathcal{O}\left(2p_i + \sum_{i'=1}^n p_{i'}\right)\right) \rightarrow \bigoplus_{i' \notin S} H^0(\mathcal{O}(p_{i'})/\mathcal{O}) \rightarrow 0,$$

condition (2) implies the vanishing of the morphism

$$H^0\left(C, \mathcal{O}\left(2p_i + \sum_{i' \in S} p_{i'}\right)\right) \rightarrow H^0(\mathcal{O}(p_j)/\mathcal{O})$$

for any $j \in S \cap J$. This shows that f_i and h_i have no poles along p_j for such j , proving our claim.

Now we construct two families of curves over $Z_{I,J}$. Let us set

$$\begin{aligned} A_I &:= \mathcal{O}(C \setminus \{p_i \mid i \in I\}), & A_J &:= \mathcal{O}(C \setminus \{p_i \mid i \in J\}), \\ C_I &:= \text{Proj } \mathcal{R}(A_I), & C_J &:= \text{Proj } \mathcal{R}(A_J), \end{aligned}$$

where $\mathcal{R}(A_I)$ (respectively, $\mathcal{R}(A_J)$) are the Rees algebras associated with the filtrations by the order of pole along $\sum_{i \in I} p_i$ (respectively, $\sum_{j \in J} p_j$). Note that the algebra A_I is a free module over $\mathcal{O}(Z_{I,J})$ with the basis

$$(f_i^m, f_i^m h_i, h_{S,i'}^{m+1}), \quad i \in S \cap I, i' \in I \setminus S, m \geq 0.$$

Indeed, this is checked easily by considering polar parts at p_i with $i \in I$, similarly to checking that (1.2.3) is a basis of $\mathcal{O}(C \setminus D)$. This implies that the generators $(f_i, h_i, h_{S,i'})$, where $i \in S \cap I, i' \in I \setminus S$, satisfy similar relations as the full set of generators of $\mathcal{O}(C \setminus D)$, and so, as in the proof of [10, Theorem 1.2.4], we get that C_I is a curve of arithmetic genus $|S \cap I| = h$, that extends to a family in $\tilde{U}_{h,I}^{ns}$. Similar argument works for C_J , so we get a morphism

$$\bar{r} : Z_{I,J} \rightarrow \tilde{U}_{h,I}^{ns} \times \tilde{U}_{k,J}^{ns}.$$

Note that we have natural morphisms $C \rightarrow C_I$ and $C \rightarrow C_J$. Let us consider the compositions

$$q : Z_{I,J} \xrightarrow{p_{j_0}} C \rightarrow C_I, \quad q' : Z_{I,J} \xrightarrow{p_{i_0}} C \rightarrow C_J,$$

for some fixed indices $i_0 \in I, j_0 \in J$. These allow to lift the morphism \bar{r} to the required morphism (1.3.3). One can immediately check that $r \circ \rho_{I,J}^{h,k}$ is the identity morphism. This implies that $\rho_{I,J}^{h,k}$ is a closed embedding. □

Example 1.3.3. In the case $g = 0$ the subscheme $Z_{I,J}$ coincides with the whole space $\tilde{\mathcal{U}}_{0,n}^{ns}$. Thus, in this case the embedding $\rho_{I,J}^{0,0}$ admits a retraction defined on $\tilde{\mathcal{U}}_{0,n}^{ns}$.

Remark 1.3.4. One can also consider the gluing morphism that associates with a curve (C, p_1, \dots, p_n) of arithmetic genus $g - 1$, equipped with extra two points $q_1 \neq q_2$ (possibly singular), different from p_1, \dots, p_n , the curve $(\bar{C}, p_1, \dots, p_n)$ of arithmetic genus g , where \bar{C} is obtained by transversally identifying q_1 and q_2 on C . In order to guarantee that $H^1(\bar{C}, \mathcal{O}(p_1 + \dots + p_n)) = 0$ one has to assume that $H^1(C, \mathcal{O}(p_1 + \dots + p_n)) = 0$ and in addition the morphism

$$\text{ev}_{q_2} - \text{ev}_{q_1} : H^0(C, \mathcal{O}(p_1 + \dots + p_n)) \rightarrow k,$$

is surjective (where ev_{q_i} is the evaluation at q_i).

1.4. Case $g = 1, n = 2$

In this section, we work over $\text{Spec}(\mathbb{Z}[1/6])$. The scheme $\tilde{\mathcal{U}}_{1,2}^{ns}$ is glued from two affine open pieces \mathcal{U}_1 and \mathcal{U}_2 determined by the conditions $H^1(C, \mathcal{O}(p_1)) = 0$ and $H^1(C, \mathcal{O}(p_2)) = 0$, respectively.

Note that we have $\mathcal{U}_1 = \tilde{\mathcal{U}}_{1,2}^{ns}(1, 0)$. The latter moduli space was described explicitly in [11, §3.1] as the affine 4-space with coordinates (a, b, e, π) . Let us rename these coordinates on \mathcal{U}_1 as $a_{12}, b_{12}, e_{12}, \pi_1$. Thus, the universal affine curve $C \setminus \{p_1, p_2\}$ over \mathcal{U}_1 is given by the equations

$$h_1^2 = f_1^3 + \pi_1 f_1 + s_1, \tag{1.4.1}$$

$$f_1 h_{12} = a_{12} h_1 + b_{12} h_{12} + a_{12} e_{12}, \tag{1.4.2}$$

$$h_1 h_{12} = a_{12} f_1^2 + e_{12} h_{12} + a_{12} b_{12} f_1 + a_{12} (\pi_1 + b_{12}^2), \tag{1.4.3}$$

where $s_1 = e_{12}^2 - b_{12}(\pi_1 + b_{12}^2)$. Note that the projection $\mathcal{U}_1 \rightarrow \mathbb{A}^1 \subset \mathbb{P}^1$ is given by the coordinate a_{12} .

Let us denote by f_2, h_2, h_{21} the generators of the algebra of functions on the universal affine curve $C \setminus \{p_1, p_2\}$ over \mathcal{U}_2 , so that $h_{21} \in H^0(C, \mathcal{O}(p_1 + p_2))$ and $h_{21} \equiv 1/t_1$ at p_1 . Let also $a_{21}, b_{21}, e_{21}, \pi_2$ be the coordinates on \mathcal{U}_2 similar to those on \mathcal{U}_1 . Note that over $\mathcal{U}_1 \cap \mathcal{U}_2$ the function $h_{21} - \frac{1}{a_{12}} h_{12}$ is constant along the fibers of the projection to the base. Hence, $h_{21} \equiv \frac{1}{a_{12} t_2}$ at p_2 , so we have

$$a_{21} = \frac{1}{a_{12}}.$$

Lemma 1.4.1. *Over $\mathcal{U}_1 \cap \mathcal{U}_2$ one has*

$$\begin{aligned} h_{21} &= a_{21} h_{12}, & f_2 &= h_{12}^2 - a_{12}^2 f_1 - a_{12}^2 b_{12}, & h_2 &= h_{12}^3 - a_{12}^3 h_1 - 3a_{12}^2 b_{12} h_{12} - 2a_{12}^3 e_{12}, \\ b_{21} &= a_{12}^2 b_{12}, & e_{21} &= a_{12}^3 e_{12}, & \pi_2 &= a_{12}^4 \pi_1, & s_2 &= a_{12}^6 s_1. \end{aligned}$$

Proof. Note that $\mathcal{U}_1 \cap \mathcal{U}_2$ is the open subset in \mathcal{U}_1 , where a_{12} does not vanish. On this open subset, we can express h_1 in terms of f_1 and f_{12} using (1.4.2):

$$h_1 = \frac{f_1 h_{12}}{a_{12}} - \frac{b_{12} h_{12}}{a_{12}} - e_{12}. \tag{1.4.4}$$

Substituting into (1.4.3) we get

$$f_1^2 - \left(\frac{h_{12}^2}{a_{12}^2} - b_{12} \right) f_1 + \frac{b_{12}h_{12}^2}{a_{12}^2} + 2e_{12} \frac{h_{12}}{a_{12}} + \pi_1 + b_{12}^2 = 0$$

which is the single equation defining the universal $C \setminus \{p_1, p_2\}$ ((1.4.1) follows from this and (1.4.4)). Note that this is a quadratic equation in f_1 . Therefore, we can define an involution of $C \setminus \{p_1, p_2\}$ over $\mathcal{U}_1 \cap \mathcal{U}_2$ by

$$(h_{12}, f_1) \mapsto \left(h_{12}, \frac{h_{12}^2}{a_{12}^2} - b_{12} - f_1 \right).$$

We claim that this involution extends to an involution τ of C permuting p_1 and p_2 . Indeed, this immediately follows from the fact that it preserves the filtration by the degree of pole along $p_1 + p_2$ (so that $\deg(h_{12}) = 1, \deg(f_1) = 2$), together with the observation that $\tau^*(f_1)$ has a pole at p_2 .

Note that

$$\tau^*(f_1) = \frac{h_{12}^2}{a_{12}^2} - b_{12} - f_1, \tag{1.4.5}$$

which has the expansion $\frac{1}{a_{12}^2 t_2^2} + \dots$ at p_2 . Similarly

$$\tau^*(h_1) = \frac{h_{12}}{a_{12}} \tau^*(f_1) - \frac{b_{12}h_{12}}{a_{12}} - e_{12} \equiv \frac{1}{a_{12}^3 t_2^3} + \dots$$

at p_2 . Now the equation

$$\tau^*(h_1)^2 = \tau^*(f_1)^3 + \pi_1 \tau^*(f_1) + s_1$$

(obtained from (1.4.1)), together with the definition of (f_2, h_2) , shows that

$$\begin{aligned} \tau^*(f_1) &= \frac{1}{a_{12}^2} f_2, & \tau^*(h_1) &= \frac{1}{a_{12}^3} h_2, \\ \pi_2 &= a_{12}^4 \pi_1, & s_2 &= a_{12}^6 s_1. \end{aligned} \tag{1.4.6}$$

We also have

$$b_{21} = f_2(p_1) = a_{12}^2 (\tau^* f_1)(p_1) = a_{12}^2 f_1(\tau(p_1)) = a_{12}^2 f_1(p_2) = a_{12}^2 b_{12}.$$

Similarly, we get that $e_{21} = a_{12}^3 e_{21}$. Finally, we get the required formulas for f_2 and h_2 by using (1.4.5), (1.4.6) and (1.4.4). \square

The above lemma shows that the functions b_{12}, e_{12}, π_1 (respectively, b_{21}, e_{21}, π_2) defined on \mathcal{U}_1 (respectively, \mathcal{U}_2) actually extend to regular functions on the entire moduli space.

Proposition 1.4.2. *Let us work over $\mathbb{Z}[1/6]$. The scheme $\tilde{\mathcal{U}}_{1,2}^{ns}$ is isomorphic as a \mathbb{P}^1 -scheme to the total space of the vector bundle $\mathcal{O}(-2) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4)$ over \mathbb{P}^1 . This isomorphism is \mathbb{G}_m^2 -equivariant, where we use the natural action of $\mathbb{G}_m^2 \subset \text{GL}_2$ on \mathbb{P}^1 and its standard lifting to $\mathcal{O}(i)$.*

Proof. The morphism $\pi : \tilde{\mathcal{U}}_{1,2}^{ns} \rightarrow \mathbb{P}^1$ is given by $(1 : a_{12})$ on \mathcal{U}_1 and by $(a_{21} : 1)$ on \mathcal{U}_2 , where $a_{12} = a_{21}^{-1}$ on the intersection. Since $\tilde{\mathcal{U}}_{1,2}^{ns}$ is glued from the open subsets \mathcal{U}_1 and

\mathcal{U}_2 , using the identifications of \mathcal{U}_1 and \mathcal{U}_2 with \mathbb{A}^4 and the transition formulas from Lemma 1.4.1, we obtain that $\tilde{\mathcal{U}}_{1,2}^{ns}$ is isomorphic to the subscheme of $\mathbb{P}^1 \times \mathbb{A}^6$ given by the equations

$$t_1^2 b_{21} = t_2^2 b_{12}, \quad t_1^3 e_{21} = t_2^3 e_{12}, \quad t_1^4 \pi_2 = t_2^4 \pi_1,$$

where $(t_1 : t_2)$ are homogeneous coordinates on \mathbb{P}^1 . Hence, b_{12}/t_1^2 , e_{12}/t_1^3 and π_1/t_1^4 extend naturally to regular sections of $\pi^* \mathcal{O}(-2)$, $\pi^* \mathcal{O}(-3)$ and $\pi^* \mathcal{O}(-4)$, respectively. This gives a morphism from $\tilde{\mathcal{U}}_{1,2}^{ns}$ to the total space of $\mathcal{O}(-2) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4)$ over \mathbb{P}^1 . Since it restricts to isomorphisms over the open subsets $t_1 \neq 0$ and $t_2 \neq 0$, it is an isomorphism. \square

Remark 1.4.3. Under the isomorphism of Proposition 1.4.2, the \mathbb{G}_m -invariant points in $\tilde{\mathcal{U}}_{1,2}^{ns}$ (see Theorem 1.2.2(ii)) get identified with the zero section in the total space of $\mathcal{O}(-2) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4)$ over \mathbb{P}^1 . Over $\mathbb{P}^1 \setminus \{0, \infty\}$ the corresponding curve is the tacnode, while at 0 and ∞ we get the union of the genus 1 cuspidal curve and of the projective line, joined to the cusp transversally.

In the remainder of this section, we work over an algebraically closed field k of characteristic $\neq 2, 3$.

Corollary 1.4.4. *The graded algebra*

$$A = \bigoplus_{n \geq 0} H^0(\tilde{\mathcal{U}}_{1,2}^{ns}, \pi^* \mathcal{O}(n)) = \bigoplus_{n \geq 0} H^0(\mathbb{P}^1, S(\mathcal{O}(2)) \otimes S(\mathcal{O}(3)) \otimes S(\mathcal{O}(4))(n))$$

can be identified with the $k[t_1, t_2]$ -subalgebra of $k[t_1, t_2, x, y, z]$ (where t_1, t_2, x, y, z are independent variables), such that the n th graded component A_n is spanned by the monomials

$$t_1^i t_2^j x^k y^l z^m \quad \text{with } i + j = 2k + 3l + 4m + n.$$

This identification is compatible with the \mathbb{G}_m^2 -actions, where x, y, z are \mathbb{G}_m^2 -invariant, while t_1 and t_2 have \mathbb{G}_m^2 -weights $(1, 0)$ and $(0, 1)$, respectively.

Proof. We use the \mathbb{G}_m^2 -invariant regular sections of $\pi^* \mathcal{O}(-2)$, $\pi^* \mathcal{O}(-3)$ and $\pi^* \mathcal{O}(-4)$,

$$x = b_{12}/t_1^2, \quad y = e_{12}/t_1^3, \quad z = \pi_1/t_1^4. \quad \square$$

Using the above description we can describe the GIT quotients $\tilde{\mathcal{U}}_{1,2}^{ns} //_{\chi} \mathbb{G}_m^2$ with respect to the \mathbb{G}_m^2 -linearizations on the line bundle $\mathcal{O}(1)$ on $\tilde{\mathcal{U}}_{1,2}^{ns}$, which differ from the standard \mathbb{G}_m^2 -equivariant structure by (rational) characters $\chi(\lambda_0, \lambda_1) = \lambda_0^u \lambda_1^v$ of \mathbb{G}_m^2 (if u and v are fractional this means that we really work with $\mathcal{O}(N)$ for some N).

We have

$$\tilde{\mathcal{U}}_{1,2}^{ns} //_{\chi} \mathbb{G}_m^2 = \text{Proj}(A(u, v)),$$

where $A(u, v) \subset A$ is the corresponding subalgebra of invariants with respect to the χ -twisted \mathbb{G}_m^2 -action on the algebra A :

$$A(u, v) := \bigoplus_{n \geq 0} (A_n \otimes \chi^{-n}) \mathbb{G}_m^2$$

(here, if χ is not integral, we have to pass to a Veronese subalgebra of A). Using Corollary 1.4.4 we see that the $A(u, v)_n$ is spanned by the monomials

$$t_1^{nu} t_2^{nv} x^k y^l z^m \quad \text{with } n(u + v - 1) = 2k + 3l + 4m. \tag{1.4.7}$$

Note that $A(u, v)$ reduces to constants if either $u < 0$ or $v < 0$ or $u + v < 1$. In the case $u \geq 0, v \geq 0, u + v = 1$, the algebra $A(u, v)$ has a basis of monomials $t_1^{nu} t_2^{nv}$, so it is isomorphic to an algebra of polynomials in one variable $t_1^{Nu} t_2^{Nv}$, where $N > 0$ is minimal such that Nu, Nv are integers. Thus, in this case the GIT quotient reduces to a point.

Proposition 1.4.5. *Assume $u \geq 0, v \geq 0, u + v > 1$. Then $A(u, v)$ is isomorphic to a Veronese subalgebra in $k[x, y, z]$, where $\deg(x) = 2, \deg(y) = 3, \deg(z) = 3$. Hence, for $u + v > 1$ the GIT quotient $\tilde{\mathcal{U}}_{1,2}^{ns} //_\chi \mathbb{G}_m^2$ is isomorphic to the weighted projective plane $\mathbb{P}(2, 3, 4)$. The χ -unstable locus in $\tilde{\mathcal{U}}_{1,2}^{ns}$ is the union of the locus of \mathbb{G}_m -invariant points and of $\pi^{-1}(t_1^{Nu} = 0) \cup \pi^{-1}(t_2^{Nv} = 0)$, where $N > 0$ is such that $Nu, Nv \in \mathbb{Z}$.*

Proof. Let $N > 0$ be minimal such that Nu, Nv are integers. We associate with each monomial (1.4.7) the corresponding monomial $x^k y^l z^m$. This gives an isomorphism of $A(u, v)$ with the subalgebra of $k[x, y, z]$ spanned by all such monomials with $n := \frac{2k+3l+4m}{u+v-1}$ belonging to $N\mathbb{Z}$. Let n_0 be minimal such $n > 0$. Then this subalgebra is precisely the Veronese subalgebra corresponding to $n_0(u + v - 1)$. The identification of the unstable locus follows from the form of the monomials (1.4.7), since the locus of \mathbb{G}_m -invariant points coincides with the locus where all the sections x, y, z vanish. \square

1.5. Case $g = 1$, arbitrary $n \geq 2$: connection to Smyth’s moduli

As before, we work over $\mathbb{Z}[1/6]$.

Recall that for integers $1 \leq m < n$, Smyth defined in [14] the moduli stack $\overline{\mathcal{M}}_{1,n}(m)$ of n -pointed m -stable curves of arithmetic genus 1, parametrizing curves (C, p_1, \dots, p_n) of arithmetic genus 1 with n distinct smooth marked points such that

- C has only nodes and elliptic l -fold points, with $l \leq m$, as singularities;
- if $E \subset C$ is any connected subcurve of arithmetic genus 1 then $|E \cap \overline{C \setminus E}| + |E \cap \{p_1, \dots, p_n\}| > m$;
- $H^0(C, \mathcal{T}_C(-p_1 - \dots - p_n)) = 0$, where \mathcal{T}_C is the tangent sheaf.

Smyth showed that $\overline{\mathcal{M}}_{1,n}(m)$ is a proper irreducible Deligne–Mumford stack.

Proposition 1.5.1. *Assume $m \geq \frac{n-1}{2}$. Then there exists a morphism*

$$\overline{\mathcal{M}}_{1,n}(m) \rightarrow \mathcal{U}_{1,n}^{ns}$$

extending the obvious map on the locus of smooth curves.

Proof. It is enough to check that $\overline{\mathcal{M}}_{1,n}(m)$ is an open substack of $\mathcal{U}_{1,n}^{ns, \prime}$ (see Definition 1.2.4) for these values of m . Indeed, then we can compose this open embedding with the morphism $\mathcal{U}_{1,n}^{ns, \prime} \rightarrow \mathcal{U}_{1,n}^{ns}$ constructed in Proposition 1.2.5.

Now we recall (see [14, Lemma 3.1]) that every m -stable curve has a fundamental decomposition

$$C = E \cup R_1 \cup \dots \cup R_k,$$

where E is the minimal *elliptic subcurve*, i.e., the connected subcurve of arithmetic genus 1 without disconnecting nodes, and each R_i is a connected nodal curve of arithmetic genus 0 meeting E in a unique point such that $E \cap R_i$ is a node of C (and $R_i \cap R_j = \emptyset$ for $i \neq j$).

We claim that there is at least one marked point p_i on E . Indeed, otherwise the m -stability of C implies that $k > m \geq \frac{n-1}{2}$, i.e., $k \geq \frac{n+1}{2}$. But each R_i contains at least two marked points (due to the last condition in the definition of m -stability), so the total number of marked points is $\geq 2k > n$, which is a contradiction.

Let $p_i \in E$. Then one has

$$H^1(C, \mathcal{O}_C(p_i)) = H^1(E, \mathcal{O}_E(p_i)) = 0.$$

Indeed, the vanishing of $H^1(E, \mathcal{O}_E(p_i))$ can be deduced from the classification of the possible minimal elliptic subcurves (see [14, Lemma 3.3]): E is either a smooth elliptic curve, or an irreducible nodal curve, or a wheel of projective lines, or an elliptic l -fold curve (which includes the cuspidal curve for $l = 1$). Hence, we have $h^1(C, \mathcal{O}(p_1 + \dots + p_n)) = 0$, as required. □

Remark 1.5.2. Recall that for each $i = 1, \dots, n$ we have an open subset $\tilde{\mathcal{U}}_{1,n}(i) = \pi^{-1}(U_i) \subset \tilde{\mathcal{U}}_{1,n}^{ns}$ consisting of $(C, p_\bullet, v_\bullet)$ such that $h^1(p_i) = 0$. The intersection $\bigcap_{i=1}^n \tilde{\mathcal{U}}_{1,n}(i)$ corresponds to the curves $(C, p_\bullet, v_\bullet)$ such that $h^1(p_i) = 0$ for each i . As shown in [11, Proposition 3.3.1], there is a natural projection

$$\bigcap_{i=1}^n \tilde{\mathcal{U}}_{1,n}(i) \rightarrow \tilde{\mathcal{U}}_{1,n}^{sns},$$

which is a \mathbb{G}_m^{n-1} -torsor, where the space $\tilde{\mathcal{U}}_{1,n}^{sns}$ classifies (C, p_\bullet, ω) , where C is of arithmetic genus 1, $h^1(p_i) = 0$ for each i and ω is a nonzero global section of the dualizing sheaf on C (and $\mathcal{O}(p_1 + \dots + p_n)$ is ample). The space $\tilde{\mathcal{U}}_{1,n}^{sns}$ was studied in [6], where we showed in particular that considering the GIT quotients of $\tilde{\mathcal{U}}_{1,n}^{sns}$ by the \mathbb{G}_m -action rescaling ω , we recover Smyth’s moduli space of $(n - 1)$ -stable curves $\overline{\mathcal{M}}_{1,n}(n - 1)$. Proposition 1.5.1 suggests that for $m \geq \frac{n-1}{2}$, there should be a close relation between Smyth’s moduli spaces $\overline{\mathcal{M}}_{1,n}(m)$ and some GIT quotients of $\tilde{\mathcal{U}}_{1,n}^{ns}$ by \mathbb{G}_m^n . In [12] we showed that for $m \geq n - 3$, the morphism $\overline{\mathcal{M}}_{1,n}(m) \rightarrow \mathcal{U}_{1,n}^{ns}$ factors through a certain open substack of GIT-semistable points in $\mathcal{U}_{1,n}^{ns}$.

2. A_∞ -moduli

2.1. Nice quotients

Below we work with schemes and group schemes over a fixed base scheme S .

Definition 2.1.1. Let G be a group scheme, X be a G -scheme. We say that a G -invariant morphism $\pi : X \rightarrow Q$ is a *nice quotient* for the G -action on X if locally over S (in Zariski topology) there exist a section $\sigma : Q \rightarrow X$ of π and a morphism $\rho : X \rightarrow G$, such that

$$x = \rho(x)\sigma(\pi(x)) \quad \text{and} \quad \rho(\sigma(x)) = 1. \tag{2.1.1}$$

We say that π is a *strict nice quotient* if ρ and σ can be defined globally over S .

In the case when S is a point we obtain precisely the situation of [10, Definition 4.2.2], where we called $\sigma(Q)$ a *nice section* for the action of G on X .

Note that a nice quotient is automatically a categorical quotient (in the category of S -schemes). Indeed, let $f : X \rightarrow Z$ be a G -invariant morphism, where Z has trivial G -action. Then $f(x) = f(\sigma(\pi(x)))$, so f is a composition of $f \circ \sigma : Q \rightarrow Z$ with π . This implies that the existence of a nice quotient is a local question in S . Namely, if $X_i \rightarrow Q_i$ are nice quotients for $X_i = p^{-1}(U_i)$, where (U_i) is an open covering of S , $p : X \rightarrow S$ is a projection, then we can glue them into a global morphism $\pi : X \rightarrow Q$.

Remark 2.1.2. If $\pi : X \rightarrow Q$ is a nice quotient for the G -action on X then π is a universal geometric quotient (see [8]). Indeed, any base change of π is still a nice quotient. The following properties are clear: π is surjective, $U \subset Q$ is open if and only if $\pi^{-1}(U)$ is open, geometric fibers are precisely the orbits of geometric points. Finally, we claim that \mathcal{O}_Q coincides with G -invariants in $\pi_*\mathcal{O}_X$. Indeed, given a G -invariant function f on $\pi^{-1}(U)$ then $f(x) = f(\sigma(\pi(x)))$, so it descends to the function $f \circ \sigma$ on U .

Let us consider the topology on the category Sch_S of S -schemes, such that open coverings of $p : T \rightarrow S$ are pull-backs under p of Zariski open coverings of S . We call this *S -Zariski topology*. Let us consider the following presheaf of sets on Sch_S :

$$T \mapsto X(T)/G(T).$$

Lemma 2.1.3. *Let $\pi : X \rightarrow Q$ be a nice quotient for the G -action then the sheafification of the above presheaf with respect to the S -Zariski topology is naturally isomorphic to the functor represented by Q . Thus, a T -point of Q can be represented by a collection of V_i -points of X , where $V_i = f^{-1}(U_i)$ for some open covering (U_i) of S , such that for any i, j , the corresponding V_{ij} -points of X , where $V_{ij} = V_i \cap V_j$, differ by $G(V_{ij})$ -action.*

Proof. We have a natural morphism from $X(T)/G(T)$ to the sheaf represented by Q , which becomes an isomorphism over an open affine covering of S (due to the existence of a decomposition (2.1.1)). This immediately implies the assertion. □

We have the following relative analog of [10, Lemma 4.2.3].

Lemma 2.1.4. *Let G be a group scheme over S acting on a scheme X over S . Assume that G fits into an exact sequence of group schemes*

$$1 \rightarrow H \rightarrow G \rightarrow G' \rightarrow 1$$

and that the projection $G \rightarrow G'$ admits a section $s : G' \rightarrow G$ which is a morphism of schemes (not necessarily compatible with the group structures). Suppose we have a scheme X' with an action of G' and a morphism $f : X \rightarrow X'$ compatible with the G -action via the homomorphism $G \rightarrow G'$. Assume that there exist a nice quotient $\pi_H : X \rightarrow Q_H$ for the H -action on X and a nice quotient $\pi' : X' \rightarrow Q'$ for the G' -action on X' . Finally, assume that the following condition holds: for any S -scheme T and any points $x \in X(T)$, $g \in G(T)$ such that $f(gx) = f(x)$ there exist an open covering $T = \cup T_i$ and a point $h_i \in H(T_i)$ for each i , such that $gx = h_i x$. Then there exists a nice quotient for the G -action on X . The same assertion holds for strict nice quotients.

Proof. It is enough to prove the assertion for strict nice quotients. Without loss of generality we can assume that the section $s : G' \rightarrow G$ satisfies $s(1) = 1$. By assumption, we have sections $\sigma_H : Q_H \rightarrow X$ and $\sigma' : Q' \rightarrow X'$ and the corresponding maps $\rho_H : X \rightarrow H$ and $\rho' : X' \rightarrow G'$ satisfying (2.1.1). Let us define morphisms $\rho_f : X \rightarrow G$ and $\pi_f : X \rightarrow X$ by

$$\rho_f = s \circ \rho' \circ f, \quad \pi_f(x) = \rho_f(x)^{-1}x.$$

One immediately checks that

$$f \circ \pi_f = \sigma' \circ \pi' \circ f.$$

In particular, $\pi_f(x) \in f^{-1}(\sigma'(Q'))$. Let us set $\tilde{Q} = f^{-1}(\sigma'(Q')) \subset X$. Note that for $x \in \tilde{Q}$ we have

$$\rho_f(x) = s(\rho'(f(x))) = s(1) = 1,$$

since $\rho'|_{\sigma'(Q')} = 1$. Hence, for $x \in \tilde{Q}$ we have $\pi_f(x) = x$. Now we set

$$Q = \sigma_H^{-1}(\tilde{Q}) \subset Q_H,$$

and define the maps $\pi : X \rightarrow Q$ and $\rho : X \rightarrow G$ required for the definition of a nice quotient by

$$\begin{aligned} \pi &= \pi_H \circ \pi_f, \\ \rho(x) &= \rho_f(x)\rho_H(\pi_f(x)). \end{aligned}$$

Note that π is well defined. Indeed, we need to show that $(\sigma_H\pi_H\pi_f)(x) \in \tilde{Q}$. But $\pi_f(x) \in \tilde{Q}$, so this follows from the identity

$$(\sigma_H\pi_H\pi_f)(x) = \rho_H(\pi_f(x))^{-1}\pi_f(x)$$

and the fact that \tilde{Q} is preserved by the action of H . We also have a section $\sigma : Q \rightarrow X$ of π given by $\sigma = \sigma_H|_Q$. As in [10, Lemma 4.2.3], we check that our data defines a strict nice quotient for the G -action on X (to prove that π is G -invariant, we observe that our assumption implies that $\pi_f(gx)$ and $\pi_f(x)$ locally in T belong to the same H -orbit). \square

2.2. General A_∞ -moduli

We follow the same conventions on the notions related to A_∞ -structures as in [10]. In particular, by *gauge transformation* of minimal A_∞ -structures (i.e., those with $m_1 = 0$) on the fixed graded space we mean the A_∞ -morphism (f_1, f_2, \dots) with $f_1 = \text{id}$ (the same terminology is used in [13]). We refer to A_∞ -morphisms, for which the map f_1 is only required to be invertible, as *extended gauge transformation*. We say that a minimal A_∞ -structure is *trivial* if it has $m_n = 0$ for all $n \geq 3$, and *homotopically trivial* if it is gauge equivalent to a trivial A_∞ -structure.

For a graded sheaf \mathcal{F} of locally free \mathcal{O} -modules over a scheme S we denote by $CH^{s+t}(\mathcal{F}/S)_t$ the sheaf of homomorphisms of \mathcal{O} -modules $\mathcal{F}^{\otimes s} \rightarrow \mathcal{F}$ of degree t . We have a natural notion of an A_n -structure (respectively, A_∞ -structure) on \mathcal{F} , given by

a collection of global sections

$$m = (m_1, \dots, m_n) \in H^0(S, CH^2(\mathcal{F}/S)_1 \times \dots \times CH^2(\mathcal{F}/S)_{2-n})$$

(respectively, $m = (m_1, m_2, \dots)$ with $m_n \in CH^2(\mathcal{F}/S)_{2-n}$), satisfying the standard A_∞ -identities involving only m_1, \dots, m_n (respectively, all A_∞ -identities). Note that in the case when 2 is invertible on S the identities for A_n -structures can be written as $\sum_{i=1}^r [m_i, m_{r+1-i}] = 0$, for $r = 1, \dots, n$, where $[\cdot, \cdot]$ is the Gerstenhaber bracket.

The action of the group of gauge transformations on the set of minimal A_n -structures also immediately generalizes to the relative context: we have a sheaf of groups \mathfrak{G} over S , where an element of $\mathfrak{G}(U)$ is a collection of sections

$$f = (f_1 = \text{id}, f_2, \dots) \in H^0(U, CH^1(\mathcal{F}/S)_{-1} \times CH^1(\mathcal{F}/S)_{-2} \times \dots),$$

with the product rule obtained by interpreting f as a coalgebra automorphism of the bar-coalgebra of \mathcal{F} (see [10, Definition 4.1.3]). We use the notation $\mathfrak{G}[2, n - 1] := \mathfrak{G}/\mathfrak{G}_{\geq n}$, introduced in [10, § 4.2], for the quotient of \mathfrak{G} acting on the set of minimal A_n -structures on \mathcal{F} . We denote the projection $\mathfrak{G} \rightarrow \mathfrak{G}[2, n - 1]$ by $u \mapsto u_{\leq n-1}$.

Remark 2.2.1. The above definition of an A_n -algebra over a scheme is a bit naive. A more flexible notion should involve defining m_i 's only over an open covering U_i of S , and the gluing should be given by a collection of higher homotopies defined on intersections $U_{i_1} \cap \dots \cap U_{i_r}$. We do not need the most general definition since we only aim at constructing the usual space as a moduli space of A_∞ -structures (in good situations), not an ∞ -stack. Even at this level we need a certain gluing procedure, but a much simpler one.

Now let us fix a scheme S and a sheaf \mathcal{E} of graded associative \mathcal{O}_S -algebras over S . We assume also that \mathcal{E} is locally free of finite rank over \mathcal{O}_S . We denote by \mathcal{E}_s the fiber of \mathcal{E} over a point $s \in S$. Roughly speaking, our goal is to classify families of minimal A_∞ -algebras, up to gauge equivalence, such that the corresponding family of graded associative algebras is induced by \mathcal{E} .

To begin with, for every $n \geq 2$ we define the functor $\mathcal{A}_n = \mathcal{A}_{n,\mathcal{E}}$ (respectively, $\mathcal{A}_\infty = \mathcal{A}_{\infty,\mathcal{E}}$) on the category of S -schemes, which associates with $f : T \rightarrow S$ the set of minimal A_n -structures (respectively, A_∞ -structures) extending the sheaf of graded associative \mathcal{O}_T -algebras $f^*\mathcal{E}$. This functor is clearly represented by an affine scheme $\mathcal{A}_n(\mathcal{E})$ over S . Namely, $\mathcal{A}_n(\mathcal{E})$ is the closed subscheme in the total space of the vector bundle $CH^2(\mathcal{E}/S)_{-1} \oplus \dots \oplus CH^2(\mathcal{E}/S)_{2-n}$ given by the A_∞ -equations. We have a natural projection

$$\pi_n : \mathcal{A}_n \rightarrow \mathcal{A}_{n-1} : m \mapsto m_{\leq n-1}. \tag{2.2.1}$$

Next, we have the sheaf of groups \mathfrak{G} of gauge transformations acting on each functor \mathcal{A}_n through the quotient $\mathfrak{G}[2, n - 1]$, and we make the following definition.

Definition 2.2.2. Let $\widetilde{\mathcal{M}}_n$ denote the quotient-functor associating to an S -scheme $f : T \rightarrow S$ the set $\mathcal{A}_n(T)/\mathfrak{G}[2, n - 1](T)$ of gauge equivalence classes of minimal A_n -structures on $f^*\mathcal{E}$. We denote by $\widetilde{\mathcal{M}}_\infty$ the similar quotient-functor for gauge equivalence classes of minimal A_∞ -structures.

Note that the sheaf of groups $\mathfrak{G}[2, n - 1]$ is representable by a unipotent affine group scheme over S which we still denote as $\mathfrak{G}[2, n - 1]$. Note also that the projection $\mathfrak{G} \rightarrow \mathfrak{G}[2, n - 1]$ admits a section (not compatible with the group structures) and so is universally surjective. However, the quotient-functor $\widetilde{\mathcal{M}}_n$ is not necessarily representable.

For a bundle V over a scheme S , and a point s of S , we denote by V_s the fiber of V over s , which is a vector space over the residue field of s .

Lemma 2.2.3. (i) *Let (V^\bullet, d) be a bounded below complex of vector bundles over S such that $H^i(V_s^\bullet) = 0$ for $i < p$ for every point $s \in S$. Then for $i < p$, the image $\text{im}(d^i)$ of the differential $d^i : V^i \rightarrow V^{i+1}$ is a subbundle of V^{i+1} .*

(ii) *If in addition S is affine then there exist decompositions of vector bundles $V^i = B^i \oplus K^i$, for $i \leq p$, such that for $i < p$ one has $d^i(K^i) = B^{i+1}$ and the map*

$$d^i|_{K^i} : K^i \rightarrow B^{i+1}$$

*is an isomorphism. In this situation for any $f : T \rightarrow S$ the complex $H^0(T, f^*V^\bullet)$ is exact in degrees $< p$.*

Proof. (i) It is enough to prove that $\text{im}(d^{p-1})$ is a subbundle in V^p . Without loss of generality we can assume that $V^i = 0$ for $i < 0$ and $p > 0$. Then the map $d_s : V_s^0 \rightarrow V_s^1$ is injective for every $s \in S$. We claim that this implies that $d : V^0 \rightarrow V^1$ is the embedding of a subbundle. Indeed, it is enough to prove the similar assertion for a morphism $f : A^m \rightarrow A^n$ of free modules over a local ring A , such that $f \bmod \mathfrak{m}$ is injective, where $\mathfrak{m} \subset A$ is a maximal ideal. But in this case we can choose a projection $p : A^n \rightarrow A^m$ to a subset of m coordinates, such that $p \circ f \bmod \mathfrak{m}$ is an isomorphism. This implies that $\det(p \circ f)$ is nonzero $\bmod \mathfrak{m}$, hence it is invertible in A . Thus, the composition $p \circ f : A^m \rightarrow A^m$ is an isomorphism, so $\text{coker}(f)$ is free.

Hence, $V^1/d(V^0)$ is a vector bundle, and we can replace our complex with

$$0 \rightarrow V^1/d(V^0) \rightarrow V^2 \rightarrow \dots$$

and iterate the same argument.

(ii) The first assertion follows from part (i): we set $B^i := \text{im}(d^{i-1})$ and let K^i be the image of any splitting of the projection $V^i \rightarrow V^i/B^i$ (which exists since S is affine). These decompositions carry over to the complex $H^0(T, f^*V^\bullet)$, which implies its exactness in degrees $< p$. □

We denote the Hochschild differential $[m_2, ?]$ on $CH^*(\mathcal{E}/S)$ by δ and its graded components as

$$\delta_t^i : CH^i(\mathcal{E}/S)_t \rightarrow CH^{i+1}(\mathcal{E}/S)_t.$$

As usual, the second grading on the complex CH^* induces the second grading on the Hochschild cohomology HH^* .

Lemma 2.2.4. *Let S be an affine scheme.*

(i) *Let us fix an integer $d \geq 2$. Assume that $HH^i(\mathcal{E}_s)_{-j} = 0$ for every point $s \in S$, for $i \leq 1$ and $j = 1, \dots, d - 2$. Now let $f : T \rightarrow S$ be a morphism of schemes, and let*

m and m' be a pair of minimal A_n -structures on $f^*\mathcal{E}$ for some $n \geq d$, such that m is gauge equivalent to m' (over T) and $m_{\leq d} = m'_{\leq d}$. Then there exists a gauge equivalence u over T , such that $u_{\leq d-1} = \text{id}$ and $m' = u \cdot m$.

(ii) Assume that $HH^i(\mathcal{E}_s)_{<0} = 0$ for every $s \in S$, for $i \leq 1$. Then the natural map

$$\widetilde{\mathcal{M}}_\infty(T) \rightarrow \varprojlim_n \widetilde{\mathcal{M}}_n(T)$$

is an isomorphism for every S -scheme T .

Proof. (i) The proof is similar to that of [10, Lemma 4.1.6]. By assumption, we have a gauge equivalence $\widetilde{u} \in \mathfrak{G}(T)$ such that $\widetilde{u}m = m'$. Then $\widetilde{u}_{\leq d}$ sends $m_{\leq d}$ to itself. Now Lemma 2.2.3(ii), applied to the Hochschild complexes $(CH^*(\mathcal{E})_{-j}, \delta)$, implies that $HH^1(H^0(T, f^*\mathcal{E})/\mathcal{O}(T))_{-j} = 0$ for $j = 1, \dots, d-2$. Hence, arguing as in [10, Lemma 4.1.6], we can correct \widetilde{u} to a gauge equivalence $u \in \mathfrak{G}(T)$ such that $u \cdot m = m'$ and $u_{\leq d-1} = \text{id}$.

(ii) The proof is identical to the argument in the proof of [10, Corollary 4.2.5], with part (i) replacing the reference to [10, Lemma 4.1.6]. □

Definition 2.2.5. Let us denote by \mathcal{M}_n (respectively, \mathcal{M}_∞) the sheafification of the functor $\widetilde{\mathcal{M}}_n$ (respectively, $\widetilde{\mathcal{M}}_\infty$) with respect to the S -Zariski topology.

The following theorem, which is a relative version of [10, Theorem 4.2.4, Corollary 4.2.5, Corollary 4.2.6], shows that under certain vanishing assumptions on the Hochschild cohomology, the functor \mathcal{M}_n (respectively, \mathcal{M}_∞) is representable by an affine S -scheme. Note that these assumptions are slightly stronger than in [10, Theorem 4.2.4].

Theorem 2.2.6. Assume that $HH^i(\mathcal{E}_s)_{-j} = 0$ for $i \leq 1$ and $1 \leq j \leq n-3$, for every point $s \in S$.

- (i) There exists a nice quotient $\mathcal{A}_n(\mathcal{E})/\mathfrak{G}[2, n-1]$ for the action of $\mathfrak{G}[2, n-1]$ on $\mathcal{A}_n(\mathcal{E})$. This quotient $\mathcal{A}_n(\mathcal{E})/\mathfrak{G}[2, n-1]$, which is affine of finite type over S , represents the functor \mathcal{M}_n . If in addition S is affine then there exists a strict nice quotient $\mathcal{A}_n(\mathcal{E})/\mathfrak{G}[2, n-1]$, and the natural map of functors $\widetilde{\mathcal{M}}_n \rightarrow \mathcal{M}_n$ is an isomorphism.
- (ii) Assume that $HH^i(\mathcal{E}_s)_{-j} = 0$ for $i \leq 1$ and $j \geq 1$, for every $s \in S$. Then the scheme $\varprojlim_n \mathcal{M}_n$, affine over S , represents the functor \mathcal{M}_∞ . In the case when S is affine, the natural map $\widetilde{\mathcal{M}}_\infty \rightarrow \mathcal{M}_\infty$ is an isomorphism.
- (iii) Assume that $HH^i(\mathcal{E}_s)_{-j} = 0$ for $i \leq 1, j \geq 1$, and in addition $HH^2(\mathcal{E}_s)_{-j} = 0$ for $j > n-2$, for every $s \in S$. Then the morphism $\mathcal{M}_\infty \rightarrow \mathcal{M}_n$ is a closed embedding. If in addition $HH^3(\mathcal{E}_s)_{-j} = 0$ for $j > n-2$, for every $s \in S$, then $\mathcal{M}_\infty \rightarrow \mathcal{M}_n$ is an isomorphism.

Proof. (i) It is enough to prove the existence of a strict nice quotient for the $\mathfrak{G}[2, n-1]$ -action on $\mathcal{A}_n = \mathcal{A}_n(\mathcal{E})$ in the case when S is affine. Indeed, then it would follow that $\widetilde{\mathcal{M}}_n$ is represented by this quotient, and hence, the map $\widetilde{\mathcal{M}}_n \rightarrow \mathcal{M}_n$ is an isomorphism.

Similarly to the proof of [10, Theorem 4.2.4] the existence of a strict nice quotient is proved by the induction on n , using Lemma 2.1.4. Assume that $n > 2$ and we already have a section S_{n-1} for the $\mathfrak{G}[2, n-2]$ -action on \mathcal{A}_{n-1} . We have an exact sequence of sheaves of groups over S ,

$$0 \rightarrow CH^1(\mathcal{E})_{2-n} \rightarrow \mathfrak{G}[2, n-1] \rightarrow \mathfrak{G}[2, n-2] \rightarrow 0.$$

We want to find a section for the $CH^1(\mathcal{E})_{2-n}$ -action on \mathcal{A}_n . By Lemma 2.2.3(ii), there exists a complement $\mathcal{K}_{2-n}^2 \subset CH^2_{2-n}$ to the subbundle $\text{im } \delta^1_{2-n}$. Let \mathcal{A}'_n denote the closed subset of \mathcal{A}_n given by the condition $m_n \in \mathcal{K}_{2-n}^2$. Since the action of $x \in CH^1(\mathcal{E})_{2-n}$ on $(m_2, \dots, m_n) \in \mathcal{A}_n$ changes m_n to $m_n + \delta^1(x)$ and does not change (m_2, \dots, m_{n-1}) , we see that \mathcal{A}'_n is a section for the $CH^1(\mathcal{E})_{2-n}$ -action on \mathcal{A}_n . Now we can apply Lemma 2.1.4 to the projection (2.2.1) and the compatible actions of $\mathfrak{G}[2, n-1] \rightarrow \mathfrak{G}[2, n-2]$. Note that to apply this Lemma we need to check that the intersection of an $\mathfrak{G}[2, n-1]$ -orbit with a fiber of π_n is a $CH^1(\mathcal{E})_{2-n}$ -orbit. But this follows from Lemma 2.2.4(i). Thus, we deduce that

$$S_n := \mathcal{A}'_n \cap \pi_n^{-1}(S_{n-1})$$

is a section for the $\mathfrak{G}[2, n-1]$ -action on \mathcal{A}_n .

(ii) First, assume that S is affine. Then, combining part (i) with Lemma 2.2.4(ii), we derive that the functor $\widetilde{\mathcal{M}}_\infty$ is represented by the scheme $\varprojlim_n \mathcal{M}_n$, affine over S . Hence, in this case the map $\widetilde{\mathcal{M}}_\infty \rightarrow \mathcal{M}_\infty$ is an isomorphism. Thus, in the case of general S the map of sheaves $\mathcal{M}_\infty \rightarrow \varprojlim_n \mathcal{M}_n$ becomes an isomorphism over an affine open covering of S , hence, it is an isomorphism.

(iii) We can assume S to be affine. For the first assertion, it is enough to prove that if $HH^{\leq 2}(\mathcal{E}_s)_{-n+1} = 0$ for all s then the projection $S_{n+1} \rightarrow S_n$ is a closed embedding. To this end we recall that \mathcal{A}_{n+1} is a closed subset of $\mathcal{A}_n \times CH^2_{1-n}$ given by the equation

$$\delta^2(m_{n+1}) = -\phi_n(m_{\leq n}),$$

where $\phi_n : \mathcal{A}_n \rightarrow CH^3_{1-n}$ is a certain morphism (see [10, Lemma 4.1.2(ii)]) such that $\delta^3 \circ \phi_n = 0$. Now by Lemma 2.2.3(ii), there exist decompositions of vector bundles

$$CH^2(\mathcal{E})_{1-n} = \mathcal{B}^2 \oplus \mathcal{K}^2, \quad CH^3(\mathcal{E})_{1-n} = \mathcal{B}^3 \oplus \mathcal{K}^3$$

such that \mathcal{B}^3 is the image of δ^2 and the restriction $\delta^2|_{\mathcal{K}^2} : \mathcal{K}^2 \rightarrow \mathcal{B}^3$ is an isomorphism. Let $(\phi_{\mathcal{B}}, \phi_{\mathcal{K}})$ be the components of ϕ_n with respect to the decomposition of CH^3_{1-n} . Then the equations defining \mathcal{A}_{n+1} become

$$\delta^2(m_{n+1}) = \phi_{\mathcal{B}}(m_{\leq n}), \quad 0 = \phi_{\mathcal{K}}(m_{\leq n}).$$

Thus, on the subscheme \mathcal{A}'_{n+1} cut out by the condition $m_{n+1} \in \mathcal{K}^2$, we can solve the first equation for m_{n+1} , which shows that the projection $\mathcal{A}'_{n+1} \rightarrow \mathcal{A}_n$ is a closed embedding. Hence, the same is true for the projection $S_{n+1} \rightarrow S_n$.

For the last statement, we observe that the assumption $HH^{\leq 3}(\mathcal{E}_s)_{-n+1} = 0$ for all s gives in addition the vanishing of the component $\phi_{\mathcal{K}}$. Thus, in this case the projection $\mathcal{A}'_{n+1} \rightarrow \mathcal{A}_n$ is an isomorphism, and hence the same is true for the projection $S_{n+1} \rightarrow S_n$. □

2.3. A_∞ -structures associated to curves

We want to study the relative moduli space of A_∞ -structures on the family of graded associative algebras (E_W) over the Grassmannian $G(n - g, n)$ (see (0.0.4)).

First, let us describe more precisely the corresponding sheaf of \mathcal{O} -algebras $\mathcal{E}_{g,n}$ over $G(n - g, n)$.

Let R be a commutative ring, and let J_0 be the two-sided ideal in the path algebra $R[Q_n]$ of Q_n generated by the elements

$$A_i B_i A_i, B_i A_i B_i, A_i B_j,$$

where $i \neq j$. Given an R -submodule $W \subset R^n$ such that R^n/W is locally free of rank r , we define $J_W \subset R[Q_n]$ as the ideal generated by J_0 and $\iota(W)$, where $\iota : R^n \rightarrow R[Q_n]$ is the embedding of R -modules given by

$$\iota(e_i) = B_i A_i, \quad i = 1, \dots, n.$$

The corresponding quotient algebra

$$E_W = R[Q_n]/J_W$$

is projective as an R -module. In the case when R^n/W is a free R -module with a basis (b_j) , the algebra E_W is also free as R -module, with the basis given by the elements $(A_i), (B_i), (\iota(b_j))$.

For an n -tuple of invertible elements $\lambda = (\lambda_1, \dots, \lambda_n) \in (R^*)^n$ we have a natural isomorphism

$$E_W \rightarrow E_{\lambda \cdot W} : A_i \mapsto A_i, B_i \mapsto \lambda_i B_i,$$

where the transformation $W \mapsto \lambda \cdot W$ is induced by the rescaling of the basis e_1, \dots, e_n of R^n .

Applying the above construction to the tautological subbundle over an affine covering of the Grassmannian $G(n - g, n)$ and gluing, we obtain a \mathbb{G}_m^n -equivariant sheaf $\mathcal{E}_{g,n}$ of \mathcal{O} -algebras over $G(n - g, n)$, where \mathbb{G}_m^n acts naturally on $G(n - g, n)$ (as diagonal matrices). We can consider the corresponding relative moduli of minimal A_∞ -structures defined in §2.2.

Definition 2.3.1. The moduli functor $\widetilde{\mathcal{M}}_\infty$ on $G(n - g, n)$ -schemes associates with $f : S \rightarrow G(n - g, n)$ the set of gauge equivalence classes of minimal A_∞ -structures on $f^* \mathcal{E}_{g,n}$. The functor \mathcal{M}_∞ is obtained from $\widetilde{\mathcal{M}}_\infty$ by the sheafification with respect to $G(n - g, n)$ -Zariski topology.

Similarly to [10, §3] we have a natural morphism of functors

$$\widetilde{\mathcal{U}}_{g,n}^{ns} \rightarrow \mathcal{M}_\infty. \tag{2.3.1}$$

Namely, for every open affine subset $U \subset \widetilde{\mathcal{U}}_{g,n}^{ns}$ we can construct a minimal A_∞ -structure on $\pi^* \mathcal{E}_{g,n}|_U$. This minimal A_∞ -structure is obtained by applying the homological perturbation procedure to the sheaf of dg-algebras over S and homotopies on it defined using some relative formal parameters along the marked points as described in [10, §3]. Note that if we choose different formal parameters with the same underlying tangent vectors, then we get an isomorphic sheaf of dg-algebras over S but it will be equipped

with different homotopies needed for the homological perturbation. However, it is well known that the minimal A_∞ -structure obtained by the homological perturbation does not depend on a choice of homotopies up to gauge equivalence. Thus, the map (2.3.1) is well defined. Furthermore, it is compatible with the \mathbb{G}_m^n -actions, where the \mathbb{G}_m^n -action on \mathcal{M}_∞ is induced by the rescalings

$$A_i \mapsto A_i, \quad B_i \mapsto \lambda_i B_i,$$

for $(\lambda_1, \dots, \lambda_n) \in \mathbb{G}_m^n$.

Next, we want to prove that \mathcal{M}_∞ is represented by an affine scheme of finite type over $G(n - g, n)$. For this we want to apply the criterion of Theorem 2.2.6, which requires some information about the Hochschild cohomology of the algebras E_W . As in [10], we will get this information geometrically by identifying $HH^*(E_W)$ with the Hochschild cohomology of the corresponding special curve.

Lemma 2.3.2. *Let $C_W = (C(\bar{h}), p_\bullet, v_\bullet) \in \tilde{U}_{g,n}^{ns}$ be the special curve corresponding to $W \in G(n - g, n)$ (see Proposition 1.1.2). Then the natural A_∞ -structure on $\text{Ext}^*(G, G)$ for G given by (0.0.3) is homotopically trivial, and hence, we have an isomorphism*

$$HH^*(C_W) \simeq HH^*(E_W),$$

where $W = \pi(C, p_1, \dots, p_n)$. The second grading on $HH^*(E_W)$ corresponds to the weights of the \mathbb{G}_m -action, coming from the natural \mathbb{G}_m -action on C_W .

Proof. This is similar to [10, Proposition 4.4.1]. □

Lemma 2.3.3. *Let C be a reduced projective curve over a field k with a \mathbb{G}_m -action, which is the union of irreducible components $C_i, i = 1, \dots, n$, joined in a single point q . Assume that $C \setminus \{q\}$ is smooth and that each normalization map $\tilde{C}_i \rightarrow C_i$ is a bijection, with $\tilde{C}_i \simeq \mathbb{P}^1$. Assume also that the action of \mathbb{G}_m on the Zariski tangent space at q has negative weights. Then*

- (i) *the action of \mathbb{G}_m on $H^1(C, \mathcal{O}_C)$ has positive weights.*
- (ii) *Assume in addition that $C = C_W$ for some subspace $W \subset k^n$, where $W = 0$ if $n = 1$. Assume also that $\text{char}(k) \neq 2$, and if $n = 1$ then also $\text{char}(k) \neq 3$. Then $H^1(C, \mathcal{T}) = 0$, and the action of \mathbb{G}_m on $H^0(C, \mathcal{T})$ has weights 0 and 1.*
- (iii) *Keep the assumptions of (ii). Let $p_i \in C_i \setminus \{q\}$ be the unique \mathbb{G}_m -invariant point, and let $D = \sum_i p_i$. Then one has*

$$H^0(C, \mathcal{T}(-D)) = H^0(C, \mathcal{T})^{\mathbb{G}_m}, \quad H^0(C, \mathcal{T}(-2D)) = 0.$$

Also, the natural map $H^0(C, \mathcal{T}(nD)) \rightarrow H^0(C, \mathcal{T}(nD)|_D)$ is surjective for $n \geq 0$.

- (iv) *For $W = k^n$, with $n \geq 2$, the assertions of (ii) and (iii) hold without any restrictions on the characteristic of k .*

Proof. (i) Let $V = C \setminus \{q\}$. We can choose a coordinate x_i on an affine part of $\tilde{C}_i \simeq \mathbb{P}^1$ containing q such that $x_i(q) = 0$ and x_i has positive weight w_i with respect to the \mathbb{G}_m -action. Let U be an affine neighborhood of q obtained by deleting on each C_i the

point where x_i has a pole. We can calculate $H^1(C, \mathcal{O}_C)$ as the quotient of $\mathcal{O}(U \setminus \{q\})$ by $\mathcal{O}(V) + \mathcal{O}(U)$. Since every x_i^n with $n \leq 0$ extends to a regular function on V , we see that $H^1(C, \mathcal{O}_C)$ is spanned by positive powers of x_i 's, so \mathbb{G}_m has only positive weights on it.

(ii) The case $n = 1$ and $W = 0$ corresponds to the cuspidal curve, for which the assertions of (ii) and (iii) are known (see e.g., [10, Lemma 4.4.2]). So we assume $n \geq 2$. We use the coordinates x_i on affine parts of the normalizations \tilde{C}_i from Definition 1.1.1. The space $H^0(C, \mathcal{T}_C)$ embeds into the space of vector fields on $V \simeq \sqcup_{i=1}^n (C_i \setminus \{q\})$, which is spanned by $x_i^m \partial_{x_i}$ with $m \leq 2$. Exactly as in the proof of [10, Lemma 4.4.2(ii)] we check that if a vector field $v = (P_i(x_i, x_i^{-1}) \partial_{x_i})$ on $U \setminus \{q\}$ extends to a derivation of $\mathcal{O}(U)$ then $P_i \in x_i k[x_i]$ for every i (this uses the assumption $\text{char}(k) \neq 2$). Thus, if v extends to a global section of \mathcal{T}_C then each P_i is a linear combination of x_i and x_i^2 , which implies that the weights of \mathbb{G}_m on $H^0(C, \mathcal{T}_C)$ are 0 and 1. Similarly, we see that if $P_i \in x_i^2 k[x_i]$ for every i then v extends to a derivation of $\mathcal{O}(U)$. Thus, $H^0(U, \mathcal{T})$ and $H^0(V, \mathcal{T})$ span $H^0(U \setminus \{q\}, \mathcal{T})$, which gives the vanishing of $H^1(C, \mathcal{T}_C)$.

(iii) A vector field on $U \setminus \{q\}$ has zero (respectively, double zero) along D if each $P_i \in x_i k[x_i^{-1}]$ (respectively, $P_i \in k[x_i^{-1}]$). Together with calculations of (ii) this immediately implies our assertions about $H^0(C, \mathcal{T}(-D))$ and $H^0(C, \mathcal{T}(-2D))$. Next, similarly to (ii) we can represent sections of $H^0(C, \mathcal{T}_C(nD))$ as vector fields $v = (P_i(x_i) \partial_{x_i})$ on $U \setminus \{q\}$ with $\text{deg}(P_i) \leq n + 2$, and the last assertion follows from the fact that v extends to a regular derivation of $\mathcal{O}(U)$ whenever $P_i \in x_i^2 k[x_i]$.

(iv) We can argue as in the proof of [10, Lemma 4.4.2(ii)], with x_i^2 replaced by x_i (complemented by zeros in all other places), to show that vector fields on V that extend to U are precisely $v = (P_i \partial_{x_i})$ with $P_i \in x_i k[x_i]$. The rest of the proof is the same as in (ii) and (iii). □

Remark 2.3.4. One can check that the assertions of Lemma 2.3.3 hold for $C = C_W$ without any restrictions on the characteristic of k , provided $n \geq g + 2$ and W is not contained in any of the coordinate hyperplanes $k^{n-1} \subset k^n$.

Corollary 2.3.5. *Let k be a field of characteristic $\neq 2$ (respectively, $\neq 2, 3$ if $n = 1$). Then for any subspace $W \subset k^n$, where $W = 0$ if $n = 1$, one has*

$$HH^0(E_W)_{<0} = HH^1(E_W)_{<0} = 0.$$

The same result holds for $W = k^n$, $n \geq 2$, with no restrictions on the characteristic.

Proof. By Lemma 2.3.2, we have $HH^1(E_W) \simeq HH^1(C_W)$, where C_W is the corresponding special curve, and the second grading is induced by the \mathbb{G}_m -action on C_W . Now $HH^0(C_W) = H^0(C_W, \mathcal{O})$ lives in degree 0. For HH^1 we use the exact sequence

$$0 \rightarrow H^1(C_W, \mathcal{O}) \rightarrow HH^1(C_W) \rightarrow H^0(C_W, \mathcal{T}) \rightarrow 0$$

(see [5, § 4.1.3]). Now the assertion follows from Lemma 2.3.3(i)(ii). □

Proposition 2.3.6. *Let us work over $\mathbb{Z}[1/2]$ if $n \geq 2$, or over $\mathbb{Z}[1/6]$ if $n = 1$, or over \mathbb{Z} if $g = 0$. Assume that either $n \geq 2$ or $g = 1$. Then the functor \mathcal{M}_∞ of A_∞ -structures (up to*

a gauge equivalence) on the family (E_W) is represented by an affine scheme of finite type over $G(n - g, n)$.

Proof. Due to Corollary 2.3.5, the criterion of Theorem 2.2.6(ii) implies that \mathcal{M}_∞ (respectively, \mathcal{M}_n) is represented by an affine scheme (respectively, of finite type) over $G(n - g, n)$. Next, we note that by Lemma 2.3.2, $HH^i(E_W)$ is finite-dimensional for every i . Hence, by Theorem 2.2.6(iii), we derive that $\mathcal{M}_\infty \simeq \mathcal{M}_n$ for sufficiently large n , so it is of finite type over $G(n - g, n)$. \square

For a scheme S over a field k we denote by L_S the cotangent complex of S over k .

Lemma 2.3.7. *Assume that either $W = k^n$ and $n \geq 2$, or k has characteristic $\neq 2$ (respectively, $\neq 2, 3$ if $n = 1$). Let $C = C_W$ be a special curve over k , where $W = 0$ if $n = 1$, and let $D = p_1 + \dots + p_n$, $U = C \setminus D$. Then the natural morphism*

$$\text{Ext}^1(L_C, \mathcal{O}_C(-2D)) \rightarrow \text{Ext}^1(L_C, \mathcal{O}_C(-D)) \tag{2.3.2}$$

is surjective, while the natural morphism

$$\text{Ext}^1(L_C, \mathcal{O}_C(-D)) \rightarrow \text{Ext}^1(L_U, \mathcal{O}_U)$$

is an isomorphism. The natural morphism

$$\text{Ext}^2(L_C, \mathcal{O}_C(-2D)) \rightarrow \text{Ext}^2(L_U, \mathcal{O}_U)$$

is an isomorphism.

Proof. The proof is almost the same as that of [10, Lemma 4.4.5], using Lemma 2.3.3. The difference is that in our case the map

$$H^0(C, \mathcal{T}(-D)) \rightarrow H^0(C, \mathcal{T}(-D)|_D)$$

is not necessarily surjective, so we cannot assert that the map (2.3.2) is an isomorphism, only that it is surjective. \square

Now we can use the results of [10] to compare the deformation theory of a special curve C_W with that of E_W , viewed as an A_∞ -algebra.

We refer to [7] for the basic deformation theory and for some terminology used below, such as *deformation functor*, *smooth (étale) map of deformation functors*, etc. Let us fix a field k and consider the category $\text{Art}(k)$ of local Artinian algebras with fixed identifications of the residue field with k . Morphisms in this category are local homomorphisms inducing the identity on the residue field. Given a curve $(C, p_1, \dots, p_n, v_1, \dots, v_n)$ with smooth distinct marked points and nonzero tangent vectors at them, we have the corresponding deformation functor

$$\text{Def}(C, p_\bullet, v_\bullet) : \text{Art}(k) \rightarrow \text{Sets}$$

associating with R the set of isomorphism classes of flat proper families of curves $\pi_R : C_R \rightarrow \text{Spec}(R)$ with sections p_1^R, \dots, p_n^R , and trivializations of the relative tangent bundle along them, such that the induced data over $\text{Spec}(k) \subset \text{Spec}(R)$ is $(C, p_\bullet, v_\bullet)$.

On the other hand, for any finite-dimensional minimal A_∞ -algebra E we have the deformation functor

$$\text{Def}(E) : \text{Art}(k) \rightarrow \text{Sets}$$

of extended gauge equivalence classes of minimal A_∞ -algebras E_R over R , reducing to E over k . Thus, for a fixed $(n - g)$ -dimensional subspace $W \subset k^n$, we have the functor $\text{Def}(E_W)$, where we equip E_W with the trivial A_∞ -structure. We also consider the relative deformation functor

$$\text{Def}(E_W/G(n - g, n)) : \text{Art}(k) \rightarrow \text{Sets}$$

associating with R the set of pairs (W_R, m_\bullet) , where W_R is an R -point of $G(n - g, n)$, reducing to W over k , and m_\bullet is a minimal A_∞ -structure on E_{W_R} , extending the given m_2 and reducing to the trivial A_∞ -structure on E_W , viewed up to a gauge equivalence reducing to the identity modulo the maximal ideal. The difference of this functor from $\text{Def}(E_W)$ is that we do not allow arbitrary variations of m_2 but only the ones coming from varying W in the Grassmannian.

As in [10, Lemma 4.5.1], one can show (using Corollary 2.3.5) that $\text{Def}(E_W/G(n - g, n))(R)$ can be identified with the fiber of the map $\mathcal{M}_\infty(R) \rightarrow \mathcal{M}_\infty(k)$ over the point corresponding to the gauge equivalence class of the trivial A_∞ -structure on E_W . It follows that the functor $\text{Def}(E_W/G(n - g, n))$ is prorepresented by the formal completion of the scheme \mathcal{M}_∞ at this point. We have a natural forgetful morphism

$$\text{Def}(E_W/G(n - g, n)) \rightarrow \text{Def}(E_W).$$

Lemma 2.3.8. *The tangent space to the functor $\text{Def}(E_W/G(n - g, n))$ can be identified with*

$$HH^2(E_W)_{<0} \oplus T_W G(n - g, n).$$

There is a complete obstruction theory for this functor with values in $HH^3(E_W)_{<0}$.

Proof. The tangent space classifies pairs (f, m_\bullet) , where $f : \text{Spec}(k[t]/(t^2)) \rightarrow G(n - g, n)$ is a morphism sending the closed point to W , and m_\bullet is a minimal A_∞ -structure on $f^*\mathcal{E}$, extending the given m_2 , reducing to the trivial one modulo (t) , up to a gauge equivalence. Then f corresponds to a tangent vector in $T_W G(n - g, n)$, while the class of (m_3, m_4, \dots) is an element in $HH^3(E_W)_{<0}$ (see e.g., [10, Lemma 4.5.2]). The obstruction theory is obtained from the usual obstruction theory for A_∞ -structures (see e.g., [10, Lemma 4.5.2]) using the fact that $G(n - g, n)$ is smooth. \square

Remark 2.3.9. In general the spaces $HH^i(E_W)_{<0}$ are given by the products of the components $HH^i(E_W)_j$ for $j < 0$. However, in our case the spaces $HH^i(E_W)$ are finite-dimensional, by Lemma 2.3.2, so for each i there is only finitely many j with $HH^i(E_W)_j \neq 0$.

For each special curve $C_W = (C_W, p_\bullet, v_\bullet) \in \tilde{\mathcal{U}}_{g,n}^{ns}$ corresponding to a subspace $W \in G(n - g, n)(k)$, where k is a field, the morphism (2.3.1) induces a morphism of deformation functors

$$\text{Def}(C_W) \rightarrow \text{Def}(E_W/G(n - g, n)). \tag{2.3.3}$$

Proposition 2.3.10. *Assume that either $n \geq 2$ and $g = 0$, or $n \geq 2$ and the characteristic of k is $\neq 2$, or $n = g = 1$ and the characteristic of k is $\neq 2, 3$. Then the morphism (2.3.3) is an isomorphism.*

Proof. Let U_W be the affine curve $C_W \setminus D$, where $D = p_1 + \dots + p_n$. Let also \mathcal{C} be the (non-full) subcategory in the A_∞ -enhancement of the derived category of $\text{Qcoh}(C_W)$ with the objects $(\mathcal{O}_C, \mathcal{O}_{p_1}, \dots, \mathcal{O}_{p_n}, \mathcal{O}_{U_W})$, all morphisms (including Ext^*) between $\mathcal{O}_C, (\mathcal{O}_{p_i})$, and all morphisms from \mathcal{O}_C to \mathcal{O}_{U_W} and from \mathcal{O}_{U_W} to \mathcal{O}_{U_W} (but we do not consider any other morphisms between these objects). As in the proof of [10, Proposition 4.5.4], we consider additional functors $\text{Def}(U_W), \text{Def}_{nc}(U_W)$ and $\text{Def}(\mathcal{C})$ (in [10] they are denoted by $F_{U_W}, F_{U_W,nc}$ and $F_{\mathcal{C}}$), corresponding to deformations of U_W , of $\mathcal{O}(U_W)$ as an associative algebra, and of \mathcal{C} as an A_∞ -category. These functors fit into a commutative diagram

$$\begin{array}{ccccc}
 \text{Def}(U_W) & \longleftarrow & \text{Def}(C_W) & \longrightarrow & \text{Def}(E_W/G(n-g, n)) \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Def}_{nc}(U_W) & \longleftarrow & \text{Def}(\mathcal{C}) & \longrightarrow & \text{Def}(E_W)
 \end{array} \tag{2.3.4}$$

It is easy to see that each functor in this diagram is a deformation functor in the sense of [7, Definition 2.5], as it describes deformations of some algebraic structure with coefficients in R (to check the first condition in the definition one can use the appropriate obstruction theory).

Step 1. The map $\text{Def}(\mathcal{C}) \rightarrow \text{Def}(E_W)$ is étale. This is proved in the same way as in Step 1 of [10, Proposition 4.5.4].

Step 2. The map $\text{Def}(C_W) \rightarrow \text{Def}(U_W)$ is smooth, while the map $\text{Def}(U_W) \rightarrow \text{Def}_{nc}(U_W)$ is étale.

First, we observe that the maps on tangent spaces induced by these maps are

$$\text{Ext}^1(L_{C_W}, \mathcal{O}(-2D)) \rightarrow \text{Ext}^1(L_{U_W}, \mathcal{O}_{U_W}) \rightarrow \text{HH}^2(U_W),$$

the first of which is surjective by Lemma 2.3.7, while the second is an isomorphism by [10, Lemma 4.4.6]. Similarly the maps of obstruction spaces are

$$\text{Ext}^2(L_{C_W}, \mathcal{O}(-2D)) \rightarrow \text{Ext}^2(L_{U_W}, \mathcal{O}_{U_W}) \rightarrow \text{HH}^3(U_W),$$

of which the first is an isomorphism by Lemma 2.3.7, while the second is injective by [10, Lemma 4.4.6]. Hence, the maps $\text{Def}(C_W) \rightarrow \text{Def}(U_W)$ and $\text{Def}(U_W) \rightarrow \text{Def}_{nc}(U_W)$ are smooth and the second is étale (see [7, Proposition 2.17]).

Step 3. The map $\text{Def}(C_W) \rightarrow \text{Def}(\mathcal{C})$ (respectively, $\text{Def}(\mathcal{C}) \rightarrow \text{Def}_{nc}(U_W)$) induces a surjection (respectively, isomorphism) on tangent spaces.

Indeed, Step 2, together with the commutativity of diagram (2.3.4), implies that $\text{Def}(\mathcal{C}) \rightarrow \text{Def}_{nc}(U_W)$ induces a surjection on tangent spaces. But

$$\text{HH}^2(U_W) \simeq \text{HH}^2(C_W) \simeq \text{HH}^2(E_W),$$

so the dimensions of tangent spaces are the same. Hence, $\text{Def}(\mathcal{C}) \rightarrow \text{Def}_{nc}(U_W)$ induces an isomorphism on tangent spaces.

It follows that the maps induced on tangent spaces by $\text{Def}(C_W) \rightarrow \text{Def}(\mathcal{C})$ and by $\text{Def}(C_W) \rightarrow \text{Def}_{nc}(U_W)$ are isomorphic, so the required surjectivity follows from Step 2.

Step 4. The map $\text{Def}(C_W) \rightarrow \text{Def}(E_W/G(n - g, n))$ (respectively, $\text{Def}(E_W/G(n - g, n)) \rightarrow \text{Def}(E_W)$) induces an isomorphism (respectively, surjection) on tangent spaces.

Note that by Steps 1 and 3, we know that the map $\text{Def}(C_W) \rightarrow \text{Def}(E_W)$ induces a surjection on tangent spaces. Hence, the same is true for $\text{Def}(E_W/G(n - g, n)) \rightarrow \text{Def}(E_W)$. We claim that there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
 k^n & \xrightarrow{\alpha} & \text{Ext}^1(L_{C_W}, \mathcal{O}(-2D)) & \xrightarrow{\beta} & \text{HH}^2(C) & \longrightarrow & 0 \\
 \downarrow = & & \downarrow \gamma & & \downarrow \gamma' & & \\
 k^n & \xrightarrow{\alpha'} & \text{HH}^2(E_W)_{<0} \oplus T_W G(n - g, n) & \xrightarrow{\beta'} & \text{HH}^2(E_W)_{\leq 0} & \longrightarrow & 0
 \end{array} \tag{2.3.5}$$

where the arrow α (respectively, α') is induced by the \mathbb{G}_m^n -action on the functor $\text{Def}(C_W)$ (respectively, $\text{Def}(E_W/G(n - g, n))$), while the right commutative square is induced by the right commutative square in (2.3.4) (flipped about the diagonal). Note that we already know that γ' is an isomorphism and β' is surjective. To see the exactness of the top row we observe that by Steps 1 and 2, the map β can be identified with the morphism

$$\text{Ext}^1(L_{C_W}, \mathcal{O}(-2D)) \rightarrow \text{Ext}^1(L_{C_W}, \mathcal{O}(-D)) \xrightarrow{\sim} \text{Ext}^1(L_{U_W}, \mathcal{O}_{U_W}),$$

where the second arrow is an isomorphism by Lemma 2.3.7. Hence, its kernel is the image of the coboundary map $H^0(C_W, \mathcal{T}(-D)|_D) \rightarrow \text{Ext}^1(L_{C_W}, \mathcal{O}(-2D))$, which can be identified with α . The exactness of the bottom row in (2.3.5) would follow from the exactness in the middle of the sequence

$$k^n \rightarrow T_W G(n - g, n) \rightarrow \text{HH}^2(E_W)_0 \rightarrow 0,$$

where the second arrow is the tangent map to the map $W \rightarrow E_W$, and the first arrow corresponds to the \mathbb{G}_m^n -action on $G(n - g, n)$. But this follows from the observation that a $k[t]/(t^2)$ -point of $G(n - g, n)$, \mathcal{W} , can be recovered from the isomorphism class of the corresponding algebra $E_{\mathcal{W}}$ up to a \mathbb{G}_m^n -action.

Note that diagram (2.3.5), together with the fact that γ' is an isomorphism, immediately implies that γ is surjective. It remains to prove that the restriction of γ to $\text{im}(\alpha)$ is injective. To this end we use the fact that each point $C_W \in \tilde{U}_{g,n}^{ns}$ lies in the section $\sigma(G(n - g, n))$ of the projection to $G(n - g, n)$, and that the \mathbb{G}_m^n -orbit of C_W still lies in $\sigma(G(n - g, n))$. Hence, the tangent space to this orbit maps injectively to $T_W G(n - g, n)$, which implies our assertion.

Step 5. The morphisms $\text{Def}(C_W) \rightarrow \text{Def}_{nc}(U_W)$, $\text{Def}(C_W) \rightarrow \text{Def}(C)$ and $\text{Def}(C_W) \rightarrow \text{Def}(E_W)$ are smooth, and the morphism $\text{Def}(C_W) \rightarrow \text{Def}(E_W/G(n - g, n))$ is an isomorphism.

The first morphism is equal to the composition

$$\text{Def}(C_W) \rightarrow \text{Def}(U_W) \rightarrow \text{Def}_{nc}(U_W),$$

where both arrows are smooth by Step 2. But it is also equal to the composition

$$\text{Def}(C_W) \rightarrow \text{Def}(C) \rightarrow \text{Def}_{nc}(U_W).$$

By Step 3, the first arrow induces a surjection on tangent spaces. Hence, by [10, Lemma 4.5.3], the morphism $\text{Def}(C_W) \rightarrow \text{Def}(C)$ is smooth. Using Step 1 again we deduce that $\text{Def}(C_W) \rightarrow \text{Def}(E_W)$ is smooth. Thus, the composition

$$\text{Def}(C_W) \rightarrow \text{Def}(E_W/G(n - g, n)) \rightarrow \text{Def}(E_W)$$

is smooth, and the first arrow induces an isomorphism of tangent spaces. Hence, by [10, Lemma 4.5.3], the morphism $\text{Def}(C_W) \rightarrow \text{Def}(E_W/G(n - g, n))$ is smooth, and hence, étale. But the functor $\text{Def}(E_W/G(n - g, n))$ is homogeneous, since it is prorepresented by the formal completion of the scheme \mathcal{M}_∞ , so our morphism is an isomorphism by [7, Corollary 2.11]. \square

Theorem 2.3.11. *Assume that either $n \geq g \geq 1, n \geq 2$ and the base is $\text{Spec}(\mathbb{Z}[1/2])$, or $n = g = 1$ and the base is $\text{Spec}(\mathbb{Z}[1/6])$, or $g = 0, n \geq 2$ and the base is $\text{Spec}(\mathbb{Z})$. Then the morphism $\tilde{U}_{g,n}^{ns} \rightarrow \mathcal{M}_\infty$ (see (2.3.1)) is an isomorphism.*

Proof. We know that both schemes are affine of finite type over $G(n - g, n)$ (by Theorem 1.2.2 and Proposition 2.3.6), and that the morphism (2.3.1) is compatible with \mathbb{G}_m -action. Furthermore, the \mathbb{G}_m -invariant loci of each scheme provide a section of the projection to $G(n - g, n)$, and over each open affine cell $U_S \subset G(n - g, n)$ our morphism corresponds to a homomorphism $f : A \rightarrow B$ of nonnegatively graded algebras such that $f_0 : A_0 \rightarrow B_0$ is an isomorphism (here we use the fact that \mathbb{G}_m acts on $\mathcal{O}(\tilde{U}_{g,n}(S))$ with nonnegative weights by Theorem 1.2.2(ii)). Furthermore, by Proposition 2.3.10, for every closed point of $\text{Spec}(A_0) \simeq \text{Spec}(B_0)$, the map f induces an isomorphism of deformation functors. Hence, applying Lemma 2.3.12 below we deduce that f is an isomorphism. \square

Lemma 2.3.12. *Let $f : A \rightarrow B$ be a morphism of degree zero of nonnegatively graded commutative algebras such that the induced map $A_0 \rightarrow B_0$ is an isomorphism. Assume that A_0 is Noetherian, A and B are finitely generated as algebras over $A_0 \simeq B_0$, and for every maximal ideal $\mathfrak{m} \subset A_0$ the map f induces an isomorphism $\hat{A} \rightarrow \hat{B}$ of the completions with respect to the maximal ideals $\mathfrak{m} + A_{>0}$ and $\mathfrak{m} + B_{>0}$, respectively. Then f is an isomorphism.*

Proof. It is enough to prove that f induces an isomorphism $A/A_{>0}^N \rightarrow B/B_{>0}^N$ for each $N > 0$. Note that $A/A_{>0}^N$ (respectively, $B/B_{>0}^N$) is a finitely generated module over A_0 (respectively, B_0). Note that for any maximal ideal $\mathfrak{m} \subset A_0 \simeq B_0$, the $(\mathfrak{m} + A_{>0})$ -adic topology on $A/A_{>0}^N$ is equivalent to the \mathfrak{m} -adic topology, and similarly on $B/B_{>0}^N$. Thus, we have a morphism

$$A/A_{>0}^N \rightarrow B/B_{>0}^N$$

of finitely generated A_0 -modules, inducing an isomorphism of \mathfrak{m} -adic completions of localizations at every maximal ideal $\mathfrak{m} \subset A_0$. Since A_0 is Noetherian, such a morphism is an isomorphism. \square

Remark 2.3.13. For $g \geq 1$ let us define the open subset $U \subset G(n - g, n)$ to be the complement to the union of the images of n embeddings $G(n - g, n - 1) \subset G(n - g, n)$ associated with the coordinate hyperplanes $k^{n-1} \hookrightarrow k^n$. It is easy to see that the

preimage $\pi^{-1}(U) \subset \tilde{\mathcal{U}}_{g,n}^{ns}$ parametrizes $(C, p_\bullet, v_\bullet)$ such that $H^1(C, \mathcal{O}(D - p_i)) = 0$ for every i (where $D = p_1 + \dots + p_n$). Using Remark 2.3.4 one can see that the analog of Proposition 2.3.6 gives a relative moduli of A_∞ -structures on the family (E_W) over U , when working over $\text{Spec}(\mathbb{Z})$, provided $n \geq g + 2$. Similarly, Proposition 2.3.10 holds without any restrictions on the characteristic, for $W \in U$ and $n \geq g + 2$. This suggests that for $n \geq g + 2$, working over $\text{Spec}(\mathbb{Z})$, one could still show that the morphism $\pi^{-1}(U) \rightarrow U$ is affine of finite type. Then the analog of Theorem 2.3.11 would give an isomorphism of $\pi^{-1}(U)$ with the corresponding relative moduli of A_∞ -structures over U .

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